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Minimum-Time Control of Systems With Coloumb Friction: Near Global Optima Via Mixed Integer Linear Programming#

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Abstract: This work presents a method of finding near global optima to minimum-time trajectory generation problem for systems that would be linear if it were not for the presence of Coloumb friction. The required final state of the system is assumed to be maintainable by the system, and the input bounds are assumed to be large enough so that they can overcome the maximum static Coloumb friction force. Other than the previous work for generating minimum-time trajectories for non redundant robotic manipulators for which the path in joint space is already specified, this work represents, to the best of our knowledge, the first approach for generating near global optima for minimum-time problems involving a nonlinear class of dynamic systems. The reason the optima generated are near global optima instead of exactly global optima is due to a discrete-time approximation of the system (which is usually used anyway to simulate such a system numerically). The method closely resembles previous methods for generating minimum-time trajectories for linear systems, where the core operation is the solution of a Phase I linear programming problem. For the nonlinear systems considered herein, the core operation is instead the solution of a mixed integer linear programming problem.

1. Introduction

The problem of generating minimum-time control for linear dynamic systems has been studied fairly extensively. The work in [1], [8], [10], [13], [14], [22], [6], [25], and [4] used a fixed-size time step. Starting with one time-step and increasing to 2, 3, 4, etc. time steps until a phase I linear programming algorithm detected that the resulting linear program was feasible, they thereby obtained the minimum-time to within roughly the size of the time step Δt. In [9], however, a binary search on the final time was used to allow the algorithm to bisect or zero in on the minimum time more efficiently.

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The minimum-time control problem with Coloumb friction actually involves a *nonlinear* dynamic system. The nonlinearity occurs due to the presence of a $\text{sign}(\dot{q})$ term in the state equation which arises from the kinetic (sliding) friction. Herein we avoid the need to model the static Coloumb friction because we restrict ourselves to the class of problems where the input bounds are large enough so that the input is always capable of overcoming the static Coloumb friction force. Other than the well-known work of Bobrow ([2]-[3]) and Shin and McKay ([15]-[18]) on minimum-time trajectory generation for path-specified non redundant robotic manipulators, this paper represents, to the best of our knowledge, the first approach for generating near global optima for minimum-time trajectory generation problems for a class of nonlinear dynamic systems. Specifically, the approach we present works for dynamic systems that would be linear if the kinetic (sliding) Coloumb friction were not present, have a specified final state that is maintainable by the system, and have input bounds that are large enough so that the input can always overcome the static Coloumb friction forces to prevention sticking.

2. Problem Statement

We are given a mechanical system whose governing equations are of the form:

$$M\ddot{q} = -C\dot{q} - Kq + F + Du \tag{1}$$

where $q \in R^n$, $u \in R^r$, and where M, C, K, and D are constant matrices and F is a Coloumb friction force with components of the form:

$$F_i = -\mu \text{sign}(\dot{q}_i) \tag{2}$$

where $F_i = -\mu \text{sign}(\dot{q}_i)$ means $F_i = -\mu$ if $\dot{q}_i > 0$, $F_i = \mu$ if $\dot{q}_i < 0$ and F_i is an unspecified static friction force if $\dot{q}_i = 0$. There is a given initial state:

$$\begin{Bmatrix} q(0) \\ \dot{q}(0) \end{Bmatrix} = \begin{Bmatrix} q_0 \\ \dot{q}_0 \end{Bmatrix} \tag{3}$$

and a specified maintainable final state that must be reached at the unknown final time t_f:

$$\begin{Bmatrix} q(t_f) \\ \dot{q}(t_f) \end{Bmatrix} = \begin{Bmatrix} q_{f,des} \\ \dot{q}_{f,des} \end{Bmatrix} \tag{4}$$

Each input is constrained to be between its lower and upper bounds:

$$u_{i,\min} \le u_i \le u_{i,\max} \tag{5}$$

The objective is to find $u(t)$ that satisfies (1)-(5) and minimizes the total trajectory execution time t_f.

3. Method of Solution

We will first bring the problem to discrete-time state-space form. Let $x_1 = q$ and $x_2 = \dot{q}$ and $x = (x_1{}^T, x_2{}^T)^T$. Then (1) and (2) combined become

$$\dot{x} = \overline{A}x + \overline{B}u + \overline{G}\text{sign}(x_2) \tag{6}$$

where $\overline{A} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}$ and $\overline{B} = \begin{bmatrix} 0 \\ M^{-1}D \end{bmatrix}$ and $\overline{G} = \begin{bmatrix} 0 \\ -\mu M^{-1} \end{bmatrix}$. State equation (6) can be brought to discrete-time form by using an Euler (or other) integration scheme:

$$x_{k+1} = Ax_k + Bu_k + G\text{sign}(x_{2,k}),\ (k=1,...,N) \tag{7}$$

where N is the number of time-steps and where $u(t)$ has been discretized into a stair-step time history and where $x_{2,k}$ denotes $\dot{q}_k$ and where A, B, and G are matrices that depend upon the sampling period $h = t_f / N$. We still have the given initial state:

$$x_1 = \begin{pmatrix} q_0 \\ \dot{q}_0 \end{pmatrix} \tag{8}$$

and the required maintainable final state:

$$x_{N+1} = \begin{pmatrix} q_{f,des} \\ \dot{q}_{f,des} \end{pmatrix} \tag{9}$$

and the input constraint (5) becomes

$$u_{\min} \le u_k \le u_{\max},\ (k=1,...,N) \tag{10}$$

We note that the state equation (7) is *not linear* due to the presence of the $G\text{sign}(x_{2,k})$ nonlinear Coloumb friction term. We now proceed nonetheless to formulate a mixed integer linear programming problem (MILP) whose solution will tell us whether the set of equations (7)-(10) has a solution. Let w_{ki}, (i=1,...,n) denote the ith component of the n-vector $-\text{sign}(x_{2,k})$, i.e., $w_{ki} = -\text{sign}(x_{2,ki})$, ($k$=1,...,$N$), ($i$=1,...,$n$). Thus (7) becomes:

$$x_{k+1} = Ax_k + Bu_k - Gw_k \tag{11}$$

We note that (8)-(11) represent a system of linear equations and inequalities and that it is necessary (but not sufficient) that this system have a solution for the chosen final time $t_f = hN$ which determined A, B, and G, in order for (7)-(10) to have a solution. It is not sufficient because (8)-(11) do not specify the requirement that $w_k = -\text{sign}(x_{2,k})$. Let the s_{ki} and v_{ki} be integer variables with the values 0 or 1, i.e.,

$$s_{ki} \in \{0,1\} \tag{12}$$

$$v_{ki} \in \{0,1\} \tag{13}$$

and impose

$$s_{ki} + v_{ki} = 1 \tag{14}$$

so that either $s_{ki} = 0$ and $v_{ki} = 1$ or $s_{ki} = 1$ and $v_{ki} = 0$. Now let

$$w_{ki} = s_{ki} - v_{ki} \tag{15}$$

Equations (12)-(15) imply that w_{ki} will be either -1 or 1, as required. To get w_{ki} to take on the *correct* value, we additionally impose

$$x_{2,ki} \le \left(\frac{x_{2,ki,\max}}{2} \right)(1 - w_{ki}) \tag{16}$$

$$x_{2,ki} \ge \left(\frac{x_{2,ki,\min}}{2} \right)(w_{ki} + 1) \tag{17}$$

where $x_{2,ki,\min}$ and $x_{2,ki,\max}$ are a priori assumed or known bounds on the velocities. Notice that if $w = -1$, (16) and (17) imply $0 \le x_{2,ki} \le x_{2,ki,\max}$ and if $w = 1$ (16) and (17) imply $x_{2,ki,\min} \le x_{2,ki} \le 0$. Thus (12)-(17) are satisfied if and only if $w_{ki} = -\text{sign}(x_{2,ki})$ as required. In summary, (8)-(11) have a solution with $w_{ki} = -\text{sign}(x_{2,ki})$ if and only if (8)-(17) have a solution. Determining whether (8)-(17) has a solution is a mixed integer linear programming problem (MILP) for which there are well established algorithms and software. These algorithms are guaranteed to find a solution to (8)-(17) if one exists; and if a solution does not exist, these algorithms return a flag saying so.

So, for a fixed final time t_f, determination of the feasibility of (8)-(11) with $w_k = -\text{sign}(x_{2,k})$ is accomplished by applying MILP to (8)-(17). Finally, a simple binary search on t_f (i.e., a bisection algorithm with repeated calls to the MILP solver) can be used to test feasibility/infeasibility of a given final time t_f. It *must* be emphasized that, so long as the terminal state $\begin{Bmatrix} q_{f,des} \\ \dot{q}_{f,des} \end{Bmatrix}$ is maintainable by the dynamic system, the above approach is *guaranteed* to produce a globally optimal solution to the minimum-time problem, within the accuracy of the discrete-time approximation (7) and the tolerance set on t_f in the bisection outer loop of the method.

4. Numerical Examples

The first example is a single mass with the equation of motion:

$$\ddot{q} = u - sign(\dot{q}) \tag{18}$$

with $u_{min} = -2$, $u_{max} = 2$, $q_0 = 0$, $\dot{q}_0 = 1$, $q_{f,des} = -1$, $\dot{q}_{f,des} = 0$. The problem was brought to discrete-time form (7) using Euler integration with 50 time steps, and the bisection tolerance on t_f was 1×10^{-4}. The resulting input history and position history are shown below in Figures 1 and 2.

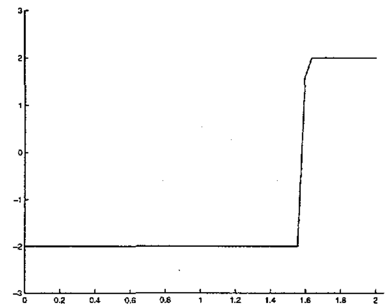

Figure 1. Input versus time, single mass example

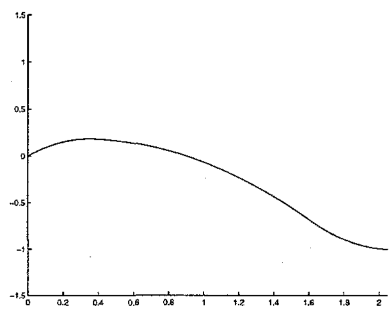

Figure 2. Position versus time for single mass example

The second example is a double spring-mass problem, illustrated below in Figure 3.

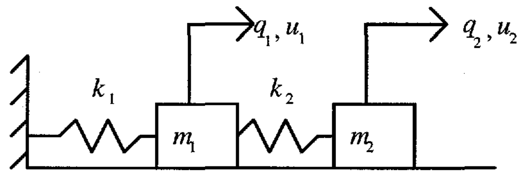

Figure 3. Schematic of double spring-mass problem

The problem parameters are $k_1 = 0.95$, $k_2 = 0.85$, $m_1 = 1.1$, $m_2 = 1.2$, $\mu = 1.0$. The equations of motion are:

$$m_1\ddot{q}_1 = (-k_1 - k_2)q_1 + k_2 q_2 - \mu\text{sign}(\dot{q}_1) + u_1 \tag{19}$$

$$m_2\ddot{q}_2 = k_2 q_1 - k_2 q_2 - \mu\text{sign}(\dot{q}_2) + u_2 \tag{20}$$

The initial condition was $q_1(0) = 0$, $q_2(0) = 0$, $\dot{q}_1(0) = -1.0$, $\dot{q}_2(0) = -2.0$, and the required final state was $q_1(t_f) = 1.0$, $q_2(t_f) = 2.0$, $\dot{q}_1(t_f) = 0$, $\dot{q}_2(t_f) = 0$. The input bounds were $u_{1,\min} = -4.0$, $u_{1,\max} = 4.0$,

$u_{2,\min} = -4.0$, $u_{2,\max} = 4.0$. The problem was brought to discrete-time form (7) using Euler integration with 50 time steps, and the bisection tolerance on t_f was 1×10^{-4}. The resulting input histories and position histories are shown in Figures 4 through 6 below.

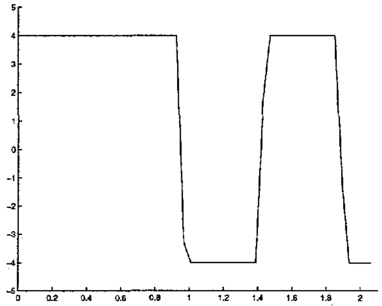

Figure 4. Input 1 versus time for double spring-mass example

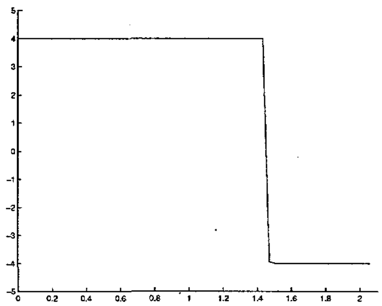

Figure 5. Input 2 versus time for double spring-mass example

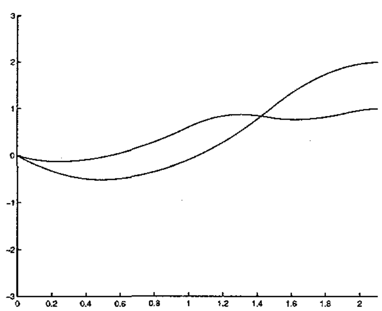

Figure 6. Positions versus time for double spring-mass example

5. Conclusion

This work presented an approach for obtaining near global optima to a class of nonlinear minimum-time trajectory generation problems. The class is those for which the dynamic system would be linear if it were not for the presence of the kinetic (sliding) Coloumb friction term, the required final state is maintainable by the system, and the input bounds are large enough that the inputs can overcome a static Coloumb friction force. Other than the previous work of Bobrow ([2]-[3]) and Shin and McKay ([15]-

[18]), who studied non redundant robotic arms whose path in joint space was completely specified, this work represents, to the best of our knowledge, the first method of obtaining near global optima to a class of minimum-time problems in which the dynamic system is *nonlinear*. The reason near global optima, instead of exactly global optima, are obtained is simply due to the fact that a discrete-time approximation is used to approximate the continuous-time system (which is usually used anyway to simulate a continuous-time system numerically).

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