A Direct Stiffness-Modification Approach to Attaining Linear Consistency Between Incompatible Finite Element Meshes[1]

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Abstract: In this work, a method is proposed for modifying the standard master-slave stiffness matrix so that linear consistency across the interface of the master and slave meshes is achieved. The existence of such a local stiffness modification is implied by the work of [Dohrmann, et al, to appear]. The present work aims at achieving the *same* linear consistency through a different method of stiffness modification that is based on simply ensuring zero residual force at the interior interface nodes for all non-zero-stress linear displacement fields and zero residual force at all interface nodes for all rigid-body linear displacement fields. These zero residuals ensure that the local stiffness modification results in an interface that passes the patch test. Numerical examples herein demonstrate that the maximum stress error at the interface goes to zero with the proposed method while it does not for the standard master-slave method.

1. Introduction

The standard master-slave method of tying two incompatible meshes together generally results in a stiffness matrix that does not satisfy linear consistency across the interface of the master and slave meshes. The work of [Dohrmann, et al] presents a method for modifying the uniform strain part of the stiffness matrices of the slave interface elements so that linear consistency across the interface is achieved. The tying method of [Dohrmann, et al] is based upon the uniform strain approach of [Flanagan and Belytschko, 1981] which has also been used to develop an element-level transition element [Dohrmann and Key, 1999]. Their method involves assigning and calculating gap volumes for each such slave element and taking partial derivatives of each such gap volume with respect to the unconstrained (independent) nodal degrees of freedom to achieve a modification to the uniform strain part of the strain displacement matrix of the element. This approach is simple in concept and appealing in concept. However, the work of the present paper aims at achieving the linear consistency through a more algebraic approach to modifying the stiffness matrix; this proposed approach avoids explicit accounting of gap volumes. The problem statement is given in Section 2. The proposed solution method is described in Section 3. Numerical examples are given in Section 4 to demonstrate convergence to zero of the maximum stress error at the interface for the proposed approach and non convergence for the standard master-slave approach. Finally concluding remarks are given in Section 5.

2. Problem Statement

We are given two dissimilar finite element meshes that we want to connect together in such a way that linear consistency across their interface is achieved. Figure 1 below illustrates such a problem in two dimensions involving two meshes of standard isoparametric 4-noded Quad4 elements. The "o" symbols denote nodes of the master Quad4s and the "x" symbols denote nodes of the slave Quad4s.

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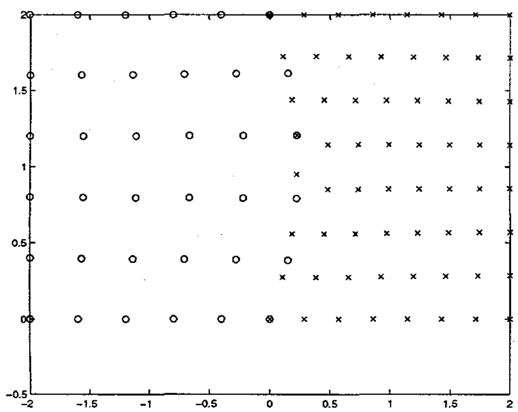

Figure 1. Illustration of a Mesh Tying Problem in Two Dimensions

In general, the elements within either the master or slave meshes do not have to be all of the same type, i.e., they do not have to be all Quad4s. It is only assumed that the master and slave meshes are individually valid, i.e., they individually pass the patch test [Zienkiewicz and Taylor, 1989].

3. Method

The first steps of the proposed approach are identical to the standard master-slave method and can be briefly summarized as follows. We move each slave interface node to the closest master edge on the interface. The nodal degrees of freedom of this slave node are then "constrained" so that the node stays on the chosen point of the master element's edge (or face in three dimensions). So, a linear multi-point constraint results which expresses the slave interface node's degrees of freedom in terms of the degrees of freedom of the master nodes on that master element's edge.

If we partition the degrees of freedom, denoted $\tilde{q}$, into unconstrained and constrained degrees of freedom as:

$$\tilde{q} = \begin{pmatrix} \hat{q} \\ q_s \end{pmatrix} \tag{3.1}$$

where the $\hat{q}$ are the unconstrained ones and the q_s the constrained slave interface nodal degrees of freedom, and if we let the constraints be denoted by

$$q_s = C\hat{q} \tag{3.2}$$

then,

$$\tilde{q} = A\hat{q} \tag{3.3}$$

where

$$A = \begin{bmatrix} I \\ C \end{bmatrix} \tag{3.4}$$

Letting $\tilde{K}$ denote the stiffness matrix associated with $\tilde{q}$, i.e., the strain energy of the combined master and slave meshes is given by $\frac{1}{2}\tilde{q}^T\tilde{K}\tilde{q}$, then combining this with (3.3) results in the stiffness matrix of the standard master-slave method:

$$K_{ms} = A^T\tilde{K}A \tag{3.5}$$

That is, the strain energy is now $\frac{1}{2}\hat{q}^T A^T\tilde{K}A\hat{q}$. Unfortunately, this stiffness matrix K_{ms} does not yield linear consistency across the interface of the master and slave meshes.

The method of [Dohrmann, et al] assigns to each slave interface element a gap volume (or gap area in two dimensions) and calculates partial derivatives of the gap volume with respect to the unconstrained nodal degrees of freedom that define this gap volume to obtain modifications to that element's uniform strain-displacement matrix, which in turn determines the modifications to the stiffness matrix contributions of this slave interface element. We propose instead to simply identify stiffness matrix elements (nodal interaction pairs) that might be affected by such a stiffness modification. Proceeding in this way through all the slave interface elements we accumulate a list of nodal interaction pairs and thereby a set P of (i,j), $j \geq i$, pairs of K_{ms} entries that might need modifying to achieve linear consistency. By definition, the nodal forces (at the interface) must be *zero* for all rigid-body linear displacement fields (of which there are 3 independent ones in two dimensions and 6 in three dimensions). Letting v_k denote such a rigid body displacement vector, we have:

$$\sum_{j\in M(I)} (K_{ms})_{Ij} v_{kj} + \sum_{j\in Q(I)} (\Delta K_{ms})_{Ij} v_{kj} = 0, \ \forall I \in S, \quad k = (1,2,...,6) \tag{3.6}$$

where S is the union of the set of master nodal degrees of freedom on the interface and the set of degrees of freedom of unconstrained slave nodes belonging to slave interface elements, $M(I)$ is the set of degrees of freedom for which $(K_{ms})_{I,M(I)} \neq 0$, and $Q(I)$ is the set of degrees of freedom for which $(I,Q(I))$ is in P or the "symmetric part" of P.

Also, for each independent non-zero-strain linear displacement field (of which there are 3 in two dimensions and 6 in three dimensions), the nodal forces at each internal interface node must be zero. Letting w_k denote such a displacement vector, we have:

$$\sum_{j\in M(I)} (K_{ms})_{Ij} w_{kj} + \sum_{j\in Q(I)} (\Delta K_{ms})_{Ij} w_{kj} = 0, \ \forall I \in S_{in}, \quad k = (1,2,...,6) \tag{3.7}$$

where S_{in} is the same as S except that it contains no degrees of freedom of nodes on the boundary of the combined master/slave domain.

It is important to note that each equation in (3.6) and (3.7) contains only a small number (O(1) in size) of unknown ΔK_{ms} values. Hence the Jacobian matrix of these constraints with respect to these unknowns is *very* sparse, having $O(N_I)$ nonzeros where N_I is the number of nodes in S. The number of constraints in (3.6)/(3.7) is essentially $12N_I$ for two-dimensional problems and $36N_I$ for three-dimensional problems. Sparsity patterns of this Jacobian will be illustrated on example two dimensional problems in the next section. In general the unknown ΔK_{ms} values that satisfy (3.6) and (3.7) are not unique, but by merely the construction of the method of [Dohrmann, et al], this set of sparse linear equations is guaranteed to have associated ΔK_{ms} entries that solve (3.6) and (3.7), i.e., a solution to the linear system of equations exists.

Although there are other heuristics for making the choice for ΔK_{ms} unique, we chose to find the ΔK_{ms} solution whose sum of the squares is smallest. Letting p denote the vector of unknown ΔK_{ms} values and J the Jacobian of (3.6)/(3.7), (each entry in J being simply composed of components of the v_k or w_k vectors), with respect to p, and g the value of (3.6)/(3.7) at $p=0$, we then must solve the following sparse linear system:

$$\begin{bmatrix} I & J^T \\ J & 0 \end{bmatrix}\begin{pmatrix} p \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ -g \end{pmatrix} \quad (3.8)$$

where λ is a vector of unknown Lagrange multipliers, one for each constraint. Again, by the method of [Dohrmann, et al], $Jp=-g$ is guaranteed to have a solution. We note that the proposed stiffness modification approach results, in general, in modification of the higher order stiffness part [Bergan and Nygard, 1984], as well as the uniform strain part, of the slave interface elements' stiffness matrices.

4. Numerical Examples

In this section, we will demonstrate that the stiffness matrix resulting from the method of the previous section yields convergence to zero of the maximum stress error on the interface, while the standard master-slave method does not. The reason, again, is that the latter does not satisfy linear consistency across the interface, i.e., does not satisfy the patch test, while the former does.

Consider the problem illustrated in Figure 1 of Section 2. We made three mesh refinements of the mesh in Figure 1 in which each Quad4 was split into 4 Quad4s, while applying both the standard master-slave method and the proposed method, and we looked at the convergence/non-convergence of the maximum centroidal stress error (over all the master interface elements). The particular boundary-value-problem used is as follows. We chose a plane stress elasticity problem with a Young's modulus of $E=2.07\times10^{11}$ and a Poisson's ratio of $\upsilon=0.3$. The boundary conditions imposed were as follows.

$$u=(\alpha_1 h_2/2)x+(-\alpha_1)x(y-h_2) \quad (4.1)$$

$$v=(\alpha_2 h_2/2)y+(-\alpha_2/2)(y-h_2)^2+(\alpha_1/2)x^2 \quad (4.2)$$

where u and v are the x and y displacements, respectively, and where

$$\alpha_1 = 1/E \quad (4.3)$$

$$\alpha_2 = -\upsilon/E \quad (4.4)$$

and $h_2=2$ is the vertical size of the rectangular domain. Displacements (4.1)/(4.2) were applied on the boundary, and the associated Dirichlet problem solved. The exact stress solution for this problem is given by

$$\sigma_x = h_2/2-y \quad (4.5)$$

$$\sigma_y = 0 \quad (4.6)$$

$$\tau_{xy} = 0 \quad (4.7)$$

Let n denote the number of master elements on an edge of the master mesh. Results for the maximum master interface element centroidal stress error versus n are given in Figure 2 below.

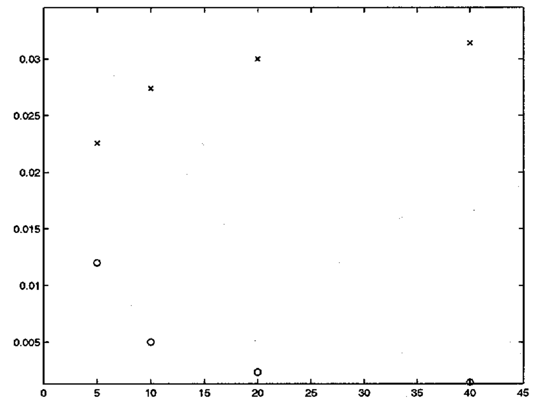

Figure 2. Maximum centroidal stress error in the master interface elements versus number of master elements on an edge of the master domain; x -- standard master-slave; o -- proposed method

Also, Table 1 and Table 2 below give tabled values for Figure 2.

Table 1. Maximum Stress Error Versus n, Standard Master-Slave

n	Max Stress Error
5	0.0226
10	0.0274
20	0.0300
40	0.0314

Table 2. Maximum Stress Error Versus n, Proposed Method

n	Max Stress Error
5	0.0120
10	0.0050
20	0.0023
40	0.0014

From Figure 2, we see that for the standard master-slave method the maximum stress error is not converging to zero. However, for the proposed approach, it is converging to zero. From Tables 1 and 2 we see that at $n=40$, we have a factor of 20 reduction in the maximum stress error for the proposed approach when compared with the standard master-slave method.

From a computational point of view, the largest cost of the proposed method is in solving the sparse linear system (3.8). The sparsity of J for $n = 40$ is illustrated in Figure 3 below:

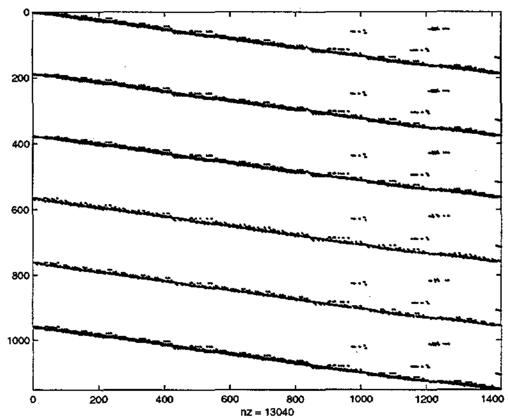

Figure 3. Illustration of the Sparsity of J, for $n = 40$

5. Conclusion

In this paper we presented a method for connecting two finite element meshes together in such a way that linear consistency across the interface of the two meshes is attained, i.e., such that the resulting stiffness matrix passes the patch test. The method starts with the stiffness matrix of the standard master-slave method. Then, it identifies a list of nodal interaction pairs across the interface that define what parts of the stiffness matrix might need to be modified to achieve linear consistency. Finally, constraints are written expressing zero residual forces at nodes on the interface (and unconstrained slave nodes belonging to slave interface elements) for independent imposed linear displacement fields. These constraints are linear in the unknown stiffness element changes. The Jacobian matrix for this set of linear constraints on these unknowns is *very* sparse and hence sparse-matrix solvers are used. The solution for the unknown stiffness element changes is not unique in general; herein, we chose to find the one whose sum of the squares is smallest. Although the existence of a stiffness modification that achieves linear consistency was implied by the method of [Dohrmann, et al], the method proposed herein does not involve any explicit gap volume assignment or accounting. Instead the method is more of an algebraic approach to obtaining the stiffness modifications that result in linear consistency across the interface. Numerical examples demonstrated that the proposed approach yields convergence to zero of the maximum stress error at the interface while the standard master-slave method does not.

References

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