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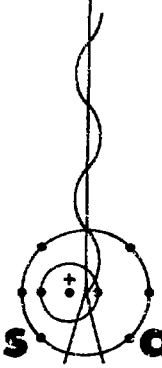
Symmetry Breaking in Fluid Dynamics: Lie Group Reducible Motions for Real Fluids

by

Darryl Dallas Holm



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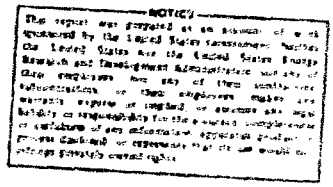

by

Darryl Dallas Holm

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PREFACE

The object of this work is to study fluid dynamics using symmetry principles. An invariance algebra technique which was originally developed by Lie is used to investigate non-ideal fluid dynamics in the Navier-Stokes approximation. A novel aspect of our approach is to regard particle interactions within the fluid as symmetry breaking mechanisms.

The fluid equations of motion must be consistent with the fundamental invariance principles of classical mechanics and thermodynamics. An operator realization of the invariance principles of fluid dynamics is derived in the first chapter and displayed in Table 1.1. The interplay between kinematical and thermodynamical symmetry principles within the Navier-Stokes description leads to a characterization of fluid motions according to subgroups of an underlying dynamical invariance group. The inequivalent subgroups of the Navier-Stokes invariance group produce "similarity variables," for which the one-dimensional unsteady Navier-Stokes equations reduce to ordinary differential equations. The group invariant coordinates for which this reduction occurs are cataloged in Table 1.3 and Table 1.4. Under the assumption that the flow variables in a group reducible motion depend on space and time in the same way as the boundary and initial conditions, the group invariants which are displayed in the text are applicable to a rather large

number of physical situations. The group invariants lead to classes of solutions of the Navier-Stokes equations and provide cases of moving boundaries for which group reducible solutions exist. The multiparameter invariants of Table 1.5 are especially useful because they contain arbitrary constants which may be adapted to the specifications of particular problems.

Two types of symmetry-breaking particle interactions are considered separately in Chapter 1: (1) Mie-Gruneisen deviations from a perfect gas equation of state; and (2) temperature dependent dissipation. In both cases the introduction of symmetry breaking reduces the invariance group of a simple isentropic gas to a subgroup, but it does not alter the invariance group structure. An empirically relevant elastic pressure interpolation function is found for which the isentropic fluid equations admit more than merely a kinematic (Galilean) symmetry group. Analytical characteristics and exact group reducible motions of isentropic Mie-Gruneisen (power-law) fluids are presented. An invariance algebra for dissipative fluids is also derived and group reduced equations of motion are given. Although no exact solutions for dissipative motion in the full Navier-Stokes description have been found in the present work, first integrals are given in a special case.

In the second chapter, dissipative fluid motions are treated at the level of Burgers' approximation to the Navier-Stokes equations. Burgers' equation governs the

final stages of weak shock wave formation in Navier-Stokes fluids. Group theoretical techniques are used to construct invariance principles and find classes of group reducible fluid motions. The symmetry analysis produces operator realizations of the invariance principles underlying the convective diffusion process. These are listed in Table 2.1. It also classifies the group-reducible motions and yields previously unpublished exact solutions for Burgers' description (see Table 2.3). Finally, the famous Cole-Hopf transformation between nonlinear and linear diffusion equations establishes itself naturally in the process of invariance analysis.

In addition to the material in I and II, explanations of peripheral topics are collected in three appendices. Primarily expository matters reside in Appendix A and Appendix C. However, further new results are included in Appendix B. Finally, the Bibliography contains an array of modern and historical articles concerning the use of symmetry principles in conjunction with dynamical laws.

Lie's invariance method for the integration of differential equations is summarized in Appendix C. The history of the development and application of Lie's theory to physical problems is surveyed. The physics background which has led to a revival of Lie's invariance analysis is given and the connection with modeling theory is described. The method of Lie is a natural generalization of the method of self-similar solutions. The deduction of

invariance principles from given dynamics is implemented in an example using the wave equation. The known conformal invariance algebra of the wave equation is constructed and realized as a differential operator algebra. Sample group reducible solutions - which include d'Alembert's solution - are derived from the operator algebra. Lie's method is also demonstrated for the boundary value problem.

The extension of invariance conditions to differential manifolds is described in Appendix A. Invariance of a differential surface element is central to the application of Lie theory to differential equations.

In Appendix B the Lie symmetry properties of Newton's equations of motion in classical mechanics are shown to lead to useful geometric invariance transformations on particle trajectories. For example, Kepler's law relating the period and size of planetary orbits appears as a scale-invariance principle for the equations of motion. Lie's method is employed in the construction of the symmetry algebra. The algebraic treatment of classical mechanics constitutes a further illustration of invariance analysis, this time for ordinary differential equations. The geometric invariance principles of classical mechanics are derived for various interparticle force laws. The associated group structures in each case are subgroups of the projective group, which is locally isomorphic to SL(3,R). A study is made of classical symmetry breaking for power law, harmonic, centrifugal, and Newtonian - $\frac{1}{r^2}$ forces.

The scaling laws of the particle orbits are obtained. The harmonic and centrifugal forces are shown to possess "accidental" symmetries. The introduction of symmetry breaking terms in the force law reduces the differential operator invariance algebra to a subalgebra. The last two statements, along with the specific algebraic structures are apparently new results.

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NOTATION

The most common symbols which denote physical quantities in the text are listed below. Wherever duplications in usage appear, the meaning of the notation is clear from the physical or mathematical context.

c	Adiabatic sound speed $c^2 = \frac{\partial P}{\partial \rho})_S = P_{\rho,S}$ (also, the speed of light in vacuum)
c_p	Specific heat at constant pressure
c_v	Specific heat at constant volume
D_{ij}	Kinematic deformation tensor
$\frac{\partial}{\partial x_j}$	j^{th} component of gradient operator
$\frac{D}{Dt}$	Material derivative $\frac{D}{Dt} = \frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j}$
δ_{ij}	Kroneker delta symbol
ε	A small parameter, often a field of infinitesimal group parameters
η	Shear viscosity coefficient
γ	Grüneisen parameter
I	Internal energy per unit mass
κ	Thermal conductivity (also, adiabatic compressibility)
κ_S	Adiabatic compressibility
L	Characteristic dimension of flow
Λ	Molecular mean free path
P_{ij}	Pressure tensor
p	Scalar pressure
Q	Infinitesimal differential generator of a Lie group

q_j	Heat flux vector
ρ	Mass density
S	Entropy per unit mass of a fluid
t	Time
θ	Temperature (^{o}K)
$\vec{v}$	Fluid velocity vector
u_i	i^{th} component of velocity vector
$u_{j,i}$	$\frac{\partial u_j}{\partial x_i}$
V	Volume per unit mass
$\vec{v}$	Constant velocity vector
$\vec{x}$	Position vector
x_i	i^{th} component of position vector
ζ	Bulk viscosity coefficient

ABSTRACT

SYMMETRY BREAKING IN FLUID DYNAMICS:
LIE GROUP REDUCIBLE MOTIONS FOR
REAL FLUIDS

by

Darryl Dallas Holm

Co-chairmen: Roy A. Axford, G. W. Ford

The physics of fluids is based on certain kinematical invariance principles, which refer to coordinate systems, dimensions, and Galilean reference frames. Other, thermodynamic, symmetry principles are introduced by the material description. In the present work, the interplay between these two kinds of invariance principles is used to solve for classes of one-dimensional non-steady isentropic motions of a fluid whose equation of state is of Mie-Gruneisen type. Also, the change in profile and attenuation of weak shock waves in a dissipative medium is studied at the level of Burgers' approximation from the viewpoint of its underlying symmetry structure.

The mathematical method of approach is based on the theory of infinitesimal Lie groups. Fluid motions are characterized according to inequivalent subgroups of the full invariance group of the flow description and exact group reducible solutions are presented.

CHAPTER I

GROUP REDUCIBLE FLUID MOTIONS

After a brief introduction to the symmetry laws of fluid dynamics, we study the invariance principles admitted by an ideal fluid with a Mie-Gruneisen equation of state. Energy dissipation by heat conduction and viscosity is put aside until the last two sections of this chapter. Empirically derived equations of state for metals are fit with symmetry - adapted functions and group reducible fluid motions are determined. In the context of fluid dynamics we determine how symmetry breaking occurs through interaction terms in the equation of state. The generators of the basic invariance principles of classical dynamics and fluid dynamics are realized in the form of differential operators. These operators are used to derive the natural group invariant variables for isentropic fluid motion with a Mie-Gruneisen equation of state. The general analytical characteristics of group reducible motions of such a fluid are presented. Similar calculations are made for a dissipative fluid obeying an ideal gas equation of state.

A. The Fundamental Invariance Principles of Fluid Dynamics

A fluid consists of an indefinitely large number of particles. Therefore it has an indefinitely large number of degrees of freedom. The structure of its motion at the molecular level is too complicated to study. However, it

has been established experimentally that the collective effects of the molecules can be adequately described by:

(1) a very small number of flow variables, e.g., the density, mean velocity, and two thermodynamic functions.

(2) a corresponding set of transport equations. These are the Navier-Stokes equations, which express the conservation laws for mass, momentum, and energy.

(3) a few constitutive functions, which express the energy and momentum flux densities in terms of the flow variables.

The view of classical fluid dynamics is that the constitutive relations are defined locally in terms of the flow variables. The material-dependent coefficients which appear in the transport equations enter as functions of local scalar state variables of the fluid, or as constants. The medium itself is assumed to stay near local thermodynamic equilibrium, so that its equilibrium equation of state applies.

Consider one-dimensional motion of an ideal fluid. Each fluid element moves isentropically. Energy dissipation between fluid elements by heat conduction and viscosity is neglected. Except where steep gradients appear, or when energy transfer mechanisms between molecular modes have long relaxation times, the neglect of energy dissipation is a good approximation.

In one dimension, the isentropic fluid dynamics equations are given by the conservation laws for mass

$$\frac{\partial \rho}{\partial t} + \rho u_x + \frac{n-1}{x} \rho u = 0 \tag{1.1}$$

momentum,

$$\frac{\partial u}{\partial t} + \frac{1}{\rho} p_x = 0 \tag{1.2}$$

and energy,

$$\theta \frac{\partial S}{\partial t} = \frac{\partial I}{\partial t} - \frac{p}{\rho^2} \frac{\partial \rho}{\partial t} = 0 \tag{1.3}$$

In the continuity equation (1.1), n = 1, 2, 3 for planar, cylindrical, or spherical geometry, respectively. Equation (1.2) is Newton's Law, F = ma. Equation (1.3) expresses the First Law of Thermodynamics for isentropic flow.

The form of the equation of state is taken to be

$$p = f(\rho, I) \tag{1.4}$$

The choice of a specific relation for the equation of state is postponed for a moment.

The energy balance or heat flow equation may be rewritten as

$$\rho \theta \frac{\partial S}{\partial t} = \rho \theta S_p \left[\frac{\partial p}{\partial t} + \frac{S_\rho}{S_p} \frac{\partial \rho}{\partial t} \right] = 0 \tag{1.5}$$

Then, remembering the definition of the sound speed and substituting the mass conservation law (1.1) we have

$$\frac{\partial p}{\partial t} + \rho c^2 \left(u_x + \frac{n-1}{x} u \right) = 0 \tag{1.6}$$

The isentropic First Law, equation (1.3), gives

$$\frac{dI}{d\rho} = \frac{p}{\rho^2} = I_p \frac{dp}{d\rho} + I_\rho = I_p c^2 + I_\rho \tag{1.7}$$

Hence, the sound speed is obtained directly from the equation of state, relation (1.4), when solved for I.

$$\rho c^2 = \frac{1}{I_p}\left(\frac{p}{\rho} - \rho I_\rho\right) = K_S^{-1}(\rho, p) \tag{1.8}$$

The inverse adiabatic compressibility K_S^{-1} is a function of ρ and p.

The isentropic fluid dynamics equations in one dimension, equations (1.1), (1.2) and (1.6), admit several types of transformations. Let the boundary and initial conditions be ignored, or assumed to admit the same transformations. First of all, the universal symmetries of nonrelativisitic mechanics are admitted: the equations are invariant under time translations, space translations, and Galilean boosts (n = 1, only. For n = 2, 3 there are preferred points). The admission of these transformations means, as usual, that the same fluid dynamical laws hold, regardless of when, where, or in which Galilean reference frame the experiments are performed. The conservation laws are also independent of the orientation of the experiment, so, in more spatial dimensions, the balance equations would also admit rigid rotations. The corresponding kinematic operations are given by:

(1) space and time translations (1.9)

$$(\vec{x}, t) \longrightarrow (\vec{x} + \vec{x}_o, t + t_o)$$

(2) rigid rotations

$$(\vec{x}, \vec{u}) \longrightarrow (R\vec{x}, R\vec{u}) \qquad (\text{same } R)$$

(3) Galilean transformations

$$(\vec{x}, \vec{u}) \longrightarrow (\vec{x} + \vec{c}\,t, \vec{u} + \vec{c})$$

These kinematic invariance transformations hold for all classical nonrelativisitic systems, and, as such, yield equivalence classes of solutions. For example, let $\rho = f(x,t)$, $p = g(x,t)$, and $u = v(x,t)$ be solutions of the fluid conservation equations for some planar fluid motion. Then a different flow, similar to the first one, but in the transformed variables, is also possible. In other words, the set of functions

$$\begin{aligned} \rho &= f(x + ct + x_o, t + t_o) \\ p &= g(x + ct + x_o, t + t_o) \\ u &= v(x + ct + x_o, t + t_o) + c \end{aligned} \tag{1.10}$$

must also be a solution set, for arbitrary constants x_o, t_o, c.

Beside these kinematic arguments, dimensional considerations also have a place in the discussion. The fluid dynamics equations contain five different dimensional quantities x, t, u, ρ, and p or I. These quantities have three

independent dimensional units, namely mass, length, and time. The basic dimensional scales may be changed provided the thermodynamic description of the fluid admits the scale transformation. Generally the equation of state and the constitutive relations place constraints on the scale invariance properties of the fluid description.

Suppose, however, that the fluid description does admit a scale transformation. Then there exists a family of different flows which are similar to one another and are derivable from a single solution by changing the basic scales of length, time, and mass.

For example, suppose that the balance equations (1.1), (1.2), and (1.6) admit a scaling transformation for some choice of equation of state, given by

$$
\begin{aligned}
x &\to x' = e^{\alpha} x \\
t &\to t' = e^{\beta} t \\
\rho &\to \rho' = e^{\delta - 3\alpha} \rho \\
p &\to p' = e^{\delta - \alpha - 2\beta} p \\
u &\to u' = e^{\alpha - \beta} u
\end{aligned}
\tag{1.11}
$$

where α, β, δ are scaling exponents for length, time, and mass. That is, let the balance equations be unchanged in form if the primed variables above are introduced. Let $\rho = f(x,t)$, $p = g(x,t)$, and $u = v(x, t)$ be solutions for a fluid motion. If the boundary and initial conditions also admit the transformation to primed variables, then a new possible motion can be described by the same functions f, g, v, but in the new variables. The solution for the

new motion is expressed by the functions

$$\rho' = e^{3\alpha-\delta} f(e^{\alpha}x, e^{\beta}t)$$
$$f' = e^{2\beta+\alpha-\delta} g(e^{\alpha}x, e^{\beta}t) \qquad (1.12)$$
$$u' = e^{\beta-\alpha} v(e^{\alpha}x, e^{\beta}t)$$

The new motion is similar to the old one, differing only in the mass, time, and coordinate scales. Note that the equation of state must follow the transformation for a new motion of the same fluid to obtain. In this way the symmetries of the fluid description yield equivalence classes of solutions.

There exist fluid motions which are invariant under the transformations admitted by the fluid description. These motions are called group reducible (see Appendix C). In a group reducible motion each of the flow variables evolves in time and changes in space in a group invariant manner. The spatial profiles of any of the flow variables, say the density, at different times during a group reducible flow can be made to coincide by performing a group transformation which leaves invariant the fluid equations of motion. The function $\rho(x,t)$ which has this group invariant property can be expressed in the form

$$\rho(x,t) = b(x,t) f(\lambda(x,t)) \qquad (1.13)$$

The argument $\lambda(x,t)$ is a group invariant quantity. Therefore the quantity $\rho/b(x,t)$ is also a group invariant.

For group reducible motions in one space dimension, the system of partial differential equations of fluid dynamics

reduces to a system of ordinary differential equations in new unknown functions of the group invariant variable λ. This of course produces a mathematical simplification. Group reducible solutions are of physical interest, as well, because they represent the asymptotic limits of solutions to non-group-reducible problems.

Group reducible motions which are derivable solely from scale invariance are called "self-similar". In a self-similar flow the spatial profiles of any of the flow variables at different times can be made to coincide by simply adjusting the scales of length and mass. Thus during a self-similar flow the spatial profiles of the flow variables retain their shapes although their dimensions may change. A self-similar density function would be expressed by

$$\rho(x,t) = D(t)\, F\left(\frac{x}{\chi(t)}\right) \qquad (1.14)$$

The argument $\frac{x}{\chi(t)}$ is called a "similarity variable".

Thus, self-similar solutions are special cases of group reducible solutions. Conditions for self-similar solutions to exist, examples of self-similar fluid motions, and references are given in the books by Landau and Lifshitz, Sedov, and Stanyukovich. Other examples of self-similar fluid motions are referenced in the bibliography.

In the first part of the present work our objective is to employ the general group invariance principles admitted by the isentropic fluid description in order to find classes of group reducible motions. The groups involved are Lie

groups - they depend continuously upon a space of parameters. In order to employ the Lie group method, the infinitesimal generators for the invariance principles admitted by the isentropic fluid description must be constructed.

Let us expand the transformations (1.9) and (1.11) in a Taylor series in the group parameters. The results of these expansions give infinitesimal variations which may be realized on the space $(\vec{x}, t, \vec{u}, \rho, p)$ as differential operators. The infinitesimal generators are given by

Space translations: $\frac{\partial}{\partial x_i}$

Time translations: $\frac{\partial}{\partial t}$

Rigid Rotations: $\varepsilon_{ijk}\left(x_j \frac{\partial}{\partial x_k} + u_j \frac{\partial}{\partial u_k}\right)$

Galilean boosts: $t \frac{\partial}{\partial x_i} + \frac{\partial}{\partial u_i}$

Length scaling: $x_i \frac{\partial}{\partial x_i}$ (no sum)

Time scaling: $t \frac{\partial}{\partial t}$

Velocity scaling: $u_i \frac{\partial}{\partial u_i}$ (no sum)

Density scaling: $\rho \frac{\partial}{\partial \rho}$

Pressure scaling: $p \frac{\partial}{\partial p}$

The combinations of infinitesimal operators admitted by the isentropic fluid equations for a specific choice of equation of state will be derived in the next section.

B. The Derivation of the Invariance Algebra for Planar Isentropic Flow

We are dealing with isentropic fluid motions. However, up 'til now the equation of state of the medium has not been specified. The medium which we shall consider

is a homogeneous metal at the pressures and temperatures produced by explosives. Under high explosive conditions in metals the experimental pressures range from about 10^5 atm to almost 10^6 atm during time intervals of tens of microseconds. The accompanying temperatures may reach $2000^{\circ}K$. Because the pressures involved greatly exceed the elastic limit of the material and because little heat exchange occurs across streamlines, the medium responds as an isentropic fluid.

A model equation of state with general validity is the Mie-Gruneisen equation of state, which can be written as

$$p = g(\rho) I + f(\rho) \tag{1.15}$$

In the Mie-Gruneisen model the pressure, p, and the specific internal energy, I, of a material are divided into two parts, an elastic part and a thermal part. The elastic pressure term $f(\rho)$ represents the modification to the equation of state due to interparticle forces within the material. The other term in the equation of state is the pressure due to thermal motion. The assumptions and physical reasoning which accompany the Mie-Gruneisen form of the equation of state are reviewed by Barron (1957). Experimental procedures for establishing Mie-Gruneisen relations at high explosive conditions in metals - i.e., thermodynamic procedures for measuring $f(\rho)$ and $g(\rho)$ above - are described in Rice, McQueen, and Walsh (1958).

For high explosive conditions in metals, the Mie-Gruneisen equation of state takes the form

$$p = \gamma \rho C_v \Theta + f(\rho) \tag{1.16}$$

where γ is a constant called the Gruneisen parameter. At the temperatures concerned, the specific heat of the lattice follows the Dulong-Petit law,

$$C_v = 3Nk \tag{1.17}$$

The specific heat is a constant, independent of volume.

The equation of state affects the isentropic fluid equations only through the adiabatic compressibility K_s. For an ideal gas, K_s^{-1} is given by

$$K_s^{-1} = \frac{c_p}{c_v} p \tag{1.18}$$

The elastic pressure term $f(\rho)$ in the Mie-Gruneisen equation of state alter the compressibility of the material. For equation of state (1.16) the compressibility is given by

$$K_s^{-1} = (\gamma+1) p + (\rho f' - f) \tag{1.19}$$

For many substances[1] the elastic pressure $f(\rho)$ obeys a single power law in the density. Thus, over the relevant range of pressure and density.

$$f(\rho) = C \rho^{\mu} \tag{1.20}$$

where C is a constant and the exponent μ is usually greater than negative one.

---

1 V. N. Zharkov and V. A. Kalinin, Equations of State for Solids at High Pressures and Temperatures, transl. by A. Tybulewics (Consultants Bureau, New York, 1971).

Let us proceed to derive the invariance algebra for planar isentropic flow of a Mie-Gruneisen fluid. Our object is to construct a differential operator realization of the invariance operations admitted by such a flow and to find the corresponding group reducible motions. The logical order of the derivation of the invariance algebra is as follows,

(1) The basic equations of motion are written down.

(2) General forms of coordinate functions for infinitesimal transformations are assumed.

(3) Covariance of the equations of motion is used to impose restrictions on the coordinate functions. The restricting equations are linear partial differential equations which determine the coordinate functions.

(4) The restricting equations are solved for the coordinate functions. The solutions yield the infinitesimal transformations which leave the equations of motion covariant. Because the coordinate transformations depend on the equation of state through the adiabatic compressibility, several analytic expressions for the compressibility are discussed.

Once the coordinate functions for the invariance algebra have been derived, group reducible fluid motions are sought by transforming the equations of motion to group invariant variables.

The basic conservation laws of isentropic fluid dynamics in one space and one time variable are:

$$\rho_t + (u\rho)_x + \frac{n-1}{x} u\rho = 0 \qquad \text{(mass)} \qquad (1.21)$$

$$u_t + u u_x + \frac{1}{\rho} p_x = 0 \qquad \text{(momentum)} \qquad (1.22)$$

$$p_t + u p_x + K_e^{-1}\left(u_x + \frac{n-1}{x} u\right) = 0 \qquad \text{(energy)} \qquad (1.23)$$

Only the planar case (n = 1) is explicitly considered, but the results for cylindrical and spherical geometry are also given. As written, the expressions in the balance equations are intended to have the following meanings and relationships,

Dependent Variables:		Independent Variables:	
Pressure	$p = p(x,t)$	Position	x
Density	$\rho = \rho(x,t)$	Time	t
Velocity	$u = u(x,t)$		

For the moment the bulk modulus $K(\rho, p) = \rho c^2$ is regarded as an arbitrary function of density and pressure.

Let a Lie group of transformations on these quantities be defined, to first order in the group parameter, by the expressions

$$\begin{aligned} \bar{x} &= x + \varepsilon X(x,t,u,\rho,p) \\ \bar{t} &= t + \varepsilon T(x,t,u,\rho,p) \\ \bar{u} &= u + \varepsilon U(x,t,u,\rho,p) \\ \bar{\rho} &= \rho + \varepsilon R(x,t,u,\rho,p) \\ \bar{p} &= p + \varepsilon P(x,t,u,\rho,p) \end{aligned} \qquad (1.24)$$

In the expressions (1.24) ε is an infinitesimal group parameter. The quantities X, T, U, R, P correspond to the coordinate functions of the infinitesimal generators in differential operator form. A general such operator Q will be given by the expression

$$Q = X\partial_x + T\partial_t + U\partial_u + [illegible] \qquad (1.25)$$

The operator Q is defined on the combined space (x, t, u, ρ, p). The coordinate functions correspond to independent variations on (x, t, u, ρ, p).

The transformation properties of the various partial derivatives appearing in the fluid dynamics equations are defined using the chain rule. A given transformation is called a symmetry operation when it leaves the conservation laws invariant up to an overall multiplicative factor. The equations which determine the coordinate functions of the symmetry operations admitted by the conservation laws for mass, momentum, and energy will be derived successively.

The order ε terms in the expressions which enter the continuity equation are given by

$$\begin{aligned} \tilde{\rho}_{\tilde{t}} &= \rho_t + \varepsilon [illegible] \\ \tilde{u}\tilde{\rho}_{\tilde{x}} &= u\rho_x + \varepsilon [illegible] \\ \tilde{\rho}\tilde{u}_{\tilde{x}} &= \rho u_x + \varepsilon [illegible] \end{aligned} \qquad (1.26)$$

The extended coordinate functions $R^{(t)}$, $R^{(x)}$, $U^{(x)}$, etc. are defined in Appendix B. In the barred variables, we have

$$\rho_{\tilde{t}} + (\tilde{\rho}\tilde{u})_{\tilde{x}} = \rho_t + (\rho u)_x + \varepsilon[R^{(t)} + uR^{(x)} + U\rho_x + \rho U^{(x)} - Ru_x] \qquad (1.27)$$

The invariance condition is that the order ε terms in (1.27) vanish when the continuity equation holds. Thus the bracketed term in (1.27) must satisfy

$$[R^{(t)} + uR^{(x)} + U\rho_x + \rho U^{(x)} + Ru_x] = C(x,t)[\rho_t + (\rho u)_x] \qquad (1.28)$$

A side calculation has eliminated the higher order dependence in the multiplier c(x,t). The bracketed term has a long expansion which is not reproduced here. Upon equating like coefficients of the bilinear products of derivatives in (1.28), a set of determining equations for the coordinate functions results,

$$
\begin{aligned}
&1: \quad R_t + uR_x + \rho U_x = 0 \\
&x: \quad uR_x + \rho U_t + \rho X_x - R = c\rho \\
&u_x: \quad R_x - \rho T_x = 0 \\
&u: \quad -X_t - uX_x + \rho T_t + \rho U_x - U = c\rho u \\
&t: \quad R_t - T_t - uT_x = 0 \\
&u_t: \quad uR_x - \rho U_x = 0 \\
&t_x: \quad R_x = 0
\end{aligned}
\Bigg\} \Rightarrow R_x = U_x = 0 \qquad (1.29)
$$

The other powers of derivatives insure only that

$$
\begin{aligned}
T &= T(x,t) \\
X &= X(x,t) \ .
\end{aligned}
\qquad (1.30)
$$

Similarly, invariance of the momentum equation implies that

$$
\left[U_t - uU_x + \rho U_x - \frac{1}{\rho}P - \frac{R}{\rho^2}P_x\right] = m(x,t)\left(u_t + uu_x + \frac{1}{\rho}P_x\right) \qquad (1.31)
$$

Hence, equating coefficients of like derivatives,

$$
\begin{aligned}
1: &\quad U_t + uU_x + \tfrac{1}{\rho}P_x = 0 \\
u_x: &\quad -X_t + uU_u - uX_x + \tfrac{1}{\rho}P_u + U = mu \\
u_t: &\quad U_u - T_t - uT_x = m \\
\rho_x: &\quad uU_\rho + \tfrac{1}{\rho}P_\rho = 0 \\
\rho_t: &\quad U_\rho = 0
\end{aligned}
\Bigg\} \Rightarrow P_\rho = U_\rho = 0 \qquad (1.32)
$$

$$
P_x: \quad \tfrac{1}{\rho}P_p - \tfrac{1}{\rho}X_x - \tfrac{1}{\rho^2}R = \tfrac{1}{\rho}m
$$

$$
P_t: \quad -\tfrac{1}{\rho}T_x = 0 \quad \Rightarrow T = T(t) \Rightarrow R_u = 0
$$

Thus the invariance criteria for the continuity and momentum balance equations lead to differential relations among the coordinate functions. The following dependence on (x, t, u, ρ, p) has been determined,

$$
\begin{aligned}
T &= T(t) \\
X &= X(x,t) \\
U &= U(x,t,u) \\
R &= R(x,t,\rho) \\
P &= P(x,t,u,p) \ .
\end{aligned}
\qquad (1.33)
$$

The invariance condition for the energy equation (1.23) is derived from

$$
\bar{P}_{\bar{t}} + \bar{u}\bar{P}_{\bar{x}} + \kappa(\bar{p},\bar{\rho})\bar{u}_{\bar{x}} = P_t + uP_x + \kappa u_x
$$

$$
+ \varepsilon\left[\Pi^{(t)} + UP_x + u\Pi^{(x)} + (\kappa_\rho R + \kappa_p \Pi)u_x + \kappa U^{(x)}\right] \qquad (1.34)
$$

The invariance condition for the energy equation, namely

$$
[\ ]_\varepsilon = \Xi(x,t)(P_t + uP_x + \kappa u_x) \qquad (1.35)
$$

requires that, for like derivatives,

$$
\begin{aligned}
1 &: \quad \mathbb{P}_t + u\mathbb{P}_x + \kappa\mathbb{U}_x = 0 \\
u_x &: \quad \kappa_\rho \mathbb{R} + \kappa_p \mathbb{P} + u\mathbb{P}_u + \kappa\mathbb{U}_u - \kappa\mathbb{X}_x = \kappa\mathcal{E} \\
u_t &: \quad \mathbb{P}_u = 0 \\
p_x &: \quad -\mathbb{X}_t + \mathbb{U} + u\mathbb{P}_p - u\mathbb{X}_x = \mathcal{E}u \\
p_t &: \quad \mathbb{R}_p - \mathbb{T}_t = \mathcal{E}
\end{aligned}
\tag{1.36}
$$

The reduced determining equations for the eight quantities $T(t)$, $X(x,t)$, $U(x,t,u)$, $R(x,t,\rho)$, $P(x,t,p)$, $c(x,t)$, $m(x,t)$, $\mathcal{E}(x,t)$, with arbitrary $\kappa(\rho,p)$, are collected below. There are twelve of them, four relations from each of the fluid dynamics equations. In the twelve determining equations, the inverse compressibility κ appears only twice.

$$
\begin{aligned}
&(1) \quad 1: && \mathbb{R}_t + u\mathbb{R}_x + \rho\mathbb{U}_x = 0 \\
&(2) \quad u_x: && \mathbb{U}_u - \mathbb{X}_x + \frac{\mathbb{R}}{\rho} = c(x,t) \\
&(3) \quad \rho_x: && \frac{1}{u}(\mathbb{U} - \mathbb{X}_t) + \mathbb{R}_\rho - \mathbb{X}_x = c(x,t) \\
&(4) \quad \rho_t: && \mathbb{R}_\rho - \mathbb{T}_t = c(x,t) \\
&(5) \quad 1: && \mathbb{U}_t + u\mathbb{U}_x + \frac{1}{\rho}\mathbb{P}_x = 0 \\
&(6) \quad u_x: && \frac{1}{u}(\mathbb{U} - \mathbb{X}_t) + \mathbb{U}_u - \mathbb{X}_x = m(x,t) \\
&(7) \quad u_t: && \mathbb{U}_u - \mathbb{T}_t = m(x,t) \\
&(8) \quad p_x: && \mathbb{P}_p - \mathbb{X}_x - \frac{\mathbb{R}}{\rho} = m(x,t) \\
&(9) \quad 1: && \mathbb{P}_t + u\mathbb{P}_x + \kappa\mathbb{U}_x = 0 \\
&(10) \quad u_x: && \frac{1}{\kappa}(\kappa_\rho \mathbb{R} + \kappa_p \mathbb{P}) + \mathbb{U}_u - \mathbb{X}_x = \mathcal{E} \\
&(11) \quad p_x: && \frac{1}{u}(\mathbb{U} - \mathbb{X}_t) - \mathbb{X}_x + \mathbb{P}_p = \mathcal{E} \\
&(12) \quad p_t: && \mathbb{P}_p - \mathbb{T}_t = \mathcal{E}
\end{aligned}
\tag{1.37}
$$

The solution of (1.37) goes as follows. Combining equations (3) and (4), we find

$$U = X_t - u(T_t - X_x) \tag{1.38}$$

The same equation for U is obtained from (6) and (7) or from (11) and (12). Now solve for $(T_t - X_x)$ separately from (2) plus (7) and from (8) plus (12). Subtracting and adding the results gives,

$$T_t - X_x = \tfrac{1}{2}(c - \varepsilon) \tag{1.39}$$

$$\frac{R}{\rho} = \left(\frac{c}{2} - m + \frac{\varepsilon}{2}\right) \tag{1.40}$$

Hence,

$$U = X_t - \tfrac{u}{2}(c - \varepsilon) \tag{1.41}$$

Equations (2) and (10) along with (1.40) imply that

$$\frac{1}{K}(K_\rho R + K_I I) = \tfrac{3}{2}\varepsilon - \tfrac{1}{2}c - m \tag{1.42}$$

If c, m, and ε are constant, then

$$U_u = -\tfrac{1}{2}(c - \varepsilon) \tag{1.43}$$

Hence, from (6) and (1.43) we find

$$X = (\varepsilon - c - m)x + f(t) \tag{1.44}$$

and as a consequence of (144), now (1.41) takes the form

$$U = \tfrac{1}{2}(\varepsilon - c)u + f'(t) \tag{1.45}$$

From (1.42) and (1.40) we have

$$\mathbb{T} K_P = \rho K_\rho \left(-\frac{c}{2} + \kappa - \frac{\varepsilon}{2}\right) + K\left(\frac{3}{2}\varepsilon - \frac{c}{2} - m\right) \tag{1.46}$$

Consequently $P_x = P_t = 0$. Since $P_x = 0$ and $X_x = \text{const}$, then $U_x = 0$. Then (5) implies that $U_t = 0$ and $f''(t) = 0$. Thus, $f(t) = a + bt$.

Finally, from (1.39) and (1.44), since $T_x = 0$, we can integrate T_t. The results so far are collected below.

$$\begin{aligned}
X(\kappa,t) &= \kappa(\varepsilon - c - m) + bt + \lambda \\
T(t) &= t\left(\frac{\varepsilon}{2} - \frac{c}{2} - m\right) + t_0 \\
\Sigma(u) &= \frac{u}{2}(\varepsilon - c) + \xi \\
R(\rho) &= \rho\left(\frac{\varepsilon}{2} + \frac{c}{2} - m\right) \\
\frac{1}{K}\left(K_\rho R + K_P \mathbb{T}\right) &= \mathbb{T}_P = \left(\frac{3}{2}\varepsilon - \frac{c}{2} - m\right)
\end{aligned} \tag{1.47}$$

The last equation in (1.47) can be rewritten as

$$\rho K_\rho\left(\frac{\varepsilon}{2} + \frac{c}{2} - m\right) = (K - \beta K_P)\left(\frac{3}{2}\varepsilon - \frac{c}{2} - m\right) + \beta K_P \tag{1.48}$$

or

$$(\alpha\beta + \beta) K_P + (\alpha + \delta)\rho K_\rho - \alpha K = 0 \tag{1.49}$$

where the new constants are defined by

$$\alpha = \frac{3}{2}\varepsilon - \frac{c}{2} - m \quad , \quad \delta = c - \varepsilon \quad , \quad \beta = \text{const.} \tag{1.50}$$

In terms of these, the coordinate functions take the form

$$
\begin{aligned}
X(x,t) &= -(\delta+m)x + bt + a \\
T(t) &= -(\tfrac{\delta}{2}+m)t + t_0 \\
U(u) &= b - \tfrac{\delta}{2}u \\
R(\rho) &= (\kappa+\delta)\rho \\
\Phi(\phi) &= \kappa\phi + \phi_0
\end{aligned}
\tag{1.51}
$$

Several expressions for the compressibility must be considered.

(1) $K(\rho,p)$ is arbitrary. Consequently, from (1.49), $\alpha = \beta = \delta = 0$, or in terms of the other constants, $\beta = 0$, $c = \varepsilon = -m$. The resulting four parameter algebra has coordinate functions,

$$
\begin{aligned}
X &= -cx + bt + a \\
T &= -ct + t_0 \\
U &= u \\
R &= \Phi = 0
\end{aligned}
\tag{1.52}
$$

Its differential operator basis is given by

$D = \partial_x$	space translations	
$H = \partial_t$	time translations	
$S = x\partial_x + t\partial_t$	x/t invariant scaling	(1.53)
$G = t\partial_x + \partial_u$	Galilean boosts	

These operators form the kinematic algebra of classical mechanics, plus scaling, in the absence of length, or time scales. In more dimensions, rigid rotations are added, as well as translations and boosts along the other spatial directions. In one-dimensional cylindrical or spherical

coordinates, the fixed axis or fixed point breaks the space translation and Galilean transformation symmetries, leaving only scale invariance and time translations.

(2) $\mathcal{K}(\rho,p)$ is derived from a Mie-Gruneisen equation of state,

$$\mathcal{K} = (\gamma+1)p + \rho f(\rho) - f(\rho) \tag{1.54}$$

For convenience equation (1.49) is rewritten

$$(\alpha+\beta)\mathcal{K}_p + (\alpha+\delta)\rho\mathcal{K}_\rho - \alpha\mathcal{K} = 0 \tag{1.49}$$

Substituting the Mie-Gruneisen form for $\mathcal{K}$ in (1.49) results in an equation for $f(\rho)$.

$$(\alpha+\delta)\rho^2 f'' - \alpha(\rho f' - f) + \beta(\gamma+1) = 0 \tag{1.55}$$

In order for the fluid motion to acquire more symmetry than (1.53), the elastic pressure $f(\rho)$ which enters the Mie-Gruneisen equation of state must satisfy the equidimensional differential relation (1.55). The solution of (1.55) for $(\alpha+\delta) \neq 0$ is given by

$$f(\rho) = \begin{cases} C\rho^{\frac{\alpha}{\alpha+\delta}} + A\rho - \frac{\beta(\gamma+1)}{\alpha}, & \delta \neq 0 \\ (C + A\ln\rho)\rho - \frac{\beta(\gamma+1)}{\alpha}, & \delta = 0 \end{cases} \tag{1.56}$$

where A and C are constants.

If $(\alpha+\delta) = 0$, but $\alpha \neq 0$, then the elastic pressure is given by

$$f(\rho) = C\rho^{1/\alpha} + \frac{\beta(\gamma+1)}{\alpha} \tag{1.57}$$

If $\alpha = 0$, but $\delta, \beta \neq 0$, then $f(\rho)$ is given by

$$f(\rho) = C\rho + \frac{\beta(\gamma+1)}{\delta}\ln\rho + f_0 \tag{1.58}$$

The parameter β may vanish in equation (1.58). Thus we have found four classes of elastic pressure interpolation functions including positive and negative powers, and logarithmi behavior, for which the fluid dynamics equations may admit a wider symmetry algebra than the purely kinematic invariance (1.53). Mie-Gruneisen equations of state which are not in one of these classes do not admit a wider algebra than (1.53).

(3) $K(\rho,p)$ is not in Mie-Gruneisen form. In a similar manner, equation (1.49) may be integrated directly for $K(\rho,p)$. The general solution is

$$K(\rho,p) = \rho^{\alpha/\kappa+\delta} F\left[(\kappa p+\beta)\rho^{-\frac{\alpha}{\kappa+\delta}}\right] \qquad (1.59)$$

where F is an arbitrary function. Since we are interested only in the Mie-Gruneisen form, results stemming from this general solution are not included.

The four classes of interpolation functions for the elastic pressure reduce to only one if logarithmic dependence in the density is ignored. Let the elastic pressure $f(\rho)$ from (1.56) appear in a Mie-Gruneisen equation of state.

$$p = \gamma\rho I + C_1\rho^{\mu} + A\rho - \beta\frac{(\kappa+1)}{\alpha} \qquad (1.60)$$

where the group parameters κ and δ are related by the exponent μ in the elastic pressure.

$$\delta = \left(\frac{1-\mu}{\mu}\right)\alpha \qquad (1.61)$$

The logic of the group analysis which follows is that the experimental value of μ imposes a relation between α and δ. This constant reduces the symmetry group of the ideal gas to

a subgroup.

Equation of state (1.60) is physically relevant. Power law interpolation functions can be used to describe the Mie-Gruneisen equations of state for most materials (see Zharkov and Kalinin) in regions of physically interesting pressures and densities. For example: $\mu = 2$ corresponds to van der Waals binary interaction correction; $\mu = -1$ yields the "harmonic" lattice model of solids; and $\mu = 0$ (or 1) results in the "stiff gas," which is applicable for a large class of solids under moderate compressions ($1 \leq \rho/\rho_0 \leq 1.2$).

The mathematical character and the invariance transformations of the resulting fluid motion depend on the particular value of the exponent μ. When μ takes on a specific value, the group space is reduced to relative values of α and δ which satisfy (1.61). Thus, the exponent μ breaks the symmetry. Rather, μ restricts or reduces the symmetry group to a subgroup one less in dimension. The other constants which appear in (1.60), C, A, and β, are integration constants. As such, they depend on the specification of the problem which is to be solved. They do not affect the mathematical character of the solution, but they do refer to physical properties of the medium. For example, a shift in energy or pressure has the nature of a change in the initial conditions. Rewrite the Mie-Gruneisen equation of state (1.60) in the form

$$p + \frac{B(\gamma+1)}{\alpha} = \gamma\rho\left(I + \frac{A}{\rho}\right) + C\rho^{\mu} \qquad (1.62)$$

Thus, $\not{p}$ and A correspond precisely to shifts in the values of the pressure and energy, respectively. Mathematically, this means that $\not{p}$ and A are inessential constants and could be set equal to zero. However, physically, $\not{p}$ and A are connected to the "stiffness" of the medium. Suppose that, for a moment, the constant C vanishes in (1.60), so that it is written,

$$p = \gamma \rho I + a^2(\rho - \rho_0) \tag{1.63}$$

The adiabatic compressibility is then given by

$$K_s = \frac{1}{(\gamma+1)p + a^2\rho_0} \tag{1.64}$$

If $(a^2\rho_0)$ is large, say 10^6 atm, then at normal pressures, the material described by (1.63) will be almost incompressible. The constants a and ρ_0 are to be interpreted as the sound speed and normal density, respectively. While the mathematical character of the solutions for nonzero $a^2\rho_0$ is not distinct from that for $a^2\rho_0 = 0$, the physical nature of the motions of a fluid whose equation of state is described by (1.63) is vastly different from an ideal gas. In fact, if the compressions are not too high, the equation of state (1.63) adequately describes the essential features of a large class of materials. In particular, it applies to metals under moderate pressure $\lesssim 10^5$ atm.

The remaining integration constant C in (1.60) multiplies ρ^{μ}. Therefore, it cannot be arbitrarily set to zero without affecting the character of the solution.

However, mathematically it could take on any nonzero real value. On the other hand, like β and A, the constant C is a parameter which depends on the physical situation. It sets the scale for the relative thermal and nonlinear ($\mu \neq 1$) elastic pressure and energy content.

Let us finally give the differential operator basis for the Lie invariance algebra for the isentropic motion of fluids governed by Mie-Gruneisen equations of state (1.60). The coordinate functions are given by

$$\begin{aligned} X &= -\left[\left(\tfrac{1-\mu}{\mu}\right)\alpha + m\right]x + bt + \lambda \\ T &= -\left[\left(\tfrac{1-\mu}{2\mu}\right)\alpha + m\right]t + \tau_0 \\ U &= b - \left(\tfrac{1-\mu}{2\mu}\right)\alpha u \\ R &= \tfrac{\alpha}{\mu}\rho \\ P &= \alpha p + \beta \end{aligned} \tag{1.65}$$

The differential operator basis is given by operators (1.53) plus the scaling operator

$$G_\mu = \tfrac{\mu-1}{2}\left(x\partial_x + u\partial_u\right) + \rho\partial_\rho + \mu p\partial_p \tag{1.66}$$

The pressure translation operator ∂_p also appears, but for now it is discounted from the mathematical discussion. The finite transformations of the operator G_μ are given by

$$\begin{aligned} x &\to \bar{x} = e^{\tau\left(\frac{\mu-1}{2}\right)}x \\ t &\to \bar{t} = t \\ u &\to \bar{u} = e^{\tau\left(\frac{\mu-1}{2}\right)}u \\ \rho &\to \bar{\rho} = e^{\tau}\rho \\ p &\to \bar{p} = e^{\mu\tau}p \end{aligned} \tag{1.67}$$

Thus, it has four inequivalent group invariants

$$t, \quad \frac{u}{x}, \quad \rho x^{-\frac{2}{\mu-1}}, \quad p x^{-\frac{2\mu}{\mu-1}} \tag{1.68}$$

The operator G_μ leaves invariant the dimension of the constant C, which has units of p/ρ^μ. It also leaves invariant the ratio of the pressure to the kinetic energy density, $p/\rho u^2$, and the relation $u^2 \sim \rho^{\mu-1}$ which pertains to the nonlinear elastic part of the sound speed squared.

$$c^2 = c_0^2 \left(\rho/\rho_0\right)^\chi + \frac{C_1 \mu(\mu-1)}{(\mu-1)-\chi} \rho^{\mu-1} \tag{1.69}$$

The operator G_μ may be written as

$$G_\mu = (1-\mu) G_0 + \mu G_1 \tag{1.70}$$

where the operators G_1 and G_0 are given by

$$G_1 = \rho \partial_\rho + p \partial_p \tag{1.71}$$

and

$$G_0 = -\tfrac{1}{2}(x\partial_x + u\partial_u) + \rho\partial_\rho + \beta\partial_p \tag{1.72}$$

when $\mu = 0$ or 1 the group parameters α and δ are decoupled and the symmetry group is given by (1.53) plus both G_0 and G_1. This occurs because when $\mu = 0$ or 1 the Mie-Gruneisen fluid is equivalent to an ideal gas.

The operator G_1 commutes with all other invariance operators, except pressure translation. Thus G_1 is in the center of the subalgebra, omitting β. If Q is a symmetry operator and $\mu = 1$, then $Q + \nu G_1$ is also a symmetry operator, where ν is an arbitrary constant. The value of ν in

some cases may be determined as an eigenvalue from the requirement that the boundary conditions be satisfied, as in the treatment of the wave equation, Appendix A.

Finally, we mention that one more operator is admitted when $\mu = 1$, $\alpha = \frac{2}{n}$ (where n = 1, 2, 3 for planar, cylindrical, spherical geometry, respectively) and the multipliers c, $\mathcal{E}$, m in (1.37) are allowed to depend linearly on the time. The additional operator is

$$F = t\partial_x + t^2\partial_t + (x-ut)\partial_u - nt\rho\partial_\rho - (2+n)t(p+\beta)\partial_p \quad (1.73)$$

Operator F was noticed by Ovsjannikov for the case n = 1 only, and for the ideal gas equation of state. The invariants of F are

$$\frac{x}{t}, \quad x-ut, \quad \rho t^n, \quad (p+\beta)t^{2+n} \quad (1.74)$$

The operator F, and the presence or absence of space translations and Galilean transformations, are the only aspects of the group analysis which depend on whether the one-dimensional geometry is planar, cylindrical or spherical.

In summary, we have proven that fluid motion admits invariance principles which correlate thermodynamic state variables with velocity and position, provided that the elastic pressure isentropes essentially follow a power law in the density. The classical kinematic Galilean symmetry laws for fluid mechanics have also been derived. Galilean invariance holds for an arbitrary equation of state because Galilean transformations do not affect the relationships

between the thermodynamic state variables. Invariance of the "internal" thermodynamic relationships in addition to the kinematical symmetry laws restricts the allowed transformations of the dynamical equations.

A differential operator basis L_{EM} for the invariance algebra of the isentropic flow equations with equation of state (1.62) is given in Table 1.1

Operator	Significance
$D = \partial_x$	Space translations
$H = \partial_t$	Time translations
$G = t\partial_x + \partial_u$	Galilean Boosts
$S = x\partial_x + t\partial_t$	x/t invariant scaling
$C_\mu = \frac{\mu-1}{2}(x\partial_x + u\partial_u) + \rho\partial_\rho + (\mu p + \beta)\partial_p$	μ -dependent scaling

Table 1.1. The Basis L_{EM} of the Lie Algebra for Euler Equations in One Planar Dimension.

The commutator table for L_{EM} is displayed in Table 1.2. The algebra L_{EM} is a semi-direct product of the Galilean algebra with the abelian scaling algebra generated by S and C_μ .

	D	H	G	S	C_μ	C_0	C_1
D	0	0	0	D	$(\mu-\frac{1}{2})D$	$-\frac{1}{2}D$	0
H	0	0	D	H	0	0	0
G	0	-D	0	0	$(\mu-\frac{1}{2})G$	$-\frac{1}{2}G$	0
S	-D	-H	0	0	0	0	0
C_μ	$-(\mu-\frac{1}{2})D$	D	$-(\mu-\frac{1}{2})G$	0	0	$-\beta\partial_\beta$	0
C_0	$\frac{1}{2}D$	0	$\frac{1}{2}G$	0	$\beta\partial_\beta$	0	0
C_1	0	0	0	0	0	0	0

Table 1.2 The Commutator Table of the Lie Algebra L_{EM} .

Let us reiterate how the Mie-Gruneisen power law term in the equation of state breaks the symmetry of perfect fluid motion. The $\mu = 0, 1$ components of C_μ apply to the case that $C = 0$ in (1.62). The two operators which take the place of C_μ when the ρ^n term vanishes in (1.62) are

$$C_1 = \rho\partial_\rho + (p + \beta)\partial_p \tag{1.75}$$

and

$$C_0 = -\frac{1}{2}(x)_x + u\partial_u) + \rho\partial_\rho + \beta\partial_p \tag{1.76}$$

The ρ^n term in (1.62) forces a relation between the group parameters for the transformations of an ideal gas and thereby reduces the symmetry group to a subgroup. The other terms β and A in (1.62) are not symmetry breaking terms because they do not change the essential character of the medium.

Let us say a word about the physical meaning of the invariance operators. The first three operators in Table 1.1 are the classical Galilean invariance operators. The operator S appears because, although no fundamental length or time enters the description, a characteristic speed appears. Namely, the sound speed of the material enters the description and must be left invariant. The operator C_μ represents a scaling transformation which preserves the form of the equations of motion as well as the relative thermal and elastic pressure contributions. The operator F in (1.73) is more complex. It correlates the fluid motion at different space-time points in a manner which depends on

the value of the time. It may also be related to the existence of a Legendre transformation which exchanges the roles of dependent and independent variables and linearizes the first two Navier-Stokes equations.

In the Mie-Gruneisen equation of state (1.62) an interpolation function has been introduced in order to describe lattice site interactions at high pressures. The group theoretical result of adding the $C\rho^{\mu}$ term in (1.62)is to couple the α and δ characteristics and group orbits, thereby breaking the symmetry of the linear interpolation function and reducing it by one in dimension. When the group parameters α and δ are independent, their orbits and characteristics are disjoint. Upon coupling them through the equation of state (1.62), the separate α and δ orbits degenerate and become identical. In order to study the effects of symmetry breaking through the equation of state, in the group analysis which follows, α and δ are treated independently and then reconnected for $\mu \neq 0, 1$, after the invariant coordinates have been found. Later on, further comments are made concerning the physical effects of the interaction term in the equation of state.

We are dealing with the isentropic motion of a fluid whose thermodynamic state is described by (1.62). The corresponding Lie invariance algebra can be realized as a set of differential operators through the coordinate functions, X, T, U, R, P, rewritten now as

$$
\begin{aligned}
X &= (d+n)x + bt + a \\
T &= nt + c \\
U &= du + b \\
R &= (\alpha - 2d)\rho \\
P &= \alpha p + \beta
\end{aligned}
\tag{1.77}
$$

where,

$$d = -\frac{\delta}{2} \quad , \quad c = t_0 \tag{1.78}$$

and

$$n = -\left(\frac{\delta}{2} + m\right) = d - m \tag{1.79}$$

For the moment, d and α are taken to be independent, as in the case of an ideal gas, also for the "stiffened gas," simplified equation of state for metals. The corresponding operator realization of the infinitesimal variations given by (1.94) is found by setting each group parameter (a, b, c, d, n, α , β) to unity in turn, while making the other six of them vanish. The resulting ideal gas operator algebra is listed below.

a: $D = \partial_x$	Space translations	
c: $H = \partial_t$	Time translations	
b: $G = t\partial_x + \partial_u$	Galilean boosts	
n: $S = x\partial_x + t\partial_t$	x/t invariant scaling	(1.80)
d: $C_0 = x\partial_x + u\partial_u - 2\rho\partial_\rho$	$(u/x, \rho u^2)$ invariant scaling	
α: $C_1 = p\partial_p + \rho\partial_\rho$	P/ρ invariant scaling	
β: $P_0 = \partial_p$	Pressure translations	

Together C_0 and C_1 leave invariant the ratio $P/\rho u^2$ of the pressure to the kinetic energy density. The same ratio is left invariant by C_μ for arbitrary μ, as well as the

other ratios $P/\rho r$ and u/x, corresponding, respectively, to the ratio of the total pressure to its elastic component and the ratio of flow velocity to a characteristic length.

C. Group Invariant Coordinates

The group invariant coordinates of the operators (1.80) are found, according to Appendix C equation (C.62) from the identities

$$\frac{dx}{(d+n)x + bt + a} = \frac{dt}{nt + c} = \frac{du}{ud + b} = \frac{d\rho}{(\alpha - 2d)\rho} = \frac{dp}{\alpha p + \beta} \quad (1.81)$$

Note that the value of α controls the character of the group reducible solutions. When the group parameter α is allowed to vanish, the nontrivial density and pressure transformations are eliminated for degenerate α and d. Therefore, in what follows it is important to distinguish between zero and nonzero values of α.

From the first identity in (1.81), the invariant independent variable $\lambda(x,t)$ is determined.

$$\frac{dx}{dt} = \frac{(d+n)x + bt + a}{nt + c} \quad (1.82)$$

or

$$\frac{d(x - x_0)}{d(t - t_0)} = \frac{(x - x_0) - v(t - t_0)}{\frac{1}{k}(t - t_0)} \quad (1.83)$$

The symmetry exponent $k = \frac{d+n}{n}$ determines the form of the invariant coordinates. In shifted coordinates (1.83) is

$$\frac{dx}{dt} = \frac{x - vt}{t/k} \quad (1.84)$$

Hence, for $k \neq 0, \infty$, we have for the group invariant coordinate,

$$\lambda(x,t) = \frac{1}{t^k}\left(x + \frac{Vt}{\gamma_k - 1}\right) \tag{1.85}$$

The function $\lambda(x,t)$ given by (1.85) contains five arbitrary parameters a, b, c, d, n for use in fitting the subsidiary conditions and physical context of a specific problem. When $V = 0$, the group invariant $\lambda(x,t)$ reduces to x/t^k, which is the customary form for similarity variables in fluid dynamics. To the author's knowledge, no solutions for $V \neq 0$ have ever been applied in the literature.

When $k = 1$, corresponding to $d = 0$, equation (1.85) becomes

$$\lambda(x,t) = \frac{x}{t} + V \log t \tag{1.86}$$

or, alternatively,

$$\lambda' = e^{\lambda} = t^V e^{\frac{x}{t}} \tag{1.87}$$

The transformation corresponding to $d = 0$ and λ given by (1.86) or (1.87) is purely kinematical when α and d are coupled through the equation of state. It is kinematical in the sense that the density and pressure do not transform. The solutions which result when the thermodynamic variables are invariant, have everywhere constant entropy.

The other special values of the symmetry exponent k occur for the vanishing of n, of (d + n), and of both d and n. The general invariants for each case are arrayed in

Table 1.3. The seven parameters a, b, c, d, n, α, β, can be used to fit a wide variety of subsidiary conditions. The physical assumption of group reducible fluid motion is that the flow variables depend on the independent variables in the same way as the boundary and initial conditions. In Table 1.3 appear the general forms of combinations of position and time for which this assumption can be true.

Special values of the general invariant quantities are obtained by setting the group parameters a, b, c, d, n equal to unity for each value of the symmetry exponent k. Examples are listed in Table 1.4.

As mentioned before, the invariants in the first row of Table 1.4 have the usual similarity form. Included among their group reducible solutions are self-similar solutions and centered simple waves.[2] In the fourth and fifth rows appear the similarity variables discussed previously by Stanyukovich and Sedov, whose solutions are called "self-similar in the limit."[3] The independent variables in row three are of similar character to those in the fourth and fifth rows of the table. The variables in rows six and nine can be found by separation of variables. They correspond in their simplest forms to steady and uniform flow respectively. However, travelling waves

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[2]L. I. Sedov, Similarity and Dimensional Methods in Mechanics (Academic Press, New York, 1959).

[3]L. I. Sedov (1959), p. 174.

(n,d)	k	Invariants λ, $U(\lambda)$, $R(\lambda)$, $P(\lambda)$, Respectively
Free	$\frac{d+n}{n}$	$\frac{1}{(nt+c)}k\left[nx+\frac{b}{d}(nt+c)+\frac{an-bc}{d+n}\right]$, $\frac{ud+b}{(nt+c)^{d/n}}$, $\rho(nt+c)^{\frac{2d-\kappa}{n}}$, $p(nt+c)^{-\frac{\kappa}{n}}$
$d=0$	1	$\frac{nx+bt+a}{nt+c}-\frac{b}{n}\log(nt+c)$, $u-\frac{b}{n}\log(nt+c)$, $\rho(nt+c)^{-\frac{\kappa}{n}}$, $p(nt+c)^{-\frac{\kappa}{n}}$
$d+n=0$	0	$x-\frac{b}{n}t+\frac{bc-an}{n^2}\log(nt+c)$, $(un+b)(nt+c)$, $\rho(nt+c)^{-(\frac{\kappa}{n}+2)}$, $p(nt+c)^{-\frac{\kappa}{n}}$
$n=0$	∞	$e^{-\frac{dt}{c}}\left[x+\frac{b}{d}(t+\frac{a}{b}+\frac{c}{d})\right]$, $(ud+b)e^{-\frac{dt}{c}}$, $\rho\exp(\frac{2d-\kappa}{c})t$, $pe^{-\frac{\kappa t}{c}}$
$n=d=0$	—	$x-\frac{a}{c}t-\frac{b}{2c}t^2$, $u-\frac{b}{c}t$, $\rho e^{-\frac{\kappa t}{c}}$, $pe^{-\frac{\kappa t}{c}}$
$c=d=n=0$	—	t, $u-\frac{bx}{bt+a}$, $\rho\exp\frac{-\kappa x}{bt+a}$, $p\exp\frac{-\kappa x}{bt+a}$

Table 1.3. General Independent Group Invariants of Fluid Dynamics.

	k	Nonzero Parameters	Zero Parameters	λ, U, R, P	Significance
1.	$1+\varepsilon$	$n=1, d=\varepsilon$	a, b, c	$\frac{x}{t^{\varepsilon+1}}$, $\frac{u}{t^{\varepsilon}}$, $\rho t^{2\varepsilon-\alpha}$, $p t^{-\alpha}$	Self-similar (SS)
2.	1	$n=1, b=\varepsilon$	a, c, d	$\frac{x}{t}+\varepsilon(1-\log t)$, $u-\frac{x}{t}-\varepsilon$, $\rho t^{-\alpha}$, $p t^{-\alpha}$	Galilean #1
3.	0	$d=-n=1$	$a, b, c, (d+n)$	x, ut, $\rho t^{\alpha-2}$, $p t^{\alpha}$	Separated variables
4.	0	$d=-n=-1$ $a=\varepsilon$	$b, c, (d+n)$	$x-\varepsilon\log t$ $(\text{or } te^{-\frac{x}{\varepsilon}})$, ut, $\rho t^{-\alpha-2}$, $p t^{-\alpha}$	SS in limit
5.	∞	$c=d=1$	a, b, n	xe^{-t}, ue^{-t}, $\rho e^{(2-\alpha)t}$, $pe^{-\alpha t}$	SS in limit
6.	—	$c=1$	a, b, d, n	x, u, $\rho e^{-\alpha t}$, $pe^{-\alpha t}$	Steady flow
7.	—	$b=\varepsilon$, $c=1$	a, d, n	$x-\varepsilon\frac{t^2}{2}$, $u-\varepsilon t$, $\rho e^{-\alpha t}$, $pe^{-\alpha t}$	Galilean #2
8.	—	$b=1$	a, c, d, n	t, $u-\frac{x}{t}$, $\rho e^{-\alpha\frac{x}{t}}$, $pe^{-\alpha t}$	Galilean #3
9.	—	$a=1$	b, c, d, n	t, u, $\rho e^{-\alpha x}$, $pe^{-\alpha x}$	Uniform flow

Table 1.4. Special Invariants of Fluid Dynamics.

are included by combining row six and row 9.[4] In rows two, seven, and eight appear invariants associated with Galilean transformations, which had not been noticed prior to Ovsjannikov (1962), and to the author's knowledge, have never been applied. These solutions are studied later on for Mie-Grüneisen equations of state.

Thus, the general multiparameter invariants of Table 1.3, upon specialization, reduce to a catalog of group invariant variables for self-similar motion. The invariants lead to the classes of solutions of the Navier-Stokes equations, and provide cases of moving boundaries for which group reducible solutions exist. The multiparameter invariants of Table 1.3 are especially useful because they contain arbitrary constants which may be adapted to the specifications of particular problems.

The effect of the nonlinear elastic pressure term in the equation of state (1.62) is to couple two of the group parameters d and α. When the exponent μ in (1.62) vanishes, or equals unity, the parameters d and α which appear in Table 1.3 and Table 1.4 will be independent of each other. Otherwise d and α are coupled, so that

$$\alpha = \frac{2\mu}{\mu - 1} d \tag{1.88}$$

For example, when d and α are connected, the last four special invariants and the second one in Table 1.4 have

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[4]Appendix C, equation (C.145) and sequel.

$\alpha = 0$, while in the remainder α is given in terms of d by (1.88). Tables 1.3 and 1.4 also contain the invariants for a general equation of state, in which case $\alpha = d = 0$ everywhere, and the flow has constant entropy.

D. Representative Group Reducible Solutions

Let us explore the effect of the power law elastic pressure on representative group reducible isentropic fluid motions. The group invariant variables generally allow a partial integration of the fluid dynamics equations. Most often, as in the case of self-similar motion, the group reduced representation reduces the problem to a phase plane analysis and quadratures.[5]

So far we have found an empirically relevant elastic pressure interpolation function for which the isentropic fluid equations admit more than merely a kinematic symmetry algebra. In the context of fluid dynamics we have determined in detail how symmetry breaking occurs through interaction terms in the equation of state. In addition, the basic invariance principles of classical mechanics and fluid dynamics have been formulated quantitatively as differential operators and have been used to derive the natural symmetry-adapted variables for isentropic fluid motion with

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[5]L. I. Sedov (1959).

Mie-Grüneisen equation of state (1.62). We shall now present the general analytical characteristics of group reducible motions. We shall see that invariance analysis unifies and systematizes previously scattered concepts. Thereby it leads to both a firmer control of existing methods of analysis and a deeper insight into their basis and possible modes of generalization and extension.

We wish to consider one-dimensional planar isentropic flows of an "interacting gas" whose equation of state is given by (1.62). Let us write the equations of motion for flows of this type. Referring to equations (1.21), (1.22) and also (1.23) we obtain the following system of equations for the density, pressure, and velocity, as functions of position and time.

$$\rho_t + u\rho_x + \rho u_x = 0 \tag{1.89}$$

$$u_t + uu_x + \frac{1}{\rho}P_x = 0 \tag{1.90}$$

$$P_t + uP_x + \left[(\gamma+1)P + C(\mu-1)\rho^{\mu}\right]u_x = 0 \tag{1.91}$$

In equation (1.91) we have reset the zero point of the pressure so that the symbol p stands for the quantity $P + \frac{P/\kappa}{\gamma+1}$.

The invariance groups for the equations (1.89) through (1.91) have already been derived and enumerated in the previous section. We have also seen how the invariance principles admitted by the fluid dynamics equations determine the natural variables of the system. The group

invariant quantities are tabulated above. In these variables the system of partial differential equations of fluid dynamics reduces to a system of ordinary differential equations for new unknown functions of the appropriate independent group invariant. Let us derive these group reduced equations and analyze them for various choices of invariant independent variables. In Table 1.3 and Table 1.4 a large number of choices are available.

In the group reduced representation of fluid dynamics the interaction term for $\mu \neq 0, 1$ changes the character of the motions. Unless μ is equal to one of its "accidentally symmetric" values, an additional connection occurs between the transformations of the kinematical variables (x, t, u) and the thermodynamical variables (ρ, p) which limits the form of the solution. The physical explanation of this group theoretical effect is that the sound speed - which connects the kinematical and thermodynamical quantities - has two components, a thermal contribution and an elastic contribution, which scale differently unless $\mu = 0$ or 1. The isentropes for the equation of state (1.62) are given by

$$P = \gamma K \rho^{\gamma+1} + C_1 \frac{\mu-1}{\mu-(\gamma+1)} \rho^{\mu} + \frac{P_0}{\gamma+1} \tag{1.92}$$

Thus the sound speed has two components which scale differently,

$$c^2 = \frac{dP}{d\rho} = c_0^2 \left(\frac{\rho}{\rho_0}\right)^{\gamma} + C_1 \frac{\mu(\mu-1)}{(\mu-1)-\gamma} \rho^{\mu-1} \tag{1.93}$$

The expression (1.93) tells us that the thermal and elastic parts of the sound speed cannot scale the same way with density, because if they did, the denominator of the second term would vanish. But for $\mu = 0, 1$, the elastic part itself vanishes and a simplification occurs. For $\mu = 0, 1$, the sound speed takes on the ideal gas form, and the isentropic Navier-Stokes description gains on extra symmetry principle.

1. Group Reducible Solutions of Self-Similar Type.

Let us derive the group reducible representation of the fluid dynamics equations for nonzero group parameters d, n, α. According to Table 1.3 the invariant coordinates are given in this case by

$$\begin{aligned} \lambda &= \frac{x}{t^{\frac{d+n}{n}}} \\ \lambda \Pi(\lambda) &= \frac{u}{t^{\alpha/n}} \\ R(\lambda) &= \rho t^{\frac{2d-\alpha}{n}} \\ \mathbb{P}(\lambda) &= p t^{-\alpha/n} \end{aligned} \tag{1.94}$$

The group reducible representation of the fluid dynamics equations is gotten by solving (1.94) for (u, ρ, p) and substituting into (1.89) through (1.91). From equations (1.94) we find

$$\begin{aligned} \lambda &= \frac{x}{t} \cdot \frac{1}{t^{\alpha/n}} \\ u &= \frac{x}{t} \Pi(\lambda) \\ \rho &= t^{\frac{\alpha-2d}{n}} R(\lambda) \\ p &= t^{\alpha/n} \mathbb{P}(\lambda) \end{aligned} \tag{1.95}$$

Substituting the group reducible expressions (1.95) for the coordinates into the starting partial differential equations for fluid dynamics, we obtain, first of all for $\mu = 1$, the group reduced representation

$$R\lambda U' + (U - \tfrac{n+d}{n})\lambda R' = -(U + \tfrac{\alpha - 2d}{n})R \tag{1.96}$$

$$(U - \tfrac{n+d}{n})\lambda U' + \frac{1}{\lambda R}P' = -U(U-1) \tag{1.97}$$

$$(\gamma+1)P\lambda U' + (U - \tfrac{n+d}{n})\lambda P' = -\left[(\gamma+1)U + \tfrac{\alpha}{n}\right]P \tag{1.98}$$

Equations (1.96), (1.97), & (1.98) comprise the group reduced representation of the balance equations for mass, momentum, and entropy, respectively.

When $\mu \neq 1, 0$ the last equation changes, and d and α become connected by (1.88). The new equations, with expressed in terms of μ, d, are given by

$$R\lambda U' + (U - \tfrac{n+d}{n})\lambda R' = -(U + \frac{2}{\mu - 1}\frac{d}{n})R \tag{1.99}$$

$$(U - \tfrac{n+d}{n})\lambda U' + \frac{1}{\lambda R}P' = -U(U-1) \tag{1.100}$$

$$\left[(\gamma+1)P + C(\mu-1)R^{\mu}\right]\lambda U' + (U - \tfrac{n+d}{n})\lambda P' = -\frac{2\mu}{\mu-1}\frac{d}{n}P - \left[(\gamma+1)P + C(\mu-1)R^{\mu}\right]U \tag{1.101}$$

We first study the case where $\mu = 1$. The mass equations (1.96) and the entropy equation (1.98) may be rewritten in the form

$$R\lambda(U-\Delta)' + (U-\Delta)\lambda R' + DR + (U-\Delta)R = 0 \tag{1.102}$$

$$(\gamma+1)P\lambda(U-\Delta)' + (U-\Delta)\lambda P' + EP + (\gamma+1)(U-\Delta)P = 0 \tag{1.103}$$

where the constants Δ, D, E are defined by

$$\Delta = \frac{n+d}{\nu}$$

$$D = \frac{\kappa - d + n}{\nu} \tag{1.104}$$

$$E = \frac{\kappa + (\gamma+1)(n+d)}{\nu} = \frac{\kappa}{\nu} + (\gamma+1)\Delta$$

These equations admit a first integral, an algebraic relation between all of the variables. From equation (1.102) multiplied by EP we subtract equation (1.103) multiplied by DR. The result appears in the form,

$$\left[E - (\gamma+1)D\right]\left(\frac{d\lambda}{\lambda} + \frac{d(W-\Delta)}{W-\Delta}\right) + E\frac{dR}{R} - D\frac{d\mathbb{P}}{\mathbb{P}} = 0 \tag{1.105}$$

A first integral is thus obtained from the mass and entropy balance equations

$$\frac{R^E\left[\lambda(W-\Delta)\right]^{E-(\gamma+1)D}}{\mathbb{P}^D} = \text{const} \tag{1.106}$$

This integral, to be called the adiabatic integral, exists because of conservation of entropy along particle paths in the flow. In general, each group reduced fluid conservation law should have a corresponding first integral.

We see that equations (1.100), (1.102), (1.103) are invariant under two scaling laws,

$$\mathbb{P} \to e^{\sigma}\mathbb{P} \qquad R \to e^{-2\tau}R$$
$$R \to e^{\sigma}R \qquad \text{and} \qquad \lambda \to e^{\tau}\lambda \tag{1.107}$$

The invariants of these scale transformations which correspond to the original invariance principles C_o and C_1 in (1.80) are given by

$$w = (U - A) \quad , \quad z = \frac{\tilde{P}}{\lambda^2 R} \tag{1.108}$$

In terms of w, z the group reducible pressure and density become, through the adiabatic integral,

$$\tilde{P} = (\text{const})\, z^{\frac{1}{1-\varepsilon}}\, \lambda^{\frac{1+(\gamma+1)\varepsilon}{1-\varepsilon}}\, w^{\frac{-1+(\gamma+1)\varepsilon}{1-\varepsilon}}$$

$$R = (\text{const})\, z^{\frac{\varepsilon}{1-\varepsilon}}\, \lambda^{\frac{-1+(\gamma+3)\varepsilon}{1-\varepsilon}}\, w^{\frac{-1+(\gamma+1)\varepsilon}{1-\varepsilon}} \tag{1.109}$$

where

$$\varepsilon = \frac{D}{E} = \frac{\alpha - d + n}{\alpha + (\gamma+1)(n+d)} \tag{1.110}$$

Substitution of (1.109) into the group reduced representation (1.100), (1.102) and (1.103) leads to complicated expressions for $\frac{dw}{d\lambda}$ and $\frac{dz}{d\lambda}$. The general expressions are omitted. They are best treated numerically.

A simplification for which the results are more tractable occurs if $\varepsilon = 1$, i.e., if $d = \frac{-\gamma}{\gamma+2} n$. In this case the adiabatic integral (1.106) becomes

$$\frac{\tilde{P}}{R} = (\text{const})(\lambda w)^{-\gamma} = z\lambda^2 \tag{1.111}$$

In terms of the invariants λ, w, z the differential condition for the adiabatic integral (1.111) to exist when E = D, is simply,

$$\frac{w'}{w} + \frac{z'}{\gamma z} = -\frac{\gamma+2}{\gamma}\frac{1}{\lambda} \tag{1.112}$$

Rewriting the momentum equation with w, z, and setting E = D, we have

$$\frac{\lambda}{w}(w^2-z)w' + \lambda z' = -\left[z+(w+\Delta)(w+\Delta-1)\right] \qquad (1.113)$$

Regard these two equations as a linear algebraic set of equations for w' and z'. Because of the choice of invariant variables w, z, an expression may be found for $\frac{dz}{dw}$ which is independent of the invariant λ. By Cramer's Rule for linear equations we find,

$$\frac{dz}{d\lambda} = \frac{\gamma z w}{\lambda} \cdot \frac{z+(w+\Delta)(w+\Delta-1)+\frac{\gamma+2}{\gamma}(w^2-z)}{(w^2-z)-\gamma w z} \qquad (1.114)$$

$$\frac{dz}{dw} = \frac{\gamma z}{w} \cdot \frac{z+(w+\Delta)(w+\Delta-1)+\frac{\gamma+2}{\gamma}(w^2-z)}{(\gamma+1)z-(w+\Delta)(w+\Delta-1)} \qquad (1.115)$$

$$\frac{dw}{d\lambda} = \frac{w^2}{\lambda} \cdot \frac{(\gamma+1)z-(w+\Delta)(w+\Delta-1)}{(w^2-z)-\gamma w z} \qquad (1.116)$$

Thus, in the general $\epsilon = 1$ case, the solution is reduced to a first order differential equation $\frac{dz}{dw} = F(w,z)$, one quadrature, and the adiabatic relation (1.111). However, the formulas are still too complicated to afford much analytical insight.

2. U = Δ Flows. If we assume in addition that $U = \Delta = \frac{2}{\gamma+2}$, then exact solution is made possible. Compatibility with the group reduced equation for $U = \Delta$ implies through (1.102) and (1.103) that

$$D = E = 0 \qquad (1.117)$$

Thus, from (1.104) we have

$$\frac{\alpha}{n} = -(\gamma+1)\Delta \tag{1.118}$$

and the momentum equation (1.100) reduces to

$$\frac{P'}{\lambda R} = -\Delta(\Delta-1) = \frac{2\gamma}{(\gamma+2)^2} \tag{1.119}$$

The solution which results under these assumptions is given by

$$u = \Delta\frac{x}{t}, \quad \rho = t^{-\Delta}R(\lambda), \quad P = t^{-(\gamma+1)\Delta}\mathbb{P}(\lambda), \quad \lambda = \frac{x}{t^{\Delta}} \tag{1.120}$$

where $R(\lambda)$ is an arbitrary function and $P(\lambda)$ is given in terms of $R(\lambda)$ by

$$\mathbb{P}(\lambda) = \frac{2\gamma}{(\gamma+2)^2}\int \lambda R(\lambda)\, d\lambda \tag{1.121}$$

For $R(\lambda) = \lambda^{-1} = \frac{t^{\Delta}}{x}$ the solution (1.120) becomes

$$u = \frac{2}{\gamma+2}\frac{x}{t}, \quad \rho = \frac{1}{x}, \quad P = \frac{2\gamma}{(\gamma+2)^2}\frac{x}{t^2} \tag{1.122}$$

At a given time the velocity and pressure have a linear spatial profile, while the density decreases inversely with distance. This flow preserves the ratio of the pressure to the kinetic energy density. From (1.122) we have

$$\frac{P}{\frac{1}{2}\rho u^2} = \gamma \tag{1.123}$$

Consider a more general $U = \Delta$ flow. Let $R(\lambda) = K\lambda^{J}$. The solution (1.120) becomes

$$u = \frac{2}{\gamma+2}\frac{x}{t}, \quad \rho = K\frac{x^{J}}{t^{(J+1)\Delta}}, \quad P = \frac{2\gamma}{(\gamma+2)^2}\frac{K}{J+2}\frac{x^{J+2}}{t^{(\gamma+J+3)\Delta}} \tag{1.124}$$

At a given instant the velocity profile is linear again, but the density and pressure profiles follow power laws. The flow (1.124) also preserves the ratio of the pressure to the kinetic energy density. From (1.124) we have

$$\frac{P}{\frac{1}{2}\rho u^2} = \frac{\gamma}{J+2} \tag{1.125}$$

Recalling the remark after equation (1.80) we realize that <u>all</u> of the group reducible solutions derived in this way, and all of those generated by operating upon them with an invariance transformation of the fluid dynamics group, preserve the quantity $P/\rho u^2$. Note that, under the transformations of the fluid dynamics group (generated by the operators (1.80)), the solutions listed above transform to four parameter families of new solutions. The new solutions derived by transformations of the rest of the group, are said to be equivalent to the old ones up to a symmetry transformation.

Let us extend our analysis of $U = \Delta$ flows to the case that the elastic pressure exponent μ is not equal to zero or unity. When $\mu \neq 0, 1$ the group reduced equations (1.99) through (1.101) take the forms

$$\lambda R(U-\Delta)' + (U-\Delta)\lambda R' = -DR - (U-\Delta)R \tag{1.126}$$

$$\lambda(U-\Delta)(U-\Delta)' + \frac{P'}{\lambda R} = -U(U-1) \tag{1.127}$$

$$[(\gamma+1)P + C(\mu-1)R^\mu]\lambda(U-\Delta)' + (U-\Delta)\lambda P' = -C(\mu-1)R^\mu\Delta - EP - C(\mu-1)R^\mu(U-\Delta) \tag{1.128}$$

where the constants Δ, D, E are given by

$$\Delta = \frac{n+d}{n}$$

$$D = \frac{2}{\mu-1}\frac{d}{n} + \Delta \qquad (1.129)$$

$$E = (\gamma+1)\Delta + \frac{2\mu}{\mu-1}\cdot\frac{d}{n}$$

and we have already used (1.88) to relate d and α. For compatibility when $U = \Delta$, we require $D = 0$ in (1.126) and, hence,

$$\Delta = \frac{2}{1+\mu}$$

$$E = \Delta(\gamma+1-\mu) \qquad (1.130)$$

The result of setting $U = \Delta$ in (1.128) is that

$$\mathbb{P} = \frac{C(\mu-1)R^{\mu}}{\mu-(\gamma+1)} \quad , \quad \frac{d\mathbb{P}}{dR} = \frac{C\mu(\mu-1)}{\mu-(\gamma+1)}R^{\mu-1} \qquad (1.131)$$

Consequently, from (1.127) with $U = \Delta$ we find

$$R = K\lambda^{\frac{2}{\mu-1}} \quad , \quad K = \left[\frac{\gamma}{(\gamma+2)^2}\frac{\mu-(\gamma+1)}{C\mu}\right]^{\frac{1}{\mu-1}} \qquad (1.132)$$

Therefore, the result of the elastic pressure term in the equation of state is to reduce the class of flows for which $U = \Delta$ to a single member, namely (1.124) with $V = \frac{2\cdot}{\mu-1}$. Note that, in (1.131) the expression $\frac{d\mathbb{P}}{dR}$ is precisely the nonlinear elastic part of the sound speed squared in (1.93). The physical reason that the entire sound speed did not appear in (1.131) is that while the group reducible motion for $U = \Delta$ has constant entropy along particle paths,

the entropy is not constant everywhere. The constant entropy relation (1.93) cannot be expected to hold for a general isentropic flow, but in this case "part of it" is satisfied.

In our analysis, we have consistently found that symmetry breaking reduces the range of available group reducible solutions. This occurs in essentially the same manner in which the number of degrees of freedom of any system is reduced by imposing a constraint.[7] The nonlinear elastic pressure could have conceivably had another type of effect: it could have completely changed the character of the motion. In fact, this would be, a priori, the most likely possibility. Except for the limiting ideal gas behavior when C is small, the nonlinear elastic component may be completely unconnected with ideal fluid flow.

3. Degenerate Self-Similar Motion: Separated Variables. Let us present another example of the group reduced representation of the fluid dynamics equations, this time for self-similar motions with group parameters d and n initially connected by $d + n = 0$. The connection between d and n forces the symmetry exponent k to vanish and changes the analytical character of the solutions. According to Table 1.3, the invariant coordinates are given by

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[7] See also Appendix C.

$$\lambda = x$$

$$U(x) = ut$$

$$R(x) = \rho t^{-(\frac{\alpha}{n}+2)} \qquad (1.133)$$

$$P(x) = p t^{-\frac{\alpha}{n}}$$

The group reduced representation is gotten by substituting the invariant forms into the fluid dynamics equations (1.89) through (1.91). In the group reducible flow variables the coordinates are separated,

$$\lambda = x$$

$$u = \frac{1}{t} U(x) \qquad (1.134)$$

$$\rho = t^{\frac{\alpha}{n}+2} R(x)$$

$$p = t^{\frac{\alpha}{n}} P(x)$$

In terms of a similarity representation, the length scale is constant and the velocity scale goes like 1/t. This is a singular case. The density scale is set by exponent $B = \frac{\alpha}{n} + 2$, which is a uniform limit of the density scale in a self-similar representation.

The results of the substitution of the group reducible flow variables (1.134) into the fluid dynamics equations are, respectively, for the mass, momentum, and entropy balance,

$$RU' + UR' = -(\tfrac{\alpha}{n}+2)R \qquad (1.135)$$

$$UU' + \frac{P'}{R} = U \qquad (1.136)$$

$$\left[(\gamma+1)\mathbb{P} + C(\mu-1)R^{\mu}\right]\mathbb{U}' + \mathbb{U}\mathbb{P}' = -\frac{\alpha}{n}\mathbb{P} \tag{1.137}$$

In these equations, when $C \neq 0$ we have, for compatibility of dimensions,

$$\frac{\alpha}{n} = -\frac{2\mu}{\mu-1} \tag{1.138}$$

When $C \neq 0$, no free parameters remain. When, however, the nonlinear elastic pressure is not present, an adiabatic integral can be gotten straight forwardly. Setting $C = 0$, from $(\frac{\alpha}{n} + 2)R$ times the entropy equation we subtract $(\frac{\alpha}{n}\mathbb{P})$ times the mass equation. The result is

$$\left(\frac{\alpha}{n}+2\right)\frac{\mathbb{P}'}{\mathbb{P}} - \frac{\alpha}{n}\frac{R'}{R} + \left[\gamma\frac{\alpha}{n} + 2(\gamma+1)\right]\frac{\mathbb{U}'}{\mathbb{U}} = 0 \tag{1.139}$$

and the adiabatic integral for $C = 0$ is

$$\frac{\mathbb{P}^{\left(\frac{\alpha}{n}+2\right)}\,\mathbb{U}^{\left[\gamma\frac{\alpha}{n}+2(\gamma+1)\right]}}{R^{\frac{\alpha}{n}}} = (\text{const}) \tag{1.140}$$

The group reduced equations (1.135) through (1.137) with $C = 0$ are invariant under the following scaling laws (see equations (1.67) and (1.68))

$$x \longrightarrow e^{\sigma}x \qquad \mathbb{U} \longrightarrow e^{\sigma}\mathbb{U} \qquad R \longrightarrow e^{-2\sigma}R \qquad \text{and} \qquad R \longrightarrow e^{\tau}R \qquad \mathbb{P} \longrightarrow e^{\tau}\mathbb{P} \tag{1.141}$$

These operations correspond to C_o and C_1, above, which are independent of time.

The invariants of these scale transformations are

$$w = \frac{U}{x} \quad , \quad z = \frac{P}{U^2 R} \tag{1.142}$$

When $C \neq 0$ a scale transformation derived from C_μ in Table 1.1 is admitted.

$$\begin{aligned} x &\to e^{(\frac{\mu}{2}-1)\tau} x \\ U &\to e^{(\frac{\mu}{2}-1)\tau} U \\ R &\to e^{\tau} R \\ P &\to e^{\mu\tau} P \end{aligned} \tag{1.143}$$

The transformation (1.143) has the same invariants (1.142)

From the adiabatic integral (1.140) we get, for $C = 0$ now,

$$\begin{aligned} R &= (\text{const})\, z^{-(\frac{\alpha}{2n}+1)} U^{-(\varepsilon+1)} \\ P &= (\text{const})\, z^{-\frac{\alpha}{2n}} U^{-\varepsilon+1} \end{aligned} \tag{1.144}$$

with

$$\varepsilon = \frac{(\alpha+2n)(\gamma+2)}{2n} \tag{1.145}$$

In terms of U and z the group reduced representation becomes

$$\varepsilon U' + (\tfrac{\alpha}{2n}+1)\frac{U}{z} z' = (\tfrac{\alpha}{n}+2) \tag{1.146}$$

$$\left[1 + (1-\varepsilon)z\right] U' - \frac{\alpha}{2n} U z' = 1 \tag{1.147}$$

$$(\gamma+2-\varepsilon) U' - \frac{\alpha}{2n}\frac{U}{z} z' = -\frac{\alpha}{n} \tag{1.148}$$

Adding the entropy equation to the mass equation, we obtain

$$(\gamma+2)\,U' + \frac{U}{z}\,z' = 2 \tag{1.149}$$

Combining this equation with the momentum equation we get an equation for U'(x) and for z'(x).

$$\frac{dU}{dx} = -\frac{1+\frac{\alpha}{n}z}{\left[\frac{\alpha}{2n}(\gamma+2)-(\varepsilon-1)\right]z-1} \tag{1.150}$$

$$\frac{dz}{dx} = \frac{z}{U}\,\frac{2(\varepsilon-1)z+\gamma}{\left[\frac{\alpha}{2n}(\gamma+2)-(\varepsilon-1)\right]z-1} \tag{1.151}$$

Hence, an equation appears for U(z)

$$\frac{dU}{dz} = -\frac{U}{z}\,\frac{1+\frac{\alpha}{n}z}{2(\varepsilon-1)z+\gamma} \tag{1.152}$$

When $\varepsilon = 1$ a simplification occurs. First of all, $\varepsilon = 1$ corresponds to

$$\frac{\alpha}{2n}(\gamma+2) = -(\gamma+1)\,, \quad \frac{\alpha}{n} = -\frac{2(\gamma+1)}{\gamma+2} \tag{1.153}$$

Therefore,

$$z' = -\frac{z}{U}\,\frac{\gamma}{1+(\gamma+1)z} \tag{1.154}$$

$$U' = \frac{1+\frac{\alpha}{n}z}{1+(\gamma+1)z} \tag{1.155}$$

$$\frac{dU}{dz} = -\frac{U}{\gamma z}\left(1+\frac{\alpha}{n}z\right) \tag{1.156}$$

Consequently, from (1.156) the expression for U(z) becomes

$$U(z) = z^{-\frac{1}{\gamma}} e^{-\frac{\alpha z}{n\gamma}} \tag{1.157}$$

and from (1.154) and the expression for U(z), we have

$$-\gamma dx = (\gamma+1)\frac{e^{-\frac{\alpha z}{n\gamma}}}{z^{1/\gamma}} dz + \frac{e^{-\frac{\alpha z}{n\gamma}}}{z^{\frac{\gamma+1}{\gamma}}} \tag{1.158}$$

The integrals involved are related to incomplete gamma functions, or degenerate hypergeometric functions. The $U \sim x/t$ solution in this case corresponds to $\gamma = 0$, i.e., it corresponds to isothermal motion. Therefore, even for the perfect gas, the isentropic solution by separation of variables (1.144) apparently can be gotten only implicitly, at least for $\varepsilon = 1$.

Further progress can be made when $\varepsilon = -\frac{\gamma+2}{\gamma}$. For this case we have

$$\frac{\alpha}{n} = \frac{2\dot{\varepsilon}}{\gamma+2} - 2 = -\frac{2}{\gamma}(\gamma+1) \tag{1.159}$$

and the dependence on U in the adiabatic integral (1.140) drops out. Consequently, (1.140) becomes

$$\frac{R^{\gamma+1}}{P} = (\text{const}) \tag{1.160}$$

and the group reduced balance equations, namely

$$(RU)' = \frac{2}{\gamma} R \tag{1.161}$$

$$U U' + \frac{P'}{R} = U \tag{1.162}$$

$$(\gamma+1) P U' + U P' = \frac{2}{\gamma}(\gamma+1) P \tag{1.163}$$

can be reduced to

$$\left[(\gamma+1) P - U^2\right] \frac{R'}{R} = \frac{\gamma-2}{\gamma} U \tag{1.164}$$

Moreover, if $\gamma = 2$ as well, two simple solutions for constant entropy flow result:

$$u = \begin{cases} \left(\frac{3}{5}x + A\right)\frac{1}{t} & , \quad R' \neq 0 \\ \frac{x}{t} & , \quad R' = 0 \end{cases} \tag{1.165}$$

$$\rho = \begin{cases} B\left(\frac{3}{5}x + A\right)^{2/3}\frac{1}{t} & , \quad R' \neq 0 \\ \frac{B}{t} & , \quad R' = 0 \end{cases} \tag{1.166}$$

$$p = \begin{cases} B\left(\frac{3}{5}x + A\right)^{2}\frac{1}{t^3} & , \quad R' \neq 0 \\ \frac{B}{t^3} & , \quad R' = 0 \end{cases} \tag{1.167}$$

where A and B are constants.

Before proceeding to a new type of fluid motion, we note that, although the transformation to the invariants (1.142) partially separates the variables in the equations

of motion for $\mu = 0, 1$, the group reduced equations (1.135) through (1.137) with arbitrary μ, even when written in the invariant combinations w, z, cannot be separated into a phase plane form: $\frac{dz}{dw} = F(w,z)$. Thus the symmetry breaking term dynamically connects P, U, and R, and changes the character of the solution. We remark that the other simplifying aspect of the motion - the adiabatic integral - has also eluded us for $\mu \neq 0, 1$.

4. Galilean Flows. Let us discuss the types of flows generated by Galilean invariance. This type of flow has been noticed only once before, by Ovsjannikov (1962) for an ideal gas. We wish to study its physical significance and to determine the effects of equation of state symmetry breaking on the analytical character of the fluid motion.

A simple Galilean flow is found by taking α and b nonzero in Table 1.3, while forcing the other five parameters (a, c, d, n, β) to vanish. The dimensionless group parameter α labels the scale transformations on p and ρ together. This thermodynamic scale change leaves invariant the quantity $\sqrt{p/\rho}$, with dimensions of speed. The other group coordinate, b, labels the Galilean transformations and also has dimensions of speed. If $\alpha = 0$, then the kinematic Galilean boost remains: $t \to t$, $x \to (x + bt)$, $u \to (u + b)$, $\rho \to \rho$, $p \to p$. The Galilean boost involves an invariant time and a family of equivalent uniform velocities (inertial frames). Under a Galilean boost, time is unchanged, velocity is translated, and position mixes linearly with time.

The first step in the symmetry analysis for Galilean flows is to write down the invariant coordinates for b nonzero. The group reducible representatives of the flow variables for Galilean flow of this type are, according to the last line of Table 1.3,

$$\lambda = t$$

$$u = \frac{x}{t} + U(t)$$

$$\rho = e^{\alpha \frac{x}{bt}} R(t) \qquad (1.168)$$

$$p = e^{\kappa \frac{x}{bt}} P(t)$$

In this flow, spatial dependence appears only in the combination x/t, with units of speed. The invariants U, R, P in (1.168) have dimensions corresponding to the notation. The velocity is linear in x/t, with an additional time dependent term which modifies its temporal behavior. At a given instant, the flow has a linear velocity profile and an exponential pressure and density shape. The pressure and density have the same profiles, except for a time-varying relative scale. The type of physical situation for which such a flow could develop would be an initial value problem (because t is invariant) with a characteristic velocity and density. For example, at t = 0 a spatial density distribution of moving gas might be set up, with a given velocity, which then expands into a vacuum. The subsequent motion

takes the form (1.168)

The Galilean/thermodynamic scaling group reduced form of the fluid dynamics equations is obtained upon substituting (1.168) into the isentropic partial differential equations, (1.89) through (1.91). The result of the substitution is

$$(Rt)' + \frac{\alpha}{b} UR = 0$$

$$(Ut)' + \frac{\alpha}{b}\frac{P}{R} = 0 \qquad (1.169)$$

$$\frac{1}{t^{\alpha}}(Pt^{\alpha+1})' + \frac{\alpha}{b} UP + C_1(\mu-1)R^{\mu}\exp\frac{(\mu-1)\alpha x}{bt} = 0$$

Immediately we notice that, because of the exponential in the last equation, nonlinear interaction pressures in the equation of state cannot be accomodated unless $\alpha = 0$. When α vanishes, the first two equations in (1.169) corresponding respectively to mass and momentum balance, give

$$R = \frac{A}{t} = \rho \qquad A,B \text{ const.} \qquad (1.170)$$

$$U = \frac{B}{t}$$

Then the last equation becomes

$$(Pt^{\alpha+1})' + C_1(\mu-1)A^{\mu}t^{\alpha-\mu} = 0 \qquad (1.171)$$

Equation (1.171) can be integrated to give the isentropic form,

$$\mathbb{P} = \frac{P_0}{t^{\gamma+1}} + \frac{C_1(\mu-1)}{\mu-(\gamma+1)}\left(\frac{A}{t}\right)^{\mu} = P = K\rho^{\gamma+1} + \frac{C_1(\mu-1)}{\mu-(\gamma+1)}\rho^{\mu} \qquad (1.172)$$

In other words, for vanishing α the group reducible motion of the purely kinematic Galilean group is given by

$$u = \frac{x+B}{t}, \quad \rho = \frac{A}{t}, \quad P = \frac{P_0}{t^{\gamma+1}} + \frac{C_1(\mu-1)}{\mu-(\gamma+1)}\left(\frac{A}{t}\right)^{\mu} \qquad (1.173)$$

Because the pressure and density are independent of position, the isentropic relation (1.93) or (1.172) holds throughout the flow. Clearly the ideal gas (or stiffened gas) result can be gotten from (1.173) by setting C = 0.

Thus the $\alpha = 0$ Galilean "flow" corresponds to a constant entropy, strongly coupled, driven pressure-density response, with a velocity response which increases linearly with position in the material. As time increases, all of the flow variables in a finite region decrease to zero.

For completeness and comparison, the $\mu = 0, 1$, $\alpha \neq 0$ Galilean flow is presented. When the nonlinear part of the elastic pressure vanishes, C = 0 and the last group reduced equation in (1.169) - the entropy balance equation - becomes

$$(\mathbb{P}t^{\gamma}\cdot t)' + \frac{n}{b}\mathbb{H}\mathbb{P}t^{\gamma} = 0 \qquad (1.174)$$

Comparing equation (1.174) and the group reduced mass balance equation in (1.169) we see that ($\mathbb{P}t^{\gamma}$) and R differ

at most by a constant factor. Consequently, in this case the adiabatic integral is given by

$$\frac{P}{R} = \frac{K}{t^{\gamma}} \quad , \quad K = \text{const} \tag{1.175}$$

and the group reduced momentum equation can be directly integrated to give

$$U = \frac{B}{t} + \frac{\alpha}{b}\frac{K}{\gamma-1}\frac{1}{t^{\gamma}} \quad , \quad B = \text{const.} \tag{1.176}$$

Rearranging the mass balance equation gives

$$d(\log Rt) = -\frac{\alpha}{b}\frac{U(t)}{t}\,dt \tag{1.177}$$

Hence, upon substituting (1.176) we have,

$$R = \frac{A}{t}\exp\frac{\alpha}{b}\left(\frac{B}{t} + \frac{\alpha}{b}\frac{K}{\gamma(\gamma-1)}\frac{1}{t^{\gamma}}\right) \tag{1.178}$$

Then from (1.175)

$$P = \frac{AK}{t^{\gamma+1}}\exp\frac{\alpha}{b}\left(\frac{B}{t} + \frac{\alpha}{b}\frac{K}{\gamma(\gamma-1)}\frac{1}{t^{\gamma}}\right) \tag{1.179}$$

When $\alpha = 0$, the solutions for U, R, P reduce to their constant entropy, C = 0 values, with P = (const) $R^{\gamma+1}$. When $\gamma = 1$, the group reducible representatives of the flow variables become

$$U = \frac{B}{t} - \frac{\alpha}{b}K\frac{\log t}{t} \tag{1.180}$$

$$R = \frac{A}{t}\exp\left[\frac{\alpha}{b}\left(B - \frac{\alpha}{b}K\right)\frac{1}{t} - \left(\frac{\alpha}{b}\right)^2 K\frac{\log t}{t}\right] \tag{1.181}$$

$$P = \frac{K}{t}R \tag{1.182}$$

Let us choose the constants α, b, K, to satisfy $K = \left(\frac{b}{\alpha}\right)^2 = 1$. Then for $\alpha \neq 0$, $\gamma \neq 1$ the flow is given by

$$u = \frac{x+B}{t} + \frac{1}{\gamma-1}\frac{1}{t^2} \tag{1.183}$$

$$\rho = \frac{A}{t}\exp\left[\frac{x+B}{t} + \frac{1}{\gamma(\gamma-1)}\frac{1}{t^2}\right] \tag{1.184}$$

$$p = \frac{A}{t^{\gamma+1}}\exp\left[\frac{x+B}{t} + \frac{1}{\gamma(\gamma-1)}\frac{1}{t^2}\right] \tag{1.185}$$

and for $\alpha \neq 0$, $\gamma = 1$ the flow is given by

$$u = \frac{x+B}{t} - \frac{\log t}{t} \tag{1.186}$$

$$\rho = \frac{A}{t}\exp\left[\frac{x+B}{t} - \frac{1+\log t}{t}\right] \tag{1.187}$$

$$p = \frac{A}{t^2}\exp\left[\frac{x+B}{t} - \frac{1+\log t}{t}\right] \tag{1.188}$$

The last six formulas were essentially given by Ovsjannikov in 1962. Ovsjannikov's work, however, is purely of a mathematical nature. He gives no discussion of physics or applications.

These solutions have the characteristics which we anticipated following the group reducible representation (1.168). In particular, the ratio p/ρ depends only on the time. However, the solution for $\alpha \neq 0$ does not have everywhere constant entropy, as evidenced by the adiabatic integral, relation (1.175). The spatial variation of the

flow variables appears only the combination x/t: linearly for the velocity; and exponentially for the pressure and density. In the case $\alpha = 0$, only the constant entropy solution comes forward.

5. Three Parameter Galilean Flows. Let us extend our analysis of Galilean flows to the case $d \neq 0$, so that we can analyze the nonlinear elastic flow with the aid of additional thermodynamic group transformations. We wish to study the case that three group parameters b, α, d are non-zero, while the rest vanish. From Table 1.3 line four, or by directly integrating (1.73), we find the group reducible representation of the flow variables,

$$\begin{aligned} \lambda &= t \\ u &= -\frac{b}{d} + (x + \tfrac{b}{d}t)\mathcal{U}(t) \\ \rho &= (x + \tfrac{b}{d}t)^{\frac{\alpha-2d}{d}} R(t) \\ p &= (x + \tfrac{b}{d}t)^{\alpha/d} \mathcal{P}(t) \end{aligned} \tag{1.189}$$

As expected, for Galilean flow, the time is the group invariant. Thus we are again dealing with an initial value problem. In the group reducible representation (1.189) the velocity retains a linear spatial behavior, but the position dependence of the density and pressure no longer track one another. The (b, α, d) group reduced equations of motion are given by

$$R' + \left(\frac{\alpha - d}{d}\right) R U = 0$$

$$U' + U^2 + \frac{\alpha}{d}\frac{P}{R} = 0 \qquad (1.190)$$

$$P' + \left(\frac{\alpha}{d} + \gamma + 1\right) U P + C(\mu - 1) R^{\mu} U = 0$$

In the group reduced balance equations (1.190) the group parameter b is absent. When the exponent μ is not equal to zero or unity, the group parameters α and d are connected throughout by the symmetry breaking relation,

$$\frac{\alpha}{d} = \frac{2\mu}{\mu - 1} \qquad (1.191)$$

Equations (1.190) have already solved themselves for the derivatives R'(t), U'(t), P'(t). After some manipulations we find

$$P R^{\gamma+1} \left(\frac{R}{P}\right)^{\alpha/d} = \text{const} \qquad (1.192)$$

This is the adiabatic integral. In addition we find,

$$\log \frac{P}{R} = -(\gamma + 2) \int U(t)\, dt \equiv -(\gamma + 2) \log y \qquad (1.193)$$

Finally, because the momentum equation is a Riccati equation for U(t) in terms of P/R, we have

$$y'' + \frac{\alpha}{d}\frac{P}{R} y = y'' + \frac{\alpha}{d} y^{-(\gamma+1)} = 0 \qquad (1.194)$$

From (1.193) and (1.194) we find a particular solution for U(t) by letting y' = z, y'' = $z\frac{dz}{dy}$, and integrating twice,

$$U(t) = \frac{2}{\gamma+2}\,\frac{1}{t} \tag{1.195}$$

Equation (1.195) holds when $\frac{\kappa}{d} = \frac{2\gamma}{(\gamma+2)^2}$.

This form for U(t) results in a familiar expression for the velocity

$$u = \frac{2}{\gamma+2}\,\frac{x}{t} - \frac{1}{d}\,\frac{\gamma}{\gamma+2} \tag{1.196}$$

Once U(t) is known, R(t) and P(t) follow by directly integrating equations (1.196). Aside from unimportant constants the results for the density and pressure are

$$\rho = \frac{\left(x + \frac{b}{d}t\right)^{\frac{\kappa-2d}{d}}}{t^{\frac{\kappa-d}{d}\cdot\frac{2}{\gamma+2}}} \tag{1.197}$$

$$P = \frac{\left(x + \frac{b}{d}t\right)^{\kappa/d}}{t^{\left(\frac{\kappa}{d}+\gamma+1\right)\frac{2}{\gamma+2}}} \tag{1.198}$$

We note, as a check, that

$$\frac{R}{P^{\gamma+1}} = t^2\,, \qquad P^{\gamma+1} = \frac{1}{t^{\frac{2\kappa}{d}}} \tag{1.199}$$

The first relation holds in general, because of the group reduced momentum equation. We see that (1.192) and (1.193) are satisfied by

$$R = \text{const}\; t^{\frac{d-\kappa}{d}\frac{2}{\gamma+2}}\,, \qquad P = \text{const}\; t^{-\left(\frac{\kappa}{d}+\gamma+1\right)\frac{2}{\gamma+2}} \tag{1.200}$$

In all of the equations (1.197) through (1.200) we must take $\frac{\alpha}{d} = \frac{2\gamma}{(\gamma+2)^2}$.

Suppose we now let $\mu \neq 0, 1$, and we employ the particular solution (1.195) for U(t). Then by (1.196) a relation is enforced between μ and γ, namely

$$\frac{1}{\mu} = 1 - \frac{(\gamma+2)^2}{\gamma} < 0 \tag{1.201}$$

For the particular value of μ given by (1.201) a solution of the group reduced equations (1.190) is possible. The analysis of the equations is presented, because the outcome of the solution is that the fluid motion for $\mu \neq 0, 1$ has a completely different character than that of the simpler cases. By solving the group reduced continuity equation for U and substituting into the group reduced entropy equation we find

$$\mathbb{P}' = D\mathbb{P}\frac{R'}{R} + E R^{\mu} R' \tag{1.202}$$

where the constants D and E are given by

$$D = \frac{d}{\alpha - d}\left(\frac{\alpha}{d} + \gamma + 1\right) \tag{1.203}$$

$$E = C_1(\mu - 1)\frac{d}{\alpha - d} \tag{1.204}$$

Thus,

$$\frac{d\mathbb{P}}{d(\log R)} = D\mathbb{P} + E e^{\mu \log R} \tag{1.205}$$

and we find that the adiabatic integral takes the form

$$\frac{\ddot{R}}{R} = (\text{const})\, R^{D-1} + \frac{E}{\mu - D} R^{\mu - 1} = \frac{1}{t^2} \qquad (1.206)$$

The relations (1.195) and (1.206) implicitly determine the flow. We note that in this case the symmetry breaking in the equation of state completely alters the analytical character of the flow.

E. Invariance Principles for the Dissipative Navier-Stokes Equations with an Ideal Gas Equation of State

We have characterized the change in isentropic Navier-Stokes flow which occurs upon introducing symmetry breaking interaction terms into the Mie-Grüneisen equation of state. However, other types of symmetry breaking interactions do occur in fluids. For example, the thermal energy may be distributed, as in a plasma, among different interacting species of particles (along with a radiation temperature field, say), so that relaxation effects occur. Or the particles may interact during the flow to dissipate energy. In the Navier-Stokes approximation, the latter type of interaction is described by transport coefficients of viscosity and thermal conductivity.[8]

In this section we briefly present the results of our derivation of the invariance principles for the full Navier-Stokes equations, with dissipation, and in one dimension. The coefficients of viscosity and thermal conduction are

---

[8] Landau and Lifshitz (1959).

allowed to depend on temperature, as in a real material. However, the equation of state is assumed to be that for the simple gas (equation (1.62) with $\mu = 1$), namely

$$P + P_0 = \gamma\rho(I + I_0), \quad I = c_v\theta \tag{1.207}$$

The medium is specified with constant Grüneisen parameter and constant specific heats. Eliminating the pressure in favor of the temperature, θ, the dissipative Navier-Stokes equations in one dimension become

$$\frac{D\rho}{Dt} + \rho u_x = 0 \tag{1.208}$$

$$\rho\frac{Du}{Dt} + (k\theta\rho)_x - \zeta'\theta_x u_x - \zeta u_{xx} = 0 \tag{1.209}$$

$$\frac{k}{2}\rho\frac{D\theta}{Dt} + (k\theta\rho)u_x - \zeta u_x^2 - K'\theta_x^2 - K\theta_{xx} = 0 \tag{1.210}$$

where $k = \gamma c_v$, and the temperature dependence has been included in the coefficients $K(\theta)$ and $\zeta(\theta)$, describing thermal conduction and viscous dissipation.

We wish to determine the maximal set of invariance principles which belong to equations (1.208) through (1.210). Let a set of transformations on the quantities x, t, u, ρ, θ be defined to first order in Taylor expansion by the expressions:

$$\begin{aligned}
\bar{x} &= x + \varepsilon X(x,t,u,\rho,\theta) \\
\bar{t} &= t + \varepsilon T(x,t,u,\rho,\theta) \\
\bar{u} &= u + \varepsilon U(x,t,u,\rho,\theta) \\
\bar{\rho} &= \rho + \varepsilon R(x,t,u,\rho,\theta) \\
\bar{\theta} &= \theta + \varepsilon \Theta(x,t,u,\rho,\theta)
\end{aligned} \tag{1.211}$$

In these expressions ε represents a field of infinitesimals. The quantities X, T, U, R, Θ, correspond to the coordinate functions for the infinitesimal generators of the transformation group, in a differential operator realization. A general such operator takes the form

$$Q = X\partial_x + T\partial_t + U\partial_u + R\partial_\rho + \Theta\partial_\theta \tag{1.212}$$

As usual, the transformation properties of the various derivatives appearing in the dissipative gas dynamics equations are defined by the chain rule of differentiation.[9] A given transformation is a symmetry operation when it leaves each of the fluid flow equations invariant, up to an overall factor. The result of a calculation which is analogous to that presented earlier for an isentropic Mie-Grüneisen fluid (but extremely tedious due to the presence of second derivatives) proves that first of all for general thermal dependence in the transport coefficients, only the kinematic operations of classical mechanics are allowed:

---

[9] See Appendix A

$$
\begin{aligned}
D &= \partial_x && \text{space translations} \\
H &= \partial_t && \text{time translations} \\
G &= t\partial_x + \partial_u && \text{Galilean boosts}
\end{aligned}
\qquad (1.213)
$$

The viscosity and conductivity introduce fundamental lengths which violate the (x/t) - invariant kinematic scaling law of isentropic flow. However, if the coefficients $\zeta(\theta)$ and $K(\theta)$ vary with temperature according to the same power law - so that $\frac{K}{\zeta}$ is a constant, as is often experimentally valid - then two more invariance principles are admitted. Namely, the scaling laws generated by,

$$
\begin{aligned}
B &= x\partial_x + 2(m-1)\rho\partial_\rho + u\partial_u + 2\theta\partial_\theta \\
C_1 &= t\partial_t - (2m-1)\rho\partial_\rho - u\partial_u - 2\theta\partial_\theta
\end{aligned}
\qquad (1.214)
$$

The viscosity and conductivity coefficients depend on temperature according to a single power law,

$$
\zeta = \zeta_0 \theta^m , \quad K = K_0 \theta^m ; \quad m, \zeta_0, K_0 \text{ const} \qquad (1.215)
$$

The exponent m is calculable from kinetic theory, given the particle interaction. Kinetic theory predicts m = 1/2 for the hard sphere model, and m = 1 for "Maxwell molecules" (with an intermolecular potential energy which has an inverse fourth power dependence on separation distance).[10]

[10] G. W. Ford and J. D. Foch, "The Dispersion of Sound in Monoatomic Gases," in Studies in Statistical Mechanics V, J. deBoer and G.E. Uhlenbeck, Eds. (North-Holland, New York 1970).

Adding and subtracting the operators in (1.214) eliminates the exponent dependence in one case,

$$\Delta = E - C_1 = x\partial_x - t\partial_t + (4m-3)\rho\partial_\rho + 2u\partial_u + 4\theta\partial_\theta$$
$$\Sigma_1 = E + C_1 = x\partial_x + t\partial_t - \rho\partial_\rho \qquad (1.216)$$

The commutator table of the invariance algebra for dissipative flow is displayed in Table 1.5. This algebra is a semi-direct product of the Galilean algebra with scaling laws (2.2).

When the coefficients ζ and K <u>both</u> vanish, the result is the algebra of operators (1.80) but with pressure replaced by $(k\theta\rho)$. For comparison we display the full invariance algebra for $\zeta = K = 0$. The results for the coordinate functions are, in planar geometry,

$$X = ht x + (d+n)x + bt + a$$
$$T = ht^2 + nt + c \qquad K = (1 - ht + g)\rho$$
$$U = h(x - ut) + du + b$$
$$\Theta = -2(ht - d)\theta \qquad (1.217)$$
$$c = -3ht + (g - n)$$
$$m = -4ht + (g - n + d)$$
$$\mathcal{E} = -5ht + (g - n + 2d)$$

with group parameters g, h, a, b, c, d, n. Translations of internal energy, although also allowed, are omitted in (1.217). An operator basis which results from the set of coordinate functions (1.217) is given by,

$$H = xt\partial_x + t^2\partial_t + (x-ut)\partial_u - t\rho\partial_\rho - 2t\theta\partial_\theta \quad \text{Projective transformations}$$

$$D = x\partial_x + u\partial_u + 2\theta\partial_\theta \quad (x,\ u, \sqrt{\theta}\ ) \text{scaling}$$

$$B = t\partial_x + \partial_u \quad \text{Galilean boosts}$$

$$A = \partial_x \quad \text{Space translations} \qquad (1.218)$$

$$N = x\partial_x + t\partial_t \quad (x,\ t) \text{ scaling}$$

$$C = \partial_t \quad \text{Time translations}$$

$$G = \rho\partial_\rho \quad \text{Density scaling}$$

The effect of symmetry breaking dissipative interactions is first of all to reduce the symmetry group of the nondissipative equations, by establishing relations among the group parameters. In particular, the group parameter g becomes related to d and n by,

$$g = 2(m-1)d - n \qquad (1.219)$$

Then, because second derivative terms appear in the resulting group reduced representations of the fluid equations, the dissipative terms actually alter the character of the flow in an essential way. A few of the analytic properties of group reducible dissipative fluid motions are discussed in the next section. In a later article, we plan to numerically investigate the fundamental group reducible dissipative motions of a Navier-Stokes fluid.

	D	H	G	Σ	Δ
D	0	0	0	D	D
H	0	0	D	H	-H
G	0	-D	0	0	2G
Σ	-D	-H	0	0	0
Δ	-D	H	-2G	0	0

Table 1.5. The Commutator Table of the Invariance Algebra for Dissipative Flow.

In the present work a group classification of dissipative flows is given. For K and ς nonzero, the group parameter h vanishes and (d, g, n) become connected by (1.219). The catalogs of group reducible representatives given in Table 1.3 and Table 1.4 still apply to the case of dissipative flow; with appropriate modifications in the last two columns of each table for the change to a thermal description, rather than a pressure description. These modifications are shown in Table 1.6 and Table 1.7. In the subsequent symmetry analysis of the group reduced equations the symmetry breaking condition on the group parameters (1.219) is imposed. The resulting ordinary differential equations are second order, rather than first order, in the group invariant independent variable $\lambda(x,t)$. Two different group reduced representations of the dissipative Navier-Stokes equations are given.

F. Self-Similar Dissipative Flows

Let us derive the group reduced representation of the dissipative fluid dynamics equations for nonzero d, n, g. According to Tables 1.3 and 1.6, the group reducible representatives of the flow variables are given by

Parameters		Invariants	
(n,d)	k	$R(\lambda)$	$\Theta(\lambda)$
Free	$\frac{d+n}{n}$	$\frac{\rho}{(nt+c)^{g/n}}$	$\frac{\theta}{(nt+c)^{2d/n}}$
$d=0$	1	$\frac{\rho}{(nt+c)^{g/n}}$	θ
$d+n=0$	0	$\frac{\rho}{(nt+c)^{g/n}}$	$\theta(nt+c)^2$
$n=0$	∞	$\rho e^{-\frac{g}{c}t}$	$\theta e^{-\frac{2d}{c}t}$
$d=n=0$	—	$\rho e^{-\frac{g}{c}t}$	θ
$c=d=n=0$	—	$\rho \exp\frac{-gx}{bt+a}$	θ

Table 1.6. Modifications of Table 1.3 for Thermal Description.

k	Nonzero Parameters	Zero Parameters	$R(\lambda), \Theta(\lambda)$
$1+\varepsilon$	$n=1$, $d=\varepsilon$	a, b, c	ρt^{-g} , $\theta t^{-2\varepsilon}$
1	$n=1$, $b=1$	a, c, d	ρt^{-g} , θ
0	$d=-n=1$	$a, b, c, (d+n)$	ρt^{g} , θt^{2}
0	$-d=n=a=1$	$b, c, (d+n)$	ρt^{-g}, θt^{2}
∞	$c=d=1$	a, b, n	ρe^{-gt} , θe^{-2t}
–	$c=1$	a, b, d, n	ρe^{-gt} , θ
–	$b=c=1$	a, d, n	ρe^{-gt} , θ
–	$b=1$	a, c, d, n	$\rho e^{-g\frac{x}{t}}$, θ
–	$a=1$	b, c, d, n	ρe^{-gx} , θ

Table 1.7. Modifications of Table 1.4 for Thermal Description.

$$\begin{aligned} \lambda &= \frac{x}{t^{1+\varepsilon}} \\ u &= t^{\varepsilon} \sqcup(\lambda) \\ \rho &= t^{g} R(\lambda) \\ \theta &= t^{2\varepsilon} \Theta(\lambda) \end{aligned} \tag{1.220}$$

Substituting (1.220) into the dissipative equations (1.208) through (1.210) we have,

$$R\sqcup' + [\sqcup - (\varepsilon+1)\lambda] R' = -gR \tag{1.221}$$

$$\{[\sqcup - (\varepsilon+1)\lambda] R\sqcup' + k\Theta R' + k\tilde{R}\Theta' + \varepsilon R\sqcup\} t^{g-\varepsilon+1}$$
$$= \xi_0 t^{2(\varepsilon m-1)-\varepsilon} \left(m\Theta^{m-1}\sqcup'\Theta' + \Theta^{m}\sqcup''\right) \tag{1.222}$$

$$\left\{k\Theta R\sqcup' + \frac{k}{2}[\sqcup - (\varepsilon+1)\lambda] R\,\Theta'\right\} t^{g+2\varepsilon-1}$$
$$= t^{2(\varepsilon m-1)}\left\{\xi_0 \Theta^{m}\sqcup'^{2} + \nu_c[m\Theta^{m-1}\Theta' + \Theta^{m}\Theta'']\right\} \tag{1.223}$$

The powers of time in these equations can only be eliminated if

$$g = 2(m-1)\varepsilon - 1 \tag{1.224}$$

which is, of course, a rewriting of the symmetry breaking relation (1.119). However, even when the factors of time are eliminated from the group reduced equations (1.221) through (1.223), a formidable set of ordinary differential

equations still remains. Equations (1.221) through (1.223) may be written as

$$\begin{aligned} dL &= \varepsilon \lambda dR \\ dM &= \varepsilon \lambda^2 d\left(\frac{RU}{\lambda}\right) \\ dN &= \varepsilon \left\{ \frac{\lambda}{2} d\left[R(k\Theta + U^2)\right] - RUU^2 d\lambda \right\} \end{aligned} \tag{1.225}$$

where L, M, N are given by

$$\begin{aligned} L &= (U - \lambda)R \\ M &= LU + k\Theta R - \tfrac{5}{2}LU' \\ N &= \tfrac{k}{2}L\Theta + MU - \tfrac{1}{2}LU^2 - K\Theta' \end{aligned} \tag{1.226}$$

when $\varepsilon = 0$ the quantities L, M, N are constants of the motion.

Let us now consider self-similar motions with group parameters d and n connected by

$$d + n = 0 \tag{1.227}$$

The connection between d and n forces the symmetry exponent k to vanish. Hence, the group reducible flow variables gotten from Tables 1.4 and 1.7 are given by

$$\begin{aligned} \lambda &= x \\ u &= \frac{1}{t} U(x) \\ \rho &= t^{-9} R(x) \\ \theta &= t^{-2} \Theta(x) \end{aligned} \tag{1.228}$$

The resulting group reduced dissipative flow equations may be expressed

$$(UR)' = (2m-1)R \tag{1.229}$$

$$URU' + (k\Theta R)' - UR = \varepsilon_0 \left[m\Theta^{m-1} U'\Theta' + \Theta^m U'' \right] \tag{1.230}$$

$$kR\Theta U' + \frac{k}{2} UR\Theta' - k\Theta R \tag{1.231}$$

$$= \zeta_0 \Theta^m U'^2 + m\kappa_0 \Theta^{m-1} \Theta'^2 + \kappa_0 \Theta^m \Theta^{m-1}$$

where the symmetry breaking relation (1.224), namely $g = 2m-1$, has already been assumed.

When $m = 1$ some simplification occurs.

$$(UR)' = R \tag{1.232}$$

$$URU' + (k\Theta R)' - UR = \zeta_0 (U'\Theta)' \tag{1.233}$$

$$kR\Theta U' + \frac{k}{2} UR\Theta - k\Theta R = \zeta_0 \Theta U'^2 + \kappa_0 (\Theta\Theta')' \tag{1.234}$$

At this point invariance principles from the previous section have been used to reduce the Navier-Stokes equations with power law temperature dependent transport coefficients to ordinary differential equations and for a special case first integrals have been obtained. Further progress with the full set of fluid equations can best be made numerically. In the next chapter we discuss the group reducible fluid motions of Burgers' approximation to the Navier-Stokes description. The fluid invariance laws established for Burgers' description lead to solvable group reduced equations of motion. The associated group reducible fluid motions in Burgers' approximation are relevant to the dissipative Navier-Stokes description in a weak shock, asymptotic limit.

CHAPTER II

WEAK DISSIPATIVE SHOCKS: BURGERS' APPROXIMATION

Burgers' equation - a prototype fluid equation - is employed to study how dissipative effects alter a wave profile as it propagates.[1] Group theoretical techniques are used to construct invariance principles and find classes of group reducible fluid motions. The analysis applies to the propagation of a weak shock in a dissipative fluid, e.g., the formation and attenuation of finite amplitude sound waves in air. In the regime of interest, both convection and dissipative effects (viscosity, self-diffusion, thermal conduction) are present. Under these conditions a balance may be struck between the convective wave-steepening tendency and the relaxation effect of dissipation. The weak shocks which result, rather than being discontinuous, have a thickness and an internal structure.

A. Burgers' Model Fluid Equation

The simplest fluid equation which combines nonlinear convection with diffusion is Burgers' equation,[2]

[1]The attenuation and dispersion law for sound waves in monatomic gases is investigated from a kinetic theory viewpoint by G. W. Ford and J. D. Foch (1970). The question of the evolution of the wave form due to diffusive effects is broached by Lighthill (1956).

[2]Burgers (1974). Equation (2.1) has received much attention in the literature. It first appeared in a paper by H. Bateman, "Some Recent Researches on the Motion of Fluids," Monthly Weather Rev. 43, 163-170 (1915). A survey of published solutions has recently been given by Benton and Platzman (1972). See also Whitham (1974).

$$u_t + uu_x = \nu u_{xx} \tag{2.1}$$

where ν is a positive constant. This equation is a simplified form of the fluid momentum balance equation. The quantity u has dimensions of velocity

$$[u] \sim \left[\frac{x}{t}\right] \tag{2.2}$$

while ν plays the role of a kinematic viscosity with dimensions

$$[\nu] \sim \left[\frac{x^2}{t}\right] \tag{2.3}$$

The Burgers' equation (2.1) may be regarded as a differential conservation law,

$$\frac{\partial D}{\partial t} + \frac{\partial F}{\partial x} = 0 \tag{2.4}$$

with density D and flux F given by

$$D = u \quad , \quad F = \left(\frac{u^2}{2} - \nu u_x\right) \tag{2.5}$$

In the conservative form it is argued that the flux F depends quadratically on the local momentum density D and linearly on the momentum density gradient. The flow is driven by the kinetic energy density $\frac{u^2}{2}$. It slows down when it encounters regions of positive u_x, but it speeds up when it meets a negative u_x. Consequently in the evolution of a Burgers wave, the negative slopes ($u_x < 0$) are steepened but the positive slopes ($u_x > 0$) slacken and

diminish to zero. If the diffusion coefficient is small, an initially sinusoidal wave train will eventually take the form of a steady sawtoothed row of triangular waves.

Burgers' equation (2.1) is not a true fluid wave equation. Because no pressure gradient term is present, its solutions do not reduce to ordinary sound waves. However it has recently been shown[3] that Burgers' equation governs the final stages of weak shock wave formation for general Navier-Stokes fluids.

The physical length and time scales for a weak shock wave are

$$L = \text{shock width}$$

$$t_c = L/u, \text{ convection time scale} \qquad (2.6)$$

$$t_d = \frac{L^2}{\nu}, \text{ diffusion time scale}$$

The time scales defined in equations (2.6) correspond to: (1) the time interval required for a signal to propagate a distance L at the convection velocity u; and (2) the average time it takes for a disturbance to diffuse a distance L, by means of one of the diffusion mechanisms which attenuate the wave. The ratio, τ, of the diffusion time scale to the convection time scale is equal to the Reynolds number

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[3]W. D. Hayes, Gasdynamic Discontinuities (Princeton University Press, 1960); J. P. Moran and S. F. Shen, "On the Formation of Weak Plane Shock Waves by Impulsive Motion of a Piston," J. Fluid Mech. 25, 705 (1966).

of the wave motion.[4]

$$\tau = \frac{t_d}{t_c} = \frac{\lambda^2}{\nu} = Re = \frac{u}{c}\frac{L}{\lambda} \qquad (2.7)$$

In equation (2.7) c refers to the local sound velocity and λ is the average diffusion mean free path,

$$\lambda \sim \frac{\nu}{c} \qquad (2.8)$$

The manner in which a wave form evolves depends on the relative magnitudes of t_c and t_d. If the diffusion time is considerably less than the convection time, then the disturbance will decay away, and nonlinear effects will not be important. However, if $\tau \sim 1$, nonlinear effects are important. The waveform then steepens and a balance may be established between shocking-up and relaxing. The result is a stable waveform.

Historically Burgers' equation (2.1) has been studied for its shock-like solutions, i.e., solutions with steep fronts of the form

$$u(x,t) = \zeta(x-ct) \quad , \quad c = \text{const} \qquad (2.9)$$

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[4]A precise definition of the Reynolds number for a Burgers' wave is given by

$$Re = \frac{1}{\nu}\int_{-\infty}^{\infty} u\,dx$$

The Reynolds number defined in this way is a constant of the motion. Thus the relative importance of convection and diffusion does not change as the wave propagates. The constancy of Re is equivalent to momentum conservation (uniform asymptotic motion).

Such solutions result from space and time translation invariance. In order to study these solutions, the shock form is substituted into the Burgers equation,

$$(s-c)s_{\bar{x}} = \nu s_{\bar{x}\bar{x}} \quad , \quad \bar{x} = x - ct \tag{2.10}$$

Integration with respect to $\bar{x}$ produces

$$\frac{1}{2}(s-c)^2 = \nu s_{\bar{x}} + \text{const.} \tag{2.11}$$

Far ahead of the front $s = s_a$, while far behind it $s = s_b \neq s_a$. Neglecting the gradient $s_{\bar{x}}$ away from the front leads to

$$s_a - c = c - s_b \tag{2.12}$$

or

$$c = \frac{1}{2}(s_a + s_b) \tag{2.13}$$

where the shock stability condition has been used,

$$s_a > c > s_b \tag{2.14}$$

The integrated form above can now be written as

$$\frac{1}{2}(s-c)^2 = \nu \ln\left(\frac{s-s_a}{s-s_b}\right) + \text{const.} \tag{2.15}$$

This equation has the solution

$$s = c - \frac{\Delta s}{2}\tanh\frac{x-ct}{L} \tag{2.16}$$

with

$$\Delta s = s_b - s_a \quad , \quad L = \frac{2\nu}{\Delta s} \tag{2.17}$$

The solution has the form of a kink which propagates at constant speed c, and acts like a shock: it changes the state s_a, ahead of the kink, to the state s_b, behind the kink. Note that its amplitude Δs and width L are interrelated via the diffusion coefficient. As ν decreases at constant amplitude, the kink gets steeper and the width decreases. Vice versa, at constant viscosity the shock thickness L decreases with increasing amplitude.

In a series of papers beginning in 1940, Burgers used equation (2.1) to develop a zeroeth order theory of turbulence. Then in the late 1940's, it attracted the interest of other physicists and mathematicians. Cole (1949) and Hopf (1950) found a linearizing transformation which relates Burgers equation to the linear diffusion equation. The Cole-Hopf transformation between Burgers waves and pure diffusion is given by

$$u(x,t) = -2\nu \partial_x [\log w(x,t)] \qquad (2.18)$$

where u(x,t) satisfies Burgers' equation and w(x,t) diffuses linearly, with the same diffusion coefficient,

$$w_t = \nu w_{xx} \qquad (2.19)$$

Thus, any solution of the diffusion equation will give a solution of Burgers' equation by applying (2.18).

From the point of view of theoretical physics, the Cole-Hopf relation is an unexpected bonus. It tells us that, in the appropriate space, Burgers waves can be added

linearly. This means, in turn, that Burgers shock interactions can be studied exactly. Burgers' wave description with Cole's and Hopf's contributions has become a testing ground for ideas about various types of nonlinear phenomena.[5]

B. The Derivation of the Burgers' Equation Invariance Algebra

In order to characterize the symmetry structure of Burgers' equation and to derive representative group reducible motions, an algebra of invariance operators must be constructed. These operators generate infinitesimal transformations

$$\begin{aligned} \bar{x} &= x + \varepsilon X(x,t) \\ \bar{t} &= t + \varepsilon T(x,t) \\ \bar{u} &= u + \varepsilon U(x,t) \end{aligned} \tag{2.20}$$

which leave invariant the Burgers' equation

$$u_t + uu_x - u_{xx} = 0 \tag{2.21}$$

Units are chosen to make the diffusion coefficient equal unity. The invariance condition is that the order ε terms vanish with (2.21) in the expression

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[5]Cole (1951) conjectured that the existence of the transformation (2.18) between static and convective diffusion equations depends on the invariance operations admitted by both equations.

$$\bar{u}_{\bar{t}} + \bar{u}\bar{u}_{\bar{x}} - \bar{u}_{\bar{x}\bar{x}} = [1 + \epsilon P(x,t,u)](u_t + uu_x - u_{xx}) \qquad (2.22)$$

or

$$U^{(t)} + U u_x + u U^{(x)} - U^{(xx)} = P(x,t,u)(u_t + uu_x - u_{xx}) \qquad (2.23)$$

The extended coordinate functions which appear in the invariance condition (2.23) are given in Appendix D. Equating like coefficients on each side of (2.23) results in an over-determined set of linear partial differential equations for the coordinate functions X, T, U, and for the multiplier P(x,t,u). A side calculation convinces us that more general dependences in X, T, U, P do not occur for Burgers' equation. The set of determining equations is quickly reduced to six members

$$U_t + uU_x - U_{xx} = 0 \qquad (2.24)$$

$$uU_u + U - X_t - uX_x - X_{xx} - 2U_{xu} = uP \qquad (2.25)$$

$$U_u - T_t - T_x u + u_t T_{xx} = P \qquad (2.26)$$

$$U_u - 2X_x = P \qquad (2.27)$$

$$T_x = 0 \qquad (2.28)$$

$$U_{uu} = 0 \qquad (2.29)$$

The last two equations imply that

$$T = T(t) \tag{2.30}$$

$$U = u f(x,t) + g(x,t) \tag{2.31}$$

Remaining are five equations for f, g, X, T, P . We use (2.30) and (2.31) to reduce the determining equations (2.26) and (2.27) to

$$f - T'(t) = P \tag{2.32}$$

$$f - 2X_x = P \tag{2.33}$$

Hence, by subtraction

$$X_x = \tfrac{1}{2} T'(t) \tag{2.34}$$

$$f - P = T'(t) \tag{2.35}$$

Equations (2.34) and (2.35) imply that $X_{xx} = 0$ and $P_u = 0$. Consequently,

$$X = \tfrac{x}{2} T'(t) + A(t) \tag{2.36}$$

Substitution of (2.30), (2.31), and (2.32) separates (2.25) into two terms.

$$u(f + \tfrac{1}{2}T') + (g - \tfrac{x}{2}T'' - A' - 2f_x) = 0 \tag{2.37}$$

Thus

$$f = -\tfrac{1}{2}T'(t), \quad f_x = 0 \tag{2.38}$$

and

$$g = \frac{x}{2} T'' + A'(t) = X_t \tag{2.39}$$

Finally, substituting (2.31) with (2.38) and (2.39) into the first determining equation (2.24) gives

$$g_t + u g_x - g_{xx} + u f'(t) = 0 \tag{2.40}$$

or

$$\frac{x}{2} T''' + A'' + u\left(\frac{1}{2} T'' + f'\right) - X_{txx} = 0 \tag{2.41}$$

The last two terms in (2.41) vanish. The first two terms must also each vanish separately. Therefore,

$$T''' = A'' = 0 \tag{2.42}$$

and

$$T = ct^2 + 2st + h \tag{2.43}$$

$$A = bt + p \tag{2.44}$$

where c, s, h, b, p are integration constants.

From (2.36) the coordinate function X(x,t) must be

$$X = cxt + sx + bt + p \tag{2.45}$$

Now

$$f = -\frac{1}{2} T' = -(ct + s) \tag{2.46}$$

$$g = X_t = cx + b \tag{2.47}$$

and finally,

$$P = -3(ct+s) \tag{2.48}$$

Collecting terms, the coordinate functions (2.20) satisfying (2.23) are given by the expressions

$$\begin{aligned} X &= cxt + sx + bt + p \\ T &= ct^2 + 2st + h \\ U &= c(x-ut) - su + b \end{aligned} \tag{2.49}$$

A differential operator basis, L_B, is found for the Burgers' algebra by individually letting each of the group parameters (p, h, s, g, c) be unity, while forcing the others to vanish. Table 2.1 lists the members of L_B along with their invariants and geometrical significance.

Appearing in the Burgers' algebra are space and time translations, Galilean transformations, scaling transformations, and conformal transformations. Space and time translations, along with Galilean transformations, are universal symmetries of planar flow. The scale transformation S leaves the units of the diffusion constant invariant. The last transformation, C, is a time dependent scaling operation. The operators in the basis L_B satisfy the commutation relations given in Table 2.2.

An interesting subalgebra of L_B is formed from $\{-G, \frac{M}{2}, H\}$ which has SU(2) commutation relations corresponding to $\{J_+, J_0, J_-\}$ respectively. Other subalgebras may also be picked out of Table 2.2. The algebra of invariance

OPERATOR	SIGNIFICANCE	INVARIANTS
$P = \partial_x$	Space translations $x(p) = x_o + p$	t, u
$H = \partial_t$	Time translations $t(h) = t_o + h$	x, u
$S = x\partial_x + 2t\partial_t - u\partial_u$	Scaling: $x(s) = e^{s}x_o$ $t(s) = e^{2s}t_o$ $u(s) = e^{-s}u_o$	$\frac{x}{\sqrt{t}}$, $\sqrt{t}\,u$
$G = t\partial_x + \partial_u$	Galilean Boosts: $x(g) = x + gt$ $u(g) = u + g$	t, x-ut
$C = xt\partial_x + t^2\partial_t + (x-ut)\partial_u$	Conformal transformations: $x(c) = \frac{x_o/t_o}{1/t_o - c}$ $t(c) = \frac{1}{1/t_o - c}$ $u(c) = u_o + c(x_o - u_o t_o)$	$\frac{x}{t}$, x-ut

Table 2.1. The Basis L_B of the Lie Invariance Algebra for Burgers' Equation.

	P	H	S	G	C
P	-	0	P	0	G
H	0	-	2H	P	S
S	-P	-2H	-	G	2C
G	0	-P	-G	-	0
C	-G	-S	-2C	0	-

Table 2.2. The Commutators of the Basis L_B for the Burgers' Equation.

operators for the linear diffusion equation is precisely the same as Table 2.2, except that $[G,P] = 1$. Thus, the linear diffusion equation admits the central extension of L_B. The relation between their invariance algebras is a clue that Burgers' equation and the diffusion equation are more deeply related.

The general families of invariant surfaces under L_B are found by integrating the characteristic equations

$$\frac{dx}{cxt + sx + gt + p} = \frac{dt}{ct^2 + 2st + h} = \frac{du}{c(x-ut) - su + g} \qquad (2.50)$$

in closed form. Rearranging the first equation and defining new constants (x_o, t_o, V, a) we have, for $c \neq 0$

$$\frac{d(x-x_o)}{d(t-t_o)} = \frac{(x-x_o)(t-t_o) + Va}{(t-t_o)^2 + a} \qquad (2.51)$$

Let x and t correspond to shifted variables. Then equation (2.51) reads,

$$\frac{dx}{dt} = \frac{xt + Va}{t^2 + a}, \quad c \neq 0 \qquad (2.52)$$

Rewriting equation (2.52) as

$$\frac{d}{dt}\frac{x}{\sqrt{t^2+a}} = \frac{Va}{(t^2+a)^{3/2}} \qquad (2.53)$$

we have by integration,

$$\frac{x}{\sqrt{t^2+a}} = \frac{Vt}{\sqrt{t^2+a}} + \lambda \qquad (2.54)$$

λ is the constant of integration. The expression $\lambda(x,t) =$ const describes invariant curves in (x,t). A four parameter family of group trajectories results.

$$\lambda(x,t) = \frac{(x-x_o) - V(t-t_o)}{\sqrt{(t-t_o)^2 + a}} \tag{2.55}$$

or

$$(x-x_o) = V(t-t_o) + \lambda\sqrt{(t-t_o)^2 + a} \tag{2.56}$$

In the case that c = 0, we can return to equation (2.50) and find other classes of invariant surfaces.

Next, the functional form of the solution corresponding to the invariant surfaces (2.56) is derived. Rearranging the second identity in equation (2.50) for $c \neq 0$ we have

$$\frac{du}{d(t-t_o)} = \frac{(x-x_o) - u(t-t_o)}{(t-t_o)^2 + a} \tag{2.57}$$

In the shifted variables x and t the determining equation (2.57) is expressed by,

$$\frac{du}{dt} = \frac{x-ut}{t^2 + a} \tag{2.58}$$

Now x may be eliminated for λ and t by substituting the relation,

$$x = \lambda\sqrt{t^2+a} + Vt \tag{2.59}$$

into equation (2.58). The result of this substitution is,

$$\frac{d(u-V)}{dt} = \frac{\lambda\sqrt{t^2+a} - (u-V)t}{t^2+a} \tag{2.60}$$

or, when the velocity is shifted, as well,

$$\frac{du}{dt} = \frac{\lambda\sqrt{t^2+a} - ut}{t^2+a} \tag{2.61}$$

Equation (2.61) may be rewritten as

$$\frac{d}{dt}\left(u\sqrt{t^2+a}\right) = \lambda \tag{2.62}$$

Consequently, the second family of invariants for $c \neq 0$ is given by

$$U(\lambda) = u\sqrt{t^2+a} - \lambda t \tag{2.63}$$

or, in terms of the original coordinates.

$$U(\lambda) = (u-V)\sqrt{(t-t_0)^2+a} - \lambda(t-t_0) \tag{2.64}$$

C. Group Reducible Burgers' Fluid Motions

A group reducible solution of Burgers' equation for $c \neq 0$ lies on the characteristic surfaces in (x, t, u) determined from (2.50). The characteristic surface may be coordinated by the families of group invariants, λ and $U(\lambda)$, given by equations (2.55) and (2.64), respectively. In terms of λ and U the Burgers' equation becomes

$$0 = u_t + uu_x - \nu u_{xx} = \frac{1}{(t^2+a)^{3/2}}\left(a\lambda + UU' - \nu U''\right) \tag{2.65}$$

As expected, on the characteristic surfaces of its

invariance group, Burgers' equation reduces to an ordinary differential equation in the invariant coordinates λ and $U(\lambda)$. Equation (2.65) integrates to

$$U' = K + \frac{1}{2\nu}(U^2 + a\lambda^2) \qquad (2.66)$$

which is a Riccati equation. Upon making the standard substitution,

$$U(\lambda) = -2\nu \frac{d}{d\lambda}(\log y(\lambda)) \qquad (2.67)$$

we find a second order <u>linear</u> differential equation for $y(\lambda)$.

$$y'' + (A - B\lambda^2)y = 0 \quad , \quad A = K, B = -\frac{a}{2\nu} \qquad (2.68)$$

Equation (2.68) can be transformed to the classical equations for either confluent hypergeometric functions, or parabolic cylinder functions.

Before we write a closed form group reducible solution, let us comment on the linearity of equation (2.68). Of course, mathematically, it is linear because (2.66) is a Riccati equation. However, the physical meaning of this linearity is that, in the right space, Burgers waves can be superimposed. Thus, equation (2.67) represents the counterpart of the Cole-Hopf transformation (2.18), taken on the $c \neq 0$ characteristic surface of the Burgers invariance group.

Equation (2.68) is related to the confluent hypergeometric equation and to Weber's equation for the

parabolic cylinder functions.[6]

(1) Let $$y(\lambda) = e^{-z/2} v(z) \quad , \quad z(\lambda) = \frac{\lambda^2}{\sqrt{B}} \tag{2.69}$$

then

$$z v'' + \left(\tfrac{1}{2} - z\right) v' - k v = 0 \quad , \quad k = \tfrac{1}{4}\left(\frac{A}{\sqrt{B}} + 1\right) \tag{2.70}$$

The solution is given in terms of the confluent hypergeometric function

$$y(\lambda) = e^{-z/2} \, {}_1F_1\left(k; \tfrac{1}{2}; z\right) \tag{2.71}$$

(2) Let $$z^2 = c\lambda^2 \quad , \quad y(\lambda) = v(z) \tag{2.72}$$

then Weber's equation appears,

$$4 v'' + (2 + 4k - z^2) v = 0 \tag{2.73}$$

where

$$2A = c(2k+1) \quad , \quad c^2 = 4B \tag{2.74}$$

The solutions for y(λ) are given in terms of parabolic cylinder functions

$$y(\lambda) = D_k(\pm z(\lambda)) \tag{2.75}$$

and

$$y(\lambda) = D_r(\pm i z(\lambda)) \quad , \quad r = -(k+1) \tag{2.76}$$

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[6] G. M. Murphy, Ordinary Differential Equations and Their Solutions (D. Van Nostrand, Princeton, 1960).

Now, transforming back to the invariant $U(\lambda)$ via relation (2.67) gives

$$U(\lambda) = V + \frac{{}_1F_1'(k;\frac{1}{2};z(\lambda))}{C_0 + {}_1F_1(k;\frac{1}{2};z(\lambda))} \qquad (2.77)$$

where C_o is a constant which is related to k and B. Finally, the relation (2.55) and (2.64) or, in the shifted coordinates,

$$u = \frac{U(\lambda) + \lambda t}{\sqrt{t^2 + a}} \quad , \quad \lambda = \frac{x - Vt}{\sqrt{t^2 + a}} \qquad (2.78)$$

restores us to the original variables.

The solution given by (2.77) and (2.78), or its equivalent in terms of parabolic cylinder functions, comprises a four-parameter family of group reducible solutions. This family of solutions corresponds to the group reducible solution generated by the infinitesimal operator C, while simultaneously incorporating the finite transformations of the rest of the invariance group. In a similar manner, other group reducible solutions may be produced. Representative examples of nontrivial Burgers waves are listed in Table 2.3. Details of the derivations are not shown. The wave form (2.77) and (2.78) can be represented in terms of various special functions of mathematical physics. The parabolic cylinder functions have already been mentioned. Hermite functions, error integrals, and exponentials in λ^2 may also be made to appear. In terms of x and t, the

Operator	Invariants	Group Reducible Form
$cP+H$	$x-ct,\ u$	$u = U(x-ct)$
S	$\frac{x}{\sqrt{t}},\ u\sqrt{t}$	$u = \frac{1}{\sqrt{t}} U(\frac{x}{\sqrt{t}})$
$H+G$	$x-\frac{t^2}{2},\ u-t$	$u = t + U(x-\frac{t^2}{2})$
C_1+S	$\frac{x}{\sqrt{t(t+2)}},\ xu-\frac{x^2}{t+2}$	$u = \frac{x}{t+2} + \frac{1}{x} U(\frac{x}{\sqrt{t(t+2)}})$
$aH+bP+C_1$	$\lambda = \frac{x-bt}{\sqrt{t^2+1}},\ (u-b)\sqrt{t^2+1} - \lambda t$	$u = b + \frac{U(\lambda)+\lambda t}{\sqrt{t^2+1}}$
$bP+C_1$	$\lambda = \frac{x}{t}+bt^2,\ (u-\lambda-\frac{b}{t^2})t$	$u = \frac{b}{t^2} + \lambda + \frac{1}{t} U(\lambda)$

Table 2.3. Representative Classes of Group Reducible Solutions for Burgers' Equation.

error function solution initially looks like a Gaussian. Then, as time progresses, its height decreases and the "hump" may be described to "lean" to the right, acquiring a steeper gradient on the side toward positive x. This is the expected character of the motion, corresponding to wave form steepening and attenuation.

In Table 2.3, the (cP + H) solution is the shock-like solution (2.16) discussed earlier. The S group reducible solution is Burgers' similarity solution for which $U(\lambda)$ is given in terms of Hermite functions. The (H + G) solution can be written in terms of Airy functions. The last three lines are special cases of the general $C \neq 0$ solution derived above. In each case the transformations of the rest of the invariance group lead to families of group equivalent solutions.

APPENDIX A

EXTENSION OF INVARIANCE TO DIFFERENTIAL MANIFOLDS

Lie's method for the study of partial differential equations depends explicitly on the transformation properties of the derivatives

$$u_{j,k}\ ,\quad u_{j,kl}\ ,\ \text{etc.} \tag{A.1}$$

under the action of a Lie group defined on (x,u) over E^N. The equations for the local action of a Lie group are given to first order by,

$$\begin{aligned}
\bar{x}_i &= x_i + \varepsilon X_i(x,u) \\
\bar{u}_j &= u_j + \varepsilon U_j(x,u) \\
\bar{u}_{j,k} &= u_{j,k} + \varepsilon U_j^{(k)}(x,u) \\
\bar{u}_{j,kl} &= u_{j,kl} + \varepsilon U_j^{(kl)}(x,u)
\end{aligned} \tag{A.2}$$

where,

$$u_{i,j} = \frac{\partial u_i}{\partial x_j}\ ,\quad \bar{u}_{j,k} = \frac{\partial \bar{u}_j}{\partial \bar{x}_k}\ ,\ \text{etc.} \tag{A.3}$$

Note: no letter distinctions are made for the appropriate ranges of the subscripts in this appendix.

The "extended" coordinate functions $U_j^{(k)}$, $U_j^{(kl)}$, etc., are calculated from the coordinate functions on E^N, directly from the chain rule. The n^{th} extension refers to how the n^{th} partial derivative transforms. In this appendix, the

forms of the first few extensions are calculated and tabulated for use in the text.[1]

Although we make no explicit notation for it, the partial derivatives of interest are always to be evaluated on a characteristic surface of the group concerned. Thus, implicit dependencies abound, because we are operating with $\frac{\partial}{\partial x_j}$ on functions F(x,u(x)). In order to simplify the notation, the total differential operator is introduced,

$$\frac{D}{Dx_\ell} = \frac{\partial}{\partial x_\ell} + u_{j,\ell}\frac{\partial}{\partial u_j} + u_{j,\ell m}\frac{\partial}{\partial u_{j,m}} + \cdots \qquad (A.4)$$

Expression (A.4) accounts for both the explicit and implicit coordinate dependence.

The derivatives transform, to first order in ε, according to,

$$\frac{D\bar{x}_i}{D\bar{x}_j} = \delta_{ij} - \varepsilon\frac{D\Xi_i}{Dx_j} \qquad (A.5)$$

$$\frac{D\bar{u}_i}{D\bar{x}_j} = u_{i,j} + \varepsilon\left[\frac{D\Pi_i}{Dx_j} - \frac{D\Xi_k}{Dx_j}u_{i,k}\right] \qquad (A.6)$$

Expanding the right side of (A.6) using (A.4) we have

$$\frac{D\bar{u}_i}{D\bar{x}_j} = u_{i,j} + \varepsilon\left[\frac{\partial\Pi_i}{\partial x_j} + \frac{\partial\Pi_i}{\partial u_k}u_{k,j} - \frac{\partial\Xi_k}{\partial x_j}u_{i,k} - \frac{\partial\Xi_k}{\partial u_\ell}u_{\ell,j}u_{i,k}\right] \qquad (A.7)$$

---

[1]See also A. Cohen (1911) p. 40 or Bluman and Cole (1969).

Expression (A.7) gives the first extension of the group (A.2).

The second extension of (A.2) is found from

$$\bar{u}_{i,jk} = u_{i,jk} + \varepsilon\left[\frac{D\Pi_i^{(j)}}{Dx_k} - \frac{DX_\ell}{Dx_k} u_{i,j\ell}\right]$$

$$\Pi_i^{(jk)} = \frac{\partial^2 \Pi_i}{\partial x_j \partial x_k} + \frac{\partial^2 \Pi_i}{\partial x_j \partial u_m} u_{m,k} + \frac{\partial^2 \Pi_i}{\partial x_k \partial u_m} u_{m,j}$$

$$+ \frac{\partial^2 \Pi_i}{\partial u_m \partial u_\ell} u_{m,j} u_{\ell,k} + \frac{\partial \Pi_i}{\partial u_m} u_{m,jk} \qquad \text{(A.8)}$$

$$- \left[\frac{\partial^2 X_m}{\partial x_j \partial x_k} + \frac{\partial^2 X_m}{\partial x_j \partial u_\ell} u_{\ell,k} + \frac{\partial^2 X_m}{\partial x_k \partial u_\ell} u_{\ell,j} + \frac{\partial^2 X_m}{\partial u_\ell \partial u_p} u_{\ell,j} u_{p,k}\right] u_{i,m}$$

$$- \frac{\partial X_m}{\partial u_\ell}\left[u_{i,m} u_{\ell,jk} + u_{\ell,j} u_{i,mk} + u_{\ell,k} u_{i,jm}\right]$$

$$- \frac{\partial X_m}{\partial x_j} u_{i,km} - \frac{\partial X_m}{\partial x_k} u_{i,jm}$$

In general the q-th extension of (A.2) is found by applying the operator

$$\frac{D}{Dx_{j_q}}$$

to the (q-1)th extension and adding the negative term.

$$- \frac{DX_m}{Dx_{j_q}} u_{i,j_1 j_2 \cdots j_{q-1} m} \qquad \text{(A.9)}$$

where

$$\begin{aligned} j_i &= 1, 2, \ldots, n \\ \text{for} \quad i &= 1, 2, \ldots, g \end{aligned} \tag{A.10}$$

For the infinitesimal transformations given by

$$\begin{aligned} \bar{x} &= x + \varepsilon X(x,t,u) \\ \bar{t} &= t + \varepsilon T(x,t,u) \\ \bar{u} &= u + \varepsilon U(x,t,u) \end{aligned} \tag{A.11}$$

the expressions (A.5), (A.6), (A.8) are tabulated below.

$$\begin{aligned} \frac{\partial x}{\partial \bar{x}} &= 1 - \varepsilon\left[X_x + X_u u_x\right] \\ \frac{\partial x}{\partial \bar{t}} &= -\varepsilon\left[X_t + X_u u_t\right] \\ \frac{\partial t}{\partial \bar{t}} &= 1 - \varepsilon\left[T_t + T_u u_t\right] \\ \frac{\partial t}{\partial \bar{x}} &= -\varepsilon\left[T_x + T_u u_x\right] \end{aligned} \tag{A.12}$$

$$U^{(x)} = \left[U_x + (U_u - X_x)u_x - T_x u_t - X_u u_x^2 - T_u u_x u_t\right] \tag{A.13}$$

$$U^{(t)} = \left[U_t + (U_u - T_t)u_t - X_t u_x - T_u u_t^2 - X_u u_t u_x\right] \tag{A.14}$$

$$\begin{aligned} U^{(xx)} = \Big[& U_{xx} + (2U_{xu} - X_{xx})u_x - T_{xx}u_t - 2T_{xu}u_x u_t \\ & + (U_{uu} - 2X_{xu})u_x^2 - X_{uu}u_x^3 - T_{uu}u_x^2 u_t \\ & + (U_u - 2X_x)u_{xx} - 2T_x u_{xt} - 3X_u u_{xx} u_x \\ & - T_u u_{xx} u_t - 2T_u u_{xt} u_x \Big] \end{aligned} \tag{A.15}$$

$$U^{(tt)} = \big[U_{tt} + (2U_{tu} - T_{tt})u_t - X_{tt}u_x - 2X_{tu}u_x u_t + (U_{uu} - 2T_{tu})u_t^2 - T_{uu}u_t^3 - X_{uu}u_t^2 u_x + (U_u - 2T_t)u_{tt} - 2X_t u_{tx} - 3T_u u_{tt}u_t - X_u u_{tt}u_x - 2X_u u_{tx}u_t\big] \tag{A.16}$$

$$U^{(xt)} = \big[U_{xt} + (U_{xu} - T_{xt})u_t + (U_{tu} - X_{xt})u_x + (U_{uu} - X_{xu} - T_{tu})u_x u_t + (U_u - X_x - T_t)u_{xt} - T_{xu}u_t^2 - X_{ut}u_x^2 - T_x u_{tt} - X_t u_{xx} - T_{uu}u_x u_t^2 - X_{uu}u_t u_x^2 - 2X_u u_x u_{xt} - 2T_u u_t u_{xt} - T_u u_x u_{tt} - X_u u_{xx}u_t\big] \tag{A.17}$$

$$U^{(xxx)} = \big[U_{xxx} + (3U_{xxu} - X_{xxx})u_x - T_{xxx}u_t - 3T_{xxu}u_x u_t + (3U_{xuu} - 3X_{xxu})u_x^2 - 3T_{uux}u_x^2 u_t + (U_{uuu} - 3X_{xuu})u_x^3 - X_{uuu}u_x^4 - 3T_{xx}u_{xt} + (3U_{xu} - 3X_{xx})u_{xx} + (3U_{uu} - 2X_{ux})u_{xx}u_x - 3T_{ux}u_{xx}u_t - 6X_{uu}u_x^2 u_{xx} + (U_u - 3X_x)u_{xxx} - T_{uuu}u_x^3 u_t - 6T_{xu}u_x u_{xt} - 3T_{uu}u_{xx}u_t u_x - 3T_x u_{xxt} - 4X_u u_{xxx}u_t - 3X_u u_{xx}^2 - 3T_{uu}u_x^2 u_{xt} - T_u u_{xxx}u_t - 3T_u u_{xx}u_{xt} - 3T_u u_x u_{xxt}\big] \tag{A.18}$$

APPENDIX B

GEOMETRICAL INVARIANCE ALGEBRAS AND SYMMETRY BREAKING IN CLASSICAL MECHANICS

The symmetry principles of nonrelativistic mechanics have traditionally been of great interest to physicists. In this work we have no intention of adding to the multitude of review articles about canonical transformations in phase space. However, we would like to point out that the symmetry principles of Newton's equations of motion lead to useful geometric invariance transformations, which refer directly to particle trajectories. In fact, the space-time transformations of particular mechanics problems often lead to constants of the motion. Occasionally they give exact solutions. Thus, the invariance principles of Newton's equations provide some insight into the character of the motion.

The nature of the symmetry principles which we discuss here is more geometrical than the phase space formalism of mechanics. First of all, we work only in configuration space, so we are interested in the mapping of a space-time flow of possible paths into itself. Our subject is the transformation properties of orbits in configuration space. In the simplest case, we are concerned with families of trajectories whose elements are geometrically similar. This is the sense in which a symmetry group in

configuration space is geometrical in character.

In what follows the free particle in uniform motion in one dimension plays a central role. All of the work is performed in one spatial dimension. The introduction of more spatial dimensions is conceptually straightforward, but it is not done here. The kinematical group of the free particle, i.e., the group of space-time transformations which maps uniform motion into uniform motion, is the planar projective group, also isomorphic to SL(3,R). Under the action of the projective group, straight lines are mapped into straight lines. When an interaction potential is present, the projective symmetry is usually broken.

Consider a particle moving in the presence of an external force which varies spatially according to a power law. The kinematical symmetry group in the general case reduces to a two parameter subgroup of the eight parameter projective group. The kinematical group decreases in size because the warping of the trajectories produced by the external force tends to make them less symmetric. However, in two special cases Newton's equation admits an accidentally larger algebra. In both cases, the extra symmetry occurs because the corresponding family of space-time trajectories is especially symmetrical. The first case is the centrifugal $1/r^3$ force, whose three parameter kinematical group of the projective group is isomorphic to SL(2,R). The second, even more symmetrical case is the linear harmonic oscillator (LHO). The one dimensional LHO

has an eight parameter kinematical algebra, which, when written using the complex normal modes as coordinates, becomes identical to the free particle projective algebra. The introduction of linear damping does not break the structure of the kinematical LHO symmetry, but damping does alter its differential operator realization. In the undamped LHO case, when centrifugal force is present, the symmetry algebra again reduces to SL(2,R).

Our justification for appending such simple examples of invariance analysis to the present work is first of all that the harmonic oscillator treatment (which was obtained in collaboration with Professor Roy Axford) is apparently a new result. Secondly, in the construction of the kinematical symmetry algebras we use the same methods as in the rest of the present article. Thus, the kinematical treatment of classical mechanics constitutes a further illustration of Lie's methods, but this time for ordinary differential equations. Thirdly, the present kinematical classical examples provide a good contrast to the analogous, but qualitatively different use of similar group methods in quantum physics.

A. Forces Obeying a Power Law

Newton's equation of motion for a particle moving under the action of an external force which varies spatially according to a power law is given by

$$y'' + G_n y^{n-1} = 0 \qquad \text{(B.1)}$$

In (B.1) G is a coupling constant and n is the spatial exponent in the potential energy $V(y) = Gy^n$. Newton's equation (B.1) is assumed to admit a Lie group whose infinitesimal variations depend on both the position y and the time x:

$$\bar{y} = y + \varepsilon Y(x,y) \qquad (B.2)$$

$$\bar{x} = x + \varepsilon X(x,y)$$

One could very well allow the coordinate functions to also depend on velocity $v \sim y'$. And indeed, when this is done, an even larger algebra results than the one which will be presented momentarily. However, for our current geometrical considerations, the simple space-time "point" transformations (B.2) rather than more general "contact" transformations depending on velocity, are sufficient. When considering transformations such as (B.2) one should worry about having violated Newton's concept of uniform time. We have admitted transformations of our time variable, called x, which depend on the motion y. In this view, space and time are still absolute (in particular, they are unaffected by the presence of mass). But the time now is somehow connected with the space coordinate. The point to remember is that the transformations (B.2) do not take a space-time point along a trajectory, they take one entire trajectory into a neighboring one. Thus the relation between corresponding points provided by (B.2) is not a dynamical relation concerning the motion along a particular path.

Rather it is a geometrical relationship among corresponding points on different possible paths.

Either by following Appendix A or by simply substituting the barred variables into the equation of motion, we find that

$$\overline{y''} + G_n \overline{y^{n-1}} = y'' + G_n y^{n-1} + \varepsilon \left[Y'' + G_{n(n-1)} Y y^{n-2} \right] \quad \text{(B.3)}$$

where

$$\begin{aligned} Y'' = {} & Y_{xx} + y'(2Y_{yx} - X_{xx}) + y''(Y_y - 2X_x) \\ & + y'^2(Y_{yy} - 2X_{xy}) - y'y''(3X_y) - y'^3 X_{yy} \end{aligned} \quad \text{(B.4)}$$

For invariance, the order ε terms in (B.3) should vanish when the equation of motion is satisfied, i.e.,

$$Y'' + G_{n(n-1)} Y y^{n-2} = \left[P(x,y) + R(x,y) y' \right] (y'' + G_n y^{n-1}) \quad \text{(B.5)}$$

The form of the multiplier in (B.5) can be derived from the classical Lie theory (see Eisenhart's book on continuous groups).

Equating coefficients of like derivatives in (B.5) we obtain the determining equations for the coordinate functions X(x,y) and Y(x,y).

$$\begin{aligned} 1: \quad & Y_{xx} + G_{n(n-1)} y^{n-2} Y = P G_n y^{n-1} \\ y': \quad & 2Y_{xy} - X_{xx} = R G_n y^{n-1} \\ y'': \quad & Y_y - 2X_x = P \\ y'^2: \quad & Y_{yy} - 2X_{xy} = 0 \\ y'y'': \quad & -3X_y = R \\ y'^3: \quad & X_{yy} = 0 \end{aligned} \quad \text{(B.6)}$$

From the last four equations in (B.6) we find the relations, valid for arbitrary exponent n,

$$\begin{aligned} X &= a(x)y + c(x) \\ R &= R(x) = -3a(x) \\ Y &= a'y^2 + (b' + P')y + C(x) \end{aligned} \tag{B.7}$$

where the functions of time a(x), b(x), c(x) are integration functions. Now using the last equation for Y in the second determining equation, we have the relation,

$$3a''y + 3b'' + 2P' = RGny^{n-1} \tag{B.8}$$

Thus, $R \equiv 0$ unless $n = 0, 1, 2$. For an arbitrary value of n, the first determining equation, along with (B.8) and the middle equation in (B.7), gives

$$\begin{aligned} a &= c = 0 \\ P &= \frac{6(n-1)}{n+2}A \quad , \quad A = \text{const} \\ b' &= -\frac{2}{3}P + A \end{aligned} \qquad n \neq -2 \tag{B.9}$$

Thus, for arbitrary exponent n the coordinate functions X and Y are given by

$$\begin{aligned} X &= -\frac{3(n-2)}{n+2}Ax + B \\ Y &= \frac{6}{n+2}Ay \end{aligned} \tag{B.10}$$

From (B.10) the operators follow by setting A, B equal to $\frac{1}{6}(n+2)$ and unity, respectively,

$$A = t\partial_t + (1-\tfrac{n}{2})x\partial_x$$
$$B = \partial_t \qquad (B.11)$$

The operators A and B are the geometrical symmetry operators for Newton's equation of motion under a spatially homogeneous force. The operator B corresponds to time translations. The operator A is a self-similar scaling rule. Namely, under

$$x \longrightarrow e^{\alpha(1-\frac{n}{2})}x$$
$$t \longrightarrow e^{\alpha}t \qquad (B.12)$$

particle trajectories transform among themselves. The invariant of the transformation A gives a relation between corresponding space-time points on geometrically similar trajectories. Namely,

$$\frac{x^{1-\frac{n}{2}}}{t} = \text{const} \qquad (B.13)$$

or

$$\frac{x}{x_0} = \left(\frac{t}{t_0}\right)^{\frac{2}{2-n}} \qquad (B.14)$$

Equation (B.14) is a scaling relation between corresponding times and lengths characterizing different motions under the same power law force. The interpretation of (B.14) for several values of n is tabulated below.

n	1-n/2	Physical Problem	Meaning of (B.14)
2	0	Harmonic motion	Period independent of amplitude
1	1/2	Constant force	Time to fall $\sim$ square root of height
0	1	Zero force	Uniform motion
-1	3/2	Kepler motion	$(\text{Period})^2$ proportional to $(\text{length})^3$
-2	2	Cotes' spirals	$\frac{\text{time}}{(\text{length})^2}$ characterizes the orbit

Of course, these particular results are standard and were known even to Newton (as cited by Whitaker, 1904, p. 47). We have, however, shown that, up to time translations, expression (B.14) represents the only relation between different orbits. The operators A and B in (B.11) close under commutation to form a two parameter Lie algebra of the third type.[1] When rewritten in terms of Lie's canonical variables the ordinary differential equation (B.1) is integrable by quadratures.[2]

Let us proceed to discuss the special values n = 0, 1, $\pm$ 2, for which extra symmetry occurs. First of all, for free particle motion (n = 0) the group analysis after (B.8) takes a different turn, resulting in coordinate functions

$$X = (A+Bx)y + Ex^2 + (2F-D)x + H$$
$$Y = By^2 + (Ex+F)y + Ix + J \qquad \text{(B.15)}$$

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[1] A. Cohen (1911) p. 159.

[2] A. Cohen (1911) p. 171.

where the eight integration constants A, B, D, E, F, H, I, J are the group parameters of the kinematic symmetry group for uniform motion. The infinitesimal generators and their commutation relations are given in Table B.2 and Table B.1 respectively. The group corresponding to the coordinate functions (B.15) has the finite transformations,

$$\bar{x} = \frac{a_1 x + a_2 y + a_3}{a_7 x + a_8 y + a_9} \quad , \quad \bar{y} = \frac{a_4 x + a_5 y + a_6}{a_7 x + a_8 y + a_9} \tag{B.16}$$

Thus, the expressions (B.15) are the coordinate functions for the classical projective group in the x-y plane. The planar projective group, which is isomorphic to SL(3,R), maps straight lines into straight lines and therefore takes the family of uniform motions into itself. Included in the projective group are operations of scaling, rotation, translation, and of course, projections in the x-y plane.

When n = 1 the external force is uniform in space and constantly accelerating motion results. The group analysis yields coordinate functions X and Y which are the same as (B.15), but with additional polynomial terms in the time variable. The eight group parameters remain the same, and the resulting algebra is the same as the projective algebra. Since no modification of the group structure occurs, the results for constantly accelerating motion are omitted.

When n = -2, corresponding to Cotes' spiral motion, where y is now a radial coordinate (Whittaker, p. 83), an intriguing extra geometrical symmetry emerges. The

	A	B	C	D	E	F	G	H
A	-	0	-B	-A	ωB	$-\frac{1}{2}(C+\frac{E}{\omega})$	$-\frac{1}{2}(D+3H)$	-A
B	0	-	-A	B	ωA	$\frac{1}{2}(D-3H)$	$\frac{1}{2}(-C+\frac{E}{\omega})$	-B
C	B	A	-	$-\frac{2E}{\omega}$	-2ωD	-G	-F	0
D	A	-B	$\frac{2E}{\omega}$	-	2ωC	F	-G	0
E	-ωB	-ωA	2ωD	-2ωC	-	ωG	-ωF	0
F	$\frac{1}{2}(C+\frac{E}{\omega})$	$-\frac{1}{2}(D-3H)$	G	-F	- G	-	0	F
G	$\frac{1}{2}(D+3H)$	$\frac{1}{2}(C-\frac{E}{\omega})$	F	G	F	0	-	G
H	A	B	0	0	0	-F	-G	-

Table B.1. Commutator Table for the Real Basis of the Harmonic Oscillator Algebra.

Table B.2

Harmonic Oscillator Algebra		Projective Algebra	
Infinitesimal Operator	**Finite Transformation**	**Infinitesimal Operator**	**Finite Transformation**
$Z = \frac{a\partial_a + b\partial_b}{2b}$	$a_\varepsilon = \frac{a_0}{b_0}(b_0 + \frac{\varepsilon}{2})$ $b_\varepsilon = (b_0 + \frac{\varepsilon}{2})$	∂_y	$x_\varepsilon = x_0$ $y_\varepsilon = y_0 + \varepsilon$
$W = 2a^2\partial_a$	$a_\varepsilon = \frac{a_0}{1-2\varepsilon a_0}$ $b_\varepsilon = b_0$	$x(x\partial_x + y\partial_y)$	$x_\varepsilon = \frac{x_0}{1-\varepsilon x_0}$ $y_\varepsilon = \frac{y_0}{1-\varepsilon x_0}$
$X = \frac{a^2}{b}\partial_a$	$a_\varepsilon = \frac{a_0}{1-\varepsilon\frac{a_0}{b_0}}$ $b_\varepsilon = b_0$	$x\partial_y$	$x_\varepsilon = x_0$ $y_\varepsilon = y_0 + \varepsilon x_0$
$Y = b\partial_b$	$a_\varepsilon = a_0$ $b_\varepsilon = e^\varepsilon b_0$	$y\partial_y$	$x_\varepsilon = x_0$ $y_\varepsilon = e^\varepsilon y_0$

Table B.2 continued

Harmonic Oscillator Algebra		Projective Algebra	
Infinitesimal Operator	Finite Transformation	Infinitesimal Operator	Finite Transformation
$Z^* = \frac{a\partial_a + b\partial_b}{2a}$	$a_\varepsilon = (a_0 + \frac{\varepsilon}{2})$ $b_\varepsilon = \frac{b_0}{a_0}(a_0 + \frac{\varepsilon}{2})$	∂_x	$x_\varepsilon = x_0 + \varepsilon$ $y_\varepsilon = y_0$
$W^* = 2b^2\partial_b$	$a_\varepsilon = a_0$ $b_\varepsilon = \frac{b_0}{1-2\varepsilon b_0}$	$y(x\partial_x + y\partial_y)$	$x_\varepsilon = \frac{x_0}{1-\varepsilon y_0}$ $y_\varepsilon = \frac{y_0}{1-\varepsilon y_0}$
$X^* = \frac{b^2}{a}\partial_b$	$a_\varepsilon = a_0$ $b_\varepsilon = \frac{b_0}{1-\varepsilon\frac{b_0}{a_0}}$	$y\partial_x$	$x_\varepsilon = x_0 + \varepsilon y_0$ $y_\varepsilon = y_0$
$Y^* = a\partial_a$	$a_\varepsilon = e^\varepsilon a_0$ $b_\varepsilon = b_0$	$x\partial_x$	$x_\varepsilon = e^\varepsilon x_0$ $y_\varepsilon = y_0$

Table B.2. The Comparison between the Complex Extended HO Algebra and the Projective Algebra.

corresponding coordinate functions are given by (B.15) with E, G, H nonzero.

$$\begin{aligned} \mathcal{X} &= Ex^2 + 2Fx + H \\ \mathcal{Y} &= (Ex + F)y \end{aligned} \tag{B.17}$$

The differential operators which appear when E = 1, F = -1/2, H = 1 are called J_-, J_0, J_+ respectively in order to suggest their commutation relations,

$$[J_0, J_\pm] = \pm J_\pm \quad , \quad [J_+, J_-] = -2J_0 \tag{B.18}$$

where

$$\begin{aligned} J_- &= xy\partial_y + x^2\partial_x && \text{Projective transformations} \\ J_0 &= -\tfrac{y}{2}\partial_y - x\partial_x && \text{Scaling: } y^2/x = \text{const} \\ J_+ &= \partial_x && \text{Time translations} \end{aligned} \tag{B.19}$$

The commutation relations (CR) in (B.18) are SL(2,R) CR which are the same as SU(2) CR, but with one wrong sign. The SU(2) CR can be obtained by relabeling (J_+, J_0, J_-) by $(J_-, -J_0, -J_+)$. In any case, the operators (B.19) generate projective transformations. The action generated by J_- (for E = 1) leaves y/x invariant. The finite transformations of J_- are given by

$$\begin{aligned} x_\varepsilon &= \frac{x_0}{1 - \varepsilon x_0} \\ y_\varepsilon &= \frac{y_0}{1 - \varepsilon x_0} \end{aligned} \tag{B.20}$$

where the subscripts refer to the value of ε. The group

parameter ε vanishes at the untransformed points. The transformations generated by J_o and J_+ are identified in (B.19).

Finally, when n = +2, in the case of harmonic vibration, the equation of motion is given by

$$y'' + \omega^2 y = 0 \tag{B.21}$$

In the group analysis of the harmonic oscillator equation the integration functions a(x), b(x), c(x), P(x), R(x) are given by

$$\begin{aligned} a(x) &= A \sin \omega x + B \cos \omega x \\ b(x) &= \frac{D}{2\omega} \sin 2\omega x - \frac{C}{2\omega} \cos 2\omega x + E \\ c(x) &= F \sin \omega x + G \cos \omega x \\ P(x) &= -\tfrac{3}{2} b'(x) + H \\ R(x) &= -3a(x) \end{aligned} \tag{B.22}$$

There are eight integration constants in the coordinate functions, corresponding to an eight parameter Lie invariance algebra with parameters A through H. A differential operator basis is given by

$$
\begin{aligned}
A &= \frac{y}{\omega}\sin\omega x\,\partial_x + y^2\cos\omega x\,\partial_y \\
B &= -\frac{y}{\omega}\cos\omega x\,\partial_x + y^2\sin\omega x\,\partial_y \\
C &= -\frac{1}{\omega}\cos 2\omega x\,\partial_x + y\sin 2\omega x\,\partial_y \\
D &= \frac{1}{\omega}\sin 2\omega x\,\partial_x + y\cos 2\omega x\,\partial_y \\
E &= \partial_x \\
F &= \sin\omega x\,\partial_y \\
G &= \cos\omega x\,\partial_y \\
H &= y\partial_y
\end{aligned}
\tag{B.23}
$$

The notation for the operators in (B.23) corresponds to the nonzero group parameter involved. The action of the operator H is to scale the amplitude of oscillation. H is admitted mathematically because Newton's equation for harmonic motion is homogeneous in the amplitude y. The operators F and G correspond to superposition of solutions, admitted because (B.1) is linear. E is the time translation symmetry operator, which arises because x does not appear explicitly in (B.1). Each of the other operators A, B, C, D, acts on the phase of the oscillation. The operators A and B, respectively, shift the cotangent and the tangent of the phase angle $\varphi = x$, and produce nonlinear transformations on the amplitude y. The invariant quantities under A and B are respectively, $\frac{y}{\sin\omega x}$ and $\frac{y}{\cos\omega x}$. Thus, A and B make phase changes, but preserve the amplitudes of the

normal modes of oscillation. The operators C and D essentially scale the tangent of the phase angle and the amplitude, but they preserve the quantities $\frac{y^2}{\cos 2\omega x}$ and $\frac{y^2}{\sin 2\omega x}$.

The eight parameter dynamical algebra of point transformations for the harmonic oscillator closes completely. Its commutator table is given as Table B.1.

It is natural to inquire about the structure of this algebra. It turns out that the complex extension of the HO dynamical algebra is isomorphic to the projective algebra of classical mathematics, i.e., the complex extended algebra is isomorphic to SL(3,R). Remember that the projective algebra is the symmetry algebra for a free particle $y'' = 0$.

A differential operator basis for the complex extension of the HO algebra is given by the eight operators.

$$
\begin{aligned}
Z_1 &= \tfrac{1}{2}(G+iF) = \tfrac{1}{2} e^{i\omega x}\partial_y \\
W &= A+iB = y e^{i\omega x}(y\partial_y - \tfrac{i}{\omega}\partial_x) \\
X &= \tfrac{1}{2}(D+iC) = \tfrac{1}{2} e^{2i\omega x}(y\partial_y - \tfrac{i}{\omega})_x) \\
Y &= \tfrac{1}{2}(H+iE) = \tfrac{1}{2}(y\partial_y + \tfrac{i}{\omega}\partial_x)
\end{aligned}
\qquad \text{(B.24)}
$$

plus their complex conjugators, Z*, W*, X*, Y*. The practical usefulness of the complex extension algebra is that its finite transformations may be calculated more easily than those of the real basis. In terms of the invariants of the complex extension algebra, namely the

normal modes,

$$a = y e^{i\omega x}$$
$$b = y e^{-i\omega x} = a^* \qquad \text{(B.25)}$$

the operator basis (B.24) takes on a simple polynomial form. For example,

$$W = z a^2 \partial_a \ , \ X = \frac{a^2}{b} \partial_a \ , \ Y^* = a \partial_a \qquad \text{(B.26)}$$

In Table B.2 the differential operator bases and the finite transformations for the complex extended HO algebra and the projective algebra are displayed side by side, according to one-parameter subalgebras.

When the operator correspondence of Table B.2 is made between the algebras, the commutator tables for the complex extended HO algebra and the projective algebra are precisely the same. The two algebras in Table B.2 also have the same set of invariant forms, namely (a, b, a/b) and (x, y, x/y). However, corresponding operators only have the same form as the invariant when they are identical. The commutator table for these two isomorphic algebras is given as Table B.3.

It is interesting to notice at this point that the SL(3,R) algebra is the same as minus the SU(3) algebra, except for four essential "wrong" signs. A correspondence can be made between the SL(3,R) algebra and the SU(3) algebra by multiplying the generators of SL(3,R) by appropriate factors of $\sqrt{-1}$. We mention the unitary algebra

	Z	W	X	Y	Z*	W*	X*	Y*
Z	-	X	0	Z	0	(2Y+Y*)	Z*	0
W	-X	-	0	0	-(2Y*+Y)	0	-W*	-W
X	0	0	-	X	-Z	W	-(Y-Y*)	-X
Y	-Z	0	-X	-	0	W*	X*	0
Z*	0	(2Y*+Y)	Z	0	-	X*	0	Z*
W*	-(2Y+Y*)	0	-W	-W*	-X*	-	0	0
X*	-Z*	W*	(Y-Y*)	-X*	0	0	-	X*
Y*	0	W	X	0	-Z*	0	-X*	-

Table B.3. The Commutator Table for the Complex Extended HO Algebra and the Projective Algebra SL(3,*R*).

SU(3) because it is usually seen in conjunction with the three dimensional harmonic oscillator, where it is the time independent phase space symmetry group of the Hamiltonian, rather than the symmetry group of the equations of motion. In contrast, the point transformation group (B.24) is a mapping of one x-y space into another, for which the change in the definition of time is crucial. In particular, the operators W and W* are nonlinear transformations for which the new value of the time depends on the motion. For example, using Table B.2, the operator W infinitesimally is seen to transform the time according to

$$x_\varepsilon = x_o + \frac{\varepsilon}{i\omega} y_o e^{-i\omega x_o} \tag{B.27}$$

where x_o, y_o refer to the original ($\varepsilon = 0$) point. The nonlinear transformations induced by W and W* cannot be found by using linear operators in the solution space. The operators W and W* operating in the (x,y) plane, replace y by $y^2 f(x)$, and so induce nonlinear transformations on y. In addition, the transformations W and W* are not canonical. Since they are special to the harmonic oscillator, they will not leave invariant every canonical equation of motion in Hamilton's form. However, because the algebra structure which includes W and W* refers directly to the dynamics of the harmonic oscillator, the energy integral for the motion of the oscillator follows directly from the algebra.

The equation of motion may be rewritten in terms of the invariants of the algebra. The invariant and 1st and

2nd differential invariant of (B.24) are

$$a = y e^{i\omega x}$$

$$a' = y' e^{i\omega x} \tag{B.28}$$

$$a'' = y'' e^{i\omega x}$$

along with their complex conjugates. The harmonic oscillator equation clearly admits (B.24), since it can be written in terms of the invariants of the algebra. Now taking the ratio of the time derivatives of a and a' we find

$$a' - i\omega a = (y' - i\omega y) e^{i\omega x} = \text{const} \tag{B.29}$$

And by complex conjugation,

$$b' + i\omega b = (y' + i\omega y) e^{-i\omega x} = \text{const} \tag{B.30}$$

Thus, the Dirac variables appear, y' ± iωy, and by multiplying together (B.29) and (B.30) the energy integral obtains,

$$y'^2 + \omega^2 y^2 = 2E = \text{const} \tag{B.31}$$

whence the solution preceeds as usual.

In the invariant phase space (a', a) ⊗ (b', b) the orbits of the solution are straight lines,

$$\frac{da'}{da} = -\frac{db'}{db} = i\omega \tag{B.32}$$

Physically, this means that the normal modes are decoupled (which follows also from the superposition principle).

During the course of the motion, as time progresses the Dirac variables change phase in opposite directions.

Of course, the derivation presented here is not rigorous, because the first integral is obtained from the solution for the normal modes. However, a completely analogous procedure is valid in more usual cases where the symmetry group is obtained without first solving the equations of motion!

B. Damped Harmonic Motion

The introduction of linear damping does not break the harmonic algebra symmetry. For the equation of motion

$$y'' + 2\beta y' + \omega^2 y = 0 \tag{B.33}$$

an exactly analogous calculation to the one presented for the homogeneous force law results in an operator algebra whose real form is given by

$$\begin{aligned}
A &= \frac{y e^{\beta x}}{\Delta} \sin \Delta x \, \partial_x + y^2 e^{\beta x} (\cos \Delta x - \tfrac{\beta}{\Delta} \sin \Delta x) \partial_y \\
B &= -\frac{y e^{\beta x}}{\Delta} \cos \Delta x \, \partial_x + y^2 e^{\beta x} (\sin \Delta x + \tfrac{\beta}{\Delta} \cos \Delta x) \partial_y \\
C &= -\frac{1}{\Delta} \cos 2\Delta x \, \partial_x + y (\sin 2\Delta x + \tfrac{\beta}{\Delta} \cos 2\Delta x) \partial_y \\
D &= \frac{1}{\Delta} \sin 2\Delta x \, \partial_x + y (\cos 2\Delta x - \tfrac{\beta}{\Delta} \sin 2\Delta x) \partial_y \\
E &= \partial_x - \beta y \partial_y \\
F &= e^{-\beta x} \sin \Delta x \, \partial_y \\
G &= e^{-\beta x} \cos \Delta x \, \partial_y \\
H &= y \partial_y
\end{aligned} \tag{B.34}$$

where $\Delta = \sqrt{\omega^2 - \beta^2}$. The basis (B.34) generates mappings of the (x,y) plane into itself, for $\omega^2 \neq \beta^2$. Each of these operators limits to a corresponding operator for the undamped harmonic oscillator when the damping coefficient, β, tends to zero. And the geometrical meaning of each operator is analogous to its undamped counterpart. Time translations are contained in the basis (B.34) in the form $\partial_x = E + \beta H$. However (B.34) is written to exhibit the connection to the algebra of the undamped harmonic motion.

The complex extension of (B.34) is defined by the operator combinations (B.24) as before. The invariants of the complex extended algebra are given by

$$a = y e^{(i\Delta+\beta)x}$$

$$b = y e^{(-i\Delta+\beta)x} = a^* \qquad \text{(B.35)}$$

In terms of the invariants a and b, the dynamical algebra of the damped harmonic oscillator has precisely the form given in Table B.2. Thus the algebras for the damped and undamped oscillators are both isomorphic to the projective transformations in the (x,y) plane and to each other.

As a proof of its invariance under (B.24), the damped oscillator equation of motion may be rewritten in terms of the invariants of its dynamical algebra. The invariants are

$$a = y e^{(i\Delta+\beta)x}$$

$$a' = y' e^{(i\Delta+\beta)x} \qquad \text{(B.36)}$$

$$a'' = y'' e^{(i\Delta+\beta)x}$$

along with their complex conjugates. The invariant expression,

$$a'' + 2\beta a' + \omega^2 a = e^{(i\Delta+\beta)x}(y'' + 2\beta y' + \omega^2 y) \qquad \text{(B.37)}$$

makes it clear that the damped harmonic oscillator equation is invariant under the complex extension of (B.34). The result of dividing $\frac{da'}{dx}$ by $\frac{da}{dx}$ is simply,

$$a' - (i\Delta+\beta)a = \left[y' - (i\Delta+\beta)y\right] e^{(i\Delta+\beta)x} = \text{const} \qquad \text{(B.38)}$$

Hence, taking the product of (B.38) with its complex conjugate, we have the constant of motion,

$$\left(\tfrac{1}{2}y'^2 + \beta y y' + \tfrac{1}{2}\omega^2 y^2\right) e^{2\beta x} = E_0 = \text{const.} \qquad \text{(B.39)}$$

The expression (B.39) is an explicitly time dependent first integral of the damped motion. According to (B.39) the oscillator energy falls exponentially with time and linearly with an effective frictional work term. For initial conditions $y(0) = A$, $y'(0) = 0$, we have $E_0 = \frac{\omega^2 A^2}{2}$. In this way, the initial potential energy of the oscillator is dissipated by friction.

We have derived an eight parameter Lie invariance algebra for the damped harmonic oscillator and have shown that it is isomorphic to the dynamical symmetry algebra of the undamped motion. In fact, the algebras are identical when written in terms of their respective invariants. In addition, a constant of the motion was found directly for the damped harmonic oscillator which is the analogue of the Hamiltonian for undamped motion. At every step the undamped case is the uniform limit of the damped motion.

Let us finally discuss the case of undamped harmonic motion including a centrifugal force. The equation of motion is

$$y'' - \frac{L}{y^3} + \omega^2 y = 0 \tag{B.40}$$

While linear damping modifies the LHO algebra but does not actually break it, the centrifugal term scales differently than the others in (B.40), and thus breaks the LHO algebra. The group analysis, which is again straight forward but tedious, produces three-parameter coordinate functions,

$$\begin{aligned} X &= \frac{D}{2\omega}\sin 2\omega x - \frac{C}{2\omega}\cos 2\omega x + E \\ Y &= \frac{D}{2} y \cos 2\omega x + \frac{C}{2} y \sin 2\omega x \end{aligned} \tag{B.41}$$

The corresponding differential operators are

$$\begin{aligned} D &= \frac{1}{\omega}\sin 2\omega x\, \partial_x + y\cos 2\omega x\, \partial_y \\ C &= -\frac{1}{\omega}\cos 2\omega x\, \partial_x + y \sin 2\omega x\, \partial_y \\ E &= \partial_x \end{aligned} \tag{B.42}$$

Clearly the eight parameter LHO algebra has been reduced to only three of its original members. The analogous reduction occurs between uniform motion and Cotes' spiral motion. In both cases the breaking of the original symmetry reduces it to a subgroup, rather than completely changing its structure, e.g., by contracting it. Referring to (B.23) we see that, with the introduction of centrifugal force, the invariances under scaling, H, and superposition, F and G, along with the motion-dependent transformations A and B, have all been lost. Remaining are time translations and phase scaling of two kinds. Identifying

$$\left\{ \frac{C/\sqrt{2}}{} \, , \, \frac{E}{2\omega} \, , \, \frac{A/\sqrt{2}}{} \right\}$$

with the SU(2) angular momentum operators $\{J_+, J_0, J_-\}$ we see that, again as in the case of free motion, including centrifugal force reduces the symmetry from SL(3,R) to SL(2,R). For ready comparison the SU(2) and SL(2,R) commutator tables are written down. Note the difference in signs at opposite corners.

	J_+	J_0	J_-
J_+	0	$-J_+$	$2J_0$
J_0	J_+	0	$-J_-$
J_-	$-2J_0$	J_-	0

SU(2) CR

	J_+	J_0	J_-
J_+	0	$-J_+$	$-2J_0$
J_0	J_+	0	$-2J_-$
J_-	$2J_0$	J_+	0

SL(2,R) CR

APPENDIX C

LIE'S INVARIANCE METHOD

Lie's invariance method for the integration of differential equations is summarized in this appendix. The history of the development and application of Lie theory to physical problems is surveyed. The physics background which has led to a revival of Lie invariance analysis is given and the connection with modeling theory is described. By referring to the Euler fluid example of Appendix A, the method of Lie is shown to be a natural generalization of the method of self-similar solutions. The deduction of invariance principles from given dynamics is implemented in an example using the wave equation. The known conformal invariance algebra of the wave equation is constructed and realized as a differential operator algebra. Sample group reducible solutions - which include d'Alembert's solution - are derived from the operator algebra. Lie's method is also demonstrated for the boundary value problem.

A. Historical Development in Physics

In this work some of the properties of real gas flows are derived from the invariance properties of their equations of motion. The infinitesimal group methods which are

used originally appeared in nineteenth century geometry. These methods stem directly from the work of the Norwegian mathematician Marius Sophus Lie. About a century ago, Lie, following Euler and in collaboration with Klein, was concerned with the use of local geometry to integrate differential equations.[1] The underlying assumption for Lie's integration method is that there should exist motions which have the same local invariance properties as the dynamics. The primary technique for integration is the method of characteristics, Lagrange's method.

Since Lie's investigations one hundred years ago, infinitesimal variations and Lie algebra methods have completely permeated physics, both in analysis and in interpretation. No attempt to chronicle the dissemination of invariance analysis in physics is presented here. However, a rather extensive bibliography has been included in order to provide a guide to the literature. The applications of symmetry analysis are manifold. In fact, subsequent developments have extended symmetry methods far beyond Lie's purely geometrical intent.

As for the line of development which <u>did</u> follow Lie's intent, namely the application of transformation groups to solve partial differential equations, little was accomplished until about 1950. Purely mathematical reviews and

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[1]S. Lie, "Allgemeine Untersuchungen über Differentialgleichungen, die eine Kontinuierliche endliche Gruppe Gestatten," Vol <u>VI</u>, Abh III, pp. 139-223.

applications along this line were given, e.g., by Cohen (1911), Dickson (1924), Eisenhart (1933), Franklin (1927), and Campbell (1903).[2] However, in the first half of the twentieth century Lie's work had little impact in physics. This happened because the most popular problems in physics then were linear, and more powerful mathematical tools than Lie's methods were used by physicists, namely spectral theory and group representation theory. General relativity, particularly Weyl's contribution,[3] is perhaps an exception, but nothing will be said about it here.

The theory of continuous groups was probably first applied to the construction of invariant solutions of specific partial differential equations in physics by Garrett Birkhoff (1950). Birkhoff discussed hydrodynamics, using one parameter invariance groups, which he found by inspection. Morgan (1951), in his doctoral dissertation, and Michal (1952), who was Morgan's thesis advisor, later gave formal methematical conditions, under which an invariance group may be used to reduce the number of independent variables in a partial differential equation. Michal and Morgan's results generalized the similarity method. Morgan applied invariance group conditions to solve two nonlinear problems in boundary layer theory.

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[2]The detailed references appear in the bibliography.

[3]H. Weyl, "Gravitation and the Electron," Proc. Nat. Acad. Sci. 15, 323 (1929).

After Morgan and Michal, little was done explicitly with Lie's methods for almost a decade. In 1962 the Soviet mathematician L. V. Ovsjannikov[4] again applied Lie algebras to the hydrodynamic equations. Ovsjannikov used Lie's invariance analysis to classify the solutions of Euler's equations for an ideal gas. There followed a flurry of Soviet publications amplifying Ovsjannikov's work and extending it to other problems, mainly in fluid flow and gas dynamics. The publications in this field after Ovsjannikov are documented in the bibliography.

In order to appreciate the modern interest in Lie's classical methods for partial differential equations, a brief digression to explain the physical background is necessary. First of all, we are dealing with nonlinear physics. In fact, it is the nonlinear experimental behavior which is most intriguing. In certain media, for example, the nonlinear response to an excitation can be tuned to the medium relaxation characteristics, thereby producing a stable propagating signal, as in the self-induced transparency effect. In other experiments, nonlinear wave "shocking up" is balanced by dispersion or dissipation in the medium, again leading to stable signal propagation. Fascinating nonlinear effects are outstanding experimental features of several modern fields of physics. Calculationally, however, the presence of important nonlinear effects

[4]L. V. Ovsjannikov, Gruppovye Svoystva Differentsialny Uravneni (Novosibirsk, 1962), (Group Properties of Differential Equations, G. Bluman, transl. 1967).

means that the available analytical tools are comparatively weak and primitive. The nonlinear effects themselves are of main interest, so linearization techniques can produce only limited results. However, because of the mathematical difficulties inherent in the nonlinear description, only a few special cases have been treated analytically. Consequently nonlinear wave phenomena are not well understood theoretically. See Scott, et. al., (1973) for an excellent review of both the theoretical and experimental status of idealized nonlinear wave descriptions.

It is traditional in physics to invoke symmetry principles as a first step toward a more complete understanding. In the case at hand Lie's invariance analysis comes forward. Knowledge of the symmetry structure of physical laws tends to reduce arbitrariness and relate previously unconnected concepts. For the nonlinear problems of current interest, the symmetry structure also provides a tool for solution of the equations of motion. Invariance analysis expedites the calculation of particular solutions by reducing the number of independent variables. As well as yielding solutions to particular problems, group invariant solutions represent general classes of motions.

B. Modeling Theory

Before explaining Lie's program for invariance analysis, we should comment on another, rather parallel, line of reasoning which appears in fluid dynamics. This is modeling theory, whose principles for hydrodynamics were first

proposed by Helmholtz (1873), almost simultaneously with the publication of Lie's theory of integration. In Helmholtz's paper all of the modeling parameters admitted by the continuity and momentum equations of (isothermal) hydrodynamics are given. These parameters were later named in honor of Reynolds, Froude, Euler and Mach.

In the present work the physical description of the fluid, which includes the transport coefficients and constitutive relations such as the equation of state, is to be invariant. The transformations of interest to us relate different possible states of the same fluid. These transformations will be used to construct solutions which have the same symmetry as the equations of motions. The transformations of modeling theory differ in this respect. They are applied to an existing solution, for a fluid flow, say, in order to find equivalent solutions, but for other fluids. General transformations of modeling theory were established by Bateman (1938, 1944), Haar (1928), Tsien (1939), and Prim (1949). Further developments have been made more recently, and are referenced by Castell and Rogers (1974).

The transformations of interest to us here relate different possible states of the same fluid flow. In fact, we are interested in solutions to the equations of motion which have the same invariance properties as the dynamics. A useful class of solutions of this kind in fluid dynamics can be found using dimensional analysis. Dimensional reasoning relies on the premise that the motion cannot

depend on the choice of units, and thus can be represented as a set of relations among dimensionless quantities. The resulting motions are described by similarity solutions. Similarity solutions exist when the physical laws are scale-invariant and the effects of the subsidiary conditions limit uniformly to zero. Often they are more easily found than the general solutions, because the similarity relationship reduces the number of independent variables by one.

C. Laplace's Equation

Another example of the simplification due to invariance properties of the dynamics, is provided by Laplace's equation for electrostatics:

$$\nabla^2 \Phi = 0 \qquad \text{(C.1)}$$

The Laplacian is invariant under rotations, so one can look for solutions which are also rotationally invariant. They will, of course, only depend on the spherical radius, r. Consequently, the angular differentiations in the Laplacian will vanish when they operate on $\Phi(r)$, and Laplace's equation will reduce to an ordinary differential equation in r. The solution $\Phi(r)$ found this way will be an invariant solution under rotations, because it does not depend on angle. It is the $\ell = 0$ radial solution for $r \neq 0$.

$$\Phi(r) = A + \frac{B}{r} \quad , \quad A, B \text{ const.} \qquad \text{(C.2)}$$

Laplace's equation is also invariant under the scaling law,

$$r \to r' = \lambda r$$
$$\Phi \to \Phi' = \lambda^{\ell} \Phi \qquad \text{(C.3)}$$

where ℓ is any constant and λ is the scale parameter. If Φ is not invariant under rotations, but it is to transform according to the radial scale change as above, it may have the functional form

$$\Phi = r^{\ell} Y(\theta, \varphi) \qquad \text{(C.4)}$$

This form leads to a partial separation of variables and evidently to the definition of the spherical functions (cf. Jackson, 1965, p. 55).

The Laplacian example is a good illustration of the difference between finding invariant solutions and another often-used tool of physicists for solving partial differential equations, namely separation of variables. Invariance analysis is generally less powerful than separation of variables, because it depends on a local rather than a global property of the solution space. The solutions of Laplace's equation form a multidimensional (linear) manifold in the coordinate metric space. Separation of variables is possible whenever the multidimensional solution manifold is really a direct-product space of one-dimensional manifolds. On the other hand, invariance analysis applies whenever an invariant manifold exists within the

solution space. The difference is between a separation of the solution space into a direct product of one-dimensional subspaces and a reduction of the solution space to an invariant manifold embedded within it. Sometimes the existence of an invariance algebra is a clue that the space of solutions may be completely separated, but this is not always true. The examples of invariant solutions cited above - similarity solutions for fluid dynamics and radial solutions of the Laplace equation - are constructed from very simple transformations on the dynamics. In one case, under simultaneous scaling of x and t the Euler equations are invariant (up to a constant). And in the other case, under rotations and a scaling transformation, Laplace's equation is invariant. The first symmetry of Laplace's equation occurs because the Laplacian operator is a dot product,

$$\nabla^2 = \underset{\sim}{\nabla} \cdot \underset{\sim}{\nabla} \qquad (C.5)$$

The second symmetry of Laplace's equation arises because Laplace's equation is homogeneous in r and Φ, and the coordinates θ and φ are independent of linear dimensions.

The remaining non-trivial property of Laplace's equation, namely conformal invariance, involves more subtle operations than scaling and rotation. Conformal invariance for Laplace's equation (even in n dimensions) is well known. The existence of conformal invariance is evidence for invariance transformations which are more general than the

simplest geometrical operations. These more general symmetries lead to other classes of invariant solutions. For their construction, a sufficiently general and systematic transformation method is useful. Such a method is provided by the classical theory of Lie.

D. Lie Groups

Lie group theory deals with first order variations of local coordinates in an N-dimensional space E^N. The coordinates of E^N are of two types,

$$x = (x_1, x_2, \ldots, x_n) \qquad \text{(n components)} \qquad \text{(C.6)}$$

and

$$u = (u_1, u_2, \ldots, u_m) \qquad \text{(m components)} \qquad \text{(C.7)}$$

where $n + m = N$, the dimension of E^N. The space E^N is a differentiable metric space, but it need not be linear. The requirement that E^N be differentiable implies that near any point (x_o, u_o), functions on E^N may be expanded in a Taylor series. The first order terms in the Taylor series define a linear vector field near (x_o, u_o). Lie's mathematical goal was to characterize all possible classes of geometry on spaces E^N according to the finite dimensional groups of motion admitted by E^N.

A Lie group G of point transformations on E^N is essentially a mapping of E^N into itself, which depends continuously on a space of parameters. The finite action of G on points in E^N is given by

$$\bar{x}_i = f_i(x,u;\alpha) \quad \text{with} \quad x_i = f_i(x,u;\alpha=0) \tag{C.8}$$

and

$$\bar{u}_j = g_j(x,u;\alpha) \quad \text{with} \quad u_j = g_j(x,u;\alpha=0) \tag{C.9}$$

where

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r) \tag{C.10}$$

denotes the general element of the group by its set of r parameters (group coordinates). The functions f_i and g_j are analytic functions of (x,u) and α. Assume now that f_i and g_j form a realization of the point transformation group, i.e., assume that the group axioms are satisfied by the system (C.8) - (C.10).

Let us look at the infinitesimal transformations of points in E^N. Expand (C.8) and (C.9) in a Taylor series in the group parameters. To first order in α, one obtains,

$$\bar{x}_i = x_i + \left.\frac{\partial F_i}{\partial \alpha_\nu}\right|_{\alpha=0} \alpha_\nu = x_i + \alpha_\nu X_{i\nu}(x,u) \tag{C.11}$$

$$\bar{u}_j = u_j + \left.\frac{\partial g_j}{\partial \alpha_\nu}\right|_{\alpha=0} \alpha_\nu = u_j + \alpha_\nu U_{j\nu}(x,u) \tag{C.12}$$

where

$$\begin{aligned} i &= 1, 2, \ldots, n \\ j &= 1, 2, \ldots, m \\ \nu &= 1, 2, \ldots, r \end{aligned} \tag{C.13}$$

and the Einstein sum rule on repeated indices is employed. Thus, the infinitesimal action of the Lie group G induces vector fields X_{iv} and U_{jv} on E^N. The functions X_{iv} and U_{jv} are called the "coordinate functions" of the group. They correspond to independent infinitesimal variations of x_i and u_j.

The coordinate functions describe the local action of the r parameter Lie group G on the N dimensional space of points (x, u), to first order in the group parameters. Equivalently, the local action of G can be realized on E^N by the differential operators

$$Q_v = X_{iv} \frac{\partial}{\partial x_i} + U_{jv} \frac{\partial}{\partial u_j} \qquad \text{(C.14)}$$

These "directional derivative" operators, also called "Lie derivatives," are the generators of the group action on functions over E^N. The characteristics of the set Q are the group trajectories of G acting on points in E^N. Thus the infinitesimal vector with components (X_{iv}, U_{jv}) is tangent at (x, u) to the group trajectory generated by Q_v which passes through (x, u). The group trajectories in E^N can be found by integrating the characteristic equations of Q_v.

$$\frac{dx_1}{X_{1v}} = \frac{dx_2}{X_{2v}} = \cdots = \frac{dx_n}{X_{nv}} = \frac{du_1}{U_{1v}} = \cdots = \frac{du_m}{U_{mv}} = d\alpha_v \qquad \text{(C.15)}$$

For a finite dimensional group, the set Q closes under commutation to form the r parameter Lie algebra L, of the

Lie group G. The Lie algebra structure of a Lie group is independent of the assignment of coordinates either in E^N or in the group space. Of course, its realization though depends on the parameterization of both spaces.

We now look at the infinitesimal transformations of surfaces in E^N. A surface J in E^N is a set of points satisfying a system of defining relations of the form

$$J: \quad g_m(x,u) = 0 \qquad (m = 1,2,\ldots,M \leq N) \tag{C.16}$$

where the functions g_m are assumed to be differentiable. When the g_m are functionally independent, the dimension of the surface J is equal to

$$\dim J = N - M$$

If M = N, then J reduces to a point. A surface $I \subset E^N$ is called a G-invariant surface if a point (x, u) initially found in I, remains in I under the action of the Lie group G. In other words, set

$$I: \quad f_m(x,u) = 0 \qquad (m = 1,2,\ldots,M \leq N) \tag{C.17}$$

I is an invariant surface under G when

$$f_m(\bar{x},\bar{u}) = 0 \quad \text{follows from} \quad f_m(x,u) = 0 \tag{C.18}$$

Expansion of this condition to first order in the group parameters gives a determining equation for the invariant surface I in terms of the group coordinate functions X_i and U_j. To first order, we have

$$f_m(\bar{x},\bar{u}) = f_m(x,u) + \varepsilon\left[X_i \frac{\partial f_m}{\partial x_i} + U_j \frac{\partial f_m}{\partial u_j}\right] \qquad (C.19)$$

or, more compactly,

$$f_m(\bar{x},\bar{u}) = (1 + \varepsilon Q) f_m(x,u) \qquad (C.20)$$

Here ε represents the group parameters, whose indices are suppressed in order to simplify the equations. It follows that $f_m(\bar{x},\bar{u})$ will vanish when $f_m(x,u)$ does, thereby satisfying condition (C.18), if

$$Q f_m(x,u) = \lambda_{ml}(x,u) f_l(x,u) \quad \text{on} \quad I \qquad (C.21)$$

This is the invariant surface condition for G acting in E^N. The multipliers λ_{ml} are auxillary functions on E^N, which also depend on the group parameters. Because each Q_j is a differential operator, it will leave invariant an entire family of surfaces in E^N, labeled by a set of constants. The members of this family are the characteristic surfaces of Q_j, in the sense of classical analysis. A surface in E^N which is invariant under Q_j, lies entirely on a characteristic surface of Q_j. Transformations between characteristic surfaces take one invariant surface into another.

Let us present an example of the invariant transformation of surfaces. The example chosen is the Lorentz transformation in one-space/one-time dimension. The Lorentz transformation serves not only to demonstrate the manipulations involved in invariance analysis. The physical

implications which are drawn from Lorentz invariance act as a model for the general use and interpretation of the invariance of laws in physics. In the example of the Lorentz transformation, the manipulation of invariance concepts is presented first rather mechanically, followed by comments on the physical meaning of Lorentz invariance and its implications to the present work.

The characteristic surfaces of Lorentz transformations in two dimensional space-time are hyperbolas, parameterized by the invariant proper time $\tau(x,t)$,

$$\tau^2 = \bar{x}^2 - c^2\bar{t}^2 = x^2 - c^2t^2 \qquad \text{(C.22)}$$

where c is the speed of light. See Figure C.1.

Under an infinitesimal Lorentz transformation the space-time variables change. The form of this change is given by

$$\begin{aligned} x \rightarrow \bar{x} &= x + \varepsilon X(x,t) \\ ct \rightarrow c\bar{t} &= ct + \varepsilon c T(x,t) \end{aligned} \qquad \text{(C.23)}$$

X(x,t) and T(x,t) are the coordinate functions of the Lorentz transformation. Invariance of the value of the proper time, according to condition (C.22) or condition (C.21) with vanishing multipliers, produces an algebraic equation which determines the coordinate functions. The relevant solution is, to first order in ε,

$$\begin{aligned} X &= ct \\ cT &= x \end{aligned} \qquad \text{(C.24)}$$

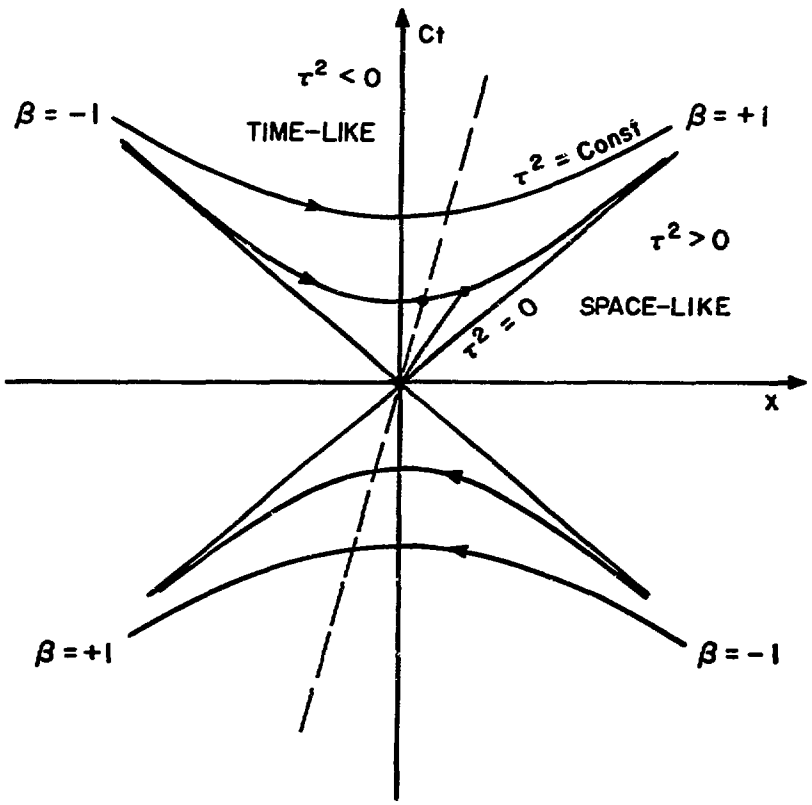

Figure C.1. The Characteristic Surfaces of the Lorentz Transformation.

Therefore the infinitesimal operator for Lorentz transforms in (x,t) in the form (C.14), is given by

$$Q = ct\,\partial_x + x\,\partial_{ct} \tag{C.25}$$

The operator Q annihilates $\hat{c}^2$. The characteristics of Q are determined from

$$\frac{dx}{d(ct)} = \frac{X}{cT} = \frac{ct}{x} \tag{C.26}$$

or, rearranging,

$$\frac{dx}{ct} = \frac{d(ct)}{x} = d\varepsilon \tag{C.27}$$

The integration constant in the solution of the first identity gives back the group invariant, the proper time. We now solve for the group parameter, ε. Obvious manipulations of equation (C.27) result in

$$d\varepsilon = \frac{(ct)dx - x\,d(ct)}{c^2t^2 - x^2} \tag{C.28}$$

so that the group parameter may be resolved as

$$\varepsilon = \tanh^{-1}\frac{x}{ct} \tag{C.29}$$

For uniform velocity frames, $\frac{x}{t}$ is the velocity v, and we have

$$\varepsilon = \tanh^{-1}\beta \quad , \quad \beta = \frac{v}{c} \tag{C.30}$$

Under Lorentz transformations generated by Q, the group parameter ε changes linearly. The differential operator

form of Q, equation (C.25), can be written as

$$Q = \frac{\partial}{\partial \varepsilon} \tag{C.31}$$

which may easily be verified. In Lie's theory, ε satisfying (C.31) is called a "canonical coordinate of the second kind." In Hamilton-Jacobi theory, ε is called an angle variable. As ε increases, an arbitrary point (x,t) moves along a τ^2 = const path toward increasing β. As ε tends toward $\pm\infty$, β limits to ± 1. The lines of constant ε coincide with lines of constant β.

Under two successive Lorentz transformations the group parameter changes according to

$$\varepsilon = \varepsilon_1 + \varepsilon_2 \tag{C.32}$$

The linearity in ε leads to velocity changes of the form,

$$\beta = \tanh(\varepsilon_1 + \varepsilon_2) = \frac{\tanh\varepsilon_1 + \tanh\varepsilon_2}{1 + \tanh\varepsilon_1 \tanh\varepsilon_2} \tag{C.33}$$

$$= \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}$$

Equation (C.33), of course, is Einstein's formula for the relativistic addition of velocities.

The Lorentz transformation example serves to dramatize the general procedure for the manipulation of invariance concepts. In other cases the procedure for deriving coordinate functions, invariants, and group parameters is analogous to this illustration. In addition, the physical implications of Lorentz invariance act as a model for the

use and interpretation of invariance principles in physics. In what follows we first recall the meaning of the proper time according to the active and passive views of invariance. Then we comment on the physical implications of invariance in general.

The proper time $\tau(x,t)$ is a correlation between two events labeled by (x,t). In the passive view, this correlation of events is regarded as being independent of the rest frame of the observer. One event is taken to be at $x_1 = t_1 = 0$, as measured in both frames. The other event is at $x_2 = x$, $t_2 = t$, as measured in the "rest" frame; and at $x_2 = \bar{x}$, $t_2 = \bar{t}$, as measured in the (uniformly) "moving" frame. The speed of light c has the same value in each frame. In the equivalent active view, which is usually taken for calculations in this work, the formulation of Lorentz invariance is slightly different. Let the states of a physical system be labeled by (x,t). Also let the laws which govern its evolution be independent of global uniform motion. Finally, let the speed of light be independent of the reference frame in which it is measured. Then, according to the active view, Lorentz transformations, i.e., transformations which preserve the value of the function $\tau(x,t)$, take given (x,t) states into states which are also physically possible under the same (invariant) laws of motion, but at different values of x and t.

In other words, if the laws of motion are Lorentz invariant, then given a physical state, Lorentz

transformations take this state into other possible states of motion. Thus each allowed solution lies entirely on a characteristic surface of the Lorentz transform operator in (x,t). This situation leads to fundamental classes of solutions admissible under Lorentz invariant equations (Wigner, 1939). Spaces of solutions are sought which are mapped into themselves by Lorentz transformations, i.e., solutions are sought which lie only on invariant surfaces of the Lorentz transform operator. In this way, the number of independent dimensions in the domain of the solutions is lowered, and the solutions themselves are automatically classified according to their transformation properties. Thus, Lorentz invariance produces conclusions about the nature of the allowed motion.

These physically important conclusions are, in fact, independent of whether the Lorentz invariant laws are linear. Only the forms of the solutions and their relationship to one another, e.g., superposition, depend upon the linearity of the equations of motion. The classification of the fundamental solutions of a physical law according to an invariance principle relies only upon the existence of the invariance principle itself. The general conclusion that one should seek the fundamental solutions on the invariant surfaces remains as a guide, even in the study of a nonlinear system. Thus as Laporte has often said, "By their transformations shall ye know them."

It turns out that a mathematical technique for both deducing the invariance properties of a given partial differential equation and for seeking its solutions on invariant surfaces was worked out by Lie. It is fortunate that Lie, in the classical era, has already developed the mathematics for precisely the task which is relevant to modern physics. After all, it is attributed to F. Klein, whose Erlanger Program (1872) was correlated with Lie's work, that "With the theory of groups, the mathematicians have finally invented something of no possible use to natural scientists."

We must go on to describe Lie's method in detail. However, even at this stage, its outline should be clear. Let a Lie group G describe an invariance principle. The coordinate functions of G define the infinitesimal operators Q. The characteristics of Q, in turn, determine invariant surfaces in E^N under G. Lie's method, as we shall use it, is to seek physical motions which lie entirely on a characteristic surface of one of the Q's. The use of Lie's method, both for the deduction of invariance principles and for the description of characteristic surfaces, relies primarily on the invariant surface condition.

The invariant surface conditions (C.21) can be viewed in two ways; according to whether the invariant surface, or its invariance group, is regarded as unknown. In the first view, equation (C.21) is a relation which determines the unknown invariant surface, I, from the given infinitesimal

generators, Q. In the second view, given the surface I in E^N, conditions (C.19) determine the coordinate functions of the Lie group which leaves it invariant. Thus, the invariant surface condition has a dual interpretation. In the example of the Lorentz transformation the second view is adopted. However, by equation (C.27) the group invariant, the proper time, may be recovered from the coordinate functions.

Lie found it possible to extend the invariance condition (C.19) to include differential manifolds, i.e., to include surfaces in the space of x, u, and derivatives of u to arbitrary order. The extension is obtained directly from the chain rule. Let the infinitesimal action of a point transformation group G be given by the expressions,

$$\bar{x}_i = x_i + \varepsilon X_i(x,u)$$
$$\bar{u}_j = u_j + \varepsilon U_j(x,u) \qquad \text{(C.34)}$$

The action of G on the partial derivative

$$u_{j,i} = \frac{\partial u_j}{\partial x_i}$$

is given by

$$\bar{u}_{j,i} = u_{j,i} + \varepsilon U_j^{(i)}(x,u,u') \qquad \text{(C.35)}$$

The extended coordinate functions $U_j^{(i)}$ (x,u,u') are calculated from the chain rule, according to

$$\frac{\partial \bar{x}^i}{\partial x^j} = \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial \bar{x}^i}{\partial x^j} \qquad \text{(C.36)}$$

The total differential operator

$$\frac{D}{Dx_k} = \frac{\partial}{\partial x_k} + u_{j,k} \frac{\partial}{\partial u_j} + u_{j,km} \frac{\partial}{\partial u_{j,m}} + \cdots \qquad \text{(C.37)}$$

accounts for both the explicit and implicit coordinate dependence. The coordinate functions for the first extension of (C.34) take the form,

$$\zeta_j^{(1)} = \frac{D\zeta_j}{Dx_i} - \frac{Dx_j}{Dx_i} u_{j,k} \qquad \text{(C.38)}$$

The calculation of higher extensions and explicit tabulations of the extended coordinate functions are given in Appendix A.

Suppose that the extension to manifolds, including higher derivatives of u has been accomplished. Let the statement of the dynamical problem of interest be (S,B,C), where typically S is a system of linear and/or nonlinear partial differential equations of order ℓ.

$$S_a(x, u, \partial_i u, \partial_{ij} u, \ldots) = 0, \quad a = 1, 2, \ldots, A \qquad \text{(C.39)}$$

B is a set of boundary conditions for S, prescribed on a region R in space and in time, bounded by curves C, i.e.,

$$B_b(x, u, \partial_i u) = 0, \quad b = 1, 2, \ldots, B \qquad \text{(C.40)}$$

on curves

$$C_c(x) = 0, \qquad c = 1, 2, \ldots, C \tag{C.41}$$

The dynamical statement is invariant under G if (S,B,C) forms an invariant manifold under the transformations of G in the extended space. The set of extended invariance conditions are analogous to the invariant surface condition (C.21). Namely, (S,B,C) is an invariant differential manifold if $(\overline{S},\overline{B},\overline{C})$ is related to (S,B,C), to first order in ε, by

$$\overline{S}_a(\bar{x}, \bar{u}, \overline{\partial_i u}, \overline{\partial_{ij} u}, \ldots) = (1 + \varepsilon K_{ad}) S_d(x, u, \partial_i u, \partial_{ij} u \ldots) \tag{C.42}$$

and

$$\overline{B}_b(\bar{x}, \bar{u}, \overline{\partial_i u}) = (1 + \varepsilon \lambda_{be}) B_e(x, u, \partial_i u) \tag{C.43}$$

on curves

$$\overline{C}_c(x) = (1 + \varepsilon \mu_{cf}) C_f(x) \tag{C.44}$$

The multipliers K, λ, μ are auxiliary functions which depend on the invariance operations. The invariance conditions mean that $(\overline{S},\overline{B},\overline{C})$ follows from (S,B,C), under infinitesimal group transformations in the extended space. These conditions produce a system of partial differential equations which determine the coordinate functions of the Lie invariance group G.

Under the invariance group of the dynamics, in E^N the solutions transform among themselves, i.e., the solution set forms an invariant surface in E^N under G. Although this

property may seem perfectly natural because of previous experience with invariance principles in physics, let us prove it in a special case. Suppose the solution of (S,B,C) is unique and given by

$$u_j = \theta_j(x) \tag{C.45}$$

Putting bars over all of the variables cannot change the form of the solution. In the new coordinates $(\bar{x},\bar{u})$, the solution is

$$\bar{u}_j = \theta_j(\bar{x}) \tag{C.46}$$

Then, to first order in ε,

$$\theta_j(\bar{x}) - \bar{u}_j = \theta_j(x) - u_j + \varepsilon\left[X_i\theta_{j,i}(x) - W_j\right] \tag{C.47}$$

Now, the left side of (C.47) must vanish when the right side does, by uniqueness. Consequently, the solution is an invariant surface under G, because the braketed term must vanish with $(\theta_j(x)-u_j)$.

In general if (S,B,C) admits G, then G leaves invariant the solution set of (S,B,C). In other words, if $u_j = \vartheta_j(x)$ is a solution, then $\bar{u}_j = \vartheta_j(\bar{x})$ is also a solution.

As an example, suppose that (S,B,C) admits the interchange of the first two independent variables, say x and t. Such a transformation is a discrete symmetry rather than a continuous one, but the same type of reasoning applies. Let

$$u_j = \theta_j(x,t) \tag{C.48}$$

be a solution. The action of (x,t) interchange is given by

$$\begin{aligned} \bar{u}_j &= u_j \\ \bar{x} &= t \\ \bar{t} &= x \end{aligned} \tag{C.49}$$

If (S,B,C) admits this operation, then for the pair of values (x,t)

$$\begin{aligned} \bar{u}_j - \theta_j(\bar{x},\bar{t}) &= \lambda\left[u_j - \theta_j(x,t)\right] \\ &= u_j - \theta_j(t,x) \end{aligned} \tag{C.50}$$

and for the pair of values (t,x)

$$\begin{aligned} \bar{u}_j - \theta_j(\bar{t},\bar{x}) &= \lambda\left[u_j - \theta_j(t,x)\right] \\ &= u_j - \theta_j(x,t) \end{aligned} \tag{C.51}$$

Consequently, for all (x,t) we have $\lambda^2 = 1$, $\lambda = \pm 1$, and

$$\bar{u}_j - \theta_j(\bar{x},\bar{t}) = \pm\left[u_j - \theta_j(x,t)\right] \tag{C.52}$$

Thus, a characterization of the solution set has been achieved. Under interchange of x and t, the solution surface either reflects through the (x,t) plane or it stays the same. There are only these two classes of solutions when (S,B,C) is invariant under (x,t) interchange.

Let the operations admitted by (S,B,C) form a Lie group, G. The coordinate functions of G define the infinitesimal operators Q. The characteristics of Q, in turn,

determine invariant surfaces in E^N under the action of G. Lie's method is to seek solutions of (S,B,C) in E^N which lie entirely on the characteristic surfaces of each distinct subgroup of G. In this way, the number of independent dimensions in the domain of the solutions is lowered, while at the same time the solutions themselves are classified according to their transformation properties.

E. Synopsis of Lie's Method for Partial Differential Equations

Let us now describe details for the actual procedure of Lie's method when the dynamical statement admits an invariance principle. A general solution surface of (S,B,C) in E^N has the form,

$$u_j = \Theta_j(x_i) \qquad \begin{matrix} j = 1, 2, \ldots, m \\ i = 1, 2, \ldots, n \end{matrix} \tag{C.53}$$

with dimension $N - m = n$. Let us seek solutions which lie entirely on a characteristic surface of an r-parameter subgroup of the main group G, admitted by (S,B,C). The characteristic surface C in E^N is defined by R relations for the invariants of Q, where Q stands for the set of $r \geq R$ generators of the subgroup H_r of G. Let P be the total number of functionally independent invariants of Q in E^N. Then C is determined from R independent relations for P invariant quantities.

$$C: \quad Q_J I_p(x,u) = 0 \quad \text{on} \quad C \tag{C.54}$$

where the subscript ranges are,

$$\begin{aligned} J &= 1, 2, \ldots, r \\ P &= 1, 2, \ldots, \mathcal{P} \end{aligned} \tag{C.55}$$

The invariant surface C has dimension

$$\dim C_1 = N - R \tag{C.56}$$

because it is the result of placing R independent first order restrictions on the N dimensional space E^N. The numbers R and P are determined from the classical theory of differential equations (Forsythe, 1906). The set $\{Q_J\}$ forms an algebra. When the Q_J are all functionally independent, they form a "complete linear system," in the sense of classical analysis. In this case R, the number of independent equations (C.54), equals the rank of the matrix of coefficients appearing in (C.54).[5]

$$R = \text{rank} \begin{vmatrix} X_1^1 \cdots X_1^n & U_1^1 \cdots U_1^m \\ \vdots & \\ X_J^1 \cdots \cdots \cdots & U_J^m \end{vmatrix} \tag{C.57}$$

Let $N > R$, i.e., suppose there are more columns than rows in equation (C.57). There are R independent equations for P invariants in an N dimensional space. Therefore,

$$\mathcal{P} \leq N - R \tag{C.58}$$

---

[5]The first question which Lie asked a prospective student was "Do you know how to solve a complete linear system?" (Private communication, Prof. R. A. Axford).

The maximum number of functionally independent invariants equals (N - R). P is less than (N - R) whenever any operators in $\{Q_j\}$ are linearly connected. If H_1 is a nontrivial one parameter subgroup of G, then R equals unity.

Suppose P equals its maximum, then

$$\dim C = N - R = P \tag{C.59}$$

and the set of independent invariants $\{I_p\}$ can be used as coordinates on C. Let condition (C.59) be satisfied, so that we can choose a set of coordinates which are invariant on C. Now suppose that the first m invariant coordinates are solvable for u_j, and that the remaining P - m invariants on C, labeled $\{\lambda_k\}$, depend only on x_i. Then a solution surface on C can be expressed in the invariant form,

$$I_j(u,x) = F_j(\lambda_k(x)) \tag{C.60}$$

where

$$\begin{aligned} j &= 1, 2, \ldots, m \\ k &= 1, 2, \ldots, P-m = n-R \end{aligned} \tag{C.61}$$

Compare (C.60) and (C.61) with (C.53). By assumption, the $\{I_j\}$ can be solved for $\{u_j\}$. Hence, the restriction of the solution surface to C essentially reduces the number of independent variables by R, provided that P equals its maximum value.

If the inequality holds in (C.58), then the solution surface has more essential dimensions than an invariant

surface under H_r. Consequently, it cannot be coordinated by the invariants $\{I_p\}$ of the particular subgroup H_r which is chosen initially. Also, if R = N, then, from (C.56), C reduces to a set of invariant points in E^N, upon which the coordinate functions vanish. In this case, no invariant functions appear at all, and P = 0.

The invariant coordinates $\{I_p\}$ on C are obtained in practice from the invariant surface condition (C.21). The characteristic form of equation (C.21) may be expressed as,

$$\frac{dx_i}{X_i(x,u)} = \frac{du_j}{U_j(x,u)} \quad (\text{no sum}) \qquad \begin{matrix} i = 1,2,\ldots,n \\ j = 1,2,\ldots,m \end{matrix} \qquad (C.62)$$

where the coordinate functions are to be evaluated on C. Note that the u dependence in the coordinate function X_i has already been factored out, according to the assumptions above. This factorization allows the first set of equalities in (C.62), namely,

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdots = \frac{dx_n}{X_n} \qquad (C.63)$$

to be solved for the group trajectories of H_r in x-space. The integrals of (C.63) are given by,

$$\lambda_k(x) = \text{const} \qquad k = 1,2,\ldots,n-R \qquad (C.64)$$

Now let equations (C.60) be resolved for u_j on the invariant surface C.

$$u_j = V_j(x, \lambda, v(\lambda)) \qquad \text{(C.65)}$$

Equation (C.65) is the form of the solution, which remains on C under H_r. Upon substitution of (C.65) into (S,B,C), a new system of equations is obtained for the set $\{v_j(\lambda)\}$.

Expressions such as (C.65) will be called "group reducible" solutions, because they can be cast into "manifestly" group invariant form. The group reducible form (C.64) includes the self-similar form as a special case. The n-R group invariant coordinates λ are analogous to similarity variables. The m functions $v(\lambda)$ in (C.65) are roughly the analogs of the "shape functions" in the self-similar solution.

Lie's program is a constructive and general mathematical technique which includes the method of self-similar solutions. The implementation of Lie's method takes the following steps:

(1) Find, or solve for G, the Lie group corresponding to the invariance principles for (S,B,C). The required result is a set of coordinate functions on E^N.

(2) Solve for the invariants of Q. Here Q is a set of infinitesimal operators on E^N corresponding to a distinct r parameter subgroup H_r of G. The required result is a complete set of invariant coordinates for the characteristic surfaces, C, of Q.

(3) Transform to invariant coordinates on C and seek solutions of the form (C.60). The result is (S,B,C) reduced to C, whose solutions give the fundamental group reducible motions on C. These motions transform reducibly under H_r.

A mathematical aspect concerning details of the group structure of G is worth mentioning. Care has been taken above to call H_r a "distinct" subgroup of G. If two subgroups $H^{(1)}$ and $H^{(2)}$ can be related set theoretically, by

$$H^{(2)} = T H^{(1)} T^{-1} \quad , \{T\} \in G \qquad (C.68)$$

then $H^{(1)}$ and $H^{(2)}$ are called "equivalent," as opposed to "distinct." The group reducible solutions φ_1 and φ_2 of equivalent subgroups $H^{(1)}$ and $H^{(2)}$, respectively, are themselves equivalent, relative to G. They are equivalent in the sense that there exists a group transformation T, such that,

$$T\varphi_1 = \varphi_2 \qquad (C.69)$$

for some transformation T∈G. Equivalent solutions are regarded as not different, because they lie on the same family of characteristic surfaces in E^N.

The equivalence relation (C.68) separates the set of all subgroups of G into distinct classes, each class containing equivalent subgroups. A distinct group reducible solution follows from each distinct subgroup of G. Under the action of separate distinct subgroups, a given group reducible solution leaves its family of characteristic surfaces in E^N, and may even cover E^N entirely (see Figure C.2). Thus, having found a group reducible solution $\varphi_j^{(r)}(x)$ for a subgroup H_rCG; other solutions, equivalent to the first one, may also be found by operating on $\varphi_j^{(r)}(x)$ with the remainder of group.

The algebraic analogs of equivalent subgroups are equivalent subalgebras. The equivalence relations for subalgebras Q_1 and Q_2 of a Lie algebra L_G are defined by the set theoretic equality,

$$[Q_1, X] = Q_2 \qquad \text{(C.70)}$$

for some element X of L_G. This relation may also be written symbolically, as

$$e^{-ad(X)} Q_1 = Q_2 \qquad \text{(C.71)}$$

where,

$$ad(X) \cdot Q_1 = [X, Q_1] \qquad \text{(C.72)}$$

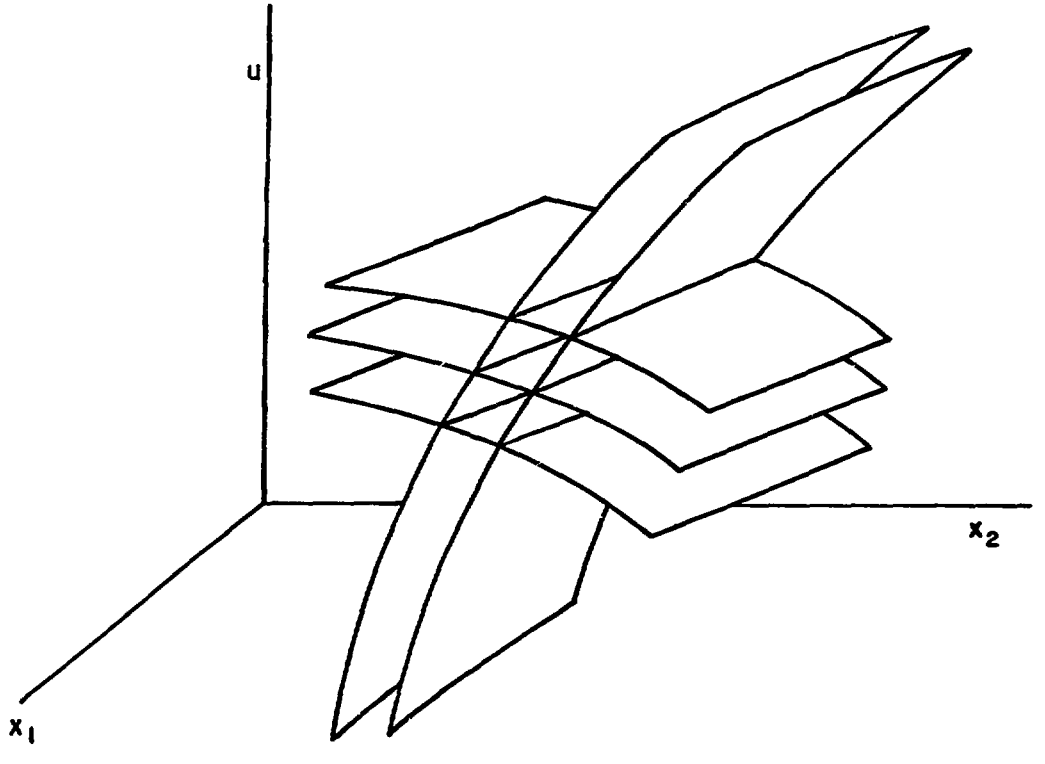

Figure C.2. The action of distinct subgroups of G. The non-parallel surfaces represent characteristic surfaces of distinct subgroups. The parallel surfaces represent characteristic surfaces of equivalent subgroups.

is an element of the adjoint group of linear inner automorphisms of the Lie algebra. The adjoint group can be used to find classes of distinct subalgebras of the main algebra L_G.[6]

F. The Wave Equation

Evidence has already been submitted that, when an invariance principle is known, Lie's method may produce useful and general results. Even for the nonlinear problem of ideal fluid flow, an invariance principle provides the key to characterizing the scaling group reducible solutions as simple waves. Previously, it has also been mentioned that the extended invariant surface conditions, (C.42) through (C.44), might be used to actually solve for the Lie group of invariance principles, which is admitted by a dynamical problem. However, the systematic deduction of nontrivial invariance principles from given dynamics has not yet been implemented in an example. Thus, as another introductory illustration of Lie's method, but this time starting immediately from the statement of the problem, we look at the linear one-dimensional wave equation.

$$u_{xx} - \frac{1}{c^2} u_{tt} = 0 \qquad \text{(C.73)}$$

The object of this illustration is to sample once again the completeness and quality of the results produced with the invariance approach; and to make its procedure even more concrete and definite.

[6] Ovsjannikov (1962). Such decompositions were first made by Cartan (1952), p. 304.

We present the wave equation as an ab initio example of Lie's method. It will be seen that Lie's method recovers naturally each of the classical results for linear wave motion. The finite part of the Lie invariance algebra for the wave equation is the conformal algebra. The operations of the conformal algebra map the solution space for the wave equation into itself. The boundary value problem for standing wave motion is also discussed from a group theoretical vantage point.

After Laplace's equation, the wave equation has been the most deeply studied partial differential equation of mathematical physics. As early as 1747, d'Alembert obtained the general solution of the one-dimensional wave equation as the superposition of two simple waves, moving with the same speed c in opposite directions,

$$u = f(x-ct) + g(x+ct) \qquad (C.74)$$

d'Alembert used his solution of the wave equation to discuss the vibrating string. He was followed by Lagrange (1781), who used the two-dimensional wave equation to study water waves; Poisson (1807), who derived the Poisson Kernel for it; Cauchy (1841), who coined the word "characteristic" to describe the wave fronts; Riemann (1860), who recognized that the zone of influence from a point in (x,t) is wedge-shaped, and who used the adjoint equation and the so-called Riemann function to solve it. Then came virtually all of

the classical mathematical physicists of the nineteenth century - Liouville, Helmholtz, Kelvin, Kirchoff, Maxwell, and Poincare.[7]

The wave equation has sufficient structure to provide an interesting illustration of Lie's method. Also, it is Lorentz invariant, so it connects well with the earlier discussion. In this section the following procedural areas will come to light:

(1) How to determine the coordinate functions X, T, and U from the invariance of the starting equation (S, in the previous notation).

(2) How the fundamental solutions are found from the set (Q_j) and the invariant surface condition.

(3) How the subsidiary conditions (B,C) restrict the invariance algebra of S.

(4) How the structure of the group enters into the classes of admissable solutions.

The starting equation, S, is the one-dimensional wave equation,

$$u_{xx} - u_{tt} = 0 \qquad \text{(C.75)}$$

where units are chosen to make the wave speed c = 1, since it is assumed not to transform. The initial and boundary conditions are put aside until later.

A preliminary calculation shows that it is sufficient

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[7]G. Birkhoff, A Source Book in Classical Analysis, (Harvard University Press, 1973), pp. 403-434.

to consider a Lie group whose coordinate functions have the following dependences,

$$\begin{aligned} \bar{x} &= x + \varepsilon X(x,t) \\ \bar{t} &= t + \varepsilon T(x,t) \\ \bar{u} &= u + \varepsilon U(x,t,u) \end{aligned} \tag{C.76}$$

The extended invariant surface condition, corresponding to equation (C.42), for the wave equation if

$$\bar{u}_{\bar{x}\bar{x}} - \bar{u}_{\bar{t}\bar{t}} = \left[1 + \varepsilon P(x,t,u)\right](u_{xx} - u_{tt}) \tag{C.77}$$

The functions X, T, U, P which are to be determined, could have depended more generally on u and its derivatives. In this case, (C.76) would represent contact transformations, instead of point transformations. However, the preliminary calculation, which is the same as the one to be shown except for including higher order dependence, has already eliminated the higher order functional dependence in X, T, U, P.[8]

The condition of invariance is that the $O(\varepsilon)$ terms in (C.77) vanish together. Expanding (C.77) to first order in ε, we find

$$U^{(xx)} - U^{(tt)} = P(x,t,u)\left(u_{xx} - u_{tt}\right) \tag{C.78}$$

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[8] Ovsjannikov (1962), p. 183.

The second-extended coordinate functions which appear in the invariance condition (C.78) are calculated in Appendix D, formulas (D.14) and (D.15). The second extensions depend on u and on its first and second derivatives. Equating like coefficients of 1, u_t, u_x, u_{xx}, u_{tt}, u_{xt}, and their bilinear combinations, on each side of the invariance condition (C.78) produces an over-determined system of linear partial differential equations for the coordinate functions X, T, U, and for the multiplier P(x,t,u). This system of equations quickly reduces to

$$U_{xx} - U_{tt} = 0 \tag{C.79}$$

$$2U_{tu} = T_{tt} - T_{xx} \tag{C.80}$$

$$-2U_{xu} = X_{tt} - X_{xx} \tag{C.81}$$

$$U_{uu} = 0 \tag{C.82}$$

$$T_x = X_t \tag{C.83}$$

$$T_t = X_x = \tfrac{1}{2}(U_u - P) \tag{C.84}$$

Equations (C.79) to (C.84) are the so-called "determining equations" for the coordinate functions. To solve them, we first combine the last two equations. The result is,

$$X_{tt} = T_{xt} = X_{xx} \tag{C.85}$$

$$T_{tt} = X_{xt} = T_{xx} \tag{C.86}$$

Thus, the coordinate functions X(x,t) and T(x,t) each satisfy the wave equation. Hence, from (C.80) to (C.82) we find

$$U_{tu} = U_{xu} = U_{uu} = 0 \tag{C.87}$$

So U_u is a constant, say a. Then from the first equation, (C.79), we have

$$U = au + \varphi(x,t) \tag{C.88}$$

where $\varphi(x,t)$ satisfies the wave equation. Equation (C.88) represents two properties which are common to the solutions of all linear equations. Namely, that they remain solutions when they are scaled and/or superposed.

Because T_t satisfies the wave equation, the function P(x,t) must also obey it. From equation (C.84) we have

$$-\tfrac{1}{2}P_{xx} = T_{txx} = T_{ttt} = -\tfrac{1}{2}P_{tt} \tag{C.89}$$

Thus U(x,t), X(x,t), T(x,t), P(x,t) each satisfies the wave equation, and the wave equation admits an infinite dimensional group, parameterized by its own solutions. The infinite dimensional group of the wave equation is a formal

mathematical property which is related to the possibility of exchanging independent and dependent variables in one space dimension. When the wave equation is written in the Cauchy-Riemann form,

$$u_t = v_x$$
$$v_t = u_x \tag{C.90}$$

we see that (x,t) could be exchanged with (u,v) without changing the form of the equations by simply inverting. The result of inverting (C.90), namely

$$t_u = x_v$$
$$t_v = x_u \tag{C.91}$$

then leads to the wave equation expressions,

$$x_{uu} - x_{vv} = 0 \tag{C.92}$$

for the position in terms of (u,v), and also

$$t_{uu} - t_{vv} = 0 \tag{C.93}$$

which is a similar wave equation for the time. This purely formal variable exchange property goes away when more spatial dimensions are present. Of course, the other infinite dimensional property - superposition - remains in higher dimensions because of linearity.

The finite part of the wave equation algebra is gotten by making P_{xx} and P_{tt} vanish separately. In this case we may take,

$$\tfrac{1}{2}(a - \mathcal{P}) = s + 2dx + 2et \qquad \text{(C.94)}$$

where s, d, e are constants. Now from (C.84) we have

$$X = sx + dx^2 + 2ext + f(t) \qquad \text{(C.95)}$$

$$T = st + 2dxt + et^2 + g(x) \qquad \text{(C.96)}$$

The integration functions f(t) and g(x) are determined from $T_x = X_t$, i.e.,

$$2dt + g'(x) = 2ex + f'(t) \qquad \text{(C.97)}$$

As a consequence of this relation, we find that

$$f(t) = dt^2 + bx + h \qquad \text{(C.98)}$$

$$g(x) = ex^2 + bt + p \qquad \text{(C.99)}$$

where b, h, p are new integration constants. Thus we have for the coordinate functions of the finite part of the wave equation algebra,

$$X = p + sx + bt + dx^2 + 2ext \qquad \text{(C.100)}$$

$$T = h + st + bx + 2dxt + et^2 \qquad \text{(C.101)}$$

$$U = au \qquad \text{(C.102)}$$

with integration constants, a, b, d, e, s, p, h.

The seven arbitrary constants are the seven independent group parameters of the finite dimensional part of the wave equation Lie invariance algebra. A differential operator basis, Q, is found by individually letting each group parameter be unity, while making the rest vanish. Table C.1 lists the members of Q, along with their invariants and geometrical significance.

The operators in the Basis Q satisfy the commutation relations given in Table C.2.

The entry which appears in the j-th row and k-th column in Table C.2 is the commutator(Q_j, Q_r).

The subsets of operators Q which form subalgebras can be picked out of the commutator table. The six parameter subalgebra (P, H, S, B, D, E) is called the conformal algebra. It is the well-known invariance algebra of the wave equation, which has seen a lot of action in relativistic particle physics.

The basis Q preserves the null space of the wave operator, W, given by

$$W = \partial_{xx} - \partial_{tt} \tag{C.103}$$

In other words, Q maps the solution space of $W\psi(x,t) = 0$ back into itself. We may check this invariance property for the elements of the conformal subalgebra of Q, by noting the commutator relation,

$$[Q, W] = P(x,t)W \tag{C.104}$$

OPERATOR	SIGNIFICANCE	INVARIANT
$P = \partial_x$	Space translations	t
$H = \partial_t$	Time translations	x
$S = x\partial_x + t\partial_t$	Space/time scaling	x/t
$B = t\partial_x + x\partial_t$	Lorentz boosts	$x^2 - t^2$
$D = 2xt\partial_x + (x^2 + t^2)\partial_t$	Conformal	$\frac{1}{x}(x^2 - t^2)$
$E = (x^2 + t^2)\partial_x + 2xt\partial_t$	Transformations	$\frac{1}{t}(x^2 - t^2)$
$A = u\partial_u$	Scaling of u	$F(x,t)$

Table C.1. The Basis Q of the Lie Invariance Algebra for the Wave Equation.

	P	H	S	B	D	E	A
P	0	0	P	H	2B	2S	0
H	0	0	H	P	2S	2B	0
S	-P	-H	0	0	D	E	0
B	-H	-P	0	0	E	D	0
D	-2B	-2S	-D	-E	0	0	0
E	-2S	-2B	-E	-D	0	0	0
A	0	0	0	0	0	0	0

Table C.2. The Commutators of the Basis Q for the Wave Equation.

where,

$$Q \in \{P, H, S, B, D, E\} \tag{C.105}$$

and P(x,t) is given by equation (C.94). Equation (C.104) implies that $WQ\psi = 0$, whenever $W\psi = 0$, which means that, under Q, solutions are mapped into solutions.

Such operators Q are the analogs of dynamical symmetry operators in quantum mechanics. In the quantum case the operator W is replaced by $K = i\hbar\partial_t - H$, where H is the Hamiltonian. The vanishing of the commutator $[K,Q]$ may be written as

$$[i\hbar\partial_t - H, Q] = i\hbar Q_t + [Q, H] = i\hbar \frac{dQ}{dt} = 0 \tag{C.106}$$

Equation (C.106) is the quantum condition for the eigenvalues of the operator Q to be constants of the motion.

Let us continue with the wave equation example. Equation (C.100) through (C.102) represent the required result for admission to the last step in Lie's program. The coordinate functions on (x, t, u) have been resolved and a basis for the Lie algebra, Q, has been

constructed. In Table C.1 the invariants of Q appear. The Lie algebra Q has been identified with the conformal algebra.

Before we proceed to construct group reducible solutions, it should be verified that the Lie algebra which appears in Tables C.1 and C.2, really does include the generators of conformal transformations. In order to make this validation, the finite transformations generated by (P + E) and (H + D) are calculated. First, let us consider the operator

$$(P+E) = \left[1 + (x^2+t^2)\right]\partial_x + 2xt\,\partial_t \qquad (C.107)$$

The group parameter corresponding to (P + E) is found by solving the characteristic equations,

$$\frac{dx}{1 + x^2 + t^2} = \frac{dt}{2xt} = d\varepsilon \qquad (C.108)$$

Let the quantities $z_\pm$ be defined by

$$z_\pm = x \pm t \qquad (C.109)$$

so that

$$z_\pm^2 = x^2 + t^2 \pm 2xt \qquad (C.110)$$

Of course, $z_\pm$ = const is a characteristic of Wu = 0. It follows from equation (C.108), that

$$d(x \pm t) = dz_\pm = (1 + z_\pm^2)\,d\varepsilon \qquad (C.111)$$

Hence,

$$d\varepsilon = d\left(\tan^{-1} z_{\pm}\right) \tag{C.112}$$

We now drop the subscript on $z_{\pm}$, because everything is symmetric under the interchange of subscripts. The result of equation (C.112) is that,

$$z(\varepsilon) = \tan(\varepsilon + \alpha) \tag{C.113}$$

where

$$\tan\alpha = z(\varepsilon = 0) = z_0 \tag{C.114}$$

Hence,

$$z(\varepsilon) = \frac{z_0 + \tan\varepsilon}{-z_0\tan\varepsilon + 1} \tag{C.115}$$

Equation (C.115) for $z(\varepsilon)$ has the form of a conformal transformation

$$z = \frac{a z_0 + b}{c z_0 + d} \tag{C.116}$$

The condition for the conformal transformation (C.116) to be nonsingular, i.e.,

$$ad - bc \neq 0 \tag{C.117}$$

corresponds to

$$1 + \tan^2\varepsilon = \frac{1}{\cos^2\varepsilon} \neq 0 \tag{C.118}$$

which restricts ε to be non-imaginary.

Clearly the operator H + D generates the same finite conformal transformation as (P + E), except for the

interchange of x and t. Thus, by calculating the appropriate finite transformations, we have verified that the wave equation algebra includes the generators of conformal transformations. Now we can proceed to derive group reducible solutions of the wave equation from its invariance algebra.

According to Lie's theory, each distinct subalgebra of Q generates a distinct group reducible solution. Let us present three examples of the construction of group reducible solutions for the wave equation.

Up until now, the boundary and initial conditions (B, C) have been put aside. However, if subsidiary conditions are present, they must also be left invariant, in order for group reducible solutions to be admitted. In principle, the invariant coordinates of the Lie invariance algebra of S. say L_G, will determine the most general "moving boundaries"

$$\zeta_i(\lambda_k(x,t)) = \text{const} \qquad \text{(C.119)}$$

upon which auxilliary conditions could be invariant. In practice, however, the requirement of invariance of (B, C) usually restricts L_G to a subalgebra, which leaves invariant both the boundary conditions, B, and the curves, C, upon which they are prescribed.

Suppose, for the wave equation example, that group reducible solutions are required for the vibrating string problem with fixed endpoints. Let the boundary values of

the amplitude and velocity vanish on the unit interval,

$$u(0,t) = u(1,t) = 0$$
$$u_t(0,t) = u_t(1,t) = 0 \qquad \text{(C.120)}$$

In order to find a group reducible solution to this problem, we must leave the boundary conditions invariant. Invariance of the endpoints, $x = 0$ and $x = 1$, requires that

$$\bar{x} = x + \varepsilon X(x,t) = x \quad \text{when} \quad x = 0, 1. \qquad \text{(C.121)}$$

Hence, referring to the coordinate functions given by (C.100) through (C.102), we have

$$a = c = b = d = e = 0 \qquad \text{(C.122)}$$

Thus, the boundary conditions restrict the non-zero constants in the coordinate functions to a two parameter subset,

$$X = 0$$
$$T = h \qquad \text{(C.123)}$$
$$U = au$$

In other words, for the vibrating string problem, (S,B,C) is invariant (up to superposition) only under time translations and scaling of the amplitude. The infinitesimal operator corresponding to these transformations is given by

$$Q = \partial_t + \omega u \partial_u \quad , \quad \omega = \frac{a}{h} \qquad \text{(C.124)}$$

We shall see that the values of the parameter ω, in equations (C.124) must be restricted, for a group reducible

solution to obtain.

A group reducible solution in (x,t,u) space lies on a characteristic surface of the differential operator Q given by equation (C.124). Such a surface is illustrated in Figure C.3. Thus, in accord with Lie's program, we seek a subspace in (x,t,u) which is invariant under Q. The invariants of Q are found from

$$Q \cdot I(x,t,u) = 0 \tag{C.125}$$

or, in characteristic form

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{\omega u} \tag{C.126}$$

The invariants are given by

$$I_1 = x \quad , \quad I_2 = e^{-\omega t} u \tag{C.127}$$

Note that for this problem, N = 3, R = 1, and P = 2, in our earlier notation, so that P attains its maximum value. Consequently, the group reducible solution can be coordinated by the invariants of the group. Group reducible solutions are now sought by transforming to the invariant coordinates. In the invariant coordinates the characteristic surface of Q is given by

$$I_2 = U(I_1) \tag{C.128}$$

See Figure C.4. In terms of the old variables, the group reducible solution is expressed by

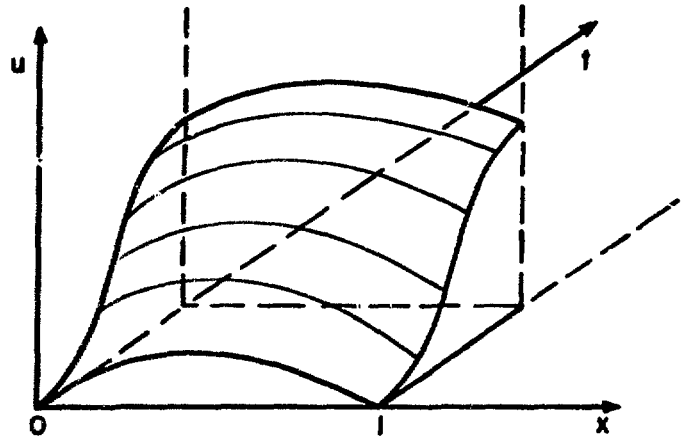

Figure C.3. The Characteristic Surface of Q Eq. (C.124) in (x,t,u) Space.

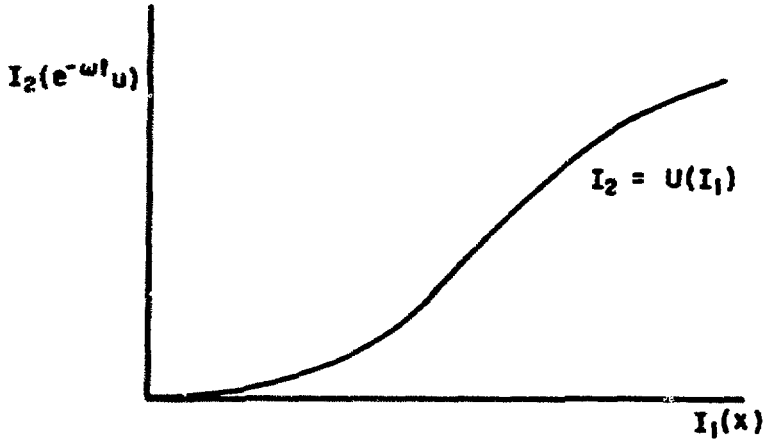

Figure C.4. The Characteristic Surface of Q Eq. (C.124) in (I_1,I_2) Space.

$$u = e^{\omega t} U(x) \tag{C.129}$$

The group reducible solution thus appears in separated form. The eigenvalue spectrum, the set of allowed values for ω, is determined from the condition that u(x,t) be a global solution, i.e., that u satisfy the boundary conditions. The rest of the problem, of course, follows the familiar pattern. However, we would still like to carry it out to its conclusion.

When the separated form is substituted into the starting equation, a differential equation in one less independent variable is obtained.

$$U''(x) - \omega^2 U(x) = 0 \tag{C.130}$$

Imposing the boundary conditions restricts the possible values of the parameter ω to a discrete spectrum

$$\omega = n\pi \qquad n = 0, 1, 2, \ldots \tag{C.131}$$

and results in the standing wave solutions,

$$u(x,t) = A e^{in\pi t} \sin(n\pi x) \tag{C.132}$$

Thus, the group reducible solutions for the vibrating string are its normal modes.

The character of this example tends to recur whenever Lie's method is applied. First, as expected, the number of independent variables decreases and a partial (or complete) separation of variables occurs. Second, the relative values

of the group parameters are restricted by imposing the boundary conditions. Third, a family of group reducible solutions appears, labeled by the (restricted) values of the group parameters.

When the problem is posed in an infinite domain, requiring only that the solution vanish at infinity, the full invariance algebra of the wave equation is admitted. In particular, space and time translations leave the wave equation invariant, because the independent variables x and t do not appear in it explicitly. Under translations of x and t,

$$\begin{aligned} x &\longrightarrow \bar{x} = x + p \\ t &\longrightarrow \bar{t} = t + h \\ u &\longrightarrow \bar{u} = u \end{aligned} \tag{C.133}$$

The solution u, itself, is invariant. The other invariant, required for group reducible solvability, is given by

$$\lambda = x - (p/h)t \tag{C.134}$$

The group reducible solution is sought in the invariant form,

$$u = F(\lambda) \tag{C.135}$$

Substituting (C.135) into the wave equation yields an ordinary differential equation,

$$\left[1 - (p/h)^2\right] F''(\lambda) = 0 \tag{C.136}$$

Therefore, the relative values of the group parameters p

and h are restricted to,

$$p/h = \pm 1 \tag{C.137}$$

i.e., space and time translations are only allowed which move u(x,t) along a characteristic. Because $u = F(\lambda)$, the wave profile does not change shape as it moves along a characteristic. For arbitrary p and h, the trivial solution

$$F = A\left[x - (p/h)t\right] + B \tag{C.138}$$

results, but it is zeroed out by the vanishing conditions at infinity. The d'Alembert travelling wave solution reappears when the group reducible solutions are superposed.

$$u = F(x-t) + G(x+t) \tag{C.139}$$

where F and G are arbitrary functions. Finally, we note that, as in the case of fluid dynamics, the similarity variables actually define the characteristics.

$$\lambda_{\pm} = x \pm t \qquad \begin{array}{l} d\lambda_+ = 0 \quad \text{on} \quad \frac{dx}{dt} = -1 \\ d\lambda_- = 0 \quad \text{on} \quad \frac{dx}{dt} = +1 \end{array} \tag{C.140}$$

The path of the propagated wave fronts follow lines of constant λ. Thus, these solutions are simple waves in the sense of fluid dynamics. However, unlike the simple waves in fluid dynamics, the solutions of the linear wave

equation retain their wave profile as they propagate.

In the travelling wave case, the relative values of the group parameters are restricted, not by the boundary conditions, but by requiring a non-trivial solution of the group-reduced differential equation (C.136). The travelling wave solution is an example of a degenerate case, which seldom occurs. Namely, the group reducible solution is functionally invariant, i.e., arbitrary functions appear in the final solution.

The object of the invariance analysis of the linear wave equation has been to sample the character of the group reducible solutions, and to make the procedure for obtaining them more definite. The types of wave motions obtained in this example - namely, standing waves, travelling waves, and centered simple waves - are representative group reducible solutions of the wave equation. In the case of standing waves, although the symmetry of the wave equation is broken by the boundary conditions, enough invariance remains to produce a nontrivial wave motion. In fact, the boundary conditions choose a discrete spectrum of allowed wave motions. A family of solutions results, labeled by the relative values of the parameters in the symmetry algebra. In the case of travelling waves, the conditions at infinity do not break the symmetry of the wave equation. The group reducible solutions which result from space and time translation invariance comprise a family of travelling waves of arbitrary shape, whose members propagate along characteris-

tics without change in form. These wave motions fall into two discrete classes - leftward and rightward moving waves. The discrete classes are again labeled by the relative allowed values of the group parameters.

It has been shown that, even when boundary conditions break the symmetry algebra of the equations of motion, group reducible solutions may still be obtainable and meaningful. However, group reducible solutions are not of interest merely because they are examples of solutions to particular problems. We have seen that the group reducible solutions can also represent the classes of motions admitted by the dynamics. This second, more general interpretation of group reducible solutions is relevant to nonlinear physics and to the present investigation.

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