ON THE DISTINCTION BETWEEN LARGE DEFORMATION AND LARGE DISTORTION FOR ANISOTROPIC MATERIALS

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ABSTRACT– A motion involves large distortion if the ratios of principal stretches differ significantly from unity. A motion involves large deformation if the deformation gradient tensor is significantly different from the identity. Unfortunately, rigid rotation fits the definition of large deformation, and models that claim to be valid for large deformation are often inadequate for large distortion. An *exact* solution for the stress in an idealized fiber-reinforced composite is used to show that conventional large deformation representations for transverse isotropy give errant results. Possible alternative approaches are discussed.

INTRODUCTION: Models that perform well for small deformations are often extemporaneously generalized to large deformations by merely applying them in the unrotated configuration. Such an approach does generate a frame indifferent model, but it does *not* necessarily produce a model that will perform well for large material distortions where the body significantly changes *shape*, not just size or orientation. We present a simple counterexample to demonstrate this claim. We also discuss some possible approaches to generalize small deformation constitutive models when large distortion measurements of material response are unavailable.

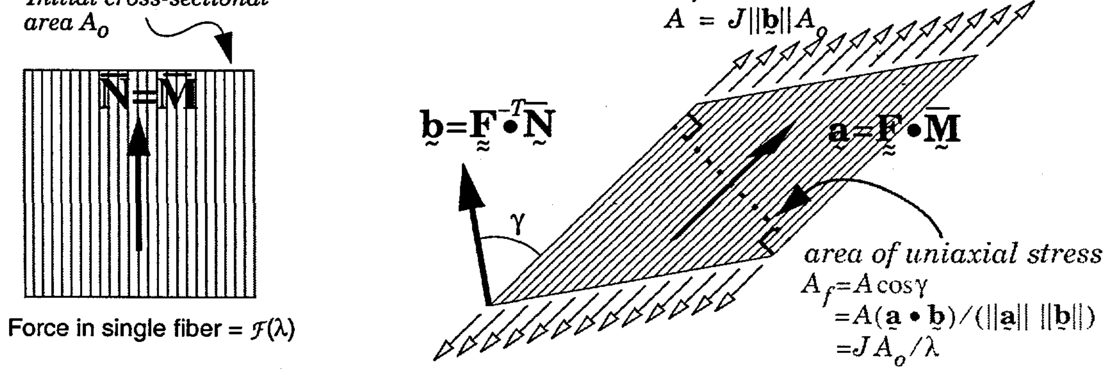

Figure 1: An idealized ensemble of fibers-in-air deforms according to a known deformation. The area that was originally normal to the fibers does not remain normal to the fibers.

PROCEDURES, RESULTS, AND DISCUSSION: Fig. 1 depicts an idealized composite consisting of fibers in a negligibly weak matrix (air). The force in a *single* fiber resulting from a fiber stretch λ (= current length / initial length) is presumed given by an arbitrary *known* function $\mathcal{F}(\lambda)$. The ensemble of fibers-in-air is subjected to an arbitrary homogeneous deformation, described by a deformation gradient tensor $\mathbf{F}$. There are ν_o fibers per unit *initial* cross-sectional area, and they are initially parallel to a unit vector $\bar{\mathbf{M}}$. The total force applied across the plane indicated in Fig. 1 is $\mathcal{F}(\lambda)\nu_o A_o$. Brannon [1998] showed that dividing this force by the area A_f gives the *exact* solutions for the Cauchy stress σ, unrotated Cauchy stress $\bar{\sigma}$, and second Piola-Kirchhoff (PK2) stress $\mathbf{s}$:

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$$\sigma_{ij} = \frac{\mathcal{G}(\lambda)}{J}a_i a_j, \quad \bar{\sigma}_{ij} = \frac{\mathcal{G}(\lambda)}{J}\bar{a}_i\bar{a}_j, \quad and \quad \bar{s}_{ij} = \mathcal{G}(\lambda)\bar{M}_i\bar{M}_j, \tag{1}$$

where

$$a_i = F_{ij}\bar{M}_j = R_{ik}\bar{a}_k, \quad \bar{a}_i = \bar{V}_{ij}\bar{M}_j, \quad \lambda \equiv \sqrt{a_k a_k}, \quad J = \det\underset{\approx}{\mathbf{F}}, \text{ and } \quad \mathcal{G}(\lambda) \equiv v_o \frac{\mathcal{F}(\lambda)}{\lambda}. \tag{2}$$

Here, $\underset{\approx}{\mathbf{R}}$ is the rotation and $\underset{\approx}{\bar{\mathbf{V}}}$ is the right stretch (more commonly denoted $\underset{\approx}{\mathbf{U}}$) from the polar decomposition of $\underset{\approx}{\mathbf{F}}$. The vector $\underset{\sim}{\mathbf{a}}$ is just the deformed vector of a material fiber that is initially coincident with the unit vector $\underset{\sim}{\bar{\mathbf{M}}}$, and $\underset{\sim}{\bar{\mathbf{a}}}$ is the vector obtained by "unrotating" $\underset{\sim}{\mathbf{a}}$ back to the reference configuration. Eqn. (1) states that the Cauchy stress must be uniaxial in the direction of $\underset{\sim}{\mathbf{a}}$, as should be self-evident from Fig. 1, which therefore lead Zheng [1992] to assert that the unrotated transverse symmetry axis (to which elastic constants are referenced) should always be parallel to $\underset{\sim}{\bar{\mathbf{a}}}$. This conclusion can be proved inadequate by considering Fig. 2 where *identical* deformations are applied to materials having different microstructures. They are both initially transversely isotropic about the same direction, but the deformed symmetry axis is different for the two cases.

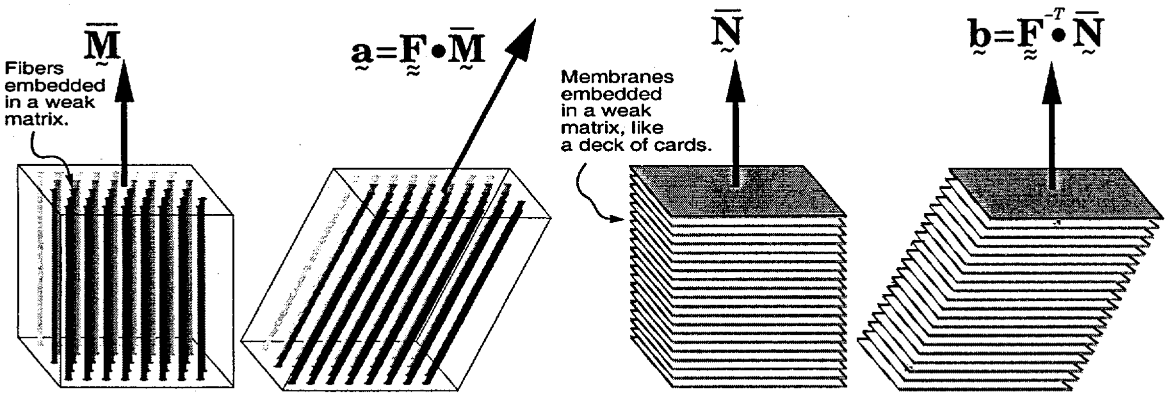

Figure 2: Two structures that are initially transversely isotropic about the *same direction*. After simple shear, the symmetry axis moves with the fibers for the fiber composite, but stays normal to the planes for the layered composite. *Neither* symmetry axis reorients according to the polar rotation.

Brannon [1998] also showed that the exact solution for the rate of the PK2 stress is

$$\dot{\bar{s}}_{ij} = \bar{E}_{ijkl}\dot{\bar{\varepsilon}}_{ij}, \text{ where } \bar{E}_{ijkl} = \frac{\mathcal{G}'(\lambda)}{\lambda}M_i M_j M_k M_l. \tag{3}$$

Here, $\underset{\approx}{\bar{\varepsilon}}$ is the Lagrange strain ($\bar{\varepsilon}_{ij} = \frac{1}{2}(F_{ki}F_{kj} - \delta_{ij})$). The PK2 tangent stiffness tensor $\bar{E}_{ijkl}$, *is nonlinear even if the fiber force function is linear*, but it has an intuitive structure. Only E_{1111} will be nonzero when the fibers are originally oriented in the 1-direction — there is no shear component of the PK2 stiffness. Taking rates of the Cauchy stress is far more complicated because neither $\underset{\sim}{\mathbf{a}}$ nor $\underset{\sim}{\bar{\mathbf{a}}}$ is constant in time. The Cauchy stress is uniaxial, but its *rate* is not uniaxial. There *does* exist a tensor L_{ijkl} such that $\dot{\bar{\sigma}}_{ij} = \bar{L}_{ijpq}\bar{D}_{pq}$, where $\bar{D}_{pq}$ is the unrotated symmetric part of the velocity gradient — however, even though the material cannot sustain shear, L_{ijkl} must contain shear terms to account for rotation of the material fibers! "Generalizing" a small distortion transversely isotropic stress-strain model by simply applying it in the unrotated Cauchy frame will not capture this kinematic effect. The structure of the exact L_{ijkl} is so complicated that it is essentially useless even for the simple fibers-in-air material. Furthermore, the Cauchy tangent stiffness L_{ijkl} is not even major symmetric (as often erroneously assumed in the literature). To ensure major symmetry, the Kirchhoff stress $\underset{\approx}{\tau} \equiv J\underset{\approx}{\sigma}$ must be used. Otherwise, rates of J from Eqn. (1) will lead to terms in the expression for L_{ijkl} that are of the form $\sigma_{ij}\delta_{kl}$, which is not major symmetric. Faults with the Cauchy stiffness might seem to suggest that the PK2 stress is a better choice for the stress measure, but then one is faced with an absolute necessity to include nonlinearity of the moduli. As explained by Brannon [1998], cavalierly using constant PK2 moduli can result in numerical instabilities under compression. In short, a constant PK2 modulus gives the right stress directions, but wrong (and potentially unstable) stress magnitudes, and vice versa for the Cauchy stress.

When data are lacking, we suggest trying the Lagrange strain and PK2 stress, but with a strain-dependent tangent modulus tensor $\bar{C}_{ijkl}$ defined by

$$\bar{C}_{ijkl} = c_1(\varepsilon_1)\bar{B}^1_{ijkl} + c_2(\varepsilon_2)\bar{B}^2_{ijkl} + c_3(\varepsilon_2)\bar{B}^3_{ijkl} + c_4(\varepsilon_4)\bar{B}^4_{ijkl} + c_5(\varepsilon_5)\bar{B}^5_{ijkl} , \tag{4}$$

where $\varepsilon^\alpha \equiv \sqrt{\bar{\varepsilon}_{ij}\bar{B}^\alpha_{ijpq}\bar{B}^\alpha_{pqkl}\bar{\varepsilon}_{kl}}$ and the five c_i are material functions for which $c_i(0)$ must equal the appropriate modulus measured under small distortions. Components of the fourth order transverse basis tensors (which can be derived using group theory) are

$$\bar{B}^1_{ijkl} = \bar{M}_i\bar{M}_j\bar{M}_k\bar{M}_l \tag{5}$$

$$\bar{B}^2_{ijkl} = \delta_{ij}\delta_{kl} - \bar{M}_i\bar{M}_j\delta_{kl} - \delta_{ij}\bar{M}_k\bar{M}_l + \bar{M}_i\bar{M}_j\bar{M}_k\bar{M}_l \tag{6}$$

$$\bar{B}^3_{ijkl} = \bar{M}_i\bar{M}_j\delta_{kl} + \bar{M}_k\bar{M}_l\delta_{ij} - 2\bar{M}_i\bar{M}_j\bar{M}_k\bar{M}_l \tag{7}$$

$$\bar{B}^4_{ijkl} = \tfrac{1}{2}(\bar{M}_i\bar{M}_k\delta_{jl} + \bar{M}_j\bar{M}_l\delta_{ik} + \bar{M}_i\bar{M}_l\delta_{jk} + \bar{M}_j\bar{M}_k\delta_{il}) - 2\bar{M}_i\bar{M}_j\bar{M}_k\bar{M}_l \tag{8}$$

$$\bar{B}^5_{ijkl} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \bar{M}_i\bar{M}_j\bar{M}_k\bar{M}_l - \tfrac{1}{2}(\delta_{ik}\bar{M}_j\bar{M}_l + \delta_{il}\bar{M}_j\bar{M}_k + \delta_{jk}\bar{M}_i\bar{M}_l + \delta_{jl}\bar{M}_i\bar{M}_k). \tag{9}$$

By considering idealized microstructures, functional forms could be derived for each of the nonlinear modulus functions. For example, defining the equivalent stretch as $\lambda^\alpha \equiv \sqrt{2\varepsilon^\alpha + 1}$, we can define the function c_1 to match the exact fibers-in-air solution in Eqn. (3).

CONCLUSIONS: Prediction of the stresses caused by large distortion remains an unsolved problem. The most popular approach of merely applying a model in the unrotated Cauchy frame is intoxicatingly robust and satisfies frame indifference, *but it is still wrong*, as was demonstrated via our simple counterexample of fibers in a weak matrix. We have offered a highly heuristic method of generating a transverse isotropy model that will give the exact answer in the limit of fibers in a weak matrix, but much more research is needed to generate models for arbitrary anisotropy that will give correct answers at large distortion as well as just large deformation.

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