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**Title:** Distribution of Nodal Vacancies in Random Graphs:  
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# Distribution of Nodal Vacancies in Random Graphs: Connectivity Management for Sensor-Fusion and Mobile Networks

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## ABSTRACT

Distributed sensor networks are systems of sensor nodes interconnected by communication networks. The networks support advanced detection algorithms that use distributed computing to detect events as emergent phenomena of the system of sensors. These systems depend on connected and reliable communication networks. The systems are characterized typically by random nodal distribution and are directly related to mobile peer-to-peer networks having similar characteristics and problems. This paper explores models of these networks based on techniques of analyses of random graphs. Through evaluation of areas within the communication network uncovered by nodes, a model for connectivity of networks is developed.

## Keywords

distributed sensor network, mobile network, connectivity, random graph

## 1. INTRODUCTION

Distributed sensor networks (DSNs) are a new type of detection, computation, and communication object. Comprised of an array of discrete sensors, a communication network that links them, and a (possibly distributed) computing environment that combines the measurements of the individual sensors, a DSN supports new types of detection algorithms that can improve event detection probability, reduce false alarms, and collect new types of scientific data.

The Anti-Personnel Landmine Alternatives (APLA) Project at the Los Alamos National Laboratory (LANL) seeks to replace certain types of military landmines with an array of sensors across a battlefield, creating a DSN to detect the presence and movement of enemy forces and direct weapons to defeat them. This is an example of a DSN in which sensors are physically distributed at random locations across a geographic area, with the detection and information collection being an emergent property of the sensor system.

The communication network is vital to the operation of the DSN. For the APLA's DSN, the sensor nodes may be linked by radio communications. The network should be connected, so messages can be routed between any pair of sensor nodes. The network should be robust, so that alternate routes for messages are available and that the criticality of individual components to the operation of the network is minimized. And, if possible, the network should possess interconnection so that routes between nodes need not be circuitous, minimizing the nodal computation and information storage

needed to complete message delivery. As an example of the management of network connectivity, consider that there will be a tradeoff between transmitter power and battery life at the sensor nodes that can be optimized. Important to the optimization of transmitter power to maximize battery life is an understanding of the relationship between transmitter power, communication range, and the connectivity necessary to adjoin the sensor nodes so that signals can be routed efficiently through the communication network.

These problems of DSN optimization are related to recent work in the field of mobile communication networks. As early as 1978, Kleinrock and Silvester [5] developed models for mobile networks that are related to the problems of DSNs. More recently, work has considered message routing by broadcast percolation [1], connectivity as a function of nodal communication distance [6], and optimization criteria for nodal communication distance [9]. Other work in clustering of random nodal phenomena [4] and modeling of mobile networks as a time series of random graphs [8] is also related to this problem.

This paper presents a model for the connectivity of random networks (such as occur in DSNs and mobile networks) based on the distribution of nodal vacancies within the area served by the network. This model is applied to the evaluation of the connectivity of networks.

## 2. NODE AND NODAL-VACANCY DISTRIBUTION OF A RANDOM GRAPH

The communication network of a DSN is a random graph comprised of a set of nodes or vertices  $V$  embedded randomly in a two-dimensional plane (a geographic map) or on a three-dimensional surface (a geographic map with terrain elevation features) such that the undirected edges  $E$  connecting these vertices exist if the distance between two vertices is less than some maximum-range parameter and (for the three-dimensional surface) the line-of-sight between the two vertices is unobstructed. Note that a distributed sensor network typically is comprised of a set of fixed vertices, whereas in mobile peer-to-peer networks the vertices are able to move to different locations over time. In either case, the set of edges and the resultant network connectivity are emergent properties of the graph that are determined by the locations and communication capability of the vertices. The maximum-range parameter,  $R$ , represents the maximum distance over which a node's transmitter can communicate effectively.

Many characteristics of random-graph communication networks can be evaluated by simulation. Figures 1 and 2 illustrate a DSN graph visualization tool that has been developed by the APLA as a Java™ applet for the world-wide web. This applet is presently available on the LANL website, at <http://public.lanl.gov/u106527/DSN/GrphDemo/GrphDemo.html>. These figures demonstrate the utility of this tool in evaluating the communication network as a function of the nodal communication distance,  $R$ . Clearly, if  $R$  is too small, then the network will not be connected, as shown in Fig. 2. The results of the simulations using this graph visualization tool have been validated through the mathematical models presented in this paper.

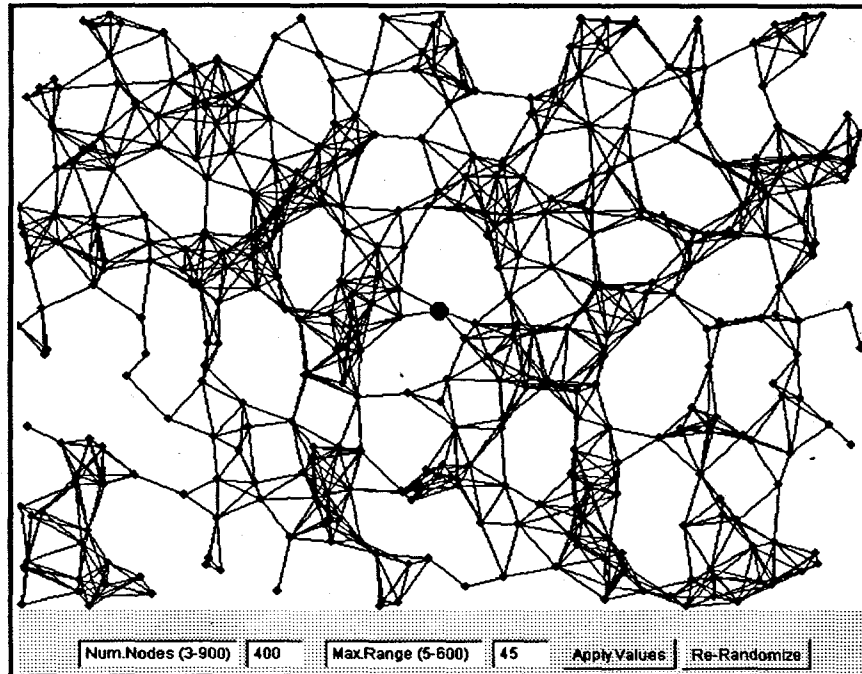


Figure 1. There is a relationship between node density and communication range for a graph to be connected with high probability.

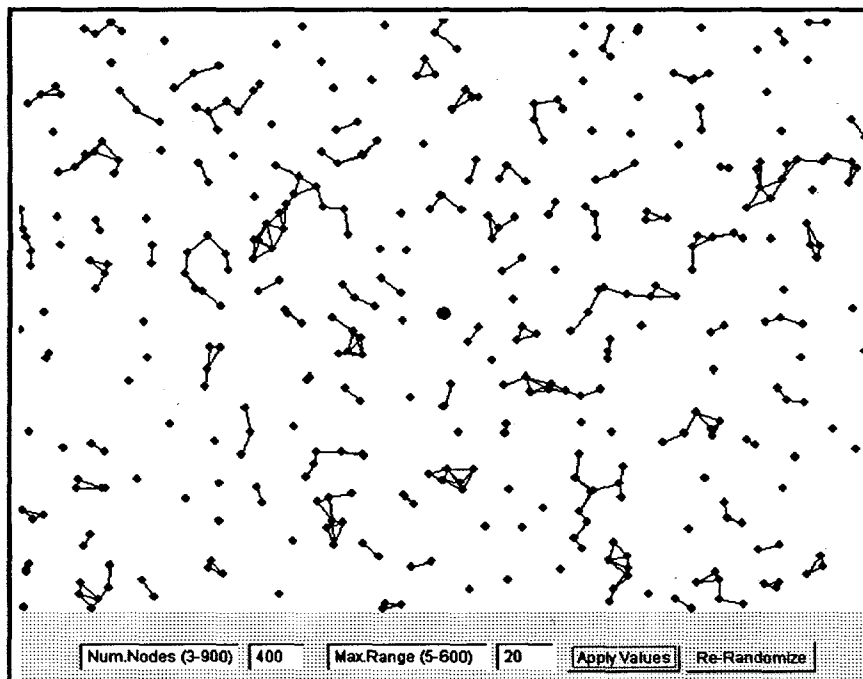


Figure 2. A graph with good node density but poor communication range may be disconnected.

The random characteristics of the node locations can be modeled by a probability density function describing nodes' locations. For example, consider the nodes to be distributed across an area  $A$  with uniform probability per unit area  $\alpha$ . The probability of the existence of the node is 1, so

$$\int_A \alpha dA = 1 \Rightarrow \alpha = \frac{1}{A} \quad (1)$$

This expression can be used to derive the probability for a node's nearest neighbor. Then the probability that the first neighbor of a node is at a distance between  $r$  and  $r+dr$  is the joint probability of observing a neighbor at this distance and the probability that there are no other nodes closer than the distance  $r$ . The expression of this probability uses a combinatorial enumeration of the possible number of nodes in the two regions. This enumeration is a function of the number of nodes,  $N$ , that are randomly placed in  $A$  to make up the DSN.

$$\begin{aligned} \Pr(1st|_r^{r+dr}) &= \Pr(0|_r) \cdot \Pr(any|_r^{r+dr}) \\ &= [1 - \Pr(any|_r)] \cdot \Pr(any|_r^{r+dr}) \\ &= \left[ 1 - \sum_{i=1}^N \binom{N}{i} (\alpha\pi r^2)^i (1 - \alpha\pi r^2)^{N-i} \right] \\ &\quad \cdot \sum_{i=1}^N \binom{N}{i} \int_r^{r+dr} (2\alpha\pi x dx)^i \int_r^{r+dr} (1 - 2\alpha\pi x dx)^{N-i} \\ &= (1 - \alpha\pi r^2)^N \cdot \left[ 1 - (1 - \alpha\pi(2rdr + dr^2))^N \right] \end{aligned} \quad (2)$$

The probability density function for the nearest-neighbor distance,  $P(r)$ , is found by using a binomial expansion of the terms involving  $dr$  and taking the limit as  $dr$  becomes vanishingly small.

$$\begin{aligned} P(r) &= \lim_{dr \rightarrow 0} \frac{\Pr(1st|_r^{r+dr})}{dr} \\ &= 2N\alpha\pi(1 - \alpha\pi r^2)^N \end{aligned} \quad (3)$$

Then the expected nearest-neighbor distance,  $\langle r \rangle$ , is found by integrating the product of the distance and the probability density function for the nearest-neighbor distance, as shown in Eq. (4).  $R(A)$  is the distance at which the entire area,  $A$ , over which the nodes are distributed has been considered.

$$\begin{aligned} \langle r \rangle &= \int_0^{R(A)} r P(r) dr = \int_0^{R(A)} 2N\alpha\pi r^2 (1 - \alpha\pi r^2)^N dr \\ &= \frac{A^{\frac{1}{2}}}{2\alpha\pi^{\frac{3}{2}}(N+1)} \sum_{k=0}^{N+1} \frac{(-1)^k}{2k+1} \end{aligned} \quad (4)$$

A nodal vacancy is a region within the area of the network containing no nodes and that is large enough to prevent communication between nodes on either side of the area. The probability that an area  $B$  within  $A$  contains no nodes is

$$\Pr(0, B) = (1 - \alpha B)^N. \quad (5)$$

If there is an area containing no nodes and the extent of that area exceeds the communication distance of the neighboring nodes, then no communication will occur across that area. The area will constitute a "hole" in the network, requiring some messages to be routed longer distances to bypass the hole or possibly disconnecting the network by dividing it into two unconnected subnetworks.

### 3. CONNECTIVITY MANAGEMENT FOR RANDOM-GRAPH NETWORKS

A pattern of holes is necessary to disconnect the graph. Figure 3 illustrates a model for computing the probability of non-trivial network disconnection (e.g., more than one node disconnected from the remainder of the graph). In this model, a node is capable of communicating with neighbors if they are within the node's communication radius. For simplicity of tiling, let this radius be approximated by a hexagon, as shown in the figure. Then a "hole" in the DSN is an area containing no nodes such that no communication edges (drawn as straight lines between adjacent neighbors) cross this area. There are several ways such holes can be described, but for simplicity let us partition the DSN's area of coverage by a pattern of regular hexagons of diameter equal to the nodes' communication radius. With this approximation, a simple description for the probability of a hole (comprised of a set of unoccupied hexagons) capable of dividing the network into two disconnected subgraphs can be derived.

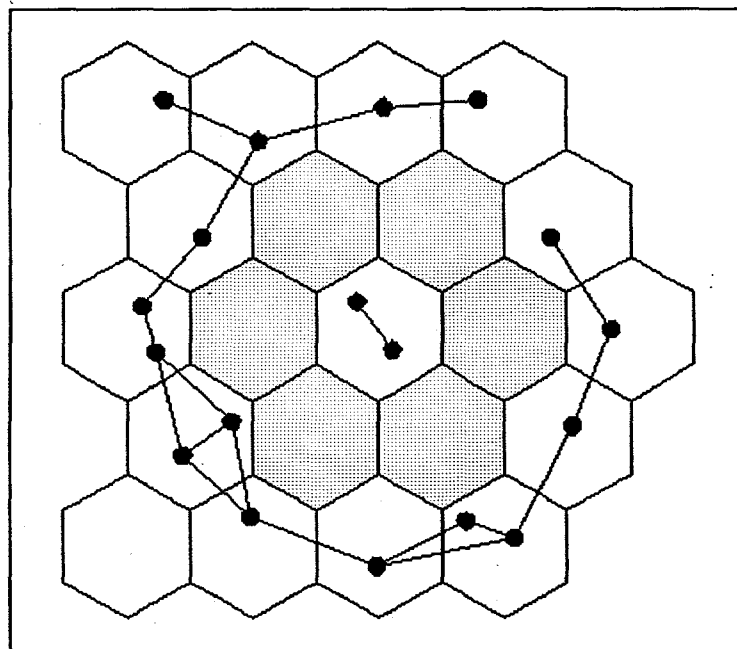


Figure 3. The presence of "holes" in a DSN (indicated by shaded hexagons) is a necessary condition for the network to be disconnected.



The probability that one hexagon is unoccupied by nodes is

$$\Pr(0) = (1 - \alpha A_H)^N = \left(1 - \frac{3\sqrt{3}}{4} \alpha R^2\right)^N, \quad (6)$$

where  $A_H$  is the area of the hexagon. Then the joint probability that six such unoccupied hexagons are arranged in the pattern shown in Fig. 5 (the minimum number of unoccupied hexagons away from the edges of the DSN area capable of causing a disconnected network) is

$$\Pr\left(\begin{array}{c} \text{six disconnecting} \\ \text{hexagons} \end{array}\right) = \left[\left(1 - \frac{3\sqrt{3}}{4} \alpha R^2\right)^N\right]^6. \quad (7)$$

Note that this joint probability becomes vanishingly small with  $R$  and  $N$  sufficiently large. (Although there are many possible locations for such a set of disconnecting holes in the DSN, the probability of the occurrence of each of them is the same and will be a function of the expression given in Eq. 7. This expression is a useful metric for management of the possibility of disconnection of a particular DSN.) Thus, for sufficient  $R$  and  $N$ , nearly all DSNs with those parameters will be expected to be connected.

Average Fraction of Graph Connected					
Nodes	Range				
	10	20	50	100	200
10	0.10	0.10	0.12	0.17	0.56
20	0.05	0.05	0.07	0.16	0.83
50	0.02	0.02	0.04	0.38	1.00
100	0.01	0.01	0.04	0.92	1.00
200	0.01	0.01	0.16	1.00	1.00
500	0.00	0.01	0.99	1.00	1.00

Table 1. Simulation reveals the relationship between the number of nodes, communication range, and probability of network connectedness.

Through simulation, we evaluated a number of networks, varying both the number of sensors and the nodal maximum communication radius to evaluate the dependence of network connectivity on these design parameters. Table 1 shows results of these simulations. As expected, the probability that all nodes in the DSN were connected increased as the nodal maximum communication radius increased. Below a finite radius threshold, few of the networks were completely connected. However, above this threshold nearly all the networks were connected. This radius threshold was larger for sparse DSNs containing fewer nodes in an area.

Table 2 compares the simulation of network connectivity with the probability that a particular "hole" exists (Eq. 6). The simulation recorded the fraction of the graph that was connected, for various values of  $N$  and  $R$ , and computed the average over 200 trials for each set of parameter values. These are the results shown in Table 1. Table 2 compares the value of the expression of Eq. 6 (the probability that a nodal vacancy exists that is capable of contributing to a network disconnection) for values of  $R$  below and above the radius threshold above which nearly all networks were connected. As shown in this table, this radius threshold corresponds to a dramatic decrease in the hole probability. When the hole probability became less than approximately 0.003, nearly all networks were connected regardless of the

values of  $N$  and  $R$ . (These simulations were performed for a 640-by-450-unit rectangular area which determined the value of  $\alpha$ .)

N	R	Probability that Network is Connected	$\left(1 - \frac{3\sqrt{3}}{4}\alpha R^2\right)^N$
50	100	0.38	0.0995
50	200	1.00	0.0000478
100	50	0.04	0.322
100	100	0.92	0.00989
200	50	0.16	0.104
200	100	1.00	0.000098
500	20	0.01	0.405
500	50	0.99	0.00345

Table 2. Comparison of the probability that a graph is connected with the probability that a particular "hole" exists (Eq. 6), as functions of  $N$  and  $R$ .

#### 4. FURTHER WORK AND CONCLUSIONS

This paper presented a mathematical model for the distribution of nodes and nodal vacancies in random graphs. It used this model in the evaluation of the connectivity of networks and demonstrated an application to connectivity management for DSNs and peer-to-peer mobile networks.

Additional work is necessary to develop further the model of nodal vacancies. This work includes the development of techniques for evaluating holes of irregular shape, combinatorics of the number of possible hole configurations for disconnecting a network, and models for the expected path length and number of retransmissions between pairs of nodes in the DSN. Further, this work can be used to determine the circumstances necessary for the use of a depth-first message-routing algorithm, offering advantages of reduced computation and storage of particular value to the limited resources of DSN and peer-to-peer mobile-network nodes.

#### 5. ACKNOWLEDGMENTS

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