

**VARIATIONAL APPROACH IN WAVELET FRAMEWORK
TO POLYNOMIAL APPROXIMATIONS OF NONLINEAR
ACCELERATOR PROBLEMS**

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Abstract. In this paper we present applications of methods from wavelet analysis to polynomial approximations for a number of accelerator physics problems. According to a variational approach in the general case we have the solution as a multiresolution (multiscales) expansion on the base of compactly supported wavelet basis. We give an extension of our results to the cases of periodic orbital particle motion and arbitrary variable coefficients. Then we consider more flexible variational method which is based on a biorthogonal wavelet approach. Also we consider a different variational approach, which is applied to each scale.

I INTRODUCTION

This is the first part of our two-part presentation in which we consider applications of methods from wavelet analysis to nonlinear accelerator physics problems. This is a continuation of results from [1]-[6], which is based on our approach to investigation of nonlinear problems – general, with additional structures (Hamiltonian, symplectic or quasicomplex), chaotic, quasiclassical, quantum, which are considered in the framework of local (nonlinear) Fourier analysis, or wavelet analysis. Wavelet analysis is a relatively novel set of mathematical methods, which gives us the possibility of working with well-localized bases in functional spaces and with the general type of operators (differential, integral, pseudodifferential) in such bases.

We consider the application of multiresolution representation to a general nonlinear dynamical system with the polynomial type of nonlinearities. In part II we

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consider this very useful approximation in the cases of orbital motion in a storage ring, a particle in the multipolar field, effects of insertion devices on beam dynamics, and spin orbital motion. Starting in part III A from variational formulation of initial dynamical problem we construct via multiresolution analysis (part III B) explicit representation for all dynamical variables in the base of compactly supported (Daubechies) wavelets. Our solutions (part III C) are parametrized by solutions of a number of reduced algebraical problems, one of which is nonlinear with the same degree of nonlinearity, and the rest are the linear problems which correspond to a particular method of calculation of scalar products of functions from wavelet bases and their derivatives. Then we consider the further extension of our previous results. In part V we consider modification of our construction to the periodic case; in part VI we consider generalization of our approach to variational formulation in the biorthogonal bases of compactly supported wavelets, and in part VII to the case of variable coefficients. In part IV we consider the different variational approach which is based on ideas of para-products (A) and approximation for a multiresolution approach, which gives us the possibility for computations in each scale separately (B).

II PROBLEMS AND APPROXIMATIONS

We consider below a number of examples of nonlinear accelerator physics problems which are from the formal mathematical point of view not more than nonlinear differential equations with polynomial nonlinearities and variable coefficients.

A Orbital Motion in Storage Rings

We consider as the main example the particle motion in storage rings in a standard approach, which is based on consideration of [7]. Starting from Hamiltonian, which described classical dynamics in storage rings,

$$\mathcal{H}(\vec{r}, \vec{P}, t) = c\{\pi^2 + m_0^2 c^2\}^{1/2} + e\phi \quad , \quad (1)$$

and using Serret–Frenet parametrization, we have the following Hamiltonian for orbital motion in machine coordinates:

$$\begin{aligned} \mathcal{H}(x, p_x, z, p_z, \sigma, p_\sigma; s) = & p_\sigma - [1 + f(p_\sigma)] \cdot [1 + K_x \cdot x + K_z \cdot z] \times \quad (2) \\ & \left\{ 1 - \frac{[p_x + H \cdot z]^2 + [p_z - H \cdot x]^2}{[1 + f(p_\sigma)]^2} \right\}^{1/2} \\ & + \frac{1}{2} \cdot [1 + K_x \cdot x + K_z \cdot z]^2 - \frac{1}{2} \cdot g \cdot (z^2 - x^2) - N \cdot xz \\ & + \frac{\lambda}{6} \cdot (x^3 - 3xz^2) + \frac{\mu}{24} \cdot (z^4 - 6x^2z^2 + x^4) \\ & + \frac{1}{\beta_0^2} \cdot \frac{L}{2\pi \cdot h} \cdot \frac{eV(s)}{E_0} \cdot \cos \left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi \right] \quad . \end{aligned}$$

Then, after standard manipulations with truncation of power series expansion of square root, we arrive at the following approximated Hamiltonian for particle motion:

$$\begin{aligned}\mathcal{H} = & \frac{1}{2} \cdot \frac{[p_x + H \cdot z]^2 + [p_z - H \cdot x]^2}{[1 + f(p_\sigma)]} + p_\sigma - [1 + K_x \cdot x + K_z \cdot z] \\ & \cdot f(p_\sigma) + \frac{1}{2} \cdot [K_x^2 + g] \cdot x^2 + \frac{1}{2} \cdot [K_z^2 - g] \cdot z^2 - N \cdot xz + \\ & \frac{\lambda}{6} \cdot (x^3 - 3xz^2) + \frac{\mu}{24} \cdot (z^4 - 6x^2z^2 + x^4) \\ & + \frac{1}{\beta_0^2} \cdot \frac{L}{2\pi \cdot h} \cdot \frac{eV(s)}{E_0} \cdot \cos \left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi \right] \quad ,\end{aligned}\quad (3)$$

and the corresponding equations of motion:

$$\begin{aligned}\frac{d}{ds}x = & \frac{\partial \mathcal{H}}{\partial p_x} = \frac{p_x + H \cdot z}{[1 + f(p_\sigma)]}; \\ \frac{d}{ds}p_x = & -\frac{\partial \mathcal{H}}{\partial x} = \frac{[p_z - H \cdot x]}{[1 + f(p_\sigma)]} \cdot H - [K_x^2 + g] \cdot x + N \cdot z + \\ & K_x \cdot f(p_\sigma) - \frac{\lambda}{2} \cdot (x^2 - z^2) - \frac{\mu}{6} (x^3 - 3xz^2); \\ \frac{d}{ds}z = & \frac{\partial \mathcal{H}}{\partial p_z} = \frac{p_z - H \cdot x}{[1 + f(p_\sigma)]}; \\ \frac{d}{ds}p_z = & -\frac{\partial \mathcal{H}}{\partial z} = -\frac{[p_x + H \cdot z]}{[1 + f(p_\sigma)]} \cdot H - [K_z^2 - g] \cdot z + N \cdot x + \\ & K_z \cdot f(p_\sigma) - \lambda \cdot xz - \frac{\mu}{6} (z^3 - 3x^2z); \\ \frac{d}{ds}\sigma = & \frac{\partial \mathcal{H}}{\partial p_\sigma} = 1 - [1 + K_x \cdot x + K_z \cdot z] \cdot f'(p_\sigma) - \\ & \frac{1}{2} \cdot \frac{[p_x + H \cdot z]^2 + [p_z - H \cdot x]^2}{[1 + f(p_\sigma)]^2} \cdot f'(p_\sigma) \\ \frac{d}{ds}p_\sigma = & -\frac{\partial \mathcal{H}}{\partial \sigma} = \frac{1}{\beta_0^2} \cdot \frac{eV(s)}{E_0} \cdot \sin \left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi \right] \quad .\end{aligned}\quad (4)$$

Then we use series expansion of function $f(p_\sigma)$ from [1]:

$$f(p_\sigma) = f(0) + f'(0)p_\sigma + f''(0)\frac{1}{2}p_\sigma^2 + \dots = p_\sigma - \frac{1}{\gamma_0^2} \cdot \frac{1}{2}p_\sigma^2 + \dots \quad (5)$$

and the corresponding expansion of RHS of equations (4). In the following we take into account only arbitrary polynomial (in terms of dynamical variables) expressions and neglect all nonpolynomial types of expressions; i.e. we consider such approximations of RHS which are not more than polynomial functions in dynamical variables and arbitrary functions of independent variable s ("time" in our case, if we consider our system of equations as a dynamical problem).

B Particle in the Multipolar Field

The magnetic vector potential of a magnet with $2n$ poles in Cartesian coordinates is

$$A = \sum_n K_n f_n(x, y), \quad (6)$$

where f_n is a homogeneous function of x and y of order n .

The real and imaginary parts of the binomial expansion of

$$f_n(x, y) = (\hat{x} + iy)^n \quad (7)$$

correspond to regular and skew multipoles. The cases $n = 2$ to $n = 5$ correspond to low-order multipoles: quadrupole, sextupole, octupole, decapole.

Then we have, in this particular case, the following equations of motion for a single particle in a circular magnetic lattice in the transverse plane (x, y) ([8] for designation):

$$\begin{aligned} \frac{d^2x}{ds^2} + \left(\frac{1}{\rho(s)^2} - k_1(s) \right) x &= \mathcal{R}e \left[\sum_{n \geq 2} \frac{k_n(s) + ij_n(s)}{n!} \cdot (x + iy)^n \right] \\ \frac{d^2y}{ds^2} + k_1(s)y &= -\mathcal{I}m \left[\sum_{n \geq 2} \frac{k_n(s) + ij_n(s)}{n!} \cdot (x + iy)^n \right] \end{aligned} \quad (8)$$

and the corresponding Hamiltonian:

$$\begin{aligned} H(x, p_x, y, p_y, s) &= \frac{p_x^2 + p_y^2}{2} + \left(\frac{1}{\rho(s)^2} - k_1(s) \right) \cdot \frac{x^2}{2} + k_1(s) \frac{y^2}{2} \\ &\quad - \mathcal{R}e \left[\sum_{n \geq 2} \frac{k_n(s) + ij_n(s)}{(n+1)!} \cdot (x + iy)^{(n+1)} \right] \end{aligned} \quad (9)$$

Then we may take into account arbitrary but finite number in expansion of RHS of Hamiltonian (9) and from our point of view the corresponding Hamiltonian equations of motion are also not more than nonlinear ordinary differential equations with polynomial nonlinearities and variable coefficients.

C Effects of Insertion Devices on Beam Dynamics

Assuming a sinusoidal field variation, we may consider, according to [9], the analytical treatment of the effects of insertion devices on beam dynamics. One of the major detrimental aspects of the installation of insertion devices is the resulting reduction of dynamic aperture. Introduction of non-linearities leads to enhancement of the amplitude-dependent tune shifts and distortion of phase space. The nonlinear fields will produce significant effects at large betatron amplitudes.

The components of the insertion device magnetic field used for the derivation of equations of motion are as follows:

$$\begin{aligned} B_x &= \frac{k_x}{k_y} \cdot B_0 \sinh(k_x x) \sinh(k_y y) \cos(kz) \\ B_y &= B_0 \cosh(k_x x) \cosh(k_y y) \cos(kz) \\ B_z &= -\frac{k}{k_y} B_0 \cosh(k_x x) \sinh(k_y y) \sin(kz), \end{aligned} \quad (10)$$

with $k_x^2 + k_y^2 = k^2 = (2\pi/\lambda)^2$, where λ is the period length of the insertion device, B_0 its magnetic field, and ρ the radius of the curvature in the field B_0 . After a canonical transformation to change to betatron variables, the Hamiltonian is averaged over the period of the insertion device, and hyperbolic functions are expanded to the fourth order in x and y (or an arbitrary order).

Then we have the following Hamiltonian:

$$\begin{aligned} H &= \frac{1}{2}[p_x^2 + p_y^2] + \frac{1}{4k^2\rho^2}[k_x^2x^2 + k_y^2y^2] \\ &+ \frac{1}{12k^2\rho^2}[k_x^4x^4 + k_y^4y^4 + 3k_x^2k^2x^2y^2] \\ &- \frac{\sin(ks)}{2k\rho}[p_x(k_x^2x^2 + k_y^2y^2) - 2k_xp_yxy] . \end{aligned} \quad (11)$$

We also have in this case nonlinear (polynomial with degree 3) dynamical system with variable (periodic) coefficients. As a related case we may consider wiggler and undulator magnets. We have in the horizontal $x - s$ plane the following equations:

$$\begin{aligned} \ddot{x} &= -\dot{s} \frac{e}{m\gamma} B_z(s) , \\ \ddot{s} &= \dot{x} \frac{e}{m\gamma} B_z(s), \end{aligned} \quad (12)$$

where the magnetic field has periodic dependence on s and hyperbolic on z .

D Spin-Orbital Motion

Let us consider the system of equations for orbital motion

$$\frac{dq}{dt} = \frac{\partial H_{orb}}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H_{orb}}{\partial q} \quad (13)$$

and the Thomas-BMT equation for classical spin vector (see [10] for designation)

$$\frac{ds}{dt} = w \times s , \quad (14)$$

Here,

$$H_{orb} = c\sqrt{\pi^2 + m_0 c^2} + e\Phi, \quad (15)$$

$$w = -\frac{e}{m_0 \gamma c} \left((1 + \gamma G) \vec{B} - \frac{G(\vec{\pi} \cdot \vec{B}) \vec{\pi}}{m_0^2 c^2 (1 + \gamma)} - \frac{1}{m_0 c} \left(G + \frac{1}{1 + \gamma} \right) [\pi \times E] \right),$$

where $q = (q_1, q_2, q_3)$, $p = (p_1, p_2, p_3)$ the canonical position and momentum, $s = (s_1, s_2, s_3)$ the classical spin vector of length $\hbar/2$, and $\pi = (\pi_1, \pi_2, \pi_3)$ is the kinetic momentum vector. We may introduce in 9-dimensional phase space $z = (q, p, s)$ the Poisson brackets

$$\{f(z), g(z)\} = f_q g_p - f_p g_q + [f_s \times g_s] \cdot s, \quad (16)$$

and the corresponding Hamiltonian equations:

$$\frac{dz}{dt} = \{z, H\}, \quad (17)$$

with Hamiltonian

$$H = H_{orb}(q, p, t) + w(q, p, t) \cdot s. \quad (18)$$

More explicitly we have

$$\begin{aligned} \frac{dq}{dt} &= \frac{\partial H_{orb}}{\partial p} + \frac{\partial (w \cdot s)}{\partial p} \\ \frac{dp}{dt} &= -\frac{\partial H_{orb}}{\partial q} - \frac{\partial (w \cdot s)}{\partial q} \\ \frac{ds}{dt} &= [w \times s] \end{aligned} \quad (19)$$

We will consider this dynamical system also in our second paper in this volume via an invariant approach, based on consideration of Lie-Poisson structures on semidirect products of groups.

But from the point of view used in this paper we may consider approximations similar to preceding examples and then also arrive to at some type of polynomial dynamics.

III POLYNOMIAL DYNAMICS

The first main part of our consideration is some variational approach to this problem, which reduces the initial problem to the problem of solving functional equations at the first stage and some algebraical problems at the second stage. We have the solution in a compactly supported wavelet basis. Multiresolution expansion is the second main part of our construction. The solution is parameterized by solutions of two reduced algebraical problems, one being nonlinear and the second being some linear problem, which is obtained from one of the next wavelet constructions: Fast Wavelet Transform (FWT), Stationary Subdivision Schemes (SSS), the method of Connection Coefficients (CC).

A Variational Method

Our problems may be formulated as the systems of ordinary differential equations

$$dx_i/dt = f_i(x_j, t), \quad (i, j = 1, \dots, n), \quad (20)$$

with fixed initial conditions $x_i(0)$, where f_i are not more than polynomial functions of dynamical variables x_j and have arbitrary dependence of time. Because of time dilation we can consider only the next time interval: $0 \leq t \leq 1$. Let us consider a set of functions,

$$\Phi_i(t) = x_i dy_i/dt + f_i y_i \quad (21)$$

and a set of functionals

$$F_i(x) = \int_0^1 \Phi_i(t) dt - x_i y_i |_0^1, \quad (22)$$

where $y_i(t)$ ($y_i(0) = 0$) are dual variables. It is obvious that the initial system and the system

$$F_i(x) = 0 \quad (23)$$

are equivalent. In the last part we consider a more general approach, which is based on the possibility of taking into account underlying symplectic structure and using a more useful and flexible analytical approach, related to bilinear structure of initial function.

Now we consider formal expansions for x_i, y_i :

$$x_i(t) = x_i(0) + \sum_k \lambda_i^k \varphi_k(t) \quad y_j(t) = \sum_r \eta_j^r \varphi_r(t), \quad (24)$$

where, because of initial conditions, we need only $\varphi_k(0) = 0$. Then we have the following reduced algebraical system of equations on the set of unknown coefficients λ_i^k of expansions (24):

$$\sum_k \mu_{kr} \lambda_i^k - \gamma_i^r(\lambda_j) = 0 \quad (25)$$

Its coefficients are

$$\mu_{kr} = \int_0^1 \varphi'_k(t) \varphi_r(t) dt, \quad \gamma_i^r = \int_0^1 f_i(x_j, t) \varphi_r(t) dt. \quad (26)$$

Now, when we solve system (25) and determine unknown coefficients from formal expansion (24) we therefore obtain the solution of our initial problem. It should be noted that, if we consider only the truncated expansion (24) with N terms, then we have from (25) the system of $N \times n$ algebraical equations; and the degree of this

algebraical system coincides with the degree of initial differential system. So, we have the solution of the initial nonlinear (polynomial) problem in the form

$$x_i(t) = x_i(0) + \sum_{k=1}^N \lambda_i^k X_k(t), \quad (27)$$

where coefficients λ_i^k are roots of the corresponding reduced algebraical problem (25). Consequently, we have a parametrization of the solution of the initial problem by solution of the reduced algebraical problem (25). The first main problem is a problem of computations of coefficients of the reduced algebraical system. As we will see, these problems may be explicitly solved in the wavelet approach.

Next we consider the construction the of explicit time solution for our problem. The obtained solutions are given in the form (27), where $X_k(t)$ are the basis functions and λ_k^i are roots of the reduced system of equations. In our first wavelet case, $X_k(t)$ are obtained via multiresolution expansions and represented by compactly supported wavelets, and λ_k^i are the roots of the corresponding general polynomial system (25) with coefficients, which are given by FWT, SSS or CC constructions. According to the variational method giving the reduction from the differential to the algebraical system of equations, we need to compute the objects γ_a^j and μ_{ji} , which are constructed from objects:

$$\begin{aligned} \sigma_i &\equiv \int_0^1 X_i(\tau) d\tau, \\ \nu_{ij} &\equiv \int_0^1 X_i(\tau) X_j(\tau) d\tau, \\ \mu_{ji} &\equiv \int X_i'(\tau) X_j(\tau) d\tau, \\ \beta_{klj} &\equiv \int_0^1 X_k(\tau) X_l(\tau) X_j(\tau) d\tau \end{aligned} \quad (28)$$

for the simplest case of Riccati systems, where the degree of nonlinearity equals two. For the general case of arbitrary n we have analogous to (28) iterated integrals with the degree of monomials in integrand, which is one bigger than the degree of the initial system.

B Wavelet Framework

Our constructions are based on a multi-resolution approach. Because affine group of translations and dilations are part of the approach, this method resembles the action of a microscope. We have a contribution to the final result from each scale of resolution from the whole infinite scale of spaces. More exactly, the closed subspace $V_j (j \in \mathbf{Z})$ corresponds to level j of resolution, or to scale j . We consider a r -regular multiresolution analysis of $L^2(\mathbf{R}^n)$ (of course, we may consider any different functional space), which is a sequence of increasing closed subspaces V_j :

$$\dots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \dots , \quad (29)$$

satisfying the following properties:

$$\begin{aligned} \bigcap_{j \in \mathbf{Z}} V_j &= 0, \quad \overline{\bigcup_{j \in \mathbf{Z}} V_j} = L^2(\mathbf{R}^n), \\ f(x) \in V_j &\Leftrightarrow f(2x) \in V_{j+1}, \\ f(x) \in V_0 &\Leftrightarrow f(x - k) \in V_0, \quad \forall k \in \mathbf{Z}^n. \end{aligned} \quad (30)$$

There exists a function $\varphi \in V_0$ such that $\{\varphi_{0,k}(x) = \varphi(x - k), k \in \mathbf{Z}^n\}$ forms a Riesz basis for V_0 .

The function φ is regular and localized: φ is C^{r-1} ; $\varphi^{(r-1)}$ is almost everywhere differentiable and for almost every $x \in \mathbf{R}^n$, for every integer $\alpha \leq r$; and for all integers p there exists constant C_p such that

$$|\partial^\alpha \varphi(x)| \leq C_p (1 + |x|)^{-p} . \quad (31)$$

Let $\varphi(x)$ be a scaling function, $\psi(x)$ a wavelet function and $\varphi_i(x) = \varphi(x - i)$. Scaling relations that define φ, ψ are

$$\varphi(x) = \sum_{k=0}^{N-1} a_k \varphi(2x - k) = \sum_{k=0}^{N-1} a_k \varphi_k(2x), \quad (32)$$

$$\psi(x) = \sum_{k=-1}^{N-2} (-1)^k a_{k+1} \varphi(2x + k). \quad (33)$$

Let indices ℓ, j represent translation and scaling, respectively and

$$\varphi_{jl}(x) = 2^{j/2} \varphi(2^j x - \ell) ; \quad (34)$$

then the set $\{\varphi_{j,k}\}, k \in \mathbf{Z}^n$ forms a Riesz basis for V_j . The wavelet function ψ is used to encode the details between two successive levels of approximation. Let W_j be the orthonormal complement of V_j with respect to V_{j+1} :

$$V_{j+1} = V_j \bigoplus W_j. \quad (35)$$

Then just as V_j is spanned by dilation and translations of the scaling function, so are W_j spanned by translations and dilation of the mother wavelet $\psi_{jk}(x)$, where

$$\psi_{jk}(x) = 2^{j/2} \psi(2^j x - k). \quad (36)$$

All expansions which we used are based on the following properties:

$$\begin{aligned} \{\psi_{jk}\}, \quad j, k \in \mathbf{Z} &\text{ is a Hilbertian basis of } L^2(\mathbf{R}) \\ \{\varphi_{j,k}\}_{j \geq 0, k \in \mathbf{Z}} &\text{ is an orthonormal basis for } L^2(\mathbf{R}), \\ L^2(\mathbf{R}) &= V_0 \overline{\bigoplus_{j=0}^{\infty} W_j}, \quad (37) \\ \text{or} \\ \{\varphi_{0,k}, \psi_{j,k}\}_{j \geq 0, k \in \mathbf{Z}} &\text{ is an orthonormal basis for } L^2(\mathbf{R}). \end{aligned}$$

C Wavelet Computations

Now we give construction for computations of objects (28) in the wavelet case. We use a compactly supported wavelet basis: an orthonormal basis for functions in $L^2(\mathbf{R})$.

Let $f : \mathbf{R} \rightarrow \mathbf{C}$ and the wavelet expansion be

$$f(x) = \sum_{\ell \in \mathbf{Z}} c_{\ell} \varphi_{\ell}(x) + \sum_{j=0}^{\infty} \sum_{k \in \mathbf{Z}} c_{jk} \psi_{jk}(x) . \quad (38)$$

If in formulae (38) $c_{jk} = 0$ for $j \geq J$, then $f(x)$ has an alternative expansion in terms of dilated scaling functions only $f(x) = \sum_{\ell \in \mathbf{Z}} c_{J\ell} \varphi_{J\ell}(x)$. This is a finite wavelet expansion, and it can be written solely in terms of translated scaling functions. Also we have the shortest possible support: scaling function DN (where N is even integer) will have support $[0, N - 1]$ and $N/2$ vanishing moments. There exists $\lambda > 0$ such that DN has λN continuous derivatives; for small N , $\lambda \geq 0.55$. To solve our second associated linear problem we need to evaluate derivatives of $f(x)$ in terms of $\varphi(x)$. Let $\varphi_{\ell}^n = d^n \varphi_{\ell}(x) / dx^n$. We consider computation of the wavelet-Galerkin integrals. If $f^d(x)$ is a d -derivative of function $f(x)$, then we have $f^d(x) = \sum_{\ell} c_{\ell} \varphi_{\ell}^d(x)$, and values $\varphi_{\ell}^d(x)$ can be expanded in terms of $\varphi(x)$,

$$\begin{aligned} \varphi_{\ell}^d(x) &= \sum_m \lambda_m \varphi_m(x), \\ \lambda_m &= \int_{-\infty}^{\infty} \varphi_{\ell}^d(x) \varphi_m(x) dx, \end{aligned} \quad (39)$$

where λ_m are wavelet-Galerkin integrals. The coefficients λ_m are 2-term connection coefficients. In general we need to find ($d_i \geq 0$),

$$\Lambda_{\ell_1 \ell_2 \dots \ell_n}^{d_1 d_2 \dots d_n} = \int_{-\infty}^{\infty} \prod_{i=1}^n \varphi_{\ell_i}^{d_i}(x) dx . \quad (40)$$

For Riccati case we need to evaluate two and three connection coefficients

$$\Lambda_{\ell}^{d_1 d_2} = \int_{-\infty}^{\infty} \varphi_{\ell}^{d_1}(x) \varphi_{\ell}^{d_2}(x) dx, \quad \Lambda_{\ell}^{d_1 d_2 d_3} = \int_{-\infty}^{\infty} \varphi_{\ell}^{d_1}(x) \varphi_{\ell}^{d_2}(x) \varphi_{\ell}^{d_3}(x) dx . \quad (41)$$

According to the CC method [11] we use the next construction. When N in the scaling equation is a finite even positive integer, the function $\varphi(x)$ has compact support contained in $[0, N - 1]$. For a fixed triple (d_1, d_2, d_3) only some $\Lambda_{\ell m}^{d_1 d_2 d_3}$ are nonzero: $2 - N \leq \ell \leq N - 2$, $2 - N \leq m \leq N - 2$, $|\ell - m| \leq N - 2$. There are $M = 3N^2 - 9N + 7$ such pairs (ℓ, m) . If $\Lambda^{d_1 d_2 d_3}$ is an M -vector, whose components are numbers $\Lambda_{\ell m}^{d_1 d_2 d_3}$, then we have the first reduced algebraical system : Λ satisfy the system of equations ($d = d_1 + d_2 + d_3$),

$$A\Lambda^{d_1 d_2 d_3} = 2^{1-d} \Lambda^{d_1 d_2 d_3}, \quad A_{\ell, m; q, r} = \sum_p a_p a_{q-2\ell+p} a_{r-2m+p} . \quad (42)$$

By moment equations we have created a system of $M + d + 1$ equations in M unknowns. It has rank M and we can obtain unique solution by combination of LU decomposition and QR algorithm. The second reduced algebraical system gives us the 2-term connection coefficients:

$$A\Lambda^{d_1 d_2} = 2^{1-d} \Lambda^{d_1 d_2}, \quad d = d_1 + d_2, \quad A_{\ell, q} = \sum_p a_p a_{q-2\ell+p} \quad (43)$$

For a nonquadratic case we have additional analogously linear problems for objects (40). Solving these linear problems, we obtain the coefficients of a nonlinear algebraical system (25), and after that we obtain the coefficients of wavelet expansion (27). As a result we obtained the explicit time solution of our problem in the base of compactly supported wavelets. We use for modelling D6, D8, and D10 functions and programs RADAU and DOPRI for testing.

In the following we consider the extension of this approach to the case of periodic boundary conditions, the case of presence of arbitrary variable coefficients and a more flexible biorthogonal wavelet approach.

IV EVALUATION OF NONLINEARITIES SCALE BY SCALE

A Para-product and Decoupling between Scales

Before we consider two different schemes of modification of our variational approach we consider different scales separately. For this reason we need to compute errors of approximations. The main problems come of course from nonlinear terms. We follow the approach from [12].

Let P_j be the projection operators on the subspaces $V_j, j \in \mathbf{Z}$:

$$\begin{aligned} P_j : L^2(\mathbf{R}) &\rightarrow V_j \\ (P_j f)(x) &= \sum_k \langle f, \varphi_{j,k} \rangle \varphi_{j,k}(x) , \end{aligned} \quad (44)$$

and Q_j are projection operators on the subspaces W_j :

$$Q_j = P_{j-1} - P_j . \quad (45)$$

So, for $u \in L^2(\mathbf{R})$ we have $u_j = P_j u$ and $u_j \in V_j$, where $\{V_j\}, j \in \mathbf{Z}$ is a multiresolution analysis of $L^2(\mathbf{R})$. It is obvious that we can represent u_0^2 in the following form:

$$u_0^2 = 2 \sum_{j=1}^n (P_j u)(Q_j u) + \sum_{j=1}^n (Q_j u)(Q_j u) + u_n^2 . \quad (46)$$

In this formula there is no interaction between different scales. We may consider each term of (46) as bilinear mappings:

$$M_{VW}^j : V_j \times W_j \rightarrow L^2(\mathbf{R}) = V_j \oplus_{j' \geq j} W_{j'} \quad (47)$$

$$M_{WW}^j : W_j \times W_j \rightarrow L^2(\mathbf{R}) = V_j \oplus_{j' \geq j} W_{j'} \quad . \quad (48)$$

For numerical purposes we need formula (46) with a finite number of scales, but when we consider limits $j \rightarrow \infty$ we have

$$u^2 = \sum_{j \in \mathbf{Z}} (2P_j u + Q_j u)(Q_j u), \quad (49)$$

which is para-product of Bony, Coifman and Meyer.

Now we need to expand (46) into the wavelet bases. To expand each term in (46) into wavelet basis, we need to consider the integrals of the products of the basis functions, e.g.,

$$M_{WW}^{j,j'}(k, k', \ell) = \int_{-\infty}^{\infty} \psi_k^j(x) \psi_{k'}^j(x) \psi_{\ell}^{j'}(x) dx, \quad (50)$$

where $j' > j$ and

$$\psi_k^j(x) = 2^{-j/2} \psi(2^{-j}x - k) \quad (51)$$

are the basis functions. If we consider compactly supported wavelets then

$$M_{WW}^{j,j'}(k, k', \ell) \equiv 0 \quad \text{for} \quad |k - k'| > k_0, \quad (52)$$

where k_0 depends on the overlap of the supports of the basis functions and

$$|M_{WW}^r(k - k', 2^r k - \ell)| \leq C \cdot 2^{-r\lambda M} \quad . \quad (53)$$

Let us define j_0 as the distance between scales so that for a given ε all the coefficients in (53) with labels $r = j - j'$, $r > j_0$ have absolute values less than ε . For the purposes of computing with accuracy ε , we replace the mappings in (47), (48) by

$$M_{VW}^j : V_j \times W_j \rightarrow V_j \oplus_{j \leq j' \leq j_0} W_{j'} \quad (54)$$

$$M_{WW}^j : W_j \times W_j \rightarrow V_j \oplus_{j \leq j' \leq j_0} W_{j'} \quad . \quad (55)$$

Since

$$V_j \oplus_{j \leq j' \leq j_0} W_{j'} = V_{j_0-1} \quad (56)$$

and

$$V_j \subset V_{j_0-1}, \quad W_j \subset V_{j_0-1} \quad , \quad (57)$$

we may consider bilinear mappings (54), (55) on $V_{j_0-1} \times V_{j_0-1}$. For the evaluation of (54), (55) as mappings $V_{j_0-1} \times V_{j_0-1} \rightarrow V_{j_0-1}$, we need significantly fewer coefficients than for mappings (54), (55). It is enough to consider only coefficients

$$M(k, k', \ell) = 2^{-j/2} \int_{-\infty}^{\infty} \varphi(x - k) \varphi(x - k') \varphi(x - \ell) dx, \quad (58)$$

where $\varphi(x)$ is the scale function. Also we have

$$M(k, k', \ell) = 2^{-j/2} M_0(k - \ell, k' - \ell), \quad (59)$$

where

$$M_0(p, q) = \int \varphi(x - p) \varphi(x - q) \varphi(x) dx. \quad (60)$$

Now, as in section (3C), we may derive and solve a system of linear equations to find $M_0(p, q)$.

B Non-regular Approximation

We use the wavelet function $\psi(x)$, which has k vanishing moments $\int x^k \psi(x) dx = 0$, or equivalently $x^k = \sum c_\ell \varphi_\ell(x)$ for each k , $0 \leq k \leq K$.

Let P_j again be the orthogonal projector on space V_j . By tree algorithm we have for any $u \in L^2(\mathbf{R})$ and $\ell \in \mathbf{Z}$, that the wavelet coefficients of $P_\ell(u)$, i.e. the set $\{\langle u, \psi_{j,k} \rangle, j \leq \ell - 1, k \in \mathbf{Z}\}$, can be computed using hierarchical algorithms from the set of scaling coefficients in V_ℓ , i.e. the set $\{\langle u, \varphi_{\ell,k} \rangle, k \in \mathbf{Z}\}$ [13]. Because for scaling function φ we have in general only $\int \varphi(x) dx = 1$, therefore we have for any function $u \in L^2(\mathbf{R})$:

$$\lim_{j \rightarrow \infty, k 2^{-j} \rightarrow x} |2^{j/2} \langle u, \varphi_{j,k} \rangle - u(x)| = 0. \quad (61)$$

If the integer $n(\varphi)$ is the largest one so that

$$\int x^\alpha \varphi(x) dx = 0 \quad \text{for} \quad 1 \leq \alpha \leq n, \quad (62)$$

then if $u \in C^{(n+1)}$ with $u^{(n+1)}$ is bounded we have for $j \rightarrow \infty$ uniformly in k :

$$|2^{j/2} \langle u, \varphi_{j,k} \rangle - u(k 2^{-j})| = O(2^{-j(n+1)}). \quad (63)$$

Such scaling functions with zero moments are very useful for us from the point of view of time-frequency localization, because we have for the Fourier component $\hat{\Phi}(\omega)$ of them, that exists some $C(\varphi) \in \mathbf{R}$, so that for $\omega \rightarrow 0$ $\hat{\Phi}(\omega) = 1 + C(\varphi) |\omega|^{2r+2}$ (remember that we consider r-regular multiresolution analysis). Using this

type of scaling functions lead to superconvergence properties for general Galerkin approximation [13]. Now we need some estimates in each scale for non-linear terms of type $u \mapsto f(u) = f \circ u$, where f is C^∞ (in previous and future parts we consider only truncated Taylor series action). Let us consider the non-regular space of approximation \tilde{V} of the form

$$\tilde{V} = V_q \oplus \sum_{q \leq j \leq p-1} \tilde{W}_j, \quad (64)$$

with $\tilde{W}_j \subset W_j$. We need an efficient and precise estimate of $f \circ u$ on \tilde{V} . Let us set for $q \in \mathbf{Z}$ and $u \in L^2(\mathbf{R})$

$$\prod f_q(u) = 2^{-q/2} \sum_{k \in \mathbf{Z}} f(2^{q/2} \langle u, \varphi_{q,k} \rangle) \cdot \varphi_{q,k} \quad . \quad (65)$$

We have the following (important for us) estimation (uniformly in q) for $u, f(u) \in H^{(n+1)}$ [13]:

$$\|P_q(f(u)) - \prod f_q(u)\|_{L^2} = O(2^{-(n+1)q}) \quad . \quad (66)$$

For non-regular spaces (64) we set

$$\prod f_{\tilde{V}}(u) = \prod f_q(u) + \sum_{\ell=q, p-1} P_{\tilde{W}_j} \cdot \prod f_{\ell+1}(u) \quad (67)$$

Then we have the following estimate:

$$\|P_{\tilde{V}}(f(u)) - \prod f_{\tilde{V}}(u)\|_{L^2} = O(2^{-(n+1)q}) \quad , \quad (68)$$

uniformly in q and \tilde{V} (64).

This estimate depends on q , not p , i.e. on the scale of the coarse grid, not on the finest grid used in definition of \tilde{V} . We have for total error

$$\|f(u) - \prod f_{\tilde{V}}(u)\| = \|f(u) - P_{\tilde{V}}(f(u))\|_{L^2} + \|P_{\tilde{V}}(f(u) - \prod f_{\tilde{V}}(u))\|_{L^2} \quad , \quad (69)$$

and since the projection error in \tilde{V} : $\|f(u) - P_{\tilde{V}}(f(u))\|_{L^2}$ is much smaller than the projection error in V_q , we have the improvement (68) of (66). In our concrete calculations and estimates it is very useful to consider our approximations in the particular case of c-structured space:

$$\tilde{V} = V_q + \sum_{j=q}^{p-1} \text{span}\{\psi_{j,k}, k \in [2^{(j-1)} - c, 2^{(j-1)} + c] \bmod 2^j\} \quad . \quad (70)$$

V VARIATIONAL WAVELET APPROACH FOR PERIODIC TRAJECTORIES

We start with an extension of our approach to the case of periodic trajectories. The equations of motion corresponding to Hamiltonians (from part II) may also be formulated as a particular case of the general system of ordinary differential equations $dx_i/dt = f_i(x_j, t)$, ($i, j = 1, \dots, n$), $0 \leq t \leq 1$, where f_i are not more than polynomial functions of dynamical variables x_j and have arbitrary dependence of time but with periodic boundary conditions. According to our variational approach we have the solution in the following form:

$$x_i(t) = x_i(0) + \sum_k \lambda_i^k \varphi_k(t), \quad x_i(0) = x_i(1), \quad (71)$$

where λ_i^k are again the roots of reduced algebraical systems of equations with the same degree of nonlinearity, and $\varphi_k(t)$ corresponds to useful types of wavelet bases (frames). It should be noted that coefficients of reduced algebraical system are the solutions of additional linear problem and also depend on a particular type of wavelet construction and type of bases.

This linear problem is our second reduced algebraical problem. We need to find in general situation objects

$$\Lambda_{\ell_1 \ell_2 \dots \ell_n}^{d_1 d_2 \dots d_n} = \int_{-\infty}^{\infty} \prod \varphi_{\ell_i}^{d_i}(x) dx, \quad (72)$$

but now in the case of periodic boundary conditions. Now we consider the procedure of their calculations in the case of periodic boundary conditions in the base of periodic wavelet functions on the interval $[0,1]$ and corresponding expansion (71) inside our variational approach. Periodization procedure gives us

$$\begin{aligned} \hat{\varphi}_{j,k}(x) &\equiv \sum_{\ell \in Z} \varphi_{j,k}(x - \ell) \\ \hat{\psi}_{j,k}(x) &= \sum_{\ell \in Z} \psi_{j,k}(x - \ell) \end{aligned} \quad (73)$$

So, $\hat{\varphi}, \hat{\psi}$ are periodic functions on the interval $[0,1]$. Because $\varphi_{j,k} = \varphi_{j,k'}$ if $k = k' \bmod(2^j)$, we may consider only $0 \leq k \leq 2^j$, and, as a consequence, our multiresolution has the form $\bigcup_{j \geq 0} \hat{V}_j = L^2[0, 1]$, with $\hat{V}_j = \text{span}\{\hat{\varphi}_{j,k}\}_{k=0}^{2^j-1}$ [14]. Integration by parts and periodicity gives useful relations between objects (72), in particular the quadratic case ($d = d_1 + d_2$):

$$\Lambda_{k_1, k_2}^{d_1, d_2} = (-1)^{d_1} \Lambda_{k_1, k_2}^{0, d_2 + d_1}, \quad \Lambda_{k_1, k_2}^{0, d} = \Lambda_{0, k_2 - k_1}^{0, d} \equiv \Lambda_{k_2 - k_1}^d. \quad (74)$$

So, any 2-tuple can be represented by Λ_k^d . Then our second additional linear problem is reduced to the eigenvalue problem for $\{\Lambda_k^d\}_{0 \leq k \leq 2^j}$ by creating a system of 2^j

homogeneous relations in Λ_k^d and inhomogeneous equations. So, if we have a dilation equation in the form $\varphi(x) = \sqrt{2} \sum_{k \in \mathbb{Z}} h_k \varphi(2x - k)$, then we have the following homogeneous relations:

$$\Lambda_k^d = 2^d \sum_{m=0}^{N-1} \sum_{\ell=0}^{N-1} h_m h_\ell \Lambda_{\ell+2k-m}^d, \quad (75)$$

or in such form $A\lambda^d = 2^d \lambda^d$, where $\lambda^d = \{\Lambda_k^d\}_{0 \leq k \leq 2^d}$. Inhomogeneous equations are:

$$\sum_{\ell} M_{\ell}^d \Lambda_{\ell}^d = d! 2^{-j/2}, \quad (76)$$

where objects $M_{\ell}^d (|\ell| \leq N - 2)$ can be computed by a recursive procedure

$$M_{\ell}^d = 2^{-j(2d+1)/2} \tilde{M}_{\ell}^d, \quad \tilde{M}_{\ell}^k = \langle x^k, \varphi_{0,\ell} \rangle = \sum_{j=0}^k \binom{k}{j} n^{k-j} M_0^j, \quad \tilde{M}_0^{\ell} = 1. \quad (77)$$

So, we reduced our last problem to a standard linear algebraical problem. Then we used the same methods as in part III C. As a result we obtained for closed trajectories of orbital dynamics described by Hamiltonians from part II the explicit time solution (71) in the base of periodized wavelets (73).

VI VARIATIONAL APPROACH IN BIORTHOGONAL WAVELET BASES

Now we consider further generalization of our variational wavelet approach. In [1]-[3] we consider different types of variational principles which give us weak solutions to our nonlinear problems.

Before this we consider the generalization of our wavelet variational approach to the symplectic invariant calculation of closed loops in Hamiltonian systems [3]. We also have the parametrization of our solution by some reduced algebraical problem; but in contrast to the general case where the solution is parametrized by construction based on scalar refinement equation, in the symplectic case we have parametrization of the solution by matrix problems – Quadratic Mirror Filters equations [3]. But because integrand of variational functionals is represented by a bilinear form (scalar product), it seems more reasonable to consider wavelet constructions [15] which take into account all advantages of this structure.

The action functional for loops in the phase space is [16],

$$F(\gamma) = \int_{\gamma} pdq - \int_0^1 H(t, \gamma(t)) dt \quad (78)$$

The critical points of F are those loops γ , which solve the Hamiltonian equations associated with the Hamiltonian H and hence are periodic orbits. By the way,

all critical points of F are the saddle points of the infinite Morse index, but surprisingly this approach is very effective. This will be demonstrated using several variational techniques starting from minimax due to Rabinowitz and ending with Floer homology. So, (M, ω) is equal to symplectic manifolds, $H : M \rightarrow \mathbb{R}$, H is Hamiltonian, X_H is the unique Hamiltonian vector field defined by

$$\omega(X_H(x), v) = -dH(x)(v), \quad v \in T_x M, \quad x \in M, \quad (79)$$

where ω is the symplectic structure. A T -periodic solution $x(t)$ of the Hamiltonian equations,

$$\dot{x} = X_H(x) \quad \text{on } M, \quad (80)$$

is a solution, satisfying the boundary conditions $x(T) = x(0), T > 0$. Let us consider the loop space $\Omega = C^\infty(S^1, \mathbb{R}^{2n})$, where $S^1 = \mathbb{R}/\mathbb{Z}$, of smooth loops in \mathbb{R}^{2n} . Let us define a function $\Phi : \Omega \rightarrow \mathbb{R}$ by setting

$$\Phi(x) = \int_0^1 \frac{1}{2} \langle -J\dot{x}, x \rangle dt - \int_0^1 H(x(t)) dt, \quad x \in \Omega. \quad (81)$$

The critical points of Φ are the periodic solutions of $\dot{x} = X_H(x)$. Computing the derivative at $x \in \Omega$ in the direction of $y \in \Omega$, we find

$$\Phi'(x)(y) = \frac{d}{d\epsilon} \Phi(x + \epsilon y)|_{\epsilon=0} = \int_0^1 \langle -J\dot{x} - \nabla H(x), y \rangle dt \quad (82)$$

Consequently, $\Phi'(x)(y) = 0$ for all $y \in \Omega$ if the loop x satisfies the equation

$$-J\dot{x}(t) - \nabla H(x(t)) = 0; \quad (83)$$

i.e., $x(t)$ is a solution of the Hamiltonian equations, which also satisfies $x(0) = x(1)$, i.e., the periodic of period 1. Periodic loops may be represented by their Fourier series:

$$x(t) = \sum_{k \in \mathbb{Z}} e^{k2\pi Jt} x_k, \quad x_k \in \mathbb{R}^{2k}, \quad (84)$$

where J is the quasicomplex structure. We give relations between the quasicomplex structure and wavelets in our second paper in this volume (see also [3]). But now we need to take into account underlying bilinear structure via wavelets.

We started with two hierarchical sequences of approximations spaces [15]:

$$\dots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \dots, \quad \dots \tilde{V}_{-2} \subset \tilde{V}_{-1} \subset \tilde{V}_0 \subset \tilde{V}_1 \subset \tilde{V}_2 \dots, \quad (85)$$

and as usual, W_0 is a complement to V_0 in V_1 , but now not necessarily an orthogonal complement. New orthogonality conditions now have the following form:

$$\tilde{W}_0 \perp V_0, \quad W_0 \perp \tilde{V}_0, \quad V_j \perp \tilde{W}_j, \quad \tilde{V}_j \perp W_j, \quad (86)$$

translates of ψ span W_0 , translates of $\tilde{\psi}$ span \widetilde{W}_0 . Biorthogonality conditions are

$$\langle \psi_{jk}, \tilde{\psi}_{j'k'} \rangle = \int_{-\infty}^{\infty} \psi_{jk}(x) \tilde{\psi}_{j'k'}(x) dx = \delta_{kk'} \delta_{jj'}, \quad (87)$$

where $\psi_{jk}(x) = 2^{j/2} \psi(2^j x - k)$. Functions $\varphi(x), \tilde{\varphi}(x - k)$ form a dual pair:

$$\langle \varphi(x - k), \tilde{\varphi}(x - \ell) \rangle = \delta_{kl}, \quad \langle \varphi(x - k), \tilde{\psi}(x - \ell) \rangle = 0 \quad \text{for } \forall k, \forall \ell. \quad (88)$$

Functions $\varphi, \tilde{\varphi}$ generate a multiresolution analysis. $\varphi(x - k), \psi(x - k)$ are synthesis functions, and $\tilde{\varphi}(x - \ell), \tilde{\psi}(x - \ell)$ are analysis functions. Synthesis functions are biorthogonal to analysis functions. Scaling spaces are orthogonal to dual wavelet spaces. Two multiresolutions are intertwining $V_j + W_j = V_{j+1}$, $\tilde{V}_j + \tilde{W}_j = \tilde{V}_{j+1}$. These are direct sums but not orthogonal sums.

So, our representation for a solution now has the form

$$f(t) = \sum_{j,k} \tilde{b}_{jk} \psi_{jk}(t), \quad (89)$$

where synthesis wavelets are used to synthesize the function. But \tilde{b}_{jk} comes from inner products with analysis wavelets. Biorthogonality yields

$$\tilde{b}_{\ell m} = \int f(t) \tilde{\psi}_{\ell m}(t) dt. \quad (90)$$

So, now we can introduce this more complicated construction into our variational approach. We have a modification only on the level of computing coefficients of a reduced nonlinear algebraical system. This new construction is more flexible. The biorthogonal point of view is more stable under the action of a large class of operators, while the orthogonal (one scale for multiresolution) is fragile all computations are much simpler and we accelerate the rate of convergence. In all types of Hamiltonian calculation, which are based on some bilinear structures (symplectic or Poissonian structures, bilinear form of integrand in variational integral), this framework leads to greater success.

VII VARIABLE COEFFICIENTS

In the case when we have a situation where our problem is described by a system of nonlinear (polynomial) differential equations, we need to consider extension of our previous approach, which can take into account any type of variable coefficients (periodic, regular or singular). We can produce such an approach if we add in our construction an additional refinement equation, which would encode all information about variable coefficients [17]. According to our variational approach we need to compute integrals of the form

$$\int_D b_{ij}(t)(\varphi_1)^{d_1}(2^m t - k_1)(\varphi_2)^{d_2}(2^m t - k_2)dx, \quad (91)$$

where now $b_{ij}(t)$ are arbitrary functions of time, where trial functions φ_1, φ_2 satisfy a refinement equation:

$$\varphi_i(t) = \sum_{k \in \mathbf{Z}} a_{ik} \varphi_i(2t - k) \quad (92)$$

If we consider all computations in the class of compactly supported wavelets, then only a finite number of coefficients do not vanish. To approximate the non-constant coefficients, we need to choose a different refinable function φ_3 , along with some local approximation scheme,

$$(B_\ell f)(x) := \sum_{\alpha \in \mathbf{Z}} F_{\ell,k}(f) \varphi_3(2^\ell t - k), \quad (93)$$

where $F_{\ell,k}$ are suitable functionals supported in a small neighborhood of $2^{-\ell}k$, and then replace b_{ij} in (91) by $B_\ell b_{ij}(t)$. In this particular case, one can take a characteristic function and can thus approximate non-smooth coefficients locally. To guarantee sufficient accuracy of the resulting approximation to (91) it is important to have the flexibility of choosing φ_3 different from φ_1, φ_2 . In the case when D is some domain, we can write

$$b_{ij}(t) |_D = \sum_{0 \leq k \leq 2^\ell} b_{ij}(t) \chi_D(2^\ell t - k), \quad (94)$$

where χ_D is a characteristic function of D . So, if we take $\varphi_4 = \chi_D$, which is again a refinable function, then the problem of the computation of (91) is reduced to the problem of calculation of the integral

$$H(k_1, k_2, k_3, k_4) = H(k) = \int_{\mathbf{R}^s} \varphi_4(2^j t - k_1) \varphi_3(2^\ell t - k_2) \varphi_1^{d_1}(2^r t - k_3) \varphi_2^{d_2}(2^s t - k_4) dx \quad (95)$$

The key point is that these integrals also satisfy some sort of refinement equation:

$$2^{-|\mu|} H(k) = \sum_{\ell \in \mathbf{Z}} b_{2k-\ell} H(\ell), \quad \mu = d_1 + d_2. \quad (96)$$

This equation can be interpreted as the problem of computing an eigenvector. Thus, we reduced the problem of the extension of our method to the case of variable coefficients to the same standard algebraical problem as in the preceding sections. So, the general scheme is the same one, and we have only one more additional linear algebraic problem by which we, in the same way, can parameterize the solutions of the corresponding problem.

An extended version and related results may be found in [1]-[6].

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