Nonlinear System Modeling Based on Experimental Data

Thomas L. Paez
Experimental Structural Dynamics Department
Sandia National Laboratories
Albuquerque, New Mexico
(505) 844-7052

Norman F. Hunter
Mechanical Testing, Engineering Sciences, and Applications Division
Los Alamos National Laboratory
Los Alamos, New Mexico
(505) 667-2099

Abstract

The canonical variate analysis technique is used in this investigation, along with a data transformation algorithm, to identify a system in a transform space. The transformation algorithm involves the preprocessing of measured excitation/response data with a zero-memory-nonlinear transform, specifically, the Rosenblatt transform. This transform approximately maps the measured excitation and response data from its own space into the space of uncorrelated, standard normal random variates. Following this transform, it is appropriate to model the excitation/response relation as linear since Gaussian inputs excite Gaussian responses in linear structures. The linear model is identified in the transform space using the canonical variate analysis approach, and system responses in the original space are predicted using inverse Rosenblatt transformation. An example is presented.

Introduction

This paper considers the modeling of nonlinear, time invariant systems based on measured excitation and response data. It is assumed that the system under consideration is excited by a weakly stationary excitation, and that the system executes a weakly stationary response. The measured data are used to train a model, and once trained, the model can be used to predict system response when an excitation is specified. There are many approaches available for modeling nonlinear systems. For example, some of these are Volterra and Wiener series [see Marmarelis and Marmarelis (1978) and Schetzen (1980)], artificial neural networks [see Urbina, Hunter, and Paez (1998), Paez and Hunter (1997), Chance, Worden, and Tomlinson (1998) and Bailer-Jones, MacKay, and Withers (1998)], Bendat's approach [see Bendat (1983, 1990, 1998)], nonlinear time series models [see Priestley (1980), Tong (1990), Packard, et al. (1980), Farmer and Sidorowich (1988), Casdagli, et al. (1990), Larimore (1991)], and many others. In principle, the parameters or characteristic functions of these models can be identified using measured data. However, in practice, some frameworks are better suited to identification with measured data, than others.

One construct that has proven to yield accurate and efficient predictions for nonlinear system behavior is local linear modeling via canonical variate analysis (CVA). CVA is a technique within which system motions are mapped into a state space using a transformation obtained from an eigen-analysis of excitation/response correlation estimated from measured data. The local linear modeling of nonlinear systems is described in detail in Hunter (1997). That approach characterizes current system behavior in terms of excitation and response records from the recent past. When a prediction of future response is sought at a given time, records from the training data that match the most recent excitation/response data as closely as possible are identified. Because the matching data come from the training set, the future responses into which current responses evolve are known. These data are used to establish a linear model that is "local" to the current data, following the CVA approach described in the following section.

DISCLAIMER

A new approach to the modeling of nonlinear systems, based on measured data, is global in nature. It is based on the well-known fact [see Wirsching, Paez, Ortiz, (1995)] that a jointly Gaussian input to a linear system excites a jointly Gaussian response. In view of this, we seek a transformation that converts data from the hyperspace of the arbitrarily distributed random system response to a Gaussian space. The Rosenblatt transform provides precisely this type of map. We describe an approximate Rosenblatt transform [see Rosenblatt (1952)] in the third section. We show how it can be used to approximately map measured excitation and response signals into the Gaussian space. In that space CVA is used to create a linear model for the system under consideration. This set of transformations and the linear model are used to predict nonlinear system response. Some numerical examples are given.

Canonical Variate Analysis

Canonical Variate Analysis (CVA) is an extension of the Auto Regressive Moving Average ARMA model technique. Originally developed by Hotelling (1936), CVA was improved by Larimore (1983). Larimore's implementation of CVA gives accurate estimates of transfer functions and mode shapes for complex systems described by noisy time series measurements. CVA is described in detail in several references. (see Larimore,1983, Hunter,1997). Figures 1a and 1b schematically illustrate the method.

Figure 1a.
Time Evolution of Sampled, Measured Variables.

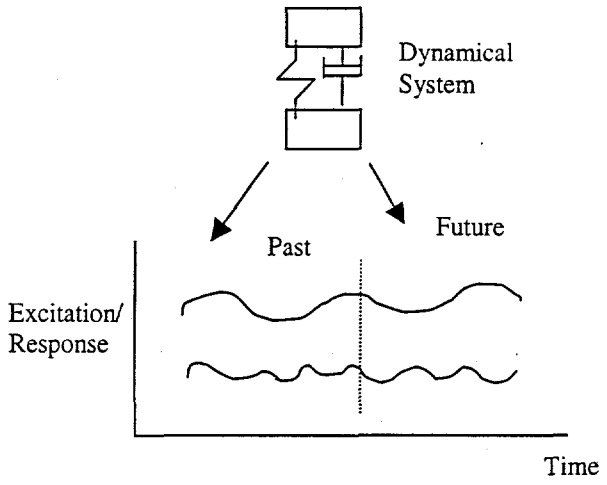

Figure 1b.
Time Series and Canonical Variate Analysis

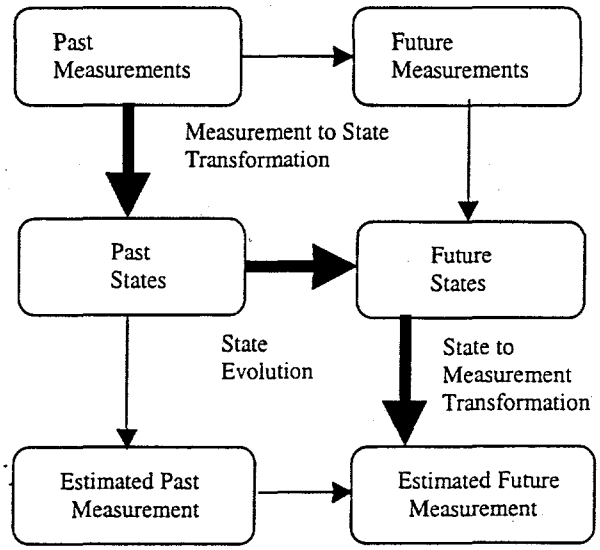

Figure 1a shows time series measured from the system illustrated. Figure 1b shows the three critical transformations involved in CVA, namely, the measurement to state transformation, the evolution of past states to future states, and the future state to estimated measurement transformation, each indicated by a dark arrow. Equations 1 through 4, below, implement the steps indicated in Figure 1b.

Let $x(t_j), j = ...,-1,0,1,...$, denote a (possibly multivariate) measured excitation, at times $t_j = j\tau, j = ...,-1,0,1,...$, and let $y(t_j), j = ...,-1,0,1,...$, denote the corresponding (possibly multivariate) measured response. The first operation of CVA transforms measurements of the system "past", $\boldsymbol{p}$, into a state space s.

$$s(t_j) = \boldsymbol{J}\boldsymbol{p}(t_j) \tag{1}$$

where the system past, relative to time t_1, is defined:

$$p(t_j) = [y(t_1 - \tau) \quad \cdots \quad y(t_1 - l\tau) \quad x(t_1) \quad \cdots \quad x(t_1 - j\tau)]^T \tag{1a}$$

The transformation matrix, $\boldsymbol{J}$, will be defined, below. The ensemble of past signals, $\boldsymbol{P}$, and the ensemble of future signals, $\boldsymbol{F}$, of the system are defined as follows, relative to time t_1, in terms of measured excitation and response.

$$\boldsymbol{P} = \begin{bmatrix} y(t_1 - \tau) \cdots y(t_1 - l\tau)\, x(t_1) \cdots x(t_1 - j\tau) \\ y(t_2 - \tau) \cdots y(t_2 - l\tau)\, x(t_2) \cdots x(t_2 - j\tau) \\ \vdots \qquad\quad \vdots \qquad \vdots \qquad\quad \vdots \\ y(t_k - \tau) \cdots y(t_k - l\tau)\, x(t_k) \cdots x(t_k - j\tau) \end{bmatrix} \tag{2a}$$

$$\boldsymbol{F} = \begin{bmatrix} y(t_1) \cdots y(t_1 + l\tau) \\ y(t_2) \cdots y(t_2 + l\tau) \\ \vdots \qquad\quad \vdots \\ y(t_k) \cdots y(t_k + l\tau) \end{bmatrix} \tag{2b}$$

Past states are evolved into future states, and future states are inverse transformed into the measured response space using the standard modal analysis framework. That framework is characterized by the equations:

$$\begin{aligned} s(t + \tau) &= \boldsymbol{A}s(t) + \boldsymbol{B}x(t) + e(t) \\ y(t) &= \boldsymbol{C}s(t) + \boldsymbol{D}x(t) + \boldsymbol{E}e(t) + w(t) \end{aligned} \tag{3}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$, are system matrices, $e(t)$ and $w(t)$ are noise vectors, and $\boldsymbol{E}$ accounts for state model noise in the state to measurement transformation. The measurement to state transform defined by the matrix $\boldsymbol{J}$ in Eq. (1) is developed as follows:

$$\begin{aligned} \boldsymbol{J} &= \boldsymbol{U}^T \left[\boldsymbol{P}^T \boldsymbol{P}\right]^{-1/2} \\ SVD\left[\left[\boldsymbol{P}^T \boldsymbol{P}\right]^{-1/2} \left[\boldsymbol{P}^T \boldsymbol{F}\right] \left[\boldsymbol{F}^T \boldsymbol{F}\right]^{-1/2}\right] &= \boldsymbol{U}\boldsymbol{W}\boldsymbol{V}^T \end{aligned} \tag{4}$$

where SVD[.] indicates the singular value decomposition. The gist of the operations carried out in these equations is to, first, establish the autocorrelation matrices of the system past, $\boldsymbol{P}$, and future, $\boldsymbol{F}$, using ensembles of segments of the measured excitation and response. These autocorrelations are decomposed using the Karhunen-Loeve expansion (see Ghanem and Spanos, 1991). The principal components are retained, and their cross-correlations are orthogonalized. This yields the transformation matrix, $\boldsymbol{J}$, a measurement to state space transformation that yields an

optimal relation between past and future principal components. Next all measured data segments are transformed into the state space using $\boldsymbol{J}$. The state transition and other state space matrices, $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$, are identified using linear least squares. The transformation and state space parameters are identified using $\boldsymbol{P}$ and $\boldsymbol{F}$, ensembles of signals. Response predictions operate on $\boldsymbol{p}$ to predict $\boldsymbol{f}$.

A global model can be created using all the data, or a local model can be created using data in the neighborhood of a particular past, $\boldsymbol{p}$. In this way, we create global and local linear models.

Equation 4 results from minimizing the mean square error in the prediction of the future $\boldsymbol{f}$ from the past $\boldsymbol{p}$. The process of transforming the measurements $\boldsymbol{p}$ to critical waveforms $\boldsymbol{s}$, transforming past states $\boldsymbol{s}$ to future states, and finally returning to the measurement domain for $\boldsymbol{f}$ may seem awkward, but in fact is considerably more stable than directly predicting $\boldsymbol{f}$ from $\boldsymbol{p}$. Intuitively, the gain comes from minimizing the number of parameters necessary in the past to future prediction process.

The Rosenblatt Transform

The Rosenblatt (1952) transform is a set of operations that permits the mapping of jointly distributed, continuous valued random variables and their realizations from the space of an arbitrary joint probability distribution into the space of uncorrelated, standard normal random variables. Let $X_1,...,X_n$, be a collection of arbitrarily, jointly distributed random variables with known marginal and conditional cumulative distribution functions (CDF), $F_{X_1}(x_1), F_{X_2|X_1}(x_2|x_1)$, etc. Then the sequence of operations:

$$\begin{aligned} U_1 &= F_{X_1}(X_1) & Z_1 &= \Phi^{-1}(U_1) \\ U_2 &= F_{X_2|X_1}(X_2|x_1) & Z_2 &= \Phi^{-1}(U_2) \\ &\vdots \\ U_n &= F_{X_n|X_1\ldots X_{n-1}}(X_n|x_1,\ldots,x_{n-1}) & Z_n &= \Phi^{-1}(U_n) \end{aligned} \tag{5}$$

transform the original random variables, first into a sequence of independent uniform[0,1] random variables, $U_1,...,U_n$, then into the sequence uncorrelated, standard normal random variables, $Z_1,...,Z_n$. The function $\Phi(.)$ is the standard normal CDF. Note that the functions operate on the random variables themselves. The top left-hand expression indicates that the functional operation transforms the random variable X_1 into a uniformly distributed random variable. This means that the realizations of X_1 are transformed into realizations of a uniformly distributed random variable.

Though it can usually be shown that the response of a nonlinear system is non-Gaussian, the joint probability distribution of the measured responses is usually unknown. Therefore, the Rosenblatt transform as written in Eq. (5) cannot be applied to measured data. One possible option is to write the kernel density estimator [see Silverman (1986)] of the probability density function (PDF) of the data, then integrate it. However, accurate representation of multivariate data requires impractical amounts of data. A compromise can be reached, though, by applying an approximate Rosenblatt transform to the data.

An approximate Rosenblatt transform can be formulated as follows. Let $y_{mj}, m = 1,...,M, j = 0,...,n-1$, be a collection of M measured signals, each containing n points. The measured signals are all assumed to emanate from weakly stationary random processes. The marginal CDF of each source random process can be approximated:

$$F_{Y_m}(y) = \frac{1}{n}\sum_{j=0}^{n-1}\Phi\left(\frac{y - y_{mj}}{\varepsilon}\right) \qquad m = 1,...,M \tag{6}$$

where ε is a width parameter that must be optimized. Operating on each of the M measured responses with Eq. (6), then taking the inverse, standard normal CDF transforms it, marginally, into standard normal random process. The operations are:

$$w_{mj} = \Phi^{-1}\left[F_{Y_m}\left(y_{mj}\right)\right] \qquad m = 1,...,M,\ j = 0,...,n-1 \tag{7}$$

These data are correlated since it is the operations below the first line in Eq. (5) that removes correlation.

To remove correlation from the signals $w_{mj,}\ m = 1,...,M,\ j = 0,...,n-1$, we first estimate the correlations among the signals, then use the information to form the correlation matrix, R_{WW}. Perform an eigenvalue analysis on the correlation matrix:

$$R_{WW} = VSV^T \cong vsv^T \tag{8}$$

and develop a reduced order approximation, as in the second step, if appropriate. S is the diagonal matrix of eigenvalues, and V is the matrix whose columns are the eigenvectors of the correlation matrix. The matrices s and v are the reduced forms of S and V with diagonal elements and columns, respectively, removed when they do not significantly contribute to the representation. Let W be the matrix with the signals $w_{mj,}\ m = 1,...,M,\ j = 0,...,n-1$, as its rows. Then the signals in the rows of

$$Z = s^{-1/2}v^T W \tag{9}$$

are realizations of uncorrelated, marginally, standard normal random processes. The random processes are not guaranteed jointly normal in time, and are only approximately jointly standard normal from signal to signal.

This transformation can be applied to the measured excitations and responses of a nonlinear system. Following transformation, a linear model for the system can be identified. This should be a fairly accurate representation because the transformed signals are approximately Gaussian, and Gaussian inputs excite Gaussian response in a linear system. Then system response to other excitations (with the same probability distribution) can be computed by (1) transforming the excitation with the procedure given above, (2) propagating the input to response using the linear model, (3) inverting the linear response into the non-Gaussian space using the inverse of the operations given above. The type of transformation used here is sometimes called a zero memory nonlinear transform.

Numerical Examples

Some numerical examples are summarized in this section. They are based on data collected from an experiment in which a system was simulated on an analog computer. The system simulated on the analog computer is one governed by the nonlinear differential equation of motion:

$$\ddot{y} + 2\zeta\omega_n(\dot{y} - \dot{x}) + \omega_n^2(y - x) + \alpha\omega_n^2(y - x)^2 + \beta\omega_n^2(y - x)|y - x| = 0 \tag{10}$$

where x denotes the excitation, y denotes the response, ζ is the system damping factor, ω_n is the system natural frequency, and α and β are nonnegative constants with $\alpha < \beta$. Dots denote differentiation with respect to time. In this particular example, the system parameters were given the values in Table 1. (The term units^{-1} indicates that the parameters have the inverse units of the displacement response.) The data were sampled at a rate of 1000 samples/second. The excitation is a band-limited white noise excitation with substantial signal content in the frequency range [30,100] Hz. The simulated excitation and response accelerations were recorded, and, of course, include noise associated with the physical electronic simulation and measurement. The estimated spectral densities of the excitation and response are shown in Figures 2 and 3. The figure makes it clear that the degree of the nonlinear response is substantial because a third harmonic of motion is present and high relative to the peak in the response spectral density associated with the fundamental.

Table 1. Parameters of the Nonlinear System

ω_n (rad/sec)	$(2\pi)76.7$
ζ	0.04
α (units^{-1})	3000
β (units^{-1})	3500

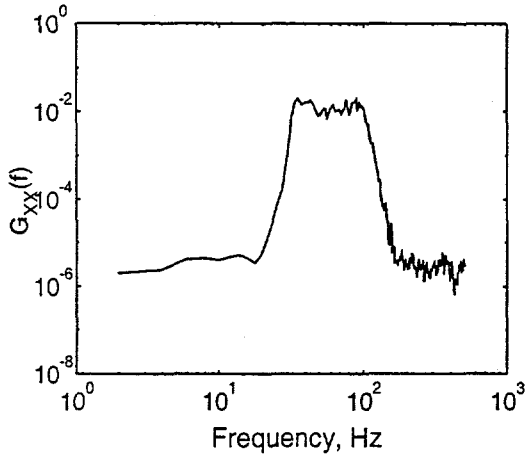

Figure 2. Estimated spectral density of excitation.

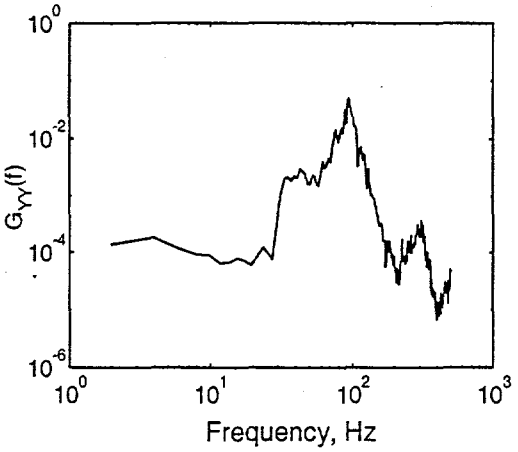

Figure 3. Estimated spectral density of response.

In all the system identification examples to follow, 4096 data points were used to identify system behavior, and an iterated prediction of the system response was made at 200 different data points. That is, the initial conditions of the system were specified for the model, then the model was used to predict the response at 200 times.

The measured system was first modeled using a local linear framework, with CVA. Excitation/response signals containing eight lag values were used in the local linear parameter identification. The number of nearest neighbors in the excitation/response signals used to create the data correlation matrices was 800, on average. Five principal components were used in the CVA transformation. The iterated prediction produced by the model is shown in Figure 4, along with the measured response from the same time period. The root-mean-square (RMS) values of the measured and predicted responses are 1.14 and 0.98, respectively.

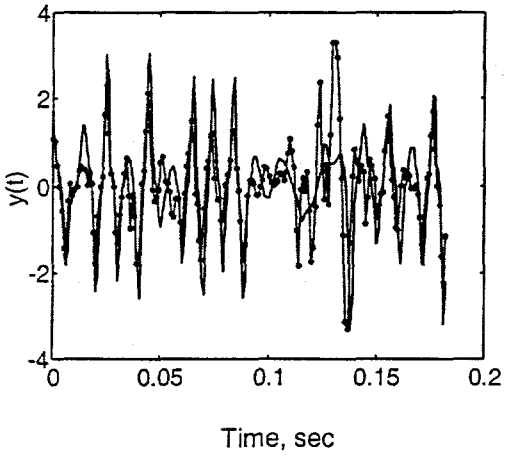

Figure 4. Measured (solid) and predicted (solid+dots) responses - local linear analysis.

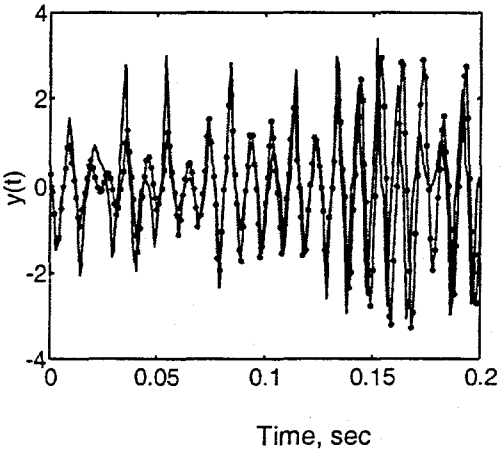

Figure 5. Measured (solid) and predicted (solid+dots) responses – Rosenblatt transform analysis.

Next, prior to system modeling, the measured excitation/response data were transformed into the Gaussian space using the approximate form of the Rosenblatt transform described above. Then the transformed excitation/response data were used to identify a global, linear model with CVA. The linear parameter identification used three lags and eight lags to characterize the excitation and response, respectively. Two and three principal components were retained in the CVA transformation of the excitation and response, respectively. The iterated prediction produced by the model is shown in Figure 5, along with the measured response from the same time period. The root-mean-square (RMS) values of the measured and predicted responses are 1.21 and 1.25, respectively. The PDF of the simulated response was estimated with a kernel density estimator, and is shown along with the estimated PDF of the measured data in Figure 6. The spectral density of the simulated response was estimated, and is shown with the estimated spectral density of the measured data in Figure 7.

The estimated PDFs indicate that the basic character of the measured response time history is simulated well by the Rosenblatt transform analysis. In particular, the distribution of peaks, indicated by the shapes of the tails, is accurately simulated. The estimated spectral densities indicate that the nonlinear character of the measured response is fairly well matched. In particular, the peak in the spectral density at the fundamental is well matched by the simulation, and the location and amplitude of the third harmonic in the simulated response matches the third harmonic in the measured data quite well. However, the width of the third harmonic peak is small.

Conclusions

A new technique for modeling nonlinear system behavior based on experimental data was developed. The technique approximately maps measured excitation and response data from the space of their own arbitrary joint probability distributions into spaces of uncorrelated standard normal random variables. It is in the latter pair of spaces that a linear model connecting transformed excitation to transformed response is identified. Nonlinear system responses to arbitrary excitations from the hyperspace of the training excitations are computed by transforming the excitation into the Gaussian space, computing the response of the linear system, then inverting the Gaussian response back into the measurement space. A numerical example shows that response computed in this way is reasonably accurate.

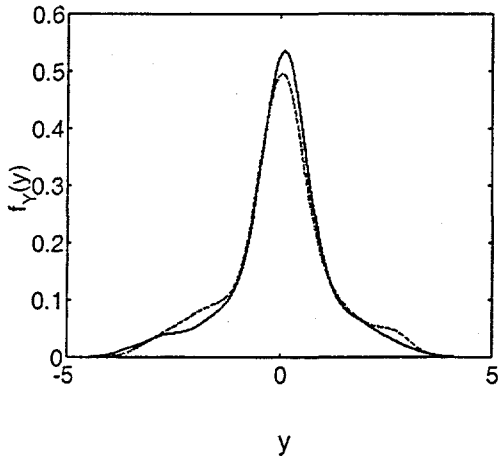

Figure 6. Estimated PDFs of measured (solid) and predicted (dashed) responses.

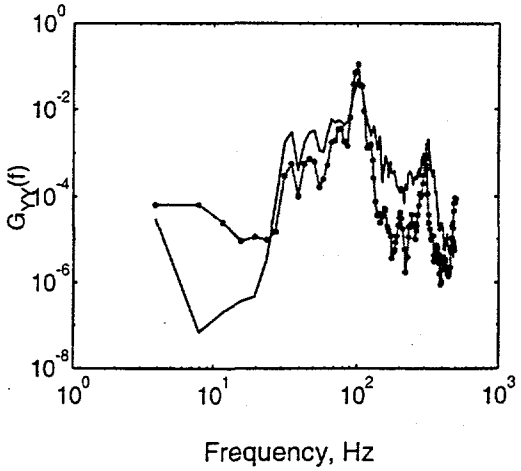

Figure 7. Estimated spectral densities of measured (solid) and predicted (solid+dots) Responses.

Several features of the approach developed here remain to be tested. First, the nearness of the transformed excitation and responses to a joint Gaussian distribution must be evaluated. It is this feature that permits characterization of a nonlinear model using a linear model in the transform space. Second, the capability of the technique to model responses to multivariate excitations must be tested. Finally, alternate procedures which, for example, transform the data in the midst of the CVA analysis should be tried.

Acknowledgment

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