HADRONIC DEFORMATION ENERGY IN THE MIT BAG MODEL*

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ABSTRACT

The MIT bag model for hadrons is treated in the static cavity approximation in three dimensions with a definite quark number. The energy of the system is computed to second order in the gluon coupling. A constrained variational method is described which permits the calculation of the energy as a function of a collective variable. The bag cavity is permitted to assume whatever shape is necessary in order to minimize the energy for a given expectation value of the collective variable. The method is well-suited for the calculation of the nuclear potential -- in particular, the short range component of the two-nucleon force. In a preliminary study presented here, the method is applied to a bag containing one quark and one anti-quark and the energy as a function of the separation of the quarks is evaluated.

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DISCLAIMER

1. Introduction

The MIT bag model of hadrons[1] in the static cavity approximation accounts remarkably well for the masses of the light hadrons[2] (the octets of pseudoscalar and vector mesons and the lowest baryon octet and decuplet). The ingredients of the model are appealingly simple:[3] the currently fashionable combination of quarks of three colors and three or four flavors and an octet of colored vector gluons, confined to a finite volume by a uniform pressure, the key innovation of the model. Non-strange quarks are massless, the hadronic mass scale being set by the confining pressure.

In the static cavity approximation[2] the quark fields are expressed in terms of the fermion creation and annihilation operators for the cavity eigenmodes

$$q(\vec{x},t) = \sum_n \left(q_n(\vec{x}) e^{-i\omega_n t} b_n + q_n^c(\vec{x}) e^{i\omega_n t} d_n^{\dagger} \right) \tag{1.1}$$

where q_n satisfies the Dirac equation for energy ω_n inside the cavity

$$(-i\vec{\alpha}\cdot\nabla + \beta m)\, q_n(\vec{x}) = \omega_n\, q_n(\vec{x}) \tag{1.2}$$

and the linear boundary condition on the surface of the cavity

$$i\vec{\alpha}\cdot\hat{n}\, q_n(\vec{x}) = -\gamma_0\, q_n(\vec{x}), \tag{1.3}$$

where $\hat{n}$ is the unit outward normal to the surface. The quark wave functions are normalized so that

$$\int q_n^{\dagger} q_m\, d^3x = \delta_{nm}. \tag{1.4}$$

They must also satisfy a quadratic boundary condition which is discussed below.

For a cavity of arbitrary shape containing only quarks, the hamiltonian to second order in the gluon coupling may be written

$$H = \int dV : q^{\dagger}(-i\alpha\cdot\nabla + \beta m) q : \; - \frac{1}{4}\int dV\, F^a_{\mu\nu} F^{a\mu\nu} + g\int dV\, j^a_{\mu} A^{a\mu} + \int B\, dV + E_0(V), \tag{1.5}$$

where the gluons behave like Maxwell fields to this order:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \tag{1.6}$$

$$j^a_\mu = g : \bar{q}\, \lambda^a \gamma_\mu\, q : \tag{1.7}$$

$$\partial^\mu F^a_{\mu\nu} = j^a_\nu \text{ in } V \text{ and } n_\mu F^{a\mu\nu} = 0 \text{ on } S. \tag{1.8}$$

In the usual notation g is the gluon coupling constant, λ^a are the 3x3 matrix generators of color SU(3) normalized so that $\mathrm{tr}(\lambda^a)^2 = 2$. Because of the linear boundary condition on $F^{a\mu\nu}$, only color singlet states can exist. The constant B is the term which provides the confining pressure and is renormalized by the zero-point energy of the fields. The finite part of the gluon and fermion zero-point energy is given by E_o, which depends on the shape of the cavity. The fermion energy term is accordingly normal-ordered. To find $E_o(V)$ it is necessary to know the sum of the eigenfrequencies for all fermion and gluon modes for a cavity of arbitrary shape. This problem is not addressed here. The order of magnitude of $E_o(V)$ is known from studies of the light hadrons, and its effect can be estimated qualitatively.[2] However, we have omitted it from our calculations of greatly distorted shapes.

The shape of the cavity is determined by requiring that the expectation value of the hamiltonian be minimized with respect to variations in its shape. This procedure results in imposing a non-linear surface boundary condition on the fields, which can be interpreted as balancing the field pressure against B, but for computational purposes it is more useful to impose the boundary condition variationally. Thus for a given state

$$|\psi(V)\rangle$$

the condition

$$\frac{\delta}{\delta V} \langle \psi(V) | H | \psi(V) \rangle = 0 \tag{1.9}$$

determines the shape of the hadron.

Without knowing in advance that the shape that minimizes

the energy is particularly simple, one faces the difficult task of solving the Dirac equation and Maxwell's equations in a cavity of arbitrary shape subject to the various linear boundary conditions -- a task which is impossible analytically. Various numerical and approximate techniques are available, however. One could set up a coordinate mesh and solve the equations numerically. A variational approach was taken instead, since it was readily adaptable to the problem of fixing a chosen collective variable. Trial fermion wave functions q_n and trial gluon vector potentials A_μ^a are constructed for the ground state. For a given fixed shape the parameters characterizing q and A_μ^a are varied and the stationary point

$$\delta \langle \Psi(V) | H | \Psi(V) \rangle = 0 \tag{1.10}$$

is located. Actually the precise form of the hamiltonian (1.5) cannot be used, but if some care is taken the stationary point of the proper variational expression provides an approximation to the solution of the Dirac and Maxwell equations and the linear boundary conditions.

Thus the problem can be regarded as being entirely variational with respect to the parameters characterizing the cavity shape and wave functions. To find the energy as a function of the expectation value of an operator $\mathcal{O}$,

$$\bar{\mathcal{O}} = \langle \Psi(V) | \mathcal{O} | \Psi(V) \rangle , \tag{1.11}$$

one simply adds the usual constraint term to the hamiltonian

$$H \rightarrow H - \lambda \mathcal{O} \tag{1.12}$$

and from the stationary point, obtains $E(\lambda)$ and $\bar{\mathcal{O}}(\lambda)$. Because $\lambda = dE/d\bar{\mathcal{O}}$ may have multiple solutions the method of Flocard et al.[4] was used instead, with the replacement

$$H \rightarrow H + c\,(\bar{\mathcal{O}} - \mathcal{O}_0)^2. \tag{1.13}$$

The parameter c is chosen to be large enough that the quadratic term as a function of $\bar{\mathcal{O}}$ has a larger curvature than $E(\bar{\mathcal{O}})$.

When the constrained hamiltonian (1.12) is minimized, the equations of motion and the boundary conditions are altered by the constraint. Since the new form of the equations of motion and boundary conditions can be rather complex, depending on the nature of the constraint, the variational approach is a convenient vehicle for the formulation of the constraint.

In the present manuscript we report the results of a preliminary calculation for a bag containing a quark and an antiquark -- in particular we have considered the state corresponding to the ρ meson with spin projection $|m_S| = 1$ on the axis of deformation. Its deformation as a function of the separation of the quarks has been studied. We obtain the expected result that the energy grows rapidly with quark separation, with a linear increase at large separation. We have also carried out the computation of the deformation of a quark-antiquark bag with

the gluon interaction switched off. Since the gluons are essential in ensuring that bag states have zero triality[1] it is expected that in the absence of gluons, a two-quark bag could be pulled apart into two bags of a single quark each. Indeed, our computation yields the expected result.

The preliminary calculation, although not providing new results, serves as a means of testing the concepts and techniques of the computation. These techniques will be applied in the calculation of the fission of a six-quark bag and a bag containing a pair of quarks and antiquarks. These applications are relevant to the study of the nuclear force and the decay of resonances.

2. Quasi-Degenerate Second Order Perturbation Theory[5]

In the present calculation the separation of the quarks is achieved by constraining the orbitals to separate into a left orbital and right orbital while preserving the spatial symmetry of quark occupation, which is present in the undistorted configurations of the low lying mesons. Thus the spatial part of the wave function is

$$\psi(1,2) = \psi_L(1)\psi_R(2) + \psi_R(1)\psi_L(2) \tag{2.1}$$

Expressing the left and right orbitals in terms of orthogonal symmetric and antisymmetric orbitals, we have

$$\psi_L = \psi_S - \mu\psi_A \;; \quad \psi_R = \psi_S + \mu\psi_A$$

$$\psi(1,2) = \psi_S(1)\psi_S(2) - \mu^2 \psi_A(1)\psi_A(2). \tag{2.2}$$

Thus we are led to consider a mixture of two orbital configurations S^2 and A^2 with the mixing parameter μ^2 ranging from 0 to 1 for maximal to minimal overlap between the orbitals. The mixing parameter μ^2 is to be determined variationally by minimizing the constrained hamiltonian.

The object of the present study is to calculate the separation energy of the quark-antiquark system to second order in the gluon coupling. If we regard the gluon interaction as a perturbation on the free fermion hamiltonian, the calculation reduces to an exercise in stationary second-order perturbation theory. Since the levels S^2 and A^2 become degenerate if the bag were to fission, it is necessary (expecially in anticipation of future fission calculations) to use quasi-degenerate perturbation theory: Given a hamiltonian H_o and perturbation V, let the perturbed ground state be expressed as a linear combination of the lowest (normalized) unperturbed states:

$$\alpha|a\rangle + \beta|b\rangle \tag{2.3}$$

where $|\alpha|^2 + |\beta|^2 = 1$. If the potential V does not connect a and b to first order, then the ground state energy to second order in V is obtained by minimizing with respect to α and β

the expression

$$E = |\alpha|^2\left(E_a + \sum_n \frac{|V_{na}|^2}{E_o - E_n}\right) + |\beta|^2\left(E_b + \sum_n \frac{|V_{nb}|^2}{E_o - E_n}\right) + 2\,\mathrm{Re}\,\alpha^*\beta \sum_n \frac{V_{na}^* V_{nb}}{E_o - E_n} \tag{2.4}$$

where

$$H_o|a\rangle = E_a|a\rangle \qquad \langle n|V|a\rangle = V_{na}$$

$$H_o|b\rangle = E_b|b\rangle \qquad \langle n|V|b\rangle = V_{nb}$$

$$E_o = |\alpha|^2 E_a + |\beta|^2 E_b . \tag{2.5}$$

This result is obtained by an elementary variational calculation. With $\beta = 0$ it reduces to the usual expression. With $E_a = E_b$ it gives the usual eigenvalue problem for degenerate perturbation theory.

In our application the perturbation V is the elementary two-quark, one-gluon vertex with fields expressed in terms of the unperturbed cavity eigenmodes. The state n consists of one gluon mode in addition to quark and anti-quark modes. We have made a number of simplifying approximations to Eq(2.4). For the diagonal terms in $|V_{na}|^2$ and $|V_{nb}|^2$ we have kept only the quark states a and b plus one gluon in the sum over n. Furthermore we neglect the difference between E_o, E_a, and E_b in the denominators of the terms second-order in V. The latter approximation has the effect of rendering all gluon fields stationary.[5] Although we are drawn to this approximation because it simplifies our calculation enormously, it can be partially justified on the grounds that when the a-b level separation is large so that the time-dependence of the transition current is important, the mixing will naturally favor the lower energy state, thereby reducing the value of β and consequently the contribution of the error in the important off-diagonal term $V^*_{na} V_{nb}$. When the a-b level separation is small, the approximation is valid anyway.

We are led finally to a calculation of the expectation value of the hamiltonian (1.5) on the state

$$|ss\rangle - \mu^2 |AA\rangle \tag{2.6}$$

with the static equations

$$\nabla^2 A^{a\mu} = -j^{a\mu} = -g:\bar{q}\lambda^a\gamma^\mu q: . \tag{2.7}$$

To make the form of the hamiltonian explicit, write

$$q = \sum_{u=S,A} (q_u b_u + q_u^c d_u^\dagger), \tag{2.8}$$

where b_S and b_A annihilate a quark in the symmetric and anti-symmetric orbitals. (The color, spin, and flavor quantum numbers have been suppressed.) The terms in $A^{a\mu}$ which contribute to second order are then

$$\begin{aligned} A^{a\mu} = {} & 2A^{a\mu}_{SS}(b_S^\dagger b_S - d_S^\dagger d_S) \\ & + 2A^{a\mu}_{AA}(b_A^\dagger b_A - d_A^\dagger d_A) \\ & + A^{a\mu}_{SA}(b_A^\dagger b_S + b_S^\dagger b_A - d_A^\dagger d_S - d_S^\dagger d_A), \end{aligned} \tag{2.9}$$

where the $A^{a\mu}_{uv}$ are c-numbers. We have assumed the corresponding currents are real for brevity. The c-number vector potentials must satisfy

$$\nabla^2 A^{a\mu}_{uv} = -g\,\bar{q}_u \lambda^a \gamma^\mu q_v , \tag{2.10}$$

with boundary condition for the corresponding electric and magnetic fields,

$$\hat{n}\cdot\vec{E}^a_{uv} = 0, \quad \hat{n}\times\vec{B}^a_{uv} = 0.$$

As will be shown below in Section 3 these conditions can be arranged to be satisfied by the variational calculation.

Let us consider the explicit expression for the expectation value of the hamiltonian on the state (2.6). The fermion contribution is simply

$$W_F = \frac{2\omega_S + 2\omega_A \mu^4}{1+\mu^4} . \tag{2.11}$$

The gluon contribution may be classified into the six diagrams of Fig. 1. It is evident that basically three types of static gluon interactions are present: one involving the S-S current coupling to the S-S current, one for the A-A current coupling to the A-A current, and the exchange one involving the A-S current. Taking into account the spin dependence, we find that each magnetic interaction either flips spins with respect to the deformation axis or does not. Corresponding to each type of interaction, we use the labels S,A,X where X denotes the exchange interaction. We use the label $\perp$ to denote spin-flip and z to denote non-flip. We use the label M and E to distinguish magnetic and electric energies.

The basic components of the static gluon interaction energies are then nine terms of the type

$$
\begin{aligned}
W_{MSz} &= \frac{1}{2}\int \vec{B}_{SSz}\cdot\vec{B}_{SSz}\,dV - \int \vec{J}_{SSz}\cdot\vec{A}_{SSz}\,dV \\
W_{MS\perp} &= \frac{1}{2}\int \vec{B}_{SS\perp}\cdot\vec{B}_{SS\perp}\,dV - \int \vec{J}_{SS\perp}\cdot\vec{A}_{SS\perp}\,dV \\
W_{ES} &= -\frac{1}{2}\int \vec{E}_{SS}\cdot\vec{E}_{SS}\,dV + \int \rho_{SS}\,\phi_{SS}\,dV.
\end{aligned}
\tag{2.12}
$$

Their contribution to the expectation value of the hamiltonian is given by

$$
\begin{aligned}
W_G = {} & \frac{2}{1+\mu^4}\left(W_{MSz}\lambda_1^a\sigma_1^z\cdot\lambda_2^a\sigma_2^z + W_{MS\perp}\lambda_1^a\sigma_1^\perp\cdot\lambda_2^a\sigma_2^\perp\right) \\
& + \frac{2}{1+\mu^4}\Big(W_{MSz}\lambda_1^a\sigma_1^z\lambda_1^a\sigma_1^z + W_{MS\perp}\lambda_1^a\sigma_1^\perp\cdot\lambda_1^a\sigma_1^\perp \\
& \qquad + W_{MXz}\lambda_1^a\sigma_1^z\lambda_1^a\sigma_1^z + W_{MX\perp}\lambda_1^a\sigma_1^\perp\cdot\lambda_1^a\sigma_1^\perp \\
& \qquad + W_{EX}\lambda_1^a\cdot\lambda_1^a\Big) \\
& + \frac{4\mu}{1+\mu^4}\left(W_{MXz}\lambda_1^a\sigma_1^z\cdot\lambda_2^a\sigma_2^z + W_{MX\perp}\lambda_1^a\sigma_1^\perp\cdot\lambda_2^a\sigma_2^\perp + W_{EX}\lambda_1^a\cdot\lambda_2^a\right) \\
& + \frac{2\mu^4}{1+\mu^4}\left(W_{MAz}\lambda_1^a\sigma_1^z\cdot\lambda_2^a\sigma_2^z + W_{MA\perp}\lambda_1^a\sigma_1^\perp\cdot\lambda_2^a\sigma_2^\perp\right) \\
& + \frac{2\mu^4}{1+\mu^4}\Big(W_{MAz}\lambda_1^a\sigma_1^z\lambda_1^a\sigma_1^z + W_{MA\perp}\lambda_1^a\sigma_1^\perp\lambda_1^a\sigma_1^\perp \\
& \qquad + W_{MXz}\lambda_1^a\sigma_1^z\lambda_1^a\sigma_1^z + W_{MX\perp}\lambda_1^a\sigma_1^\perp\lambda_1^a\sigma_1^\perp \\
& \qquad + W_{EX}\lambda_1^a\lambda_1^a\Big)
\end{aligned}
\tag{2.13}
$$

In this expression the terms have been grouped so as to correspond to the six graphs of Fig. 1. (The third and fourth graph contribute equally.)

For the ρ meson with spin projection $|m_S| = 1$

$$\lambda_1^a \lambda_2^a = -16/3 \qquad \lambda_1^a \lambda_1^a = 16/3$$

$$\lambda_1^a \sigma_1^z \cdot \lambda_2^a \sigma_2^z = -16/3$$

$$\lambda_1^a \sigma_1^\perp \cdot \lambda_2^a \sigma_2^\perp = 0.$$

$$\lambda_1^a \sigma_1^z \cdot \lambda_1^a \sigma_1^z = 16/3$$

$$\lambda_1^a \sigma_1^\perp \cdot \lambda_1^a \sigma_1^\perp = 32/3. \tag{2.14}$$

In the following section we discuss the mechanics of the variational computation.

3. The Variational Approach to the Quark and Gluon Energies

In this section we discuss the way the variational principle is put into practice by considering first the component parts of the computation as though they were independent of each other, and finally, the result of combining them into a simultaneous variation of all parameters at once.

a. The Bag Geometry

In the present study a three-parameter azimuthally symmetric surface has been considered, defined in cylindrical coordinates by

$$\rho^2 = g^2(z) = n^2(1 - z^2/d^2)(1 + a\, z^2/d^2), \tag{3.1}$$

where n is the cylindrical radius at z = 0, d is the length of extension in z and

$a = 0$	ellipse
$-1 < a < 0$	distorted ellipse - bulge in middle
$0 < a < 1$	distorted ellipse - flattened in middle
$1 < a$	peanut shape
$a \to \infty,\ n \to 0$	fission
$-\infty < a < -1$	two bags (3.2)

A considerable variety of shapes can be studied with such a parameterization, although it has a distinct limitation in that at the point of fission the two bag components have a tear-drop configuration. This leads to a slight over-estimate of the two-bag energy at this point which could be remedied by adding terms cubic in z^2 and higher.

b. Fermion Wavefunction in Isolation

The variational approach to finding the ground state wave function for the Dirac equation is complicated by the fact that the hamiltonian is not positive definite. Thus we have chosen to minimize the expectation value of the square of the Dirac hamiltonian

$$\omega^2 = \int q^\dagger (i\alpha\cdot\overleftarrow{\nabla} + \beta m)(-i\alpha\cdot\overrightarrow{\nabla} + \beta m)\, q\, dV \Big/ \int q^\dagger q\, dV. \tag{3.3}$$

Since the linear boundary condition is not reproduced by an unconstrained variation of this expression with respect to q, it is imposed explicitly in the construction of the trial expression for q:

$$i\vec{\alpha}\cdot\hat{n}\, q(\vec{x}) = -\gamma^0 q(\vec{x}) \quad \text{on } S. \tag{3.4}$$

Variation of ω^2 with q so constrained leads to the boundary condition

$$\vec{\alpha}\cdot\hat{n}\left(-i\alpha\cdot\vec{\nabla}+\beta m\right) q = -\gamma^0\left(-i\alpha\cdot\vec{\nabla}+\beta m\right) q \tag{3.5}$$

which is compatible with (3.4) when q satisfies the Dirac equation.[6]

Although minimization of (3.3) leads to an upper bound on the true value of the square of the ground state energy, it does not determine the eigenfunction uniquely, since particle and antiparticle wave functions give the same value for ω^2. Thus given any trial function q_0 which minimizes ω^2, there is a one-parameter family

$$q_\lambda = e^{\frac{1}{2}\lambda\gamma_0\gamma_5} q_0 = \cos\tfrac{\lambda}{2}\, q_0 + \sin\tfrac{\lambda}{2}\,\gamma_0\gamma_5\, q_0, \tag{3.6}$$

which has the same value of ω^2. To select the function which most nearly represents a state of positive energy, it suffices to maximize

$$\omega_\lambda = \int dV q_\lambda^\dagger\left(-i\alpha\cdot\vec{\nabla}+\beta m\right) q_\lambda \Big/ \int dV\, q^\dagger q \tag{3.7}$$

with respect to λ. Our choice of trial function has the feature that it maximizes ω_λ automatically, so this step is unnecessary.

For the present study the trial function is

$$q = \begin{pmatrix} \varphi\, u \\ i\sigma\cdot\vec{s}\,\varphi\, u \end{pmatrix}, \tag{3.8}$$

where u is a Pauli spinor and

$$\begin{aligned} \varphi &= \beta(z) + \alpha(z) R^2 \\ \vec{s} &= \nabla R/\lambda \;; \quad \lambda = |\nabla R|_{\rho = q(z)} \\ R^2 &= R_0^2 + \rho^2 - q^2(z) \\ R_0^2 &= \max_z q^2(z). \end{aligned} \tag{3.9}$$

The quantity R^2 is a generalization of the square of the spherical radius. For the sphere it is just r^2, and if q^2 were sufficiently general, for a cylinder, it would be ρ^2; and for two spherical bags it would be r_1^2 and r_2^2, the radius squared of the individual spheres. The vector $\vec{s}$ is constructed so that it is the unit normal to the surface when evaluated on the surface $[\rho = q(z)]$. For the sphere it is $\vec{r}/R_0$ and so vanishes at the origin.

The function φ depends on two functions of z, which were parameterized by

$$\beta_S = 1 \qquad \alpha_S = C_1$$
$$\beta_A = \tanh \eta z/d \qquad \alpha_A = C_1 \tanh \eta z/d, \tag{3.10}$$

for the symmetric and antisymmetric states considered in the present study. Because $\vec{s}$ becomes the unit normal on the surface the trial state satisfies the linear boundary condition explicitly. A more general function

$$q = \begin{pmatrix} \varphi u \\ i\sigma\cdot\vec{s}\,\chi u \end{pmatrix} \tag{3.11}$$

might be used to improve the calculation. It would satisfy the linear boundary condition if $\varphi = \chi$ on the surface.

c. Magnetic field

For the calculation of the gluon energy to second order in the gluon coupling, the self-interaction of the gluons may be neglected. The magnetic field obeys Maxwell's equations and the linear boundary condition

$$\nabla \times \vec{B}^a_{uv} = \vec{J}^a = q_u^\dagger \vec{\alpha} \lambda^a q_v \quad \text{in } V$$
$$\hat{n} \times \vec{B}^a_{uv} = 0 \quad \text{on } S \tag{3.12}$$

where q_u is the trial fermion wave function for the symmetric or antisymmetric orbital. Both conditions may be achieved varia-

tionally by writing $\vec{B}^a = \nabla\times\vec{A}^a$ and minimizing

$$W_M = \frac{1}{2}\int \vec{B}^a\cdot\vec{B}^a\,dV - \int \vec{J}^a\cdot\vec{A}^a\,dV \tag{3.13}$$

with respect to variations in $\vec{A}$. This procedure is quite straightforward, except for the added complication that the current $\vec{J}^a$, being composed of only approximate solutions to the Dirac equation, is only approximately conserved. Thus by replacing $\vec{A}^a \to \vec{A}^a + \nabla\psi$ one can make the energy W arbitrarily negative. A simple remedy is to fix the gauge of $\vec{A}^a$ by adding the term

$$W_M \to W_M + \frac{1}{2}\int (\nabla\cdot\vec{A}^a)^2\,dV. \tag{3.14}$$

If we write $\chi^a = \nabla\cdot A^a$, then minimizing E_B with respect to arbitrary variations in $\vec{A}^a$ yields the equations

$$\nabla\times\vec{B}^a = \vec{J}^a + \nabla\chi^a \quad \text{in } V \tag{3.15a}$$
$$\hat{n}\times\vec{B}^a = 0 \quad \text{and} \quad \chi^a = 0 \quad \text{on } S. \tag{3.15b}$$

The first equation yields

$$\nabla^2\chi^a = -\nabla\cdot\vec{J}^a, \tag{3.16}$$

and so for a conserved current, gives together with the boundary condition (3.15b) $\chi^a = 0$, i.e., no change in W_M. If the current is not conserved, the minimum value of W_M is

$$W_M^0 = -\frac{1}{2}\int \vec{B}^a\cdot\vec{B}^a dV - \frac{1}{2}\int (\nabla\cdot A^a)^2 dV, \tag{3.17}$$

where the second term compensates to some extent for the lack of current conservation.

The trial vector potential was patterned after its exact expression for the spherical and cylindrical cases:

$$\vec{A}^a(\rho,z) = \lambda^a\, \vec{\sigma}\times\vec{\xi}\, A(\rho,z)$$
$$A(\rho,z) = a_0 + a_1 R^2 + a_2 R^4 + a_3 R^6, \tag{3.18}$$

where the a_i are odd functions of z or constants depending on whether the current represents a transition between symmetric and antisymmetric orbitals or it is diagonal in the orbitals, respectively.

d. Electric field

The electric field must satisfy

$$\left.\begin{aligned} \vec{E}^a_{uv} &= -\nabla\phi^a_{uv} \\ \nabla\cdot\vec{E}^a_{uv} &= \rho^a_{uv} \end{aligned}\right\} \text{ in } V$$
$$\hat{n}\cdot\vec{E}^a_{uv} = 0 \quad \text{in } S. \tag{3.19}$$

This condition can be satisfied by demanding that the expression

$$W_E = -\frac{1}{2}\int \vec{E}^a\cdot\vec{E}^a dV + \int \rho^a\phi^a dV \tag{3.20}$$

be stationary with respect to variations in ϕ^a. Actually the stationary point in this example is a maximum rather than a minimum.[7]

The trial parameterization was chosen in analogy to the exact solution in the spherical configuration:

$$\phi = b_0 R^2 + b_1 R^4 + b_2 R^6 + (b_3 R^2 + b_4 R^4 + b_5 R^6)|\vec{\xi}|^2$$
$$+ w_1 r P_1(\cos\theta) + w_3 r^3 P_3(\cos\theta), \tag{3.21}$$

where the parameters $b_0 - b_5$ are odd functions of z or constants depending on whether the corresponding charge density is a transition charge density or a diagonal charge density. (In actual practice only one or two parameters were used to characterize $b_0 - b_5$.)

e. Constraint

In the present calculation the constraint is imposed directly on the quark wave function. We suppose that left and right orbitals can be distinguished by an operator z_R which has the property

$$z_R \psi_R(1) = z_1 \psi_R(1) \; ; \; z_R \psi_R(2) = z_2 \psi_R(2). \tag{3.22}$$

Then using (2.2)

$$\delta = \langle z_R - z_L \rangle = \frac{2\mu(1+\mu^2)}{1+\mu^4} \int \psi_s(\vec{x}) \psi_A(\vec{x}) z \, dV. \tag{3.23}$$

Thus δ measures the average separation of left and right orbitals. It is naturally linear in μ.

The constraint is implemented by adding to the variational hamiltonian the term

$$H \rightarrow H + c_\delta (\delta - \delta_0)^2 \tag{3.24}$$

f. Combined Variational Calculation

Combining the several components of the variational expression and varying with respect to all parameters at once leads to a number of interactions among the terms. The equations of motion for the fermion are altered by the constraint term. They are also modified by the presence of the gluon field. In the latter case the modification can be interpreted as an approximation to the introduction of the minimally coupled vector potential into the Dirac equation. The gluon terms interact among themselves to some extent with the gluon fields responding to the total currents to which they couple, as they indeed should.

4. Preliminary Results and Conclusions

a. Ground States without Constraint

Exact results have been obtained in Ref. 2 for the light hadrons, assuming a spherical cavity shape. It is interesting to compare them with the results of the variational calculation. If the cavity is forced to remain spherical in shape we obtain a value for the energy of the massless fermion which agrees extremely well (1.5% high) with the exact result. Our result is $E_Q \cdot R = 2.071$, whereas the result of Ref. 2 is 2.043. The parameter M_{00} which measures the strength of the color magnetic contribution to the energy from massless quarks is not in as good agreement (25% high). We obtain $M_{00} = 0.322$, compared with 0.259 from Ref.2. Since the variational expression yields an exact solution to Maxwell's equations for the trial current in the spherical cavity, the discrepancy is undoubtably due to the fact that errors in trial fermion wave function are reflected in the form of the fermion current. This problem could be diminished by the use of an improved parameterization of the fermion wave function.

When the shape of the ρ meson is permitted to deviate from sphericity, it is found that the energy is reduced. This is to be expected, since the magnetic dipole field exerts a non-uniform pressure on the bag surface. In the case of the ρ meson the deviation from sphericity is prolate for the state with spin projection $|m_S| = 1$ onto the axis of deformation and oblate for the state with spin projection $m_S = 0$. With the zero point energy omitted and gluon coupling $\alpha = .5$ the ratio of the polar radius to equatorial radius is $1.11 \pm .01$ for the ρ meson with $|m_S| = 1$. The reduction in energy for the same state is 0.4%. Thus the energy levels computed for the light hadrons in Ref. 2 are expected to remain essentially unaffected by treating non-spherical shapes so as to satisfy the non-linear boundary condition for the gluon field. The π meson is still spherical since its gluon fields exert a uniform pressure on the surface.

b. Deformed States with Constraint

In the present preliminary calculation the gluon self-energy terms in Eq. (2.13) have been omitted, except where they must be kept in order to satisfy the boundary conditions for the fields. Six parameters were used to characterize the quark and gluon fields in addition to the three parameters specifying the surface and the parameter μ specifying the orbitals. In Fig. 2 the deformation energy as a function of the expectation value of the separation parameter δ (3.23) is presented for two values of the gluon coupling constant, $\alpha = 0.5$ and $\alpha = 0.0$. The energy is given in units of $B^{1/4}$ and separation in units of $B^{-1/4}$. Calculations of the masses of the light hadrons[2] give $B^{1/4} \approx 145$ MeV or $B^{-1/4} \approx 1.29$ fm.

In the presence of gluons the energy rises quite steeply as the quark orbitals are separated. The slope at large separation agrees quite well with the result obtained from the cylinder approximation[8,9].

The slope expected from the cylinder approximation is

$$dE/d\delta = \sqrt{4\pi \cdot 16/3\,\alpha} = 5.79$$

The value from this computation is 6.00. The radius in the cylinder approximation is

$$\rho = \sqrt[4]{16/3\,\alpha/\pi} = .807.$$

The value obtained in this computation is in the range .800 to .833. In Fig. 3 are presented the shapes of the hadron corresponding to the various points indicated in Fig. 2. The contribution to the energy from the gluon electric and magnetic fields are shown in Fig. 4. The electric field obviously contributes the major portion of the energy at large separation with the volume term giving most of the rest.

In the absence of gluons ($\alpha = 0$) the energy rises briefly and then plateaus at a value about 10% higher than the actual energy of two bags with one quark each. If we consider the shape of the bag, shown in Fig. 3, we see that the n ck radius shrinks to zero in the bag without gluons, indicating that it undergoes fission. The 10% energy difference can be attributed to the inadequacy of the three-parameter description of the surface, which causes the two bags to assume a tear-drop shape at the point of fission.

c. Conclusion

The calculation presented here represents the first effort at treating the deformation properties of bags containing massless quarks. Kuti and co-workers pioneered with a calculation involving point-like massive quarks.[8] Many things have been of necessity omitted from the calculation in the interest of simplicity, including the zero-point energy of the fields, the center of mass motion, surface fluctuations, the projection onto states of definite total angular momentum, terms higher order in the gluon coupling, etc. With time these improvements may be incorporated. That our results correspond favorably to our naive theoretical expectations supports our optimism that the techniques presented here are sound and may be applied with some confidence to the more difficult problems of resonance decay and the nuclear force, where confrontation with experiment is possible.

Acknowledgment

This work began in collaboration with several members of the Center for Theoretical Physics, each of whom contributed significantly to its early theoretical development -- in particular Alan Chodos, Barry Freedman, Kenneth Johnson, Joseph Kiskis, Ernest Moniz, and Charles Thorn. Special thanks are also due Arthur Kerman and John Negele for discussions of related calculations in nuclear physics. It should be mentioned that Alan Chodos and Ernest Moniz are currently pursuing an alternate formulation of the bag deformation problem and we have benefitted mutually by comparing observations on nearly all details of the calculation.

Figure Captions

1. Diagrams for the gluon coupling to second order in perturbation theory.
2. Total energy of the bag in units of $B^{1/4} \approx 145$ MeV as a function of the average separation of the quark orbitals δ in units of $B^{-1/4}$ for two values of the gluon coupling constant. The shape of the bag at points A,B, and C is indicated in Fig. 3.
3. Shape of one quadrant of the bag in longitudinal section at the four points A,B, and C of Fig. 2. The z-axis is the symmetry axis.
4. Contribution to the total bag energy from the color electric and color magnetic field in units of $B^{1/4}$ versus the average separation of the quark orbitals δ, for two values of the gluon coupling constant.

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Footnotes and References

1. A. Chodos, R.L. Jaffe, K. Johnson, C.B. Thorn, and V.F. Weisskopf, Phys. Rev. D9, 3471 (1974); A. Chodos, R.L. Jaffe, K. Johnson, and C.B. Thorn, Phys. Rev. D10, 2599.
2. T. Degrand, R.L. Jaffe, K. Johnson, and J. Kiskis, Phys. Rev. D12, 2060 (1975).
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4. H. Flocard, P. Quentin, A.K. Kerman, and D. Vautherin, Nucl.Phys. A203, 433 (1973).
5. The advantages of appealing to quasi-degenerate perturbation theory and the approximations leading to a calculation with static fields have been stressed by A. Chodos and E. Moniz (private communication).
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7. This feature can be traced to the fact that ϕ is not a canonical variable -- the variational method is merely a convenient device for obtaining the equation of constraint (3.19) which it satisfies.
8. P. Hasenfratz, J. Kuti, and A. Szalay, in proceedings of the Xth Rencontre de Moriond, ed. by Tran Thanh Van, March 2, 1975.
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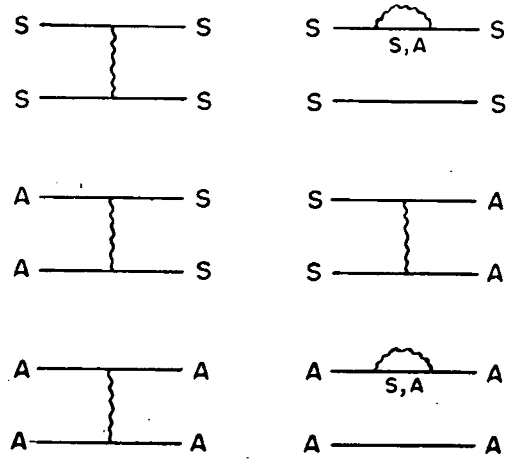
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Fig. 1

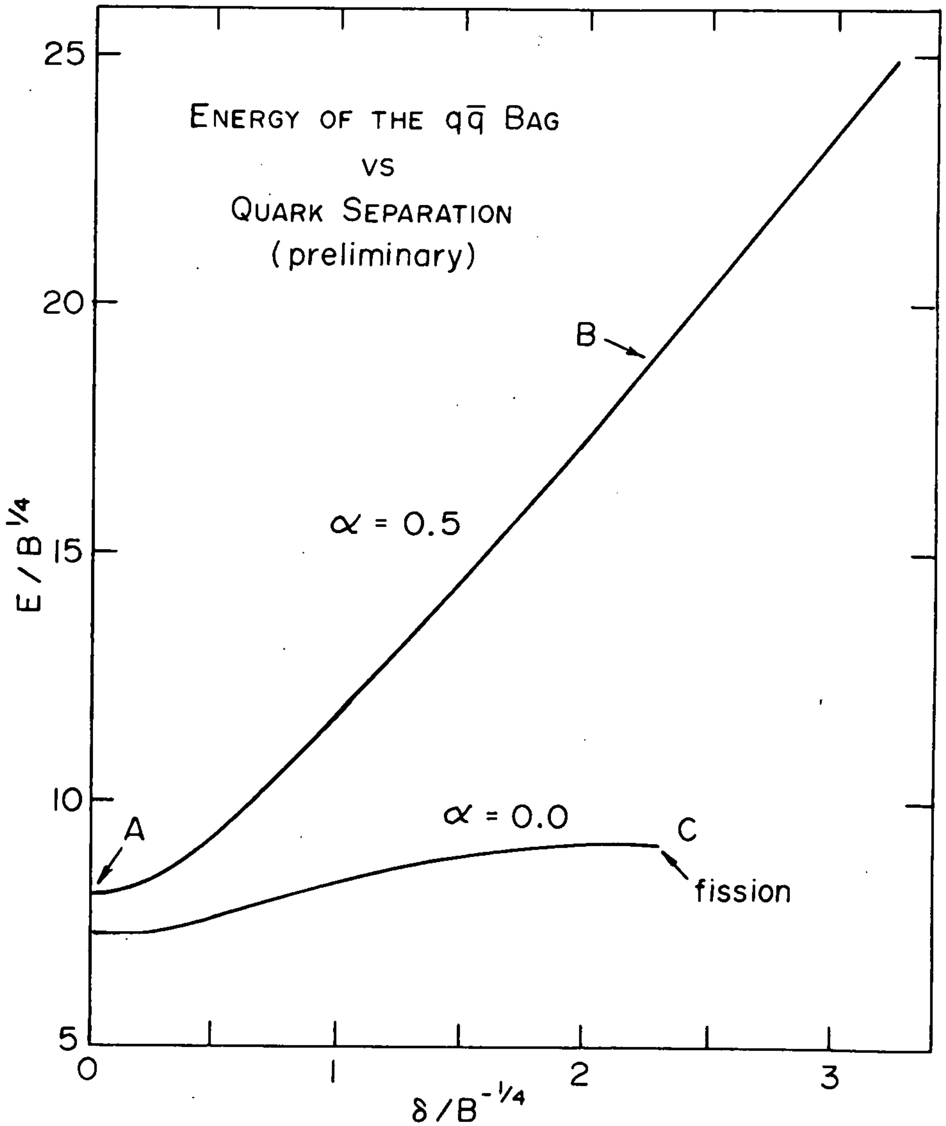
ENERGY OF THE $q\bar{q}$ BAG
VS
QUARK SEPARATION
(preliminary)
$E/B^{1/4}$
$\delta/B^{-1/4}$
B
A
C
fission
$\alpha = 0.5$
$\alpha = 0.0$
25
20
15
10
5
0
1
2
3

Fig. 2

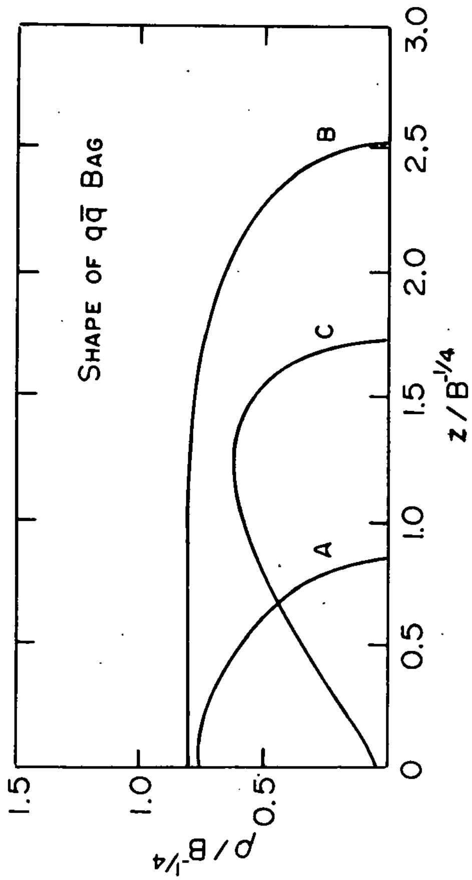
Shape of $q\bar{q}$ Bag
A
B
C
$z / B^{-1/4}$
$\rho / B^{-1/4}$
0
0.5
1.0
1.5
2.0
2.5
3.0

Fig. 3

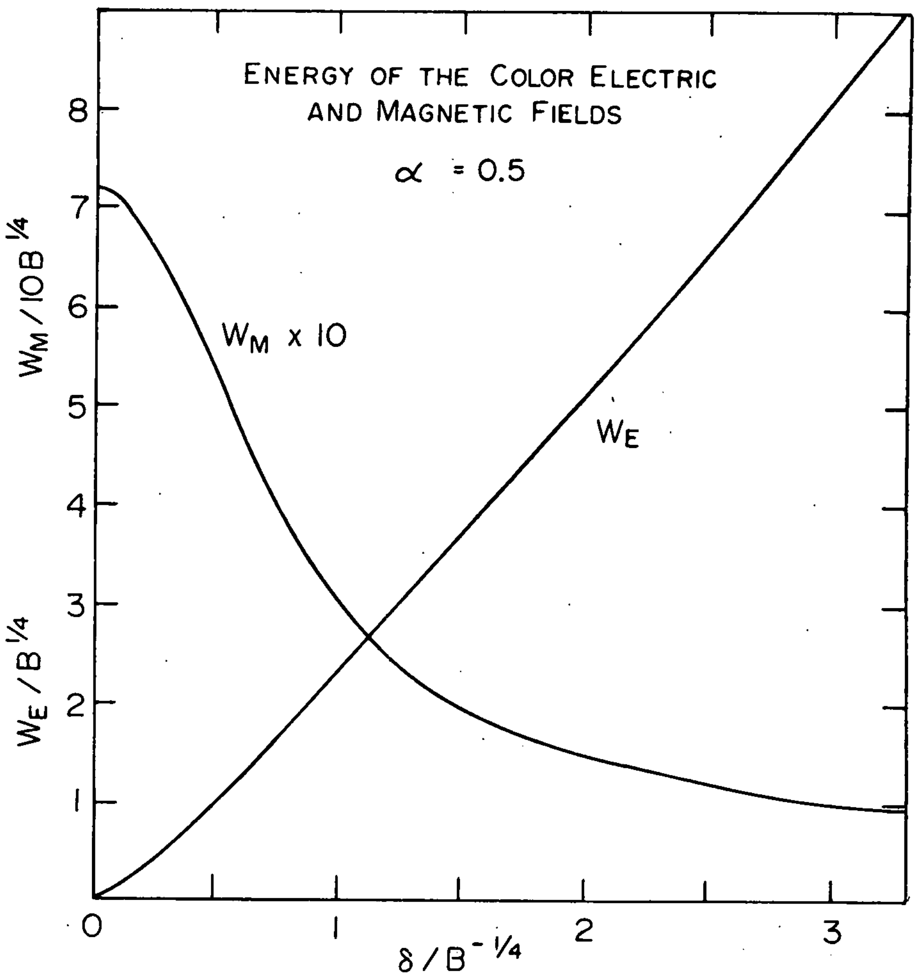
Energy of the Color Electric and Magnetic Fields
$\alpha = 0.5$
$W_M \times 10$
W_E
$W_M / 10B^{1/4}$
$W_E / B^{1/4}$
$\delta / B^{-1/4}$
0
1
2
3
4
5
6
7
8

Fig. 4