SAND2000-1358C

On the Discontinuity of the Costates for Optimal Control Problems With Coloumb Friction[1]

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Abstract: This work points out that the costates are actually discontinuous functions of time for optimal control problems with Coloumb friction. In particular these discontinuities occur at the time points where the velocity of the system changes sign. To our knowledge, this has not been noted before. This phenomenon is demonstrated on a minimum-time problem with Coloumb friction and the consistency of discontinuous costates and switching functions with respect to the input switches is shown.

1. Introduction

Most optimal control research work has dealt with systems in which the state equation is continuous in the states. Exceptions include the work of [Willigenburg and Loop, 1991] who included Coloumb friction when generating minimum-time trajectories of a rigid two-link robotic arm. Since the costates for such a problem are discontinuous, as we shall see, the switching functions should also be discontinuous. However, no such discontinuities could be seen in the plots given in [Willigenburg and Loop, 1991]. It is conceivable that the discrepancies between the input switches and the switching functions observed by these researchers could be explained by the fact that they did not include the "jumps" or discontinuities in the costates.

Herein we will explain why one might expect discontinuities in the costates, and we will demonstrate that the calculated discontinuities yield consistency between the switching functions and the input switches for a minimum-time problem.

2. Motivation for Discontinuous Costates

The reason to expect discontinuities in the costates is best explained with a simple example. Consider the following optimal control problem.

$$\text{Min}\, t_f \qquad (1)$$

subject to

$$\ddot{q} = u - \text{sign}(\dot{q}) \qquad (2)$$

$$q(0) = q_0 \qquad (3)$$

$$\dot{q}(0) = v_0 < 0 \qquad (4)$$

$$q(t_f) = q_f > q_0 \qquad (5)$$

$$\dot{q}(t_f) = 0 \qquad (6)$$

$$-u_{\max} \le u \le u_{\max} > 1 \qquad (7)$$

Letting $x_1 = q$ and $x_2 = \dot{q}$, the state equation is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ u - \text{sign}(x_2) \end{pmatrix} \qquad (8)$$

The Hamiltonian is

$$H = 1 + \lambda_1 x_2 + \lambda_2 u - \lambda_2 sign(x_2) \qquad (9)$$

where λ_1 and λ_2 are the costates. Then,

$$\frac{\partial H}{\partial x_2} = \lambda_1 - 2\lambda_2 \delta(x_2) \qquad (10)$$

where $\delta(\cdot)$ is the dirac delta function, which appears because of the infinite derivative of the signum function. The factor of 2 in (10) appears because the magnitude of

[1] Sandia is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the United States Department of Energy under Contract DE-AC04-94AL85000.

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the instantaneous change in the signum function is 2. Since

$$\dot{\lambda} = -\frac{\partial H}{\partial x} \tag{11}$$

we have

$$\dot{\lambda}_1 = 0 \tag{12}$$

$$\dot{\lambda}_2 = -\lambda_1 + 2\lambda_2\delta(x_2) \tag{13}$$

3. Numerical Demonstration of Consistency of Discontinuous Costates

In this section we will solve the optimal control problem of section 2. Consider the case of $u_{max} = 2$, $v_0 = -1$, and $q_f = 1$. Figure 1 below shows the minimum-time input, states, and costates.

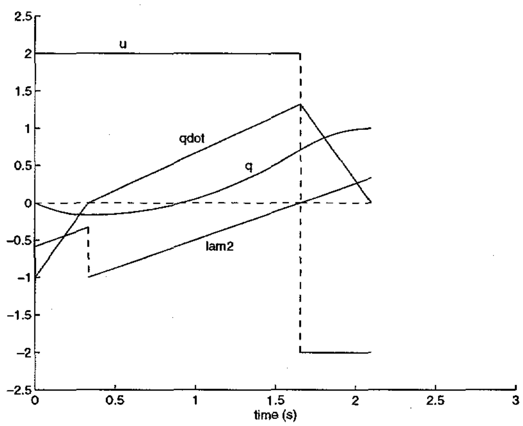

Figure 1. Input, States, and Costates Versus Time

We see in Figure 1 that the input is bang-bang, switching from its maximum value to its minimum value. Let t_1 be the time the velocity $\dot{q}$ changes sign and t_s the time the input switches. From time zero to the time $t_1 = 0.3333s$ at which the velocity changes sign, the acceleration is at its maximum possible magnitude of $|-u_{max} - 1| = |-3|$, where the friction force actually helps the input bring the velocity to zero.

From t_1 to the input switch time $t_s = 1.656s$, the acceleration is only $u_{max} - 1 = 1$, where the friction force is now opposing the input. Finally, after t_s, the friction force aids the input again in bringing the velocity to zero at the final time t_f.

Using the fact that the value of $\lambda(0)$ is given by the derivative of the cost-to-go with the initial state, one can calculate $\lambda(0)$ to be:

$$\lambda(0) = \begin{pmatrix} -0.7559 \\ -0.5853 \end{pmatrix} \tag{14}$$

Hence, we have from (13):

$$\dot{\lambda}_2 = 0.7559 + 2\lambda_2\delta(t - t_1) \tag{15}$$

and

$$\lambda_2(0) = -0.5853 \tag{16}$$

The value of $\lambda_2(t_1^-)$ just before the velocity changes sign is from (15) and (16):

$$\lambda_2(t_1^-) = -1/3 \tag{17}$$

which is half the value of the jump in λ_2. In Figure 1, we observe this discontinuity in λ_2. Clearly, without this jump in λ_2, λ_2 would not be zero at the switching time t_s.

4. Conclusion

We demonstrated that the costates and switching functions are not continuous functions of time for problems with Coloumb friction, because the derivative of the signum function is $2\delta(t - t_1)$ where t_1 is the time at which the argument of the signum funrction changes sign. This may explain discrepancies observed in previous work in which switching function appeared to have been calculated to be continuous despite the presence of Coloumb friction and changes in the velocity's sign.

References

[1] Willigenbug, L. and Loop, R., "Computation of Time-Optimal Controls Applied to Rigid Manipulators With Friction," *International Journal of Control*, Vol. 54, 1991, pp. 1097-1117.