A NONDETERMINISTIC SHOCK AND VIBRATION APPLICATION USING POLYNOMIAL CHAOS EXPANSIONS

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Abstract

In the current study, the generality of the key underpinnings of the Stochastic Finite Element (SFEM) method is exploited in a nonlinear shock and vibration application where parametric uncertainty enters through random variables with probabilistic descriptions assumed to be known. The system output is represented as a vector containing Shock Response Spectrum (SRS) data at a predetermined number of frequency points. In contrast to many reliability-based methods, the goal of the current approach is to provide a means to address more general (vector) output entities, to provide this output as a random process, and to assess characteristics of the response which allow one to avoid issues of statistical dependence among its vector components.

Introduction

Consider a general framework for nondeterministic analyses, shown graphically in Fig. 1. Here, M is an analytical model defining the map between input f and output u where, in general, both M and f are nondeterministic entities. In addition, the model is assumed to contain numerous parameters, some of which can only be quantified to within some level of uncertainty. These will be denoted by an n-dimensional vector Φ. A reasonable approach sufficient to address the uncertainty of the output, u, is to express it in a statistical form, $E[g(u)]$, where $E[\cdot]$ is the operator of mathematical expectation and $g(\cdot)$ is an appropriate deterministic function. Bearing in mind that issues relating to other sources of uncertainty remain, the following defines the so-called uncertainty propagation problem,

$$E[g(u)] = E\{E[g(u)|\Phi]\}, \tag{1}$$

where the goal is to determine the response statistics in terms of those of the input parameters.

Problem Formulation

The application under consideration is the penetration of a vehicle into a stratified soil medium. A schematic of this system as it impacts a target is shown in Fig. 2, where $\boldsymbol{v}$ and γ are the velocity vector and impact angle of the system, respectively, taken to be deter-

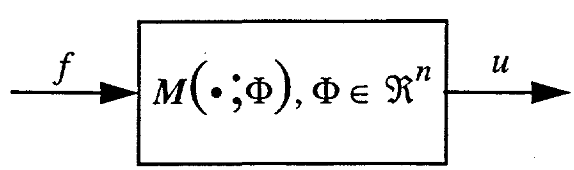

Figure 1. General framework for nondeterministic analyses.

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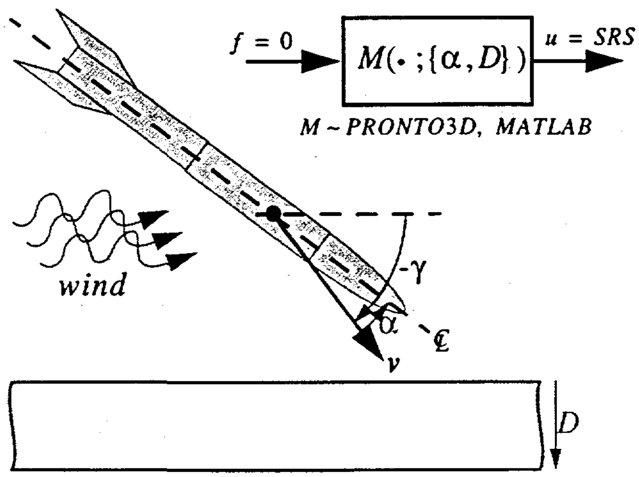

Figure 2. Problem formulation for the penetration system.

ministic. A considerable amount of uncertainty exists, however, in the knowledge of a particular soil depth parameter D. In addition, the angle-of-attack α is nondeterministic due to uncertainty in the knowledge of the wind conditions. Hence, for this application $\Phi = \{\alpha, D\} \in \Re^2$, where α and D were modeled as independent random variables, with normal and lognormal distributions, respectively.

The engineering question of interest is whether or not the internal components will survive the shock environment induced by the penetration event. For historical reasons, the response of interest is the shock response spectrum (SRS) of the acceleration of the centroid of a given internal component. One can then compare the predicted response to the shock specifications of the internal component of interest. PRONTO3D (Attaway *et. al* (1998)), a nonlinear, transient dynamics finite element code developed at Sandia National Laboratories, coupled with a spherical cavity expansion model of the soil-structure interaction (Warren and Tabbara (1997)), was used to predict the internal component response during the penetration event. In addition, filtering routines in MATLAB (The MathWorks (1998)) were used to compute the SRS.

Each function evaluation using this complex system model required over 33 CPU hours on a SUN Ultra II workstation. This fact motivated the use of approximations to the design space, which were developed using the design of experiment methods of Box and Behnken (1960). Hence, in the notation of Fig. 1, the model, $\boldsymbol{M}$, is a quadratic response surface that approximates the cascaded system of PRONTO3D, the cavity expansion model of the soil, and the MATLAB filtering routines.

The overall goal is to estimate $P(S)$, the probability of internal component survival. In previous work (see Field *et. al*, (2000a)), the authors' approach was to define a series of outputs

$$u_i = SRS(f_i),\ i = 1, 2, \ldots, n_f, \tag{2}$$

where f_i denotes the i^{th} natural frequency of the computed shock response spectrum. Reliability- and sampling-based methods were then employed to compute corresponding probability density functions (PDFs) $f_{U_i}(u_i)$, $i = 1, 2, \ldots, n_f$. Upon comparing to a reference, denoted SRS_{ref}, at each frequency line one could then make a statement regarding the probability of exceeding SRS_{ref}. However, because the correlation structure of the output is unknown with this approach, statements pertaining to system reliability cannot be made. As an alternative, in Field *et. al* (2000b) the authors defined the scalar output

$$\bar{u} = \min_f \ (SRS_{ref} - SRS), \tag{3}$$

where SRS denotes the calculated shock response spectrum from the computational model, and employed the identical methodologies to estimate $f_{\bar{U}}(\bar{u})$. While this approach facilitates a system reliability assessment, the response metric of Eq. (3) can be overly conservative. In addition, both of the schemes outlined in Eqs. (2) and (3) require *a priori* knowledge of SRS_{ref}, a somewhat ad hoc, test-based failure criterion. In the sequel, polynomial chaos expansions will be employed to address the shortcomings of each of the previous approaches to this problem. In addition, because these techniques provide an approximation to the response process, they may be used to construct an analytically-based failure criterion.

Polynomial Chaos Expansions

The stochastic finite element method, as developed in Ghanem and Spanos (1991), relies on the notion that random processes are mathematically well-defined mappings assumed to satisfy certain criteria. Among them is the notion that a real random variable (r.v.) is a deterministic measurable function which maps the sample space of random events to the real line. It is this attribute of measurability that provides the foundation for defining a Hilbert space, $\mathcal{H}$, of square-integrable, measurable functions and subsequently following function approximation theory in $\mathcal{H}$ in a way that directly parallels the path taken in a deterministic finite element approach with, in the stochastic case, an inner product operator that is given by mathematical expectation and the norm generated by this inner product (see Friedman (1982)).

The result of this analytical background is that finite-dimensional series approximations can be made for both the input and the output random processes that can be shown to converge weakly to the functions they replace. In the following, the special case where the input and output are vectors of random variables is considered. As introduced in Fig. 1, let Φ and u represent the internal and output quantities, respectively, and consider the following orthogonal decompositions

$$\Phi(\theta) \approx \Phi(\theta)^N \equiv \sum_{i=0}^{N} \Phi_i \Gamma_i(\{\xi_j\}), \text{ and} \tag{4}$$

$$u(x, \theta) \approx u(x, \theta)^N \equiv \sum_{i=0}^{N} u_i(x)\Gamma_i(\{\xi_j\}), \qquad (5)$$

where $\Gamma_i(\{\xi_j\})$ are defined to be multi-variate Hermite polynomials in the sequence of standard normal r.v.'s, $\{\xi_j\}$. The $\{\xi_j\}$ are defined on a sample space Ω with elements θ_j here and throughout; the explicit notation of dependence on θ is suppressed to simplify notation.

It can be shown that the $\Gamma_i(\{\xi_j\}) = \Gamma_i$ comprise an orthogonal set relative to the Hermite kernel; and, further that

$$E[\Gamma_i\Gamma_j] = \langle\Gamma_i\Gamma_j\rangle = 0,\ i \neq j. \qquad (6)$$

There are a number of methodologies that can be employed to determine the coefficients for the internal r.v.'s in Eq. (4). For example, if only statistical information is known about the components of Φ, moment matching, using either an exact or a least squares fit, is one choice. For the case under consideration, a general Fourier analysis for evaluation of the coefficients scaling the Γ_i was implemented. Here, a projection of both sides of Eq. (4) onto Γ_j is performed

$$\langle\Phi\Gamma_j\rangle = \Phi_j\langle\Gamma_j^2\rangle, \qquad (7)$$

where the orthogonality of the collection of Γ_i has been exploited. Thus, the Fourier coefficients can be computed

$$\Phi_j = \langle\Phi\Gamma_j\rangle / \langle\Gamma_j^2\rangle. \qquad (8)$$

The expectations in Eq. (8) were approximated using a Monte Carlo scheme, where joint probability distributions for Φ were assumed known, and arithmetic means were substituted for the expectations.

Once the input approximation has been computed, a similar procedure is followed for Eq. (5); specifically, terms in the relation

$$u_i(x) = \langle u(x)\Gamma_i\rangle / \langle\Gamma_i^2\rangle \qquad (9)$$

are approximated using Monte Carlo sampling. First, the input r.v.'s are approximated using the right-hand side of Eq. (4). Second, these so-called realizations are run directly through the analytical model, depicted by M in Fig. 2. Finally, each output realization is scaled by the current realization of Γ_i and added to the previous terms in the sum for each i.

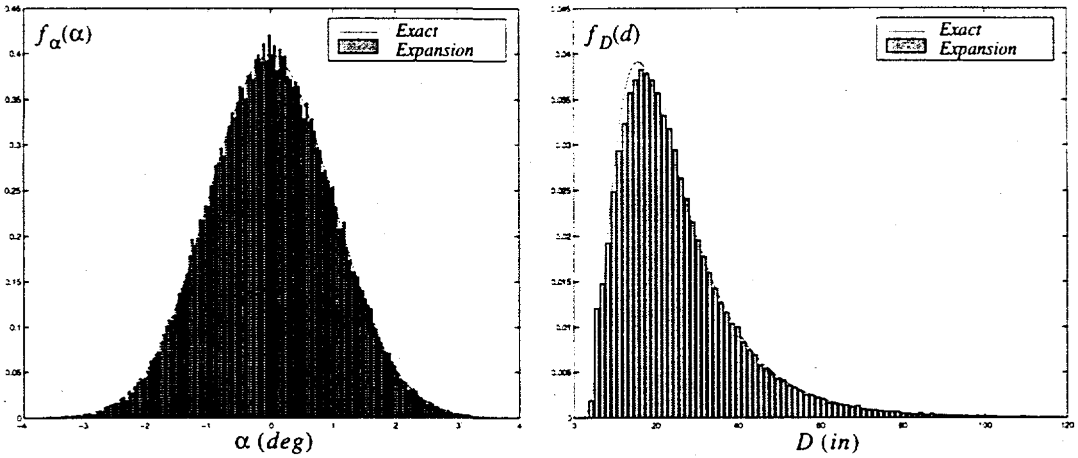

Figure 3. Polynomial chaos expansion of input random variables computed with $N = 4$, $n_s = 100000$.

Results

Figure 3 illustrates the approximation to the two input r.v.'s α and D using the polynomial chaos expansion. Here $N = 4$, and 100,000 Monte Carlo samples were used to compute the Fourier coefficients. In both cases, the relative mean square error is small. Note that while the two input random variables were assumed uncorrelated, the methods presented are not limited to problems with zero correlation. The approximation to the output stochastic process u is computed in a similar manner. Fifty realizations of the output SRS are shown in Fig. 4a, where it is evident that significant variability exists in the amplitude of the response.

When employing Eq. (1), the expectation operator does not commute, *i.e.*,

$$E[u] \neq M(\cdot;E[\Phi]) . \quad (10)$$

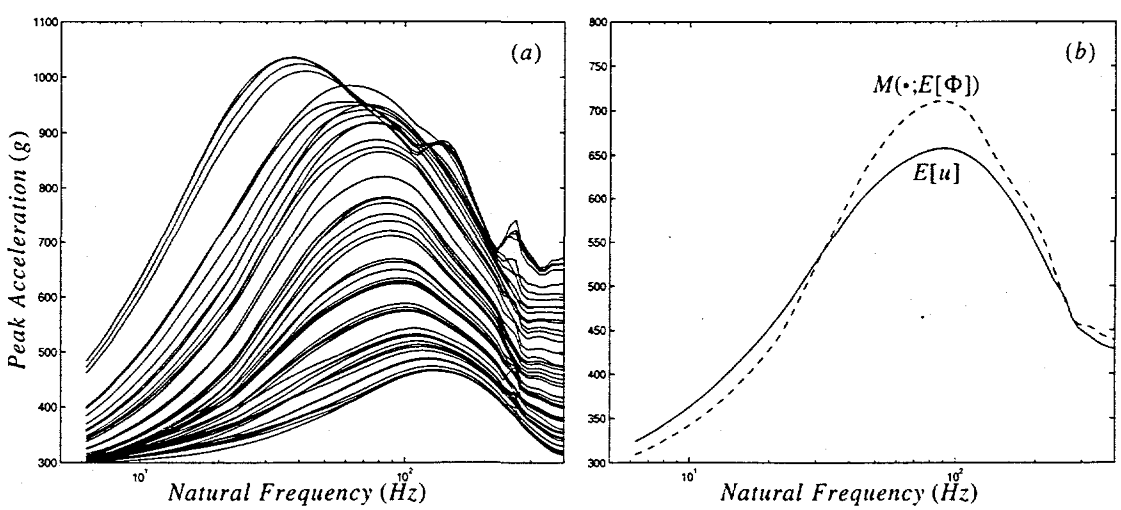

Figure 4. Approximation of output random process: (a) 50 realizations, (b) illustration that expectation operator does not commute.

This is a common misconception in industry, as an analyst may attempt to simulate the "worst-case" system response by performing a single deterministic analysis at the perceived "worst-case" input conditions. The large disparity between the two curves in Fig. 4b illustrates the danger in making this assumption.

Conclusions

The use of one of the key elements of the Stochastic Finite Element Method, namely the polynomial chaos expansion, has been utilized in a nonlinear shock and vibration application. Finite dimensional series approximations were made for both the input and output variables using Monte Carlo simulations and arithmetic means to estimate the expected values associated with each set of Fourier coefficients scaling the polynomial chaos terms. As a result, the computed response was expressed as a random process, which is an approximation to the true solution process, and can be thought of as a generalization to solutions given as statistics only. This important characteristic facilitates the use of this technique in, for example, multi-physics applications which require a link between different analysis codes. In addition, the authors were able to demonstrate by example that the operator of mathematical expectation does not commute, which is a known result when the input/output relationship is nonlinear. One corollary is that an analysis run at "worst-case" input conditions cannot be guaranteed to yield a "worst-case" response.

Acknowledgements

Sandia National Laboratories is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the United States Department of Energy under Contract DE-AC04-94AL85000.

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