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Monopoles and Dyons in the Pure Einstein-Yang-Mills Theory

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Abstract

In the pure Einstein-Yang-Mills theory in four dimensions there exist monopole and dyon solutions. The spectrum of the solutions is discrete in asymptotically flat or de Sitter space, whereas it is continuous in asymptotically anti-de Sitter space. The solutions are regular everywhere and specified with their mass, and non-Abelian electric and magnetic charges. In asymptotically anti-de Sitter space a class of monopole solutions have no node in non-Abelian magnetic fields, and are stable against spherically symmetric perturbations.

1. Introduction

In flat space there cannot be any static solution in the pure Yang-Mills theory in four dimensions[1]. Only with scalar fields monopole solutions exist, the topology of the scalar field playing a crucial role there. The inclusion of the gravity opens a possibility of having a soliton solution. The gravitational force, being always attractive, may balance the repulsive force of the non-Abelian gauge fields. Such configurations were indeed found in asymptotically flat and de Sitter space some time ago [2]-[9]. Unfortunately all of them turned out unstable against small perturbations [10]-[13].

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The situation drastically changes in asymptotically anti-de Sitter space. The negative cosmological constant ($\Lambda < 0$) provides negative pressure and energy density, making a class of monopole and dyon configurations stable [14]-[17]. We review the current status of such solutions.

2. Equations

The equations of motion in the Einstein-Yang-Mills theory are

$$\begin{aligned} &R^{\mu\nu} - \frac{1}{2} g^{\mu\nu}(R - 2\Lambda) = 8\pi G\, T^{\mu\nu} \\ &F^{\mu\nu}{}_{;\mu} + e[A_\mu, F^{\mu\nu}] = 0 \quad . \end{aligned} \tag{1}$$

To find soliton solutions we make a spherically symmetric, static ansatz:

$$\begin{aligned} ds^2 &= -\frac{H(r)}{p(r)^2}\, dt^2 + \frac{dr^2}{H(r)} + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \\ A &= \frac{\tau^j}{2e} \left\{ u(r)\, \frac{x_j}{r}\, dt - \frac{1 - w(r)}{r^2}\, \epsilon_{jkl} x_k\, dx_l \right\} \end{aligned} \tag{2}$$

with boundary conditions $u(0) = 0$ and $H(0) = p(0) = w(0) = 1$. It is convenient to parameterize

$$H(r) = 1 - \frac{2m(r)}{r} - \frac{\Lambda r^2}{3} \quad , \tag{3}$$

where $m(r)$ is the mass contained inside the radius r. $p(r) = 1$ and $m(r) = 0$ corresponds to the Minkowski, de Sitter, or anti-de Sitter space for $\Lambda = 0, > 0$, or < 0, respectively. $u(r)$ and $w(r)$ represent the electric and magnetic Yang-Mills fields, respectively.

The equations in (1) reduce to

$$\begin{aligned} (r^2 p u')' &= \frac{2p}{H}\, u w^2 \\ \left(\frac{H}{p}\, w'\right)' &= \frac{w(w^2 - 1)}{r^2 p} - \frac{p}{H}\, u^2 w \\ m' &= \frac{4\pi G}{e^2} \left\{ \frac{1}{2} r^2 p^2 u'^2 + \frac{(1 - w^2)^2}{2r^2} + \frac{p^2}{H}\, u^2 w^2 + H w'^2 \right\} \\ p' &= -\frac{8\pi G}{e^2}\, \frac{p}{rH} \left\{ \frac{p^2}{H}\, u^2 w^2 + H w'^2 \right\} . \end{aligned} \tag{4}$$

Near the origin $u = ar$, $w = 1 - br^2$, $m = v(a^2 + 4b^2) r^3/2$, and $p = 1 - v(a^2 + 4b^2) r^2$, where (a, b) are two parameters to be fixed and $v = 4\pi G/e^2$. With given (a, b) the equations in (4) are numerically integrated from $r = 0$ to $r = \infty$.

In general, solutions blow up at finite r, unless (a, b) take special values. We are looking for everywhere regular soliton configurations with a finite ADM mass $M = m(\infty)$.

3. Conserved charges

Non-Abelian solitons are characterized by the ADM mass and non-Abelian electric and magnetic charges. The kinematical identities $(F^{\mu\nu}{}_{;\mu})_{;\nu} = 0$ and $(\tilde{F}^{\mu\nu}{}_{;\mu})_{;\nu} = 0$ lead to conserved charges given by $\int dS_k \sqrt{-g}\,(F^{k0}, \tilde{F}^{k0})$. They are gauge variant, and therefore there are infinitely many conserved charges. Most of them vanish for solutions under consideration. Non-vanishing charges are

$$\begin{pmatrix} Q_E \\ Q_M \end{pmatrix} = \frac{e}{4\pi} \int dS_k \sqrt{-g}\, \mathrm{Tr}\; \tau_r \begin{pmatrix} F^{k0} \\ \tilde{F}^{k0} \end{pmatrix} \quad , \qquad \tau_r = \frac{x^j \tau^j}{r}$$

$$= \begin{pmatrix} u_1 p_0 \\ 1 - w_0^2 \end{pmatrix} \quad . \tag{5}$$

In the second equality the coefficients u_1, p_0, and w_0 are defined by the asymptotic expansion $u \sim u_0 + (u_1/r) + \cdots$ etc.. These two charges are conserved as there exists a unitary matrix S satisfying $\tau_r = S\tau_3 S^{-1}$.

4. Solutions in the $\Lambda = 0$ or $\Lambda > 0$ case

It has been shown in [18] and [17] that solutions are electrically neutral ($a = 0$, $u(r) = 0$). Solutions exist only for a discrete set of values of b. In the $\Lambda = 0$ case, $w_0 = w(\infty) = \pm 1$ so that $Q_M = 0$. In the $\Lambda > 0$ case, $Q_M \neq 0$. The Bartnik-McKinnon solution in the $\Lambda = 0$ case is depicted in fig. 1.

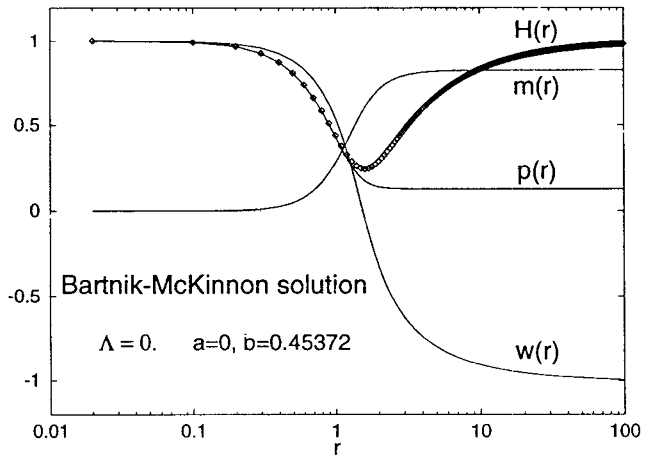

Figure 1. Bartnik-McKinnon solution in the $\Lambda = 0$ theory.

Solutions are characterized by the number of nodes, n, in $w(r)$; $n = 1, 2, 3, \cdots$. All of these solutions are shown to be unstable against small spherically symmetric perturbations [10]-[12].

5. Solutions in the $\Lambda < 0$ case

5.1. Configurations

The $\Lambda < 0$ case is qualitatively different in many respects from the $\Lambda = 0$ or $\Lambda > 0$ case. First there are dyonic solutions in which electric fields are non-vanishing; $u(r) \neq 0$. Secondly solutions exist in a finite continuum region in the parameter space (a, b) as opposed to discrete points. Thirdly there exist solutions with no node $(n = 0)$ in $w(r)$.

A typical monopole solution with no node in w is depicted in fig. 2. Depending on the value of b, the asymptotic value $w(\infty)$ can be either greater than 1, or between 0 and 1, or negative.

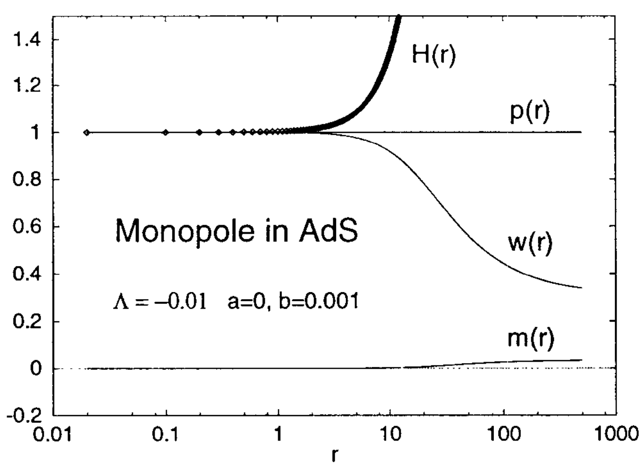

Figure 2. Monopole solution in the $\Lambda < 0$ theory. There are a continuum of solutions. Dyon solutions have similar behavior, with the additional $u(r)$ monotonically increasing from zero to the asymptotic value in the range $(0 \sim 0.2)$.

5.2. Monopole and dyon spectrum

When $a = 0$ and b is varied, a continuum of monopole solutions are generated. With Λ given, solutions appear in a finite number of branches. The number of branches increases as $\Lambda \to 0$. For $\Lambda = -0.01$ there are only two branches, which are displayed in fig. 3.

The upper branch ends near the point $(Q_M = 1, M = 1)$. The end point corresponds to the critical spacetime geometry discussed in the next subsection.

Solutions with no node in $w(r)$ are special. They are stable against small perturbations as discussed in Section 6. They exist in a limited region in the parameter space as depicted in fig. 4.

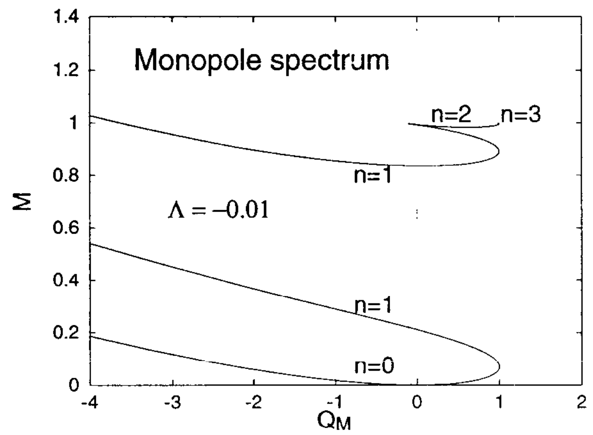

Figure 3. Monopole spectrum at $\Lambda = -0.01$. n is the number of the nodes in $w(r)$.

5.3. Critical spacetime

When the parameter b is increased, the solution either blows up or reaches a critical configuration. The end point in the upper branch in fig. 3 represents such a critical spacetime with $b = b_c = 0.70$. $H(r)$ vanishes at $r = r_h$. However, $r = r_h$ is not a standard event horizon appearing in black hole solutions.

When b is very close to b_c, $H(r)$ almost vanishes at $r \sim r_h$. One of such configurations is displayed in fig. 5.

At $b = b_c$, $H(r)$ becomes tangent to the axis at r_h. Further $p(r)$ vanishes. In other words the space ends at $r = r_h$. It is an open question whether such configurations really represent possible spacetime.

These critical spacetimes have universal behavior. Their magnetic charge is quantized, $Q_M = 1$, whereas their electric charge Q_E is not. There are two additional parameters, Λ and $v = 4\pi G/e^2$. When $v|\Lambda| \ll 1$, the critical spacetime is described near r_h by

$$\begin{aligned} r_h &= \frac{1}{2|\Lambda|}\left(\sqrt{1+4v|\Lambda|}-1\right) \sim \sqrt{v} \\ w(r) &\sim 2y^{1/2} \\ H(r) &\sim 4y^2 \end{aligned}$$

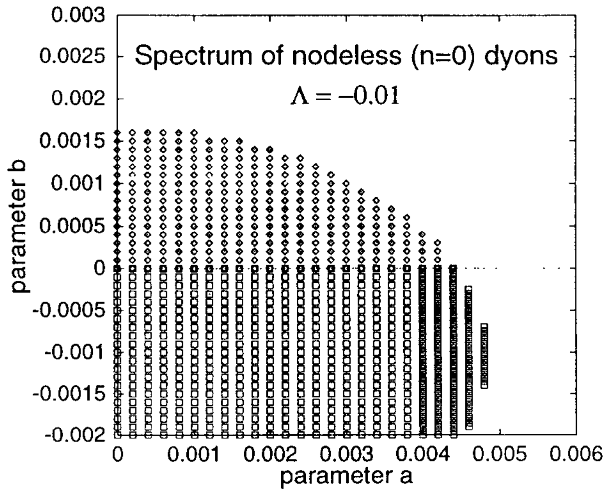

Figure 4. Spectrum of nodeless dyons.

$$\begin{aligned} m(r) &\sim \frac{1}{2}\sqrt{v}(1-y) \\ p(r) &\sim p_0 y^2 \\ &\quad \text{where} \quad y = 1 - \frac{r}{r_h} \geq 0 \ . \end{aligned} \tag{6}$$

Except for p_0 all coefficients and critical exponents are independent of Q_E and $|\Lambda|(\ll 1/v)$.

6. Stability

The stability of the solutions is examined by considering small perturbations. If they exponentially grow in time, the solutions are unstable, whereas if they remain small, they are stable. In the linearized theory, the problem is reduced to finding eigenvalues of a Schrödinger equation.

The analysis is simplified in the tortoise coordinate ρ defined by $d\rho/dr = p/H$. The range of ρ is finite for $\Lambda < 0$; $0 \leq \rho \leq \rho_{\rm max}$. For monopole solutions

$$\left\{ -\frac{d^2}{d\rho^2} + U(\rho) \right\} \chi = \omega^2 \chi \tag{7}$$

(i) Odd parity perturbations

$$U_{\rm odd} = \frac{H}{r^2 p^2}(1 + w^2) + \frac{2}{w^2}\left(\frac{dw}{d\rho}\right)^2 \quad , \tag{8}$$

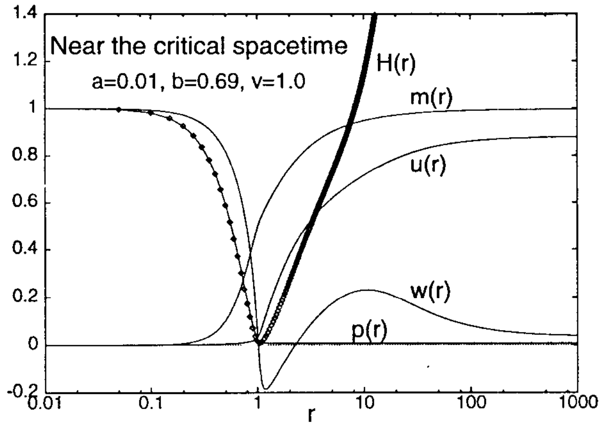

Figure 5. Dyon solution very close to the critical spacetime. At the critical value b_c the space ends at r_h. $p(r) = 0$ for $r \geq r_h$.

$$\delta\nu = \frac{w}{r^2 p}\chi_{\rm odd} \quad , \quad \delta\tilde{w} = -\frac{1}{2w}\frac{d}{d\rho}(r^2 p\,\delta\nu)$$

(ii) Even parity perturbations

$$U_{\rm even} = \frac{H}{r^2p^2}(3w^2 - 1) + 4v\frac{d}{d\rho}\left(\frac{Hw'^2}{pr}\right) \tag{9}$$

$$\delta w = \chi_{\rm even} \quad , \quad \frac{\delta H}{H} = -\frac{4v}{r}w'\delta w \quad , \quad \left(\frac{\delta p}{p}\right)' = -\frac{4v}{r}w'\delta w'$$

Here $'$ denotes a r-derivative. The boundary condition for odd parity perturbations is given by $\chi_{\rm odd} = 0$ at $\rho = 0$ and $d(w\chi_{\rm odd})/d\rho = 0$ at $\rho = \rho_{\rm max}$. For even parity perturbations $\chi_{\rm even} = 0$ at both ends. If Eq. (7) admits no boundstate ($\omega^2 < 0$), then the solution is stable.

Although $U_{\rm odd}(\rho)$ is positive definite, the corresponding eigenvalues may not be positive due to the nontrivial boundary condition imposed on $\chi_{\rm odd}$. For solutions with no node in $w(r)$, both $U_{\rm odd}$ and $U_{\rm even}$ behave as $2/\rho^2$ near the origin, but are regular elsewhere. One can prove that all $\omega^2_{\rm odd}, \omega^2_{\rm even} > 0$. In other words nodeless monopole solutions are stable against spherically symmetric perturbations.

When w has n nodes at ρ_k ($k = 1, \cdots, n$), $U_{\rm odd}$ become singular there. There appear n negative $\omega^2_{\rm odd}$ modes, which generally diverges at the singularities. $U_{\rm even}$ also becomes negative in the vicinity of the nodes, and there appear negative $\omega^2_{\rm even}$. Solutions with nodes in $w(r)$ are unstable.

7. The $\Lambda \to 0$ limit

When the cosmological constant approaches zero, more and more branches of monopole solutions emerge. In the $\Lambda \to 0$ limit the spectrum becomes discrete, there appearing infinitely many (unstable) solutions.

How is it possible? The nodeless, stable solutions must disappear. One parameter family of solutions must collapse into one point in the moduli space of solutions. In fig. 6 we have plotted the monopole spectrum with various values of Λ. One can see that as Λ approaches zero, new branches of solutions appear, and each branch collapses to a point in the $\Lambda \to 0$ limit. The nodeless solutions disappear as their ADM mass vanishes.

The result is indicative of a fractal structure in the moduli space of the solutions.

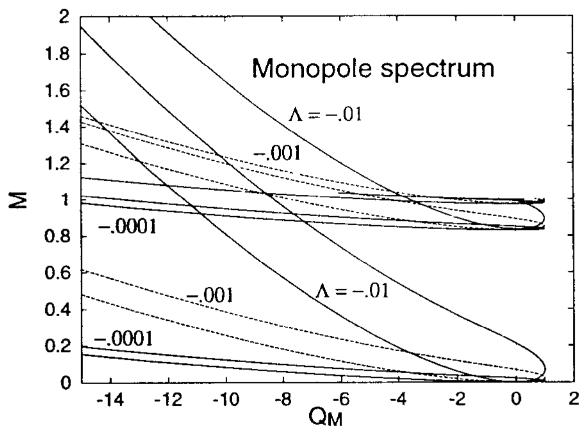

Figure 6. Monopole spectrum with varying Λ.

8. Summary

We have shown that there exist stable monopole and dyon solutions in the Einstein-Yang-Mills theory in asymptotically anti-de Sitter space. They have a continuous spectrum. Their implication to physics is yet to be examined.

Acknowledgments

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References

[1] S. Deser, *Phys. Lett.* **B64** (1976) 463.

[2] R. Bartnik and J. McKinnon, *Phys. Rev. Lett.* **61** (1988) 141.

[3] P. Bizon, *Phys. Rev. Lett.* **64** (1990) 2844.

[4] H. Künzle and A. Masood-ul-Alam, *J. Math. Phys.* **31** (1990) 928.

[5] K. Lee, V. Nair and E. Weinberg, *Phys. Rev. Lett.* **68** (1992) 1100; B. Greene, S. Mathur, and C. O'Niell, *Phys. Rev.* **D47** (1993) 2242.

[6] T. Torii, K. Maeda and T. Tachizawa, *Phys. Rev.* **D52** (1995) R4272.

[7] M.S. Volkov, N. Straumann, G. Lavrelashvili, M. Huesler and O. Brodbeck, *Phys. Rev.* **D54** (1996) 7243.

[8] J.A. Smoller and A. G. Wasserman, *Comm. Math. Phys.* **194** (1998) 707.

[9] For review, see: M. S. Volkov and D. Gal'tsov, hep-th/9810070 (1998).

[10] N. Straumann and Z. Zhou, *Phys. Lett.* **B237** (1990) 353; *Phys. Lett.* **B243** (1990) 33; Z. Zhou and N. Straumann, *Nucl. Phys.* **B360** (1991) 180.

[11] O. Brodbeck, M. Huesler, G. Lavrelashvili, N. Straumann and M.S. Volkov, *Phys. Rev.* **D54** (1996) 7338.

[12] M.S. Volkov, O. Brodbeck, G. Lavrelashvili, and N. Straumann, *Phys. Lett.* **B349** (1995) 438; O. Brodbeck and N. Straumann, *J. Math. Phys.* **37** (1996) 1414.

[13] P. Kanti and E. Winstanley, gr-qc/9910069.

[14] E. Winstanley, *Class. Quant. Grav.* **16** (1999) 1963.

[15] J. Bjoraker and Y. Hosotani, gr-qc/9906091, to appear in *Phys. Rev. Lett.* Feb, 2000.

[16] Y. Hosotani and J. Bjoraker, in the Proceedings of the 8th Canadian Conference on *"General Relativity and Relativistic Astrophysics"*, page 285 (AIP, 1999). (gr-qc/9907091)

[17] J. Bjoraker and Y. Hosotani, *"Monopoles, Dyons, and Black Holes in the Four-Dimensional Einstein-Yang-Mills Theory"*, (UMN-TH-1838/00, in preparation.)

[18] A. Ershov and D. Galt'sov, *Phys. Lett.* **A138** (1989) 160; *Phys. Lett.* **A150** (1990) 159.