TECHNICAL MEMORANDUM

ANL-CT-76-30

MASTER

DYNAMIC RESPONSES OF HEAT EXCHANGER TUBE BANKS

S. S. Chen and J. A. Jendrzejczyk

Components Technology Division

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PREFACE

The work reported herein was performed as part of the base technology activity under the Flow Induced Vibration Programs (189a Nos. CA054 and CA070) sponsored by ERDA/RDD. The overall objective of the activity is to develop new and/or improved analytical methods and guidelines for designing LMFBR components to avoid detrimental flow induced vibration.

Heat exchanger tubes and reactor fuel pins are long, slender, beam-like components typically arranged in bundles and immersed in a flowing liquid. As such, they are susceptible to flow induced vibration. The excitation mechanism may be associated with vortex-shedding, fluidelastic interaction, or random pressure fluctuations in the turbulent flow. Designing to avoid large amplitude motion, that is, to avoid a resonance condition or instability condition, and the prediction of component response, require knowledge of the dynamic behavior of the components. However, cylinders in a closely spaced bundle do not respond as single cylinders immersed in a liquid, rather, interaction with the liquid causes coupled motion of groups of cylinders. The fundamental natural frequency of the coupled system will be lower than that of a single cylinder immersed in a liquid.

Understanding and modeling fluid/structure interaction in cylinder bundles is a basic requirement in the development of analytical methods and guidelines for designing heat exchanger and reactor fuel assemblies that are free from component vibration problems. As a step toward satisfying this requirement, in this report, an analytical and experimental study of tube banks vibrating in liquids is presented. Firstly, a general method of analysis is presented for free and forced vibrations of tube banks including tube/fluid interaction; numerical results are given for tube banks subjected to various types of excitations. Secondly, two cantilevered tubes are tested in a water tank; the natural frequencies and forced responses of coupled motion are measured. Experimental data and analytical results are found to be in reasonably good agreement.

The analytical method presented is currently being extended to account for the flowing fluid in tube banks and will be used in the development of the mathematical models for crossflow- and parallel-flow-induced vibrations of tube bundles. Those models will be useful in predicting the response of tube bundles and in design to avoid detrimental vibration.

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NOMENCLATURE

a_{ni}	Coefficient given by Eq. (10)
$\bar{a}_{nm}$	Coefficient given by Eq. (A.5)
A	Column vector describing the initial displacement, see Eq. (9)
b_{ni}	Coefficient given by Eq. (10)
$\bar{b}_{nm}$	Coefficient given by Eq. (A.5)
B	Column vector describing the initial velocity, see Eq. (9)
c_i	Viscous damping coefficient
c_{ij}	Damping coefficient given in Eq. (10)
C	Damping matrix
D	Tube diameter
e_i	Excitation force per unit length acting on the tube
E_i	Modulus of elasticity
f	Excitation frequency
f_n	Natural frequency of a solitary tube given by Eq. (21)
f_n^ℓ	The lower bounding frequency of the nth passing band given by Eq. (22)
f_n^u	The upper bounding frequency of the nth passing band given by Eq. (22)
f_v	Vortex shedding frequency
F_i	The hydrodynamic force acting on the tube given by Eq. (2)
$\bar{F}_1, \bar{F}_2$	Concentrated forces acting on tubes 1 and 2
g_i	Initial displacement of the tube
h_i	Initial velocity of the tube
H_{ni}	Frequency response function given by Eq. (18)
I	Identity matrix
I_i	Moment of inertia
$I_{nms\ell}$	Cross spectral density given by Eq. (20)

NOMENCLATURE (Contd.)

k	Number of tubes
$\bar{k}, k_1, k_2$	Spring constant
k_{nij}	Matrix element given by Eq. (10)
K	Stiffness matrix
ℓ	Tube length
m	Tube mass per unit length
m_i	Mass per unit length of tube i
m_{ij}	Mass matrix element given by Eq. (10)
M	Mass matrix
M', M_1', M_2'	Concentrated masses
p	Dimensionless parameter given by Eq. (B.6)
p_{ni}	Generalized force given by Eq. (10)
P	Tube pitch
q_i	Response of tube i
q_{ni}	Generalized coordinates
Q	Column vector describing tube response
R, R_i	Tube radius
S	Weighted modal matrix
S_t	Strouhal number
$S_{n\ell j}$	Elements of the weighted modal matrix
t	Time
u, u_i	Tube displacement
$\bar{u}_i$	RMS tube displacement
U	Flow velocity
w_{nj}	Modal response of tube given by Eq. (18)
x, y, z	Cartesian coordinates

NOMENCLATURE (Contd.)

α_1,α_2	Coefficients given by Eq. (A.5)
$\bar{\alpha}_1,\bar{\alpha}_2,$ $\bar{\alpha}_3,\bar{\alpha}_4$	Coefficients given by Eq. (B.12)
β	Mass ratio given by Eq. (B.6)
β_i	Mass ratio given by Eq. (A.5)
γ	Eigenvalue obtained from Eq. (B.13)
γ_{ij}	Added mass matrix
δ_{ij}	Kronecker's delta
ζ_{nj}	Modal damping ratio
λ	1 for in-plane motion and -1 for out-of-plane motion
Λ	A diagonal matrix
μ_i	Effective added mass
μ_i'	Effective added mass coefficient
$\bar{\mu}_1,\bar{\mu}_2,\bar{\mu}_3$	Added mass coefficients
ν_{ij}	Added mass coefficient matrix
ρ	Fluid density
ϕ_n	Orthonormal function of tubes in vacuo
ϕ_{in}	Orthonormal function of tube i in vacuo
$\Phi_{u_iu_i}$	Power spectral density of the displacement
ψ_{in}	Adjoint orthonormal function of tube i in vacuo
ω_n,ω_{ni}	Circular frequency of nth mode in vacuo
Ω_1,Ω_2	Coupled frequencies of two tubes
Ω_{ni}	Frequencies of coupled modes

ABSTRACT

Flow-induced vibration in heat exchanger tube banks is of great concern, particularly in high performance heat exchangers used in nuclear reactor systems. In this report, an analytical and experimental study of the general dynamic characteristics of tube banks in liquids is presented. First, a method of analysis is presented for free and forced vibrations of tube banks including tube/fluid interaction; numerical results are given for tube banks subjected to various types of excitations. Second, two cantilevered tubes are tested in a water tank; the natural frequencies and forced responses of coupled motion are measured. Experimental and theoretical results are in reasonably good agreement.

I. INTRODUCTION

Flow-induced vibrations of heat exchanger tubes have been studied extensively, either because of excessive noise or because of resultant damage [1,2]. The problems become more serious in high performance heat exchangers used in nuclear reactor systems. In the past, analytical studies were primarily made based on a single tube consideration; i.e., considering the motion of a single tube in a tube bank to establish the critical flow velocity at which large oscillations occur. Little effort has been made to include the fluid coupling effect.

In a tube bank, the motion of a tube will excite the surrounding ones due to fluid coupling; therefore, the tubes in a tube bank will respond as a group, rather than as a single tube. Unfortunately, the existing mathematical models for tube-bank vibrations are not based on the coupled vibration modes. The objective of this report is to provide some additional insight into the general dynamic characteristics of coupled tube/fluid vibration such that a better mathematical model incorporating fluid coupling effect may be developed.

First, a general method of analysis is presented for free and forced vibrations of tube banks including tube/fluid interaction; numerical results are given for tube banks subjected to various types of excitations. Second, two cantilevered tubes are tested in a water tank; the natural frequencies and forced responses of coupled motion are measured. Experimental and theoretical results are in reasonably good agreement.

II. ANALYSIS

A. Tube/Fluid Interaction

An elastic tube bank subject to a fluid flow experiences complicated fluid forces. Those forces may be classified as steady-state and fluctuating components, or force components as functions of displacement, velocity, and acceleration. It is beyond the scope of this paper to consider all forces associated with tube banks vibrating in a flowing fluid; rather, this study is restricted to the case of tube banks vibrating in a stationary fluid.

Consider a tube bank consisting of k tubes whose axes are parallel to the z-axis (Fig. 1). Each tube can move in the x and y directions. Let u_i and u_{k+i} designate the displacement components of tube i in the x and y directions respectively. The fluid inertia forces acting on tube i are given by F_i in the x direction and F_{k+i} in the y direction. F_i is given by [3]

$$F_i = \sum_{j=1}^{2k} \gamma_{ij} \frac{\partial^2 u_j}{\partial t^2} \quad , \tag{1}$$

$$i = 1, 2, 3, \ldots 2k \quad ,$$

where t is time and γ_{ij} is the added mass matrix. Equation (1) may also be written

$$F_i = \sum_{j=1}^{2k} \left(\frac{R_i + R_j}{2}\right)^2 \rho \nu_{ij} \frac{\partial^2 u_j}{\partial t^2} \quad , \tag{2}$$

where R_i is the radius of tube i, ρ is fluid density and ν_{ij} is the added mass coefficient matrix. When all tubes are of the same radius, γ_{ij} is equal to ν_{ij} multiplied by the fluid mass displaced by the tube.

Since γ_{ij} is symmetric [3], for a group of k tubes, it possesses 2k eigenvalues, designated by μ_i (i = 1 to 2k). The eigenvalues of the matrix γ_{ij} for a tube bank play the same role as the added mass for a solitary tube; those eigenvalues are called effective added masses.

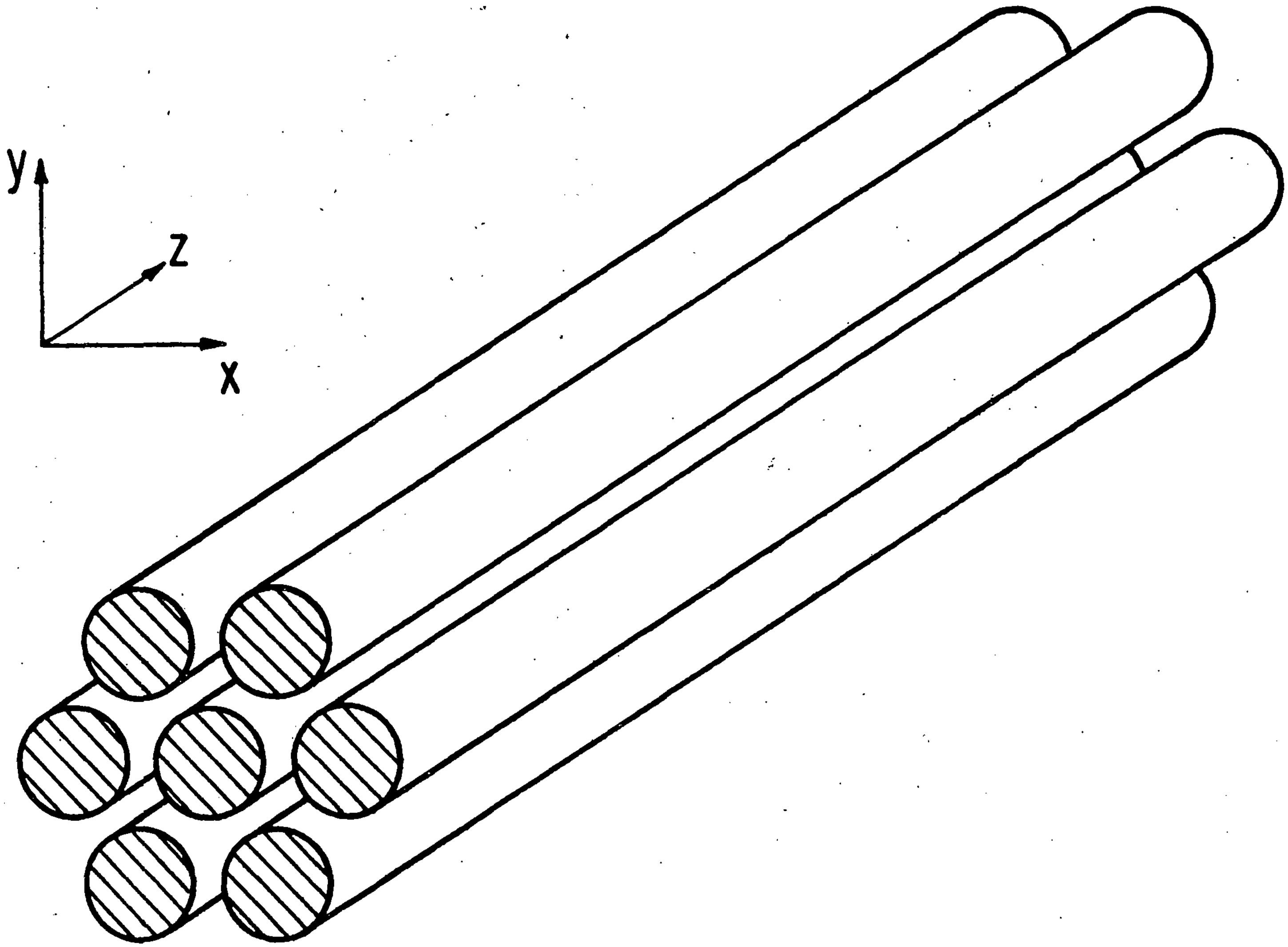

Fig. 1. Schematic of a group of tubes vibrating in a liquid

The computational procedure for γ_{ij} is given in Ref. 3 based on the potential flow theory. It must be kept in mind the limitation of a potential flow analysis when applied to real-fluid flow analysis.

B. Equations of Motion

Consider the motion of a group of k tubes in a liquid. The equations of motion are [3]

$$E_iI_i \frac{\partial^4 u_i}{\partial z^4} + c_i \frac{\partial u_i}{\partial t} + m_i \frac{\partial^2 u_i}{\partial t^2} + \sum_{j=1}^{2k} \gamma_{ij} \frac{\partial^2 u_j}{\partial t^2} = e_i \quad , \tag{3}$$

where m_i is tube mass per unit length, E_iI_i is flexural rigidity, c_i is damping coefficient, e_i is excitation force, and the other variables have been defined previously. Note that the variables with the index i (or j) from 1 to k are associated with the motion in the x direction while from k+1 to 2k in the y direction. For simplicity, the simply-supported end condition is considered; i.e., at x = 0 and ℓ,

$$u_i = 0 \quad ,$$

and (4)

$$\frac{\partial^2 u_i}{\partial z^2} = 0 \quad ,$$

where ℓ is the length of the tubes. Without loss of generality, the initial state of the tubes may be assumed as follows:

$$u_i(z,t)\Big|_{t=0} = g_i(z) \quad ,$$

and (5)

$$\frac{\partial u_i(z,t)}{\partial t}\Big|_{t=0} = h_i(z) \quad .$$

Equations (3), (4), and (5) are the complete mathematical statement of the problem.

Let

$$u_i(z,t) = \sum_{n=1}^{\infty} q_{ni}(t)\phi_n(z) \quad , \tag{6}$$

where ϕ_n is the nth orthonormal function of the tube in vacuo; i.e.,

$$\frac{1}{\ell}\int_0^\ell \phi_m(z)\phi_n(z)dz = \delta_{mn} \quad . \tag{7}$$

Using Eqs. (3), (5), (6) and (7) yields

$$[M]\{\ddot{Q}\} + [C]\{\dot{Q}\} + [K]\{Q\} = \{P\} \quad , \tag{8}$$

and

$$\begin{aligned} \{Q\}_{t=0} &= \{A\} \quad , \\ \{\dot{Q}\}_{t=0} &= \{B\} \quad , \end{aligned} \tag{9}$$

where [M], [C], and [K] are symmetric matrices with elements m_{ij}, c_{ij}, and k_{nij}, and {P}, {A}, and {B} are generalized force, initial displacement, and initial velocity with elements p_{ni}, a_{ni}, and b_{ni}, in which

$$\begin{aligned} m_{ij} &= m_i\delta_{ij} + \gamma_{ij} \quad , \\ c_{ij} &= c_i\delta_{ij} \quad , \\ k_{nij} &= m_i\omega_{ni}^2\delta_{ij} \quad , \\ p_{ni} &= \frac{1}{\ell}\int_0^\ell e_i\phi_n dz \quad , \\ a_{ni} &= \frac{1}{\ell}\int_0^\ell g_i\phi_n dz \quad , \\ b_{ni} &= \frac{1}{\ell}\int_0^\ell h_i\phi_n dz \quad , \end{aligned} \tag{10}$$

and ω_{ni} is the nth natural frequency (in radian) of tube i in vacuo.

C. Free Vibration

For free vibration, neglect damping and forcing terms and let

$$\{Q\} = \{\bar{Q}\}\exp(i\Omega t) \quad . \tag{11}$$

Natural frequencies and mode shapes of coupled tube/fluid vibration are computed from the undamped homogeneous equations

$$[K]\{\bar{Q}\} = \Omega^2[M]\{\bar{Q}\} \quad . \tag{12}$$

Let [S] be the weighted modal matrix formed from the columns of eigenvectors, it is easily shown that

$$[S^T K S] = [\Lambda] \quad ,$$

and (13)

$$[S^T M S] = [I] \quad ,$$

where [I] is an identity matrix and [Λ] is a diagonal matrix formed from the eigenvalues Ω^2.

In many cases, the tubes in a heat-exchanger tube bank may be considered to be identical. Furthermore, it is assumed that all tubes have the same mass m ($m_i = m$), natural frequency ω_n ($\omega_{ni} = \omega_n$). In this case, Eqs. (12) become

$$[\gamma_{ij}]\{\bar{Q}\} - \left[m \left(\frac{\omega_n^2}{\Omega^2} - 1 \right) \delta_{ij} \right] \{\bar{Q}\} = \{0\} \quad . \qquad (14)$$

It can be shown that the solutions of Eq. (14) are given by

$$\Omega_{ni} = \left(\frac{m}{m + \mu_i} \right)^{1/2} \omega_n \quad , \qquad (15)$$

$$i = 1, 2, 3, \ldots 2k \quad ,$$

$$n = 1, 2, 3, \ldots \infty \quad ,$$

where μ_i is the effective added mass, and Ω_{ni} is the natural frequency of a coupled mode. Equation (15) shows that the natural frequency of a coupled mode can be calculated from the natural frequency of a single tube and the effective added mass.

D. Forced Vibration

The responses of tube banks to an excitation can readily be calculated from Eqs. (6), (8), and (9). In many practical situations, the damping matrix may be assumed to be proportional to the stiffness matrix K or the mass matrix M. Let

$$\{Q\} = [S]\{W\} \quad . \tag{16}$$

Using Eqs. (8), (13), and (16) yields a set of uncoupled modal equations:

$$\ddot{w}_{nj} + 2\zeta_{nj}\Omega_{nj}\dot{w}_{nj} + \Omega_{nj}^2 w_{nj} = \sum_{\ell=1}^{2k} S_{n\ell j} p_{n\ell} \quad , \tag{17}$$

$i = 1, 2, 3, \ldots 2k$,

$n = 1, 2, 3, \ldots \infty$.

ζ_{ni} is the modal damping ratio of the coupled modes. Equations (17) are easily solved and the tube responses can readily be computed.

E. Response to Random Force Inputs

The method developed above gives the tube response to deterministic force inputs. In many cases, the forces are not completely predictable. The response to a random input will, in general, be random itself. If the average values in the random process are constant with time it is said to be stationary. The general solution for the steady state random case is considered.

The steady-state solution for w_{ni} obtained from Eq. (17) is

$$w_{ni} = \sum_{j=1}^{2k} S_{nji} H_{ni}(\Omega) p_{nj} \quad ,$$

where (18)

$$H_{ni}(\Omega) = [(\Omega_{ni}^2 - \Omega^2) + 2\sqrt{-1}\,\zeta_{ni}\Omega_{ni}\Omega]^{-1} \quad .$$

Using Eqs. (6), (10), (17), and (18), we can show that the mean-square displacement $\bar{u}_i^2(z)$ is

$$\bar{u}_i^2(z) = \int_0^\infty \Phi_{u_i u_i}(z,z',\Omega)\,d\Omega \quad , \tag{19}$$

where $\Phi_{u_i u_i}$ is the power spectral density of the displacement and is given by

$$\Phi_{u_i u_i}(z,z',\Omega) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{j=1}^{2k} \sum_{\ell=1}^{2k} \sum_{r=1}^{2k} \sum_{s=1}^{2k} S_{nim} S_{n\ell j} S_{mir} S_{msr} \left| H_{nj}(\Omega) H_{mr}(\Omega) \right|$$

$$\cdot \phi_n(z) \phi_m(z') I_{nms\ell}(\Omega) \quad , \tag{20}$$

$$I_{nms\ell}(\Omega) = \frac{1}{\ell^2} \int_0^{\ell} \int_0^{\ell} \Phi_{e_s e_\ell}(z,z',\Omega) \phi_n(z) \phi_m(z') dz dz' \quad .$$

$I_{nms\ell}$ is the cross spectral density of the generalized force and $\Phi_{e_s e_\ell}$ is the cross spectral density of the excitation. Based on those results, the quantities of interest can be calculated once the excitation properties are known.

III. EXPERIMENTAL METHOD

A system consisting of two tubes is one of the simplest systems having coupled motion. It can be used to demonstrate the essential characteristics of coupled vibration. Therefore, tests are made with two cantilevered tubes.

The test rig is shown in Figure 2. The rig is fabricated of a four side lucite box fastened to a brass plate to form a water tank allowing easy observation of motion.

The two specimen tubes are constructed of brass tubes, 30.48 cm (12") long, 1.27 cm (0.5") O.D., and 0.159 cm (1/16") wall thickness. Each tube is silver soldered to a brass base. The brass bases are then fastened to the tank base with provision to vary the gap between the bases. The desired gap is set by use of shims of the required thickness between the brass bases, which are then removed after securing the bases to the tank plate.

The details inside each tube are shown in Figure 3. Two small permanent magnets are mounted near the top of the tube in two perpendicular directions and two accelerometers are mounted to a small aluminum mounting block at 90° to each other and their sensitive axis in line with the magnets. The leads to accelerometers are routed through the tubes, brass bases, and tank plate.

The excitation is produced by the permanent magnets and external coils. The coils are supported by aluminum bars fastened to the brass base. The amplitude of the exciting force is held constant as the exciting frequency is uniformly increased by the Hewlett-Packard 5451/71A analyzer. The tube acceleration is continuously monitored.

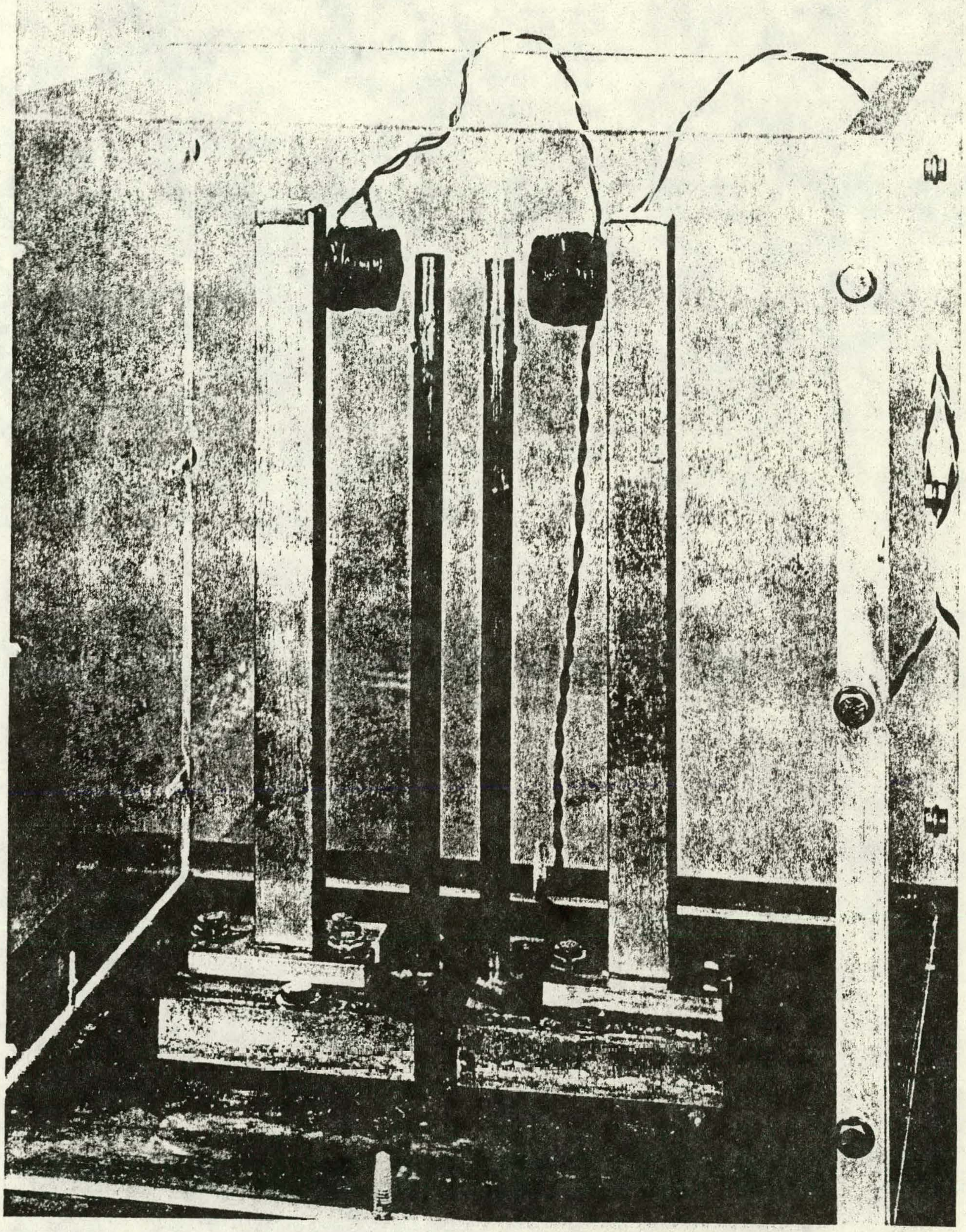

Fig. 2. Test assembly used to study coupled vibration of tubes in a liquid

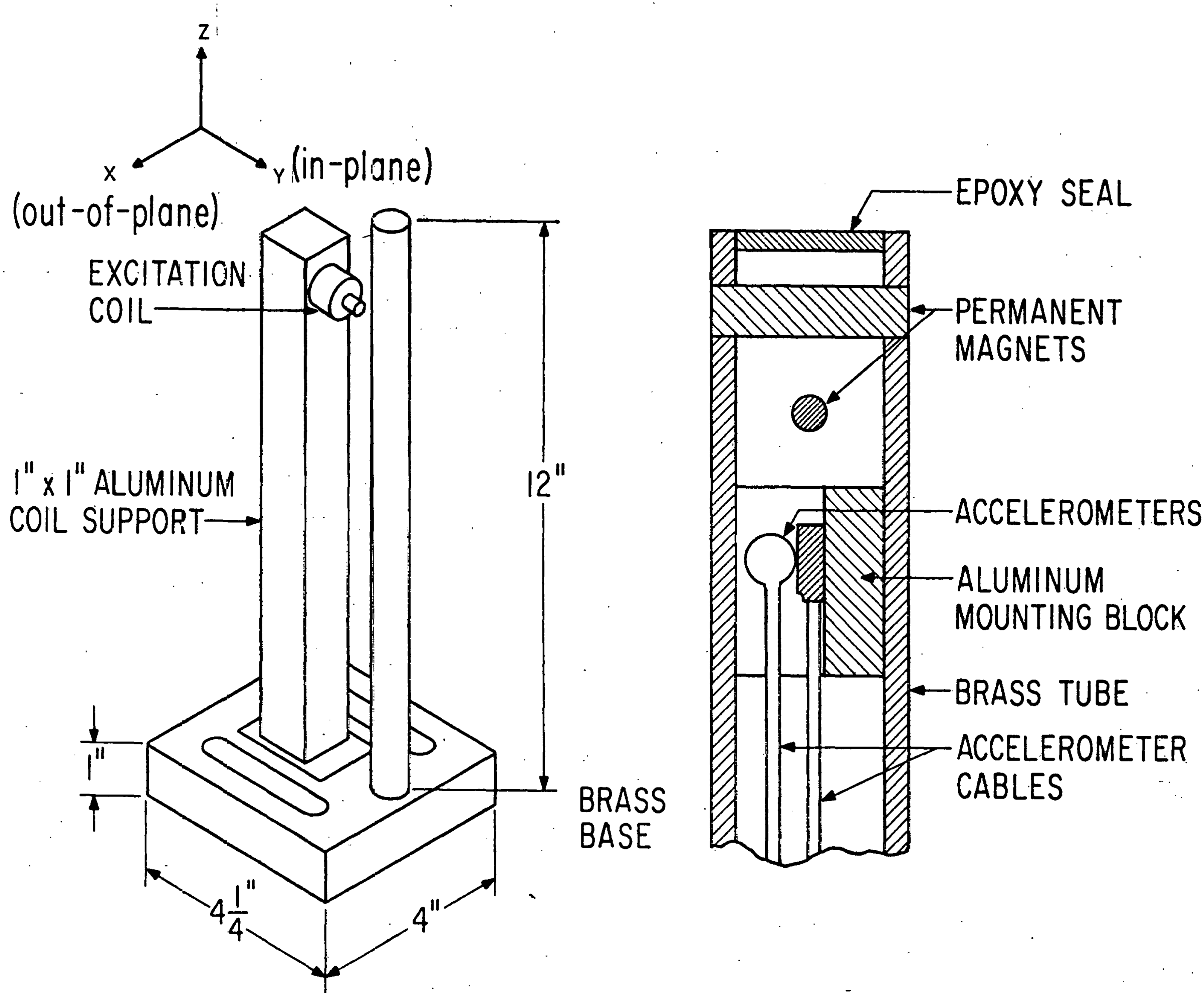

Fig. 3. Test element

Tests were made for four different gaps between the two tubes. In each case, the frequency response of each tube is measured in air. Then the response is measured in water by exciting one tube and restraining the other to eliminate the coupling effect. Finally, the restraint is removed and the coupled response is obtained by exciting one of the tubes.

In addition to the tests for steady-state responses, transient responses were also measured for out-of-plane motion. In this case, one of the tubes is given a small disturbance by plucking it. The accelerations of the two tubes are measured.

IV. ANALYTICAL RESULTS

For presentation, stainless steel tubes with the following properties are considered: outside diameter = 2.223 cm (0.875 in.), wall thickness = 0.114 cm (0.045 in.), and length = 76.2 cm (30 in.). The tubes are arranged in several patterns: row, array, and hexagonal arrangement. The pitch of tube banks is assumed to be 3.332 cm (1.312 in.) in all cases except in the study of effective added mass where the pitch is varied. All tubes are assumed to be simply-supported at both ends and submerged in a sodium and containing sodium at 516°C (960°F). The viscous damping coefficient c_i is assumed to be 4.1 kg-sec/M^2 (0.0059 lb-sec/in.^{2}).

A. Effective Added Masses and Effective Added Mass Coefficients

Many tube banks are arranged in a hexagonal pattern as shown in Fig. 4. The added mass coefficient ν_{ij}, and the effective added mass coefficient μ_i', which is the eigenvalue of $[\nu_{ij}]$, are computed for three tube banks consisting of seven, nineteen, and thirty-seven tubes.

Figure 5 shows the variation of ν_{ij} with P/D where P is tube pitch and D is tube diameter. Note that ν_{ii} is positive and approaches one as P/D becomes very large, while ν_{ij} for $i \neq j$ may be negative and approaches zero as P/D becomes infinite.

Figure 6 shows the upper and lower bounds of μ_i' as functions of P/D. For a group of k tubes, there are 2k effective added mass coefficients distributed between the upper and lower bounds. All μ_i' approach one as P/D is increased, while the upper bound increases and lower bound decreases as P/D is reduced.

From Figs. 5 and 6, it is seen that as the number of tubes increases, μ_i' and the absolute values of ν_{ij} increase. The results for tube 1 and 2 obtained from 19 tubes and 37 tubes do not differ significantly. The implication is that the coupling between a tube and other tubes which are not immediately

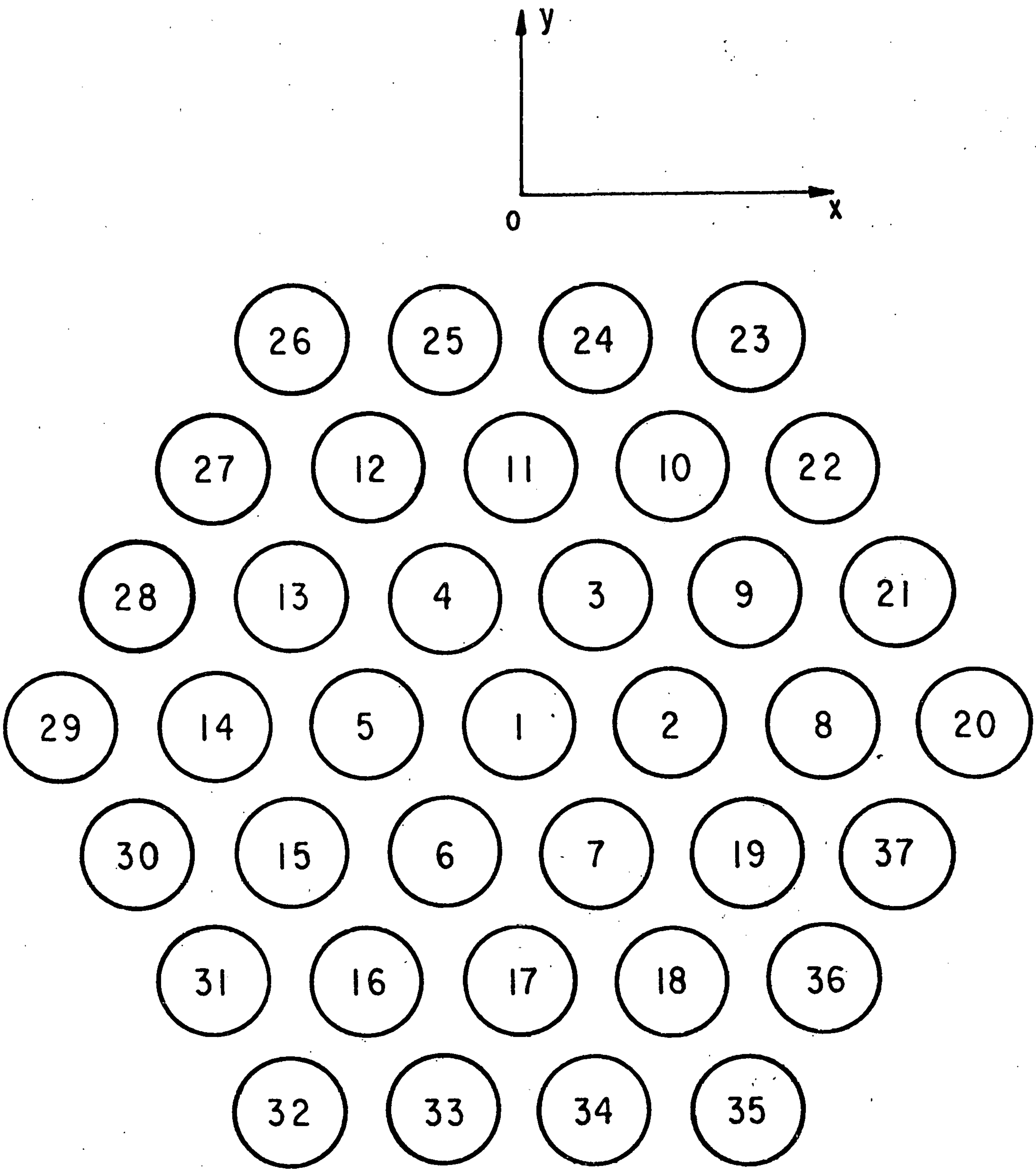

Fig. 4. Schematic of a tube bank arranged in a hexagonal pattern

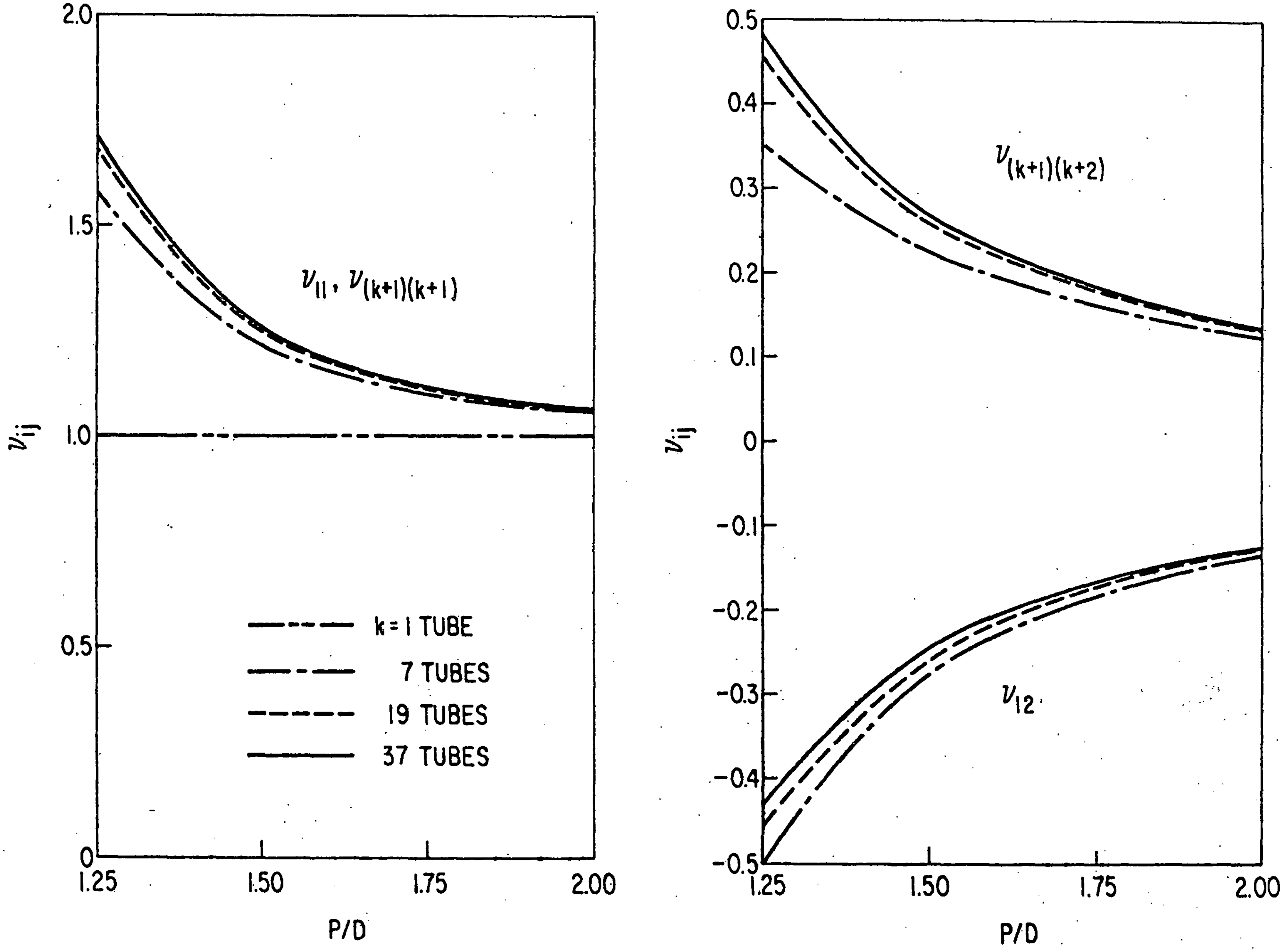

Fig. 5. Added mass coefficients as functions of the pitch-to-diameter ratio

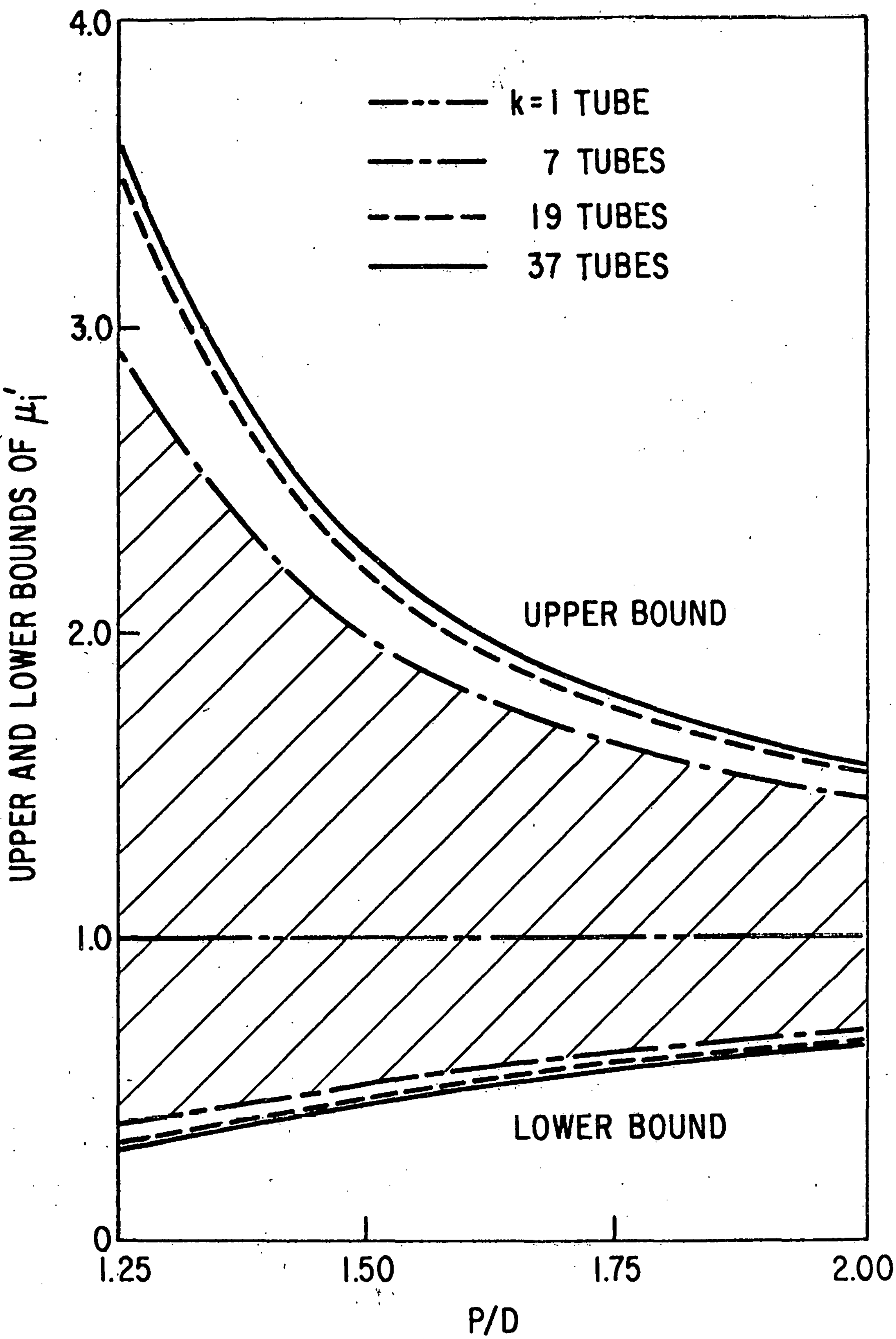

Fig. 6. Upper and lower bounds of the effective added mass coefficients as functions of the pitch-to-diameter ratio

surrounding it can be neglected. For example, in the hexagonal arrangement, only the effects of the six tubes surrounding the central tube have to be considered if the motion of the central tube is of interest.

B. Natural Frequencies and Mode Shapes

In the case of a solitary tube, the natural frequencies (in Hz) of the tube is given by

$$f_n = \left(\frac{1}{2\pi}\right)\left(\frac{\gamma_n}{\ell}\right)^2\left(\frac{EI}{m + m'}\right)^{1/2} , \tag{21}$$

$$n = 1, 2, 3, \ldots \infty ,$$

where γ_n is a mode constant, and m' is the displaced mass of fluid. The values of f_n are well separated; for example, for a simply-supported tube $f_n/f_1 = n^2$.

In the case of a group of k tubes, there are 2k natural frequencies corresponding to a single frequency f_n for a solitary tube. Those 2k frequencies are distributed near the frequency f_n. More precisely, the distribution of the natural frequencies for a group of tubes in liquids can be represented in Fig. 7. f_n (n = 1 to ∞) shows the natural frequencies of a solitary tube. Corresponding to each f_n, there is a frequency band with the lower and upper bounding frequencies f_n^ℓ and f_n^u, which are given by

$$f_n^\ell = \left(\frac{m + m'}{m + \mu_i\big|_{max}}\right)^{1/2} f_n ,$$

and (22)

$$f_n^u = \left(\frac{m + m'}{m + \mu_i\big|_{min}}\right)^{1/2} f_n .$$

$\mu_i\big|_{max}$ and $\mu_i\big|_{min}$ denote the maximum and minimum values of the effective added masses. The 2k natural frequencies are distributed in the frequency band from f_n^ℓ to f_n^u. For example, for a group of ten tubes, there are twenty natural frequencies distributed between f_n^ℓ and f_n^u. Those frequencies

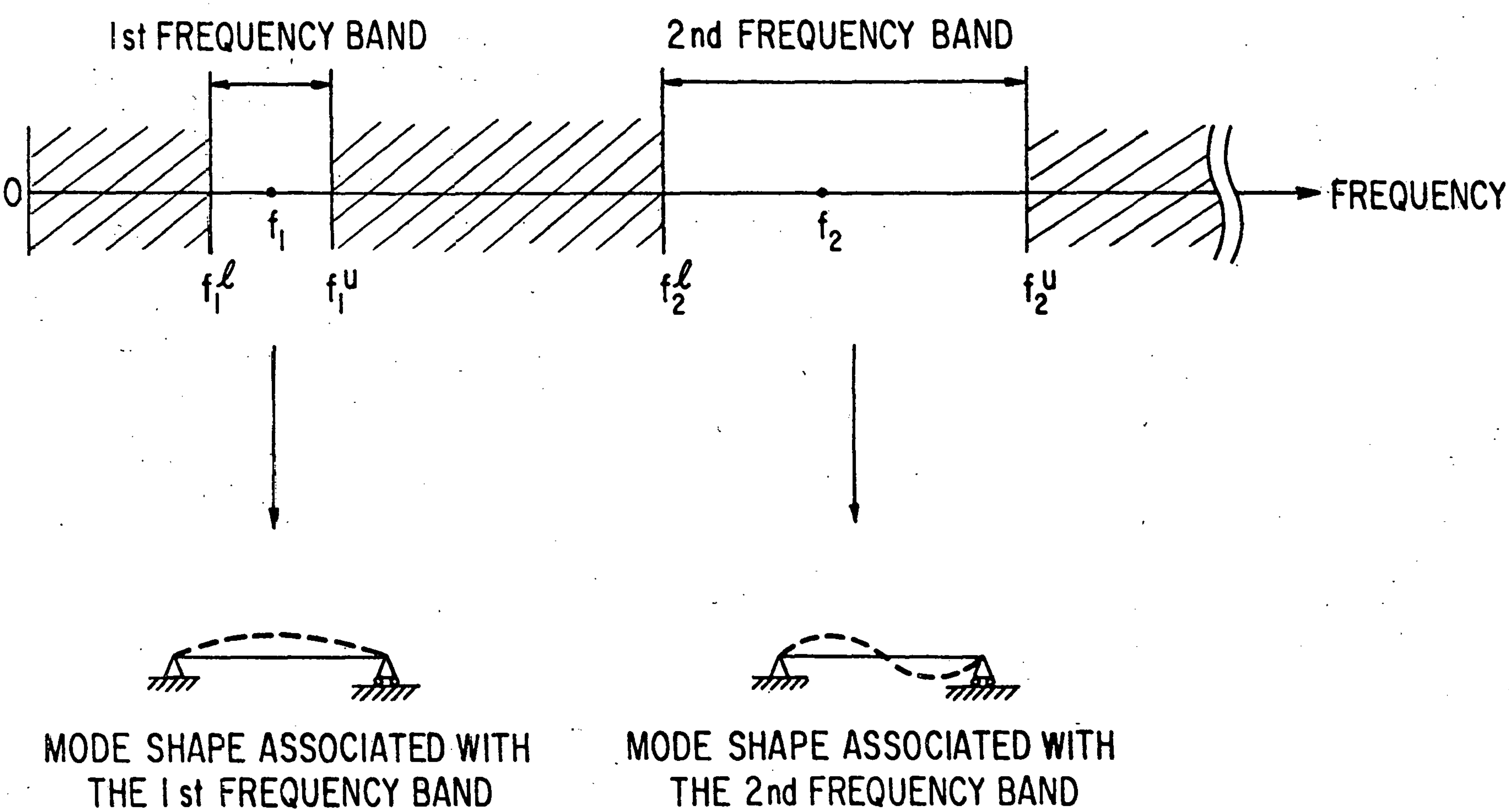

Fig. 7. Frequency bands for tube banks in a liquid

are the natural frequencies of coupled modes including tube/fluid interaction effect.

The mode shapes associated with each natural frequency are, of course, different. For example, consider a group of tubes simply-supported at both ends. The axial variations of the motion are shown in Fig. 7. In the nth frequency band, the mode shapes associated with the axial variations are the same as the nth mode of a solitary tube.

In a group of tubes, each tube will move in different directions. The modes for a group of nine tubes are shown in Fig. 8 for the natural frequencies in the first frequency band, where the arrows denote the directions of motion. Since the group consists of nine tubes, there are eigenteen modes in the first frequency band. It can be seen clearly that the frequencies are relatively close to one another.

It should be noted that, if all tubes have the same end conditions, the mode shapes associated with the natural frequencies in the other frequency bands are the same as those in the first frequency band except the axial variation.

C. Steady-State Response of a Row of Tubes

It is of interest to know how the effect of an excitation propagating to the neighboring tubes. Consider a row of tubes where, except the middle k tubes, all others are assumed to be rigid (see Fig. 9). It has been shown that for a row of tubes in a stationary liquid, in-plane and out-of-plane motions are independent, and furthermore, the hydrodynamic coupling can be neglected except the two neighboring tubes [4]. Therefore, the hydrodynamic force acting on tube i is

$$F_i = \gamma_{11} \frac{\partial^2 u_i}{\partial t^2} + \gamma_{12} \left(\frac{\partial^2 u_{i-1}}{\partial t^2} + \frac{\partial^2 u_{i+1}}{\partial t^2} \right) . \tag{23}$$

Assume that tube 1 is subjected to an excitation $e_1 \exp(i2\pi ft)$. Let

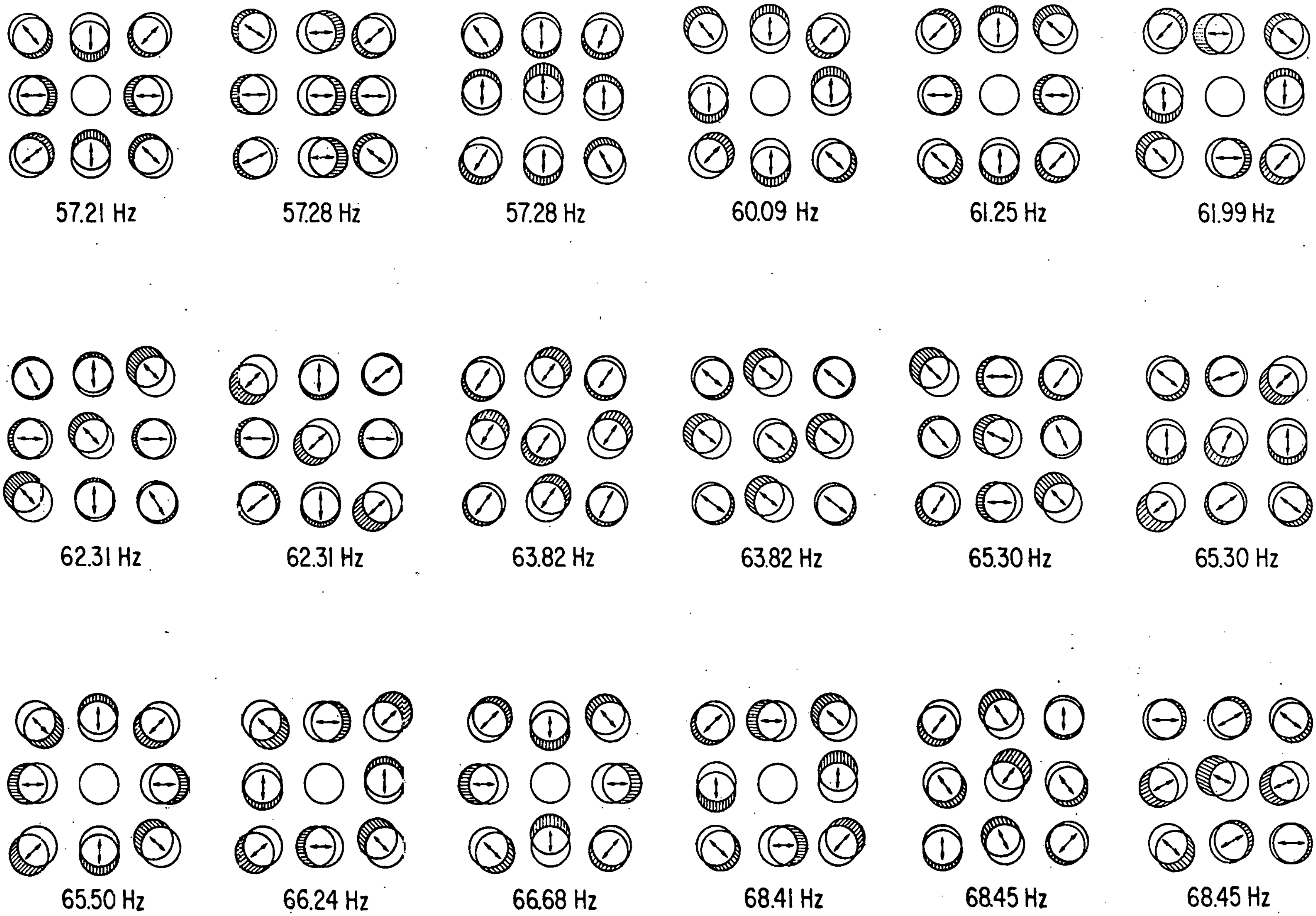

Fig. 8. Mode shapes for an array of nine tubes

$q_o \exp(i2\pi ft)$ be the response for the case of a solitary tube. The responses for a group of k tubes are easily obtained:

$$q_i = q_o a_i \exp(i2\pi ft) \quad ,$$

$$a_i = \frac{\sin(k - i + 1)\theta}{\sin(k + 1)\theta} \quad , \tag{24}$$

$$\theta = \cos^{-1}\left[\frac{m}{2\gamma_{12}m'}\left(\frac{f^2}{f_n^2} - 1\right) - \frac{\gamma_{11}}{\gamma_{12}}\right] \quad ,$$

where m is tube mass, m' is the displaced mass of fluid and f_n is the natural frequency of the tube in vacuo.

The values of the response ratio a_i are given in Fig. 9 for a row of five elastic tubes. The responses are strongly dependent on the driving frequency f:

(1) For $f < f_n^\ell$ or $f > f_n^u$, the response of tube 1 is close to that of a single tube subject to the same excitation, and the responses of other tubes are small; that is, the effect of excitation deteriorates rapidly. Therefore, a single-tube assumption is applicable.

(2) For $f_n^\ell < f < f_n^u$, the response of tube 1 is different from that of a single tube, and the responses of other tubes can be very large; the effect of an excitation will propagate to others without decay. In this range of frequency, only the coupled-mode method will give accurate results.

D. <u>A Row of Tubes Subject to Vortices</u>

A common cause of flow-induced vibration is an alternating force in the lift direction caused by the shedding of Karman vortices. The vortex shedding frequency is given as

$$f_v = \frac{S_t U}{D} \quad , \tag{25}$$

where S_t is the Strouhal number, U is the flow velocity and D is tube diameter. For a single tube, S_t is approximately equal to 0.2 for subcritical Reynolds number. For tube banks, the value of S_t is highly

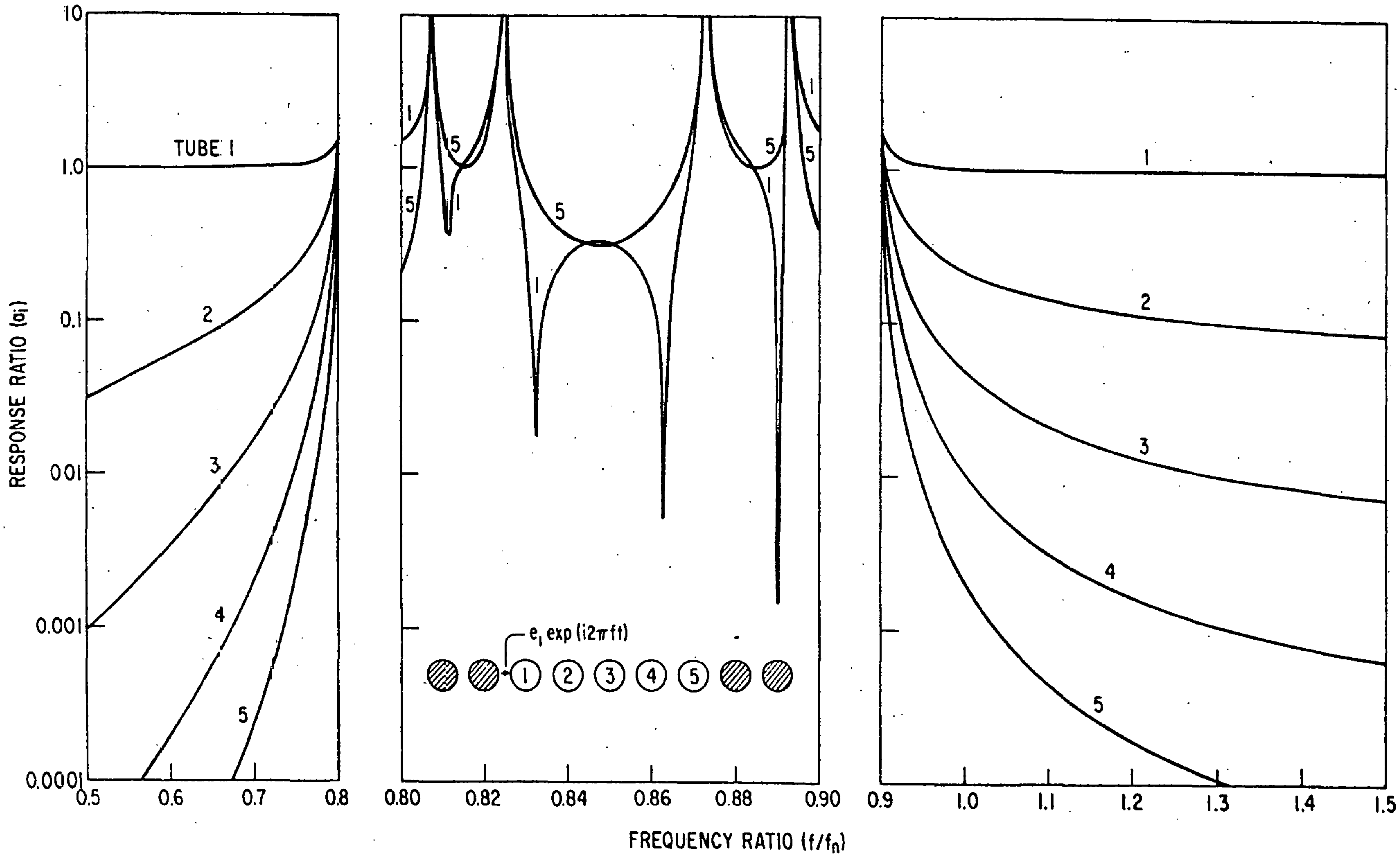

Fig. 9. Steady-state response of a row of tubes in liquid

dependent on tube spacing and pattern, and multiple values of S_t have been found. A summary of the experimental results is compiled by Fitz-Hugh [5].

Responses of a row of five tubes subject to vortices are presented in Fig. 10, where the damping ratios for the first five modes in the in-plane motion are 0.041, 0.041, 0.042, 0.044, and 0.045. Solid lines are calculated for $S_t = 0.14$ and all vortices being out-of-phase, while dotted lines are for $S_t = 0.38$ and all vortices being in-phase. The values of Strouhal number are based on the experimental results obtained by Dye [7].

In the actual situation, the responses of the tubes are probably to follow the curve ABCDE as the flow velocity is increased. The modes of response also change with increasing flow velocity. For example, at B, the tubes vibrate predominantly in the out-of-phase mode, while at D, it is predominantly in the in-phase mode. In the other range of flow velocity, it will involve other modes.

E. Transient Responses of Tube Banks to Impingement Loads

If a tube rupture occurs, thrust and jet forces caused by fluid discharge may cause further damage to tube banks. Discharge fluid creates a thrust reaction on the ruptured tube itself and impingement loads on neighboring tubes by the jet. It is desirable to know the response of the tubes due to those impingement loads. As an example, a seven-tube bank arranged in a hexagonal pattern is considered.

The magnitude and duration of tube rupture forces in a tube bank must be available before the analysis can be made. For the purpose of illustrating the general characteristics, it is assumed that the jet and thrust are of equal magnitude and can be represented by the step function. The responses of the tubes are presented in Fig. 11 when tube 1 is ruptured with a jet acting on tube 2 in the x direction. The magnification factor is defined

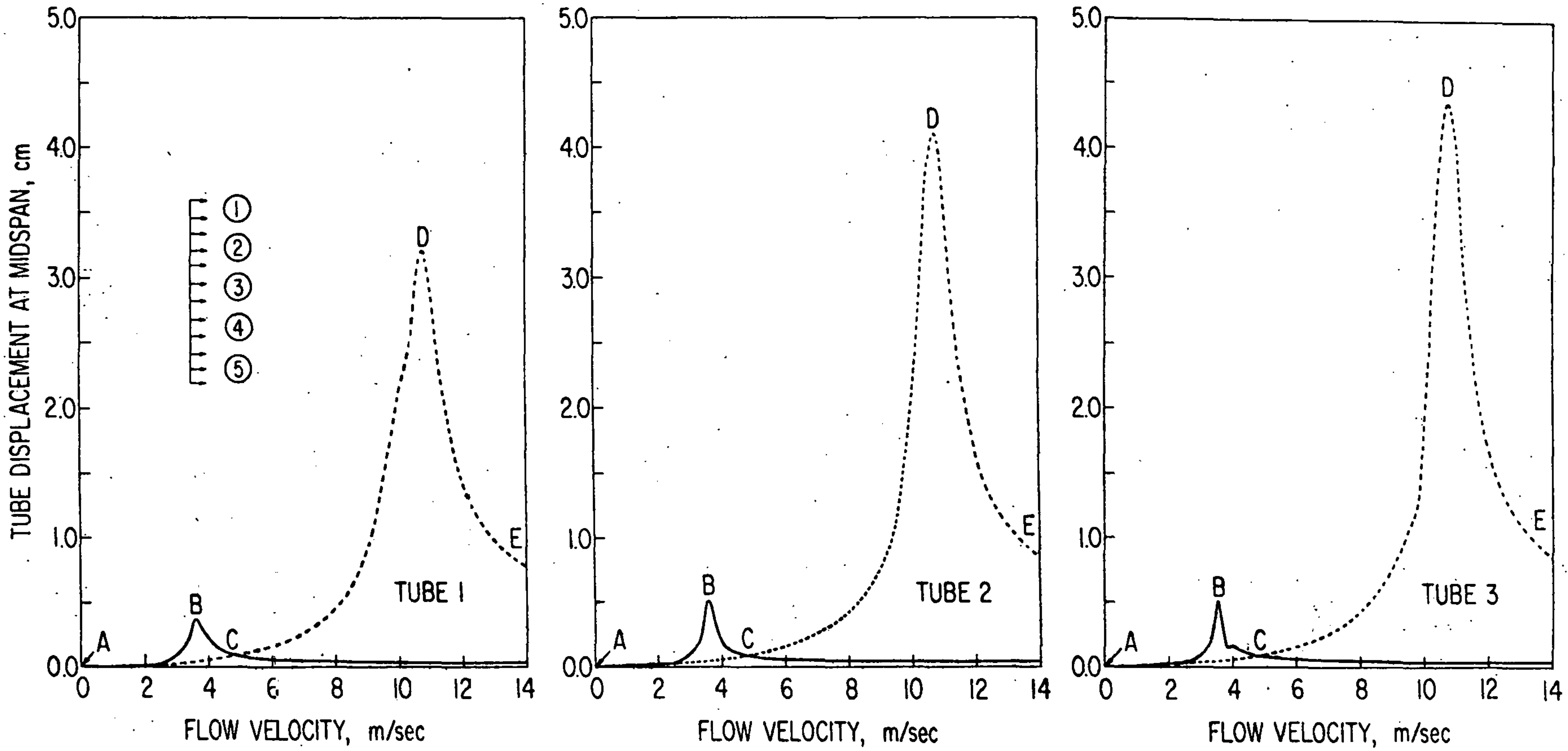

Fig. 10. Tube responses to vortices

as the ratio of the displacement at midspan to that of the deflection of tube 1 in the x direction to a static load of the same magnitude. The damping ratios for the natural frequencies in the first frequency band are found to be ranging from 0.038 to 0.046.

Due to symmetry, tubes 1, 2, and 5 respond in the x direction only. The other tubes vibrate with whirling motions. The absolute value of the magnification factor for tubes 1 and 2 is close to 2 and is smaller for other tubes. As the time increases, as expected, the absolute value of the magnification factor approaches 1 for tubes 1 and 2 and zero for others.

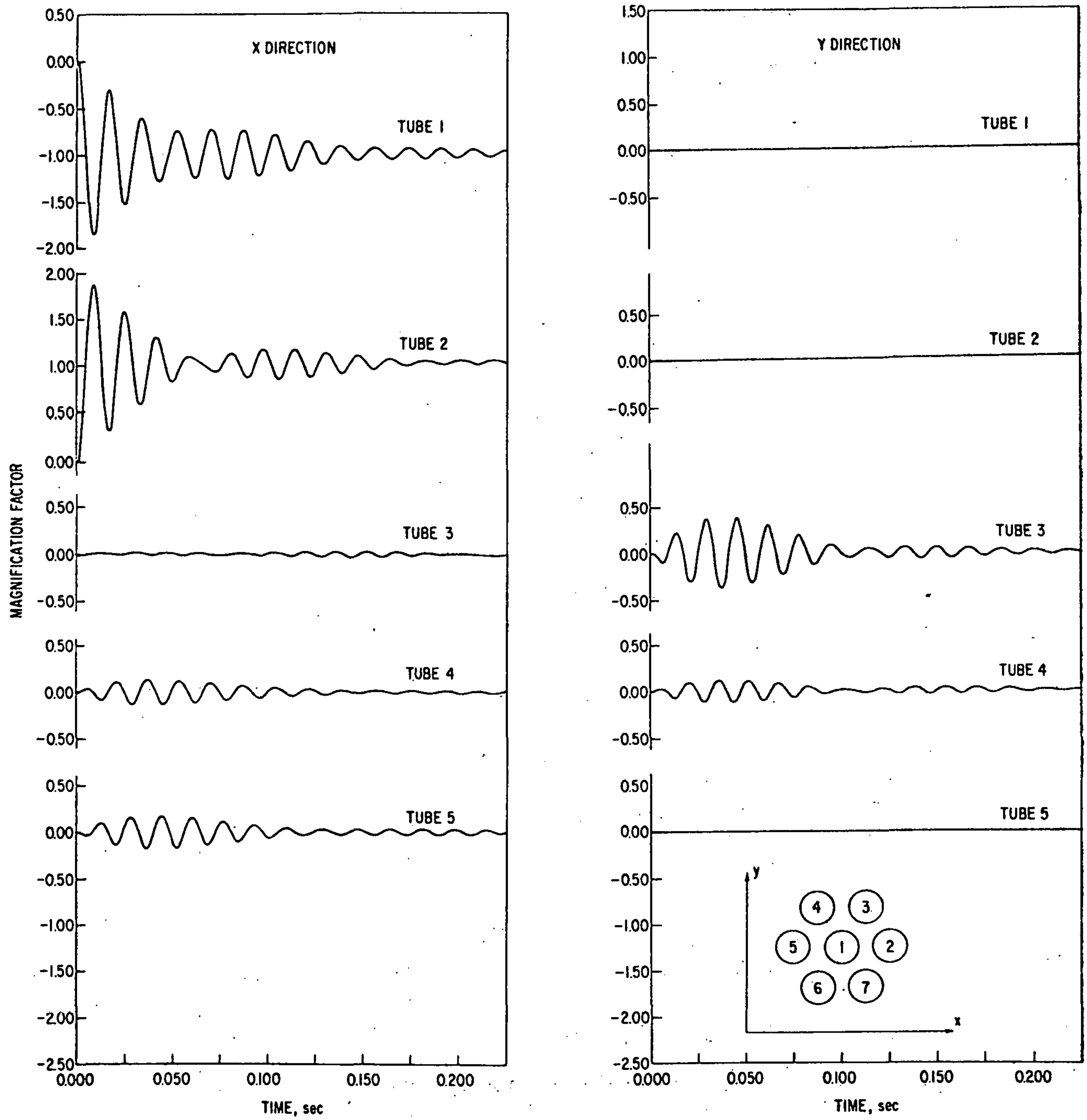

Fig. 11. Transient responses of a tube bank

V. COMPARISON OF THEORETICAL AND EXPERIMENTAL RESULTS FOR TWO CANTILEVERED TUBES

The method of analysis presented in Section II can be applied to a group of tubes vibrating in a liquid. In order to compare the analytical result with the experimental data, the detailed analysis for two cantilevered tubes is given in Appendix A. The numerical results for two tubes are based on the model given in Fig. 16 on page 48.

The frequencies measured in air are used to establish the torsional spring constant at the support. Then the frequencies of uncoupled modes can readily be calculated. The natural frequencies of uncoupled vibration obtained experimentally and analytically are given in Table 1 where the measured values of damping ratio are also given. It is seen that experimental data are in reasonably good agreement with the analytical results.

The natural frequencies of the coupled modes are presented in Table 2. In every case, there are two natural frequencies; one corresponds to the in-phase mode and the other to the out-of-phase mode. The general characteristics are similar to those of simply-supported tubes [8].

Figures 12 and 13 show the frequency response curves. Figure 12 shows the response of tube 1 to an excitation acting on itself and Figure 13 the response of tube 2 to an excitation acting on tube 1. There are two peaks: one associated with the out-of-phase mode and the other with the in-phase mode. Note that the response of tube 2 is larger than tube 1 in the frequency range between the two peaks. This means that although tube 2 is not directly excited, its response can be larger than the other, which is directly excited, in certain frequency ranges.

TABLE 1. Theoretical and experimental values of uncoupled frequencies

Tube No.	Direction of Motion	Gap-to-Radius Ratio	Dimensionless Spring Constant, p	Measured Natural Frequency, Hz		Measured Damping Ratio		Calculated Natural Frequency in Water, Hz
				In Air	In Water	In Air	In Water	
1	In-plane	0.2	78.0	77.63	69.05	0.0022	0.0059	68.51
		0.45	76.7	77.60	69.32	0.0022	0.0031	68.97
		1.2	71.1	77.45	69.65	0.0026	0.0036	69.26
		4.2	76.7	77.60	69.42	0.0024	0.0029	69.54
	Out-of-plane	0.2	68.1	77.36	69.11	0.0014	0.0037	68.28
		0.45	52.9	76.76	69.10	0.0016	0.0045	68.23
		1.2	70.0	77.42	69.24	0.0012	0.0055	69.23
		4.2	81.1	77.70	69.80	0.0026	0.0065	69.64
2	In-plane	0.2	112.6	77.20	69.10	0.0037	0.0048	68.33
		0.45	113.4	77.21	68.97	0.0017	0.0042	68.82
		1.2	86.3	76.81	69.30	0.0020	0.0032	68.87
		4.2	113.4	77.21	68.95	0.0028	0.0043	69.37
	Out-of-plane	0.2	75.4	76.57	69.08	0.0020	0.0058	67.77
		0.45	129.7	77.37	69.18	0.0025	0.0043	68.96
		1.2	82.3	76.73	69.52	0.0025	0.0067	68.79
		4.2	84.7	76.78	69.13	0.0025	0.0054	68.99

Table 2. Theoretical and experimental values of coupled frequencies

Direction of Motion	Gap-to-Radius Ratio	Measured Frequency, Hz	Calculated Frequency, Hz
In-plane	0.2	66.78 71.38	65.31 72.02
	0.45	67.30 71.33	66.47 71.60
	1.2	68.23 70.63	67.65 70.57
	4.2	69.37 70.28	69.07 69.86
Out-of-plane	0.2	66.58 71.27	64.92 71.61
	0.45	67.53 71.31	66.16 71.32
	1.2	68.63 70.83	67.59 70.52
	4.2	69.35 69.80	68.48 69.54

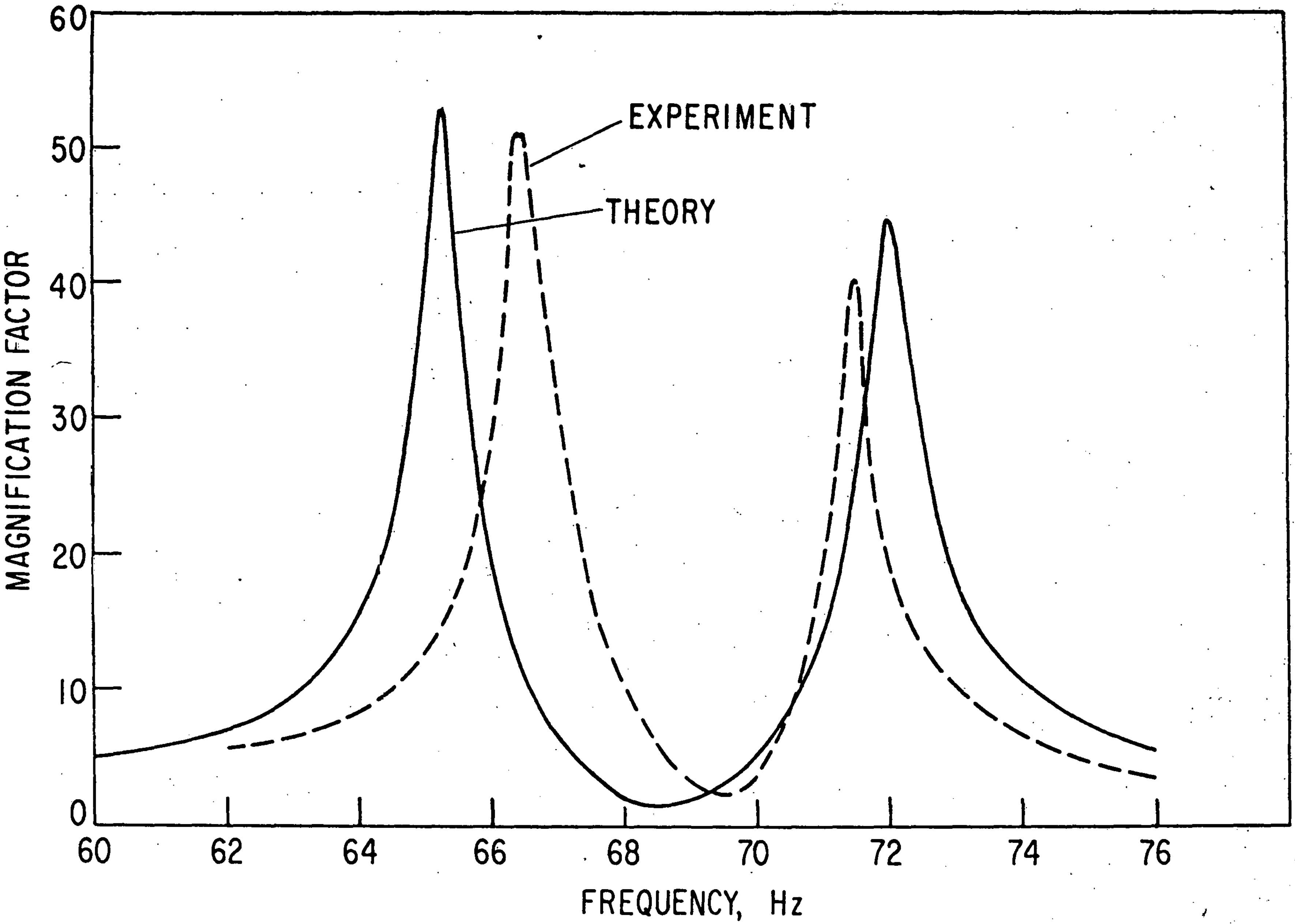

Fig. 12. Frequency response of tube 1 to an excitation acting on itself

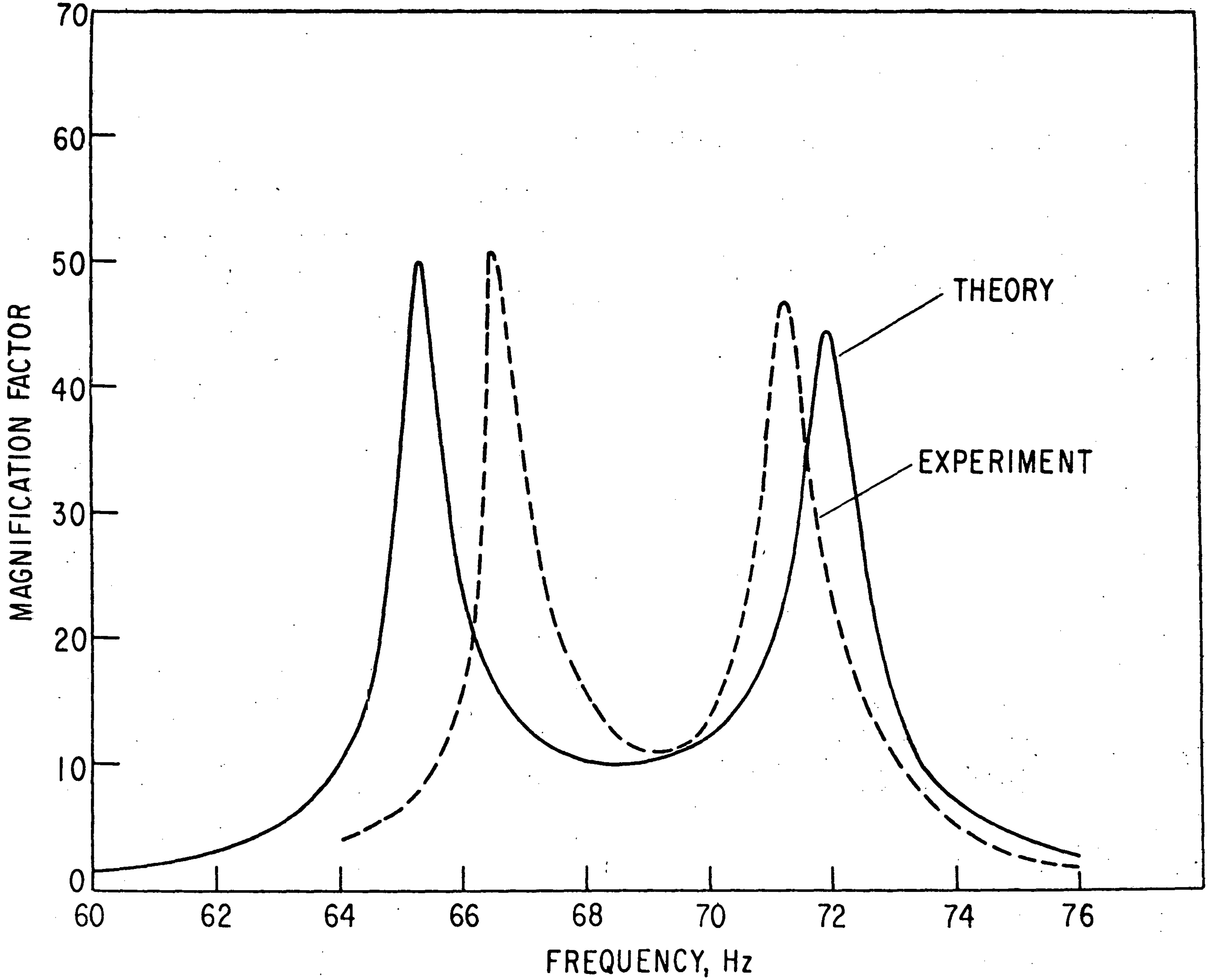

Fig. 13. Frequency response of tube 2 to an excitation acting on tube 1

The transient responses in the out-of-plane motion of the two tubes are shown in Figs. 14 and 15. In Fig. 14, tube 1 is given a small disturbance, while in Fig. 15, tube 2 is given the disturbance. In this test, the coupled frequencies are 68.02 Hz (in-phase mode) and 71.46 Hz (out-of-phase mode) for the gap-to-radius ratio equal to 0.45 and 69.72 Hz (in-phase mode) and 70.56 Hz (out-of-phase mode) for the gap-to-radius ratio equal to 4.2. The beat phenomenon can be seen clearly from the response curves. The amplitude-modulated accelerations have a frequency equal to the average frequency of the in-phase and out-of-phase modes, while the beat frequency is equal to the difference of the two coupled frequencies.

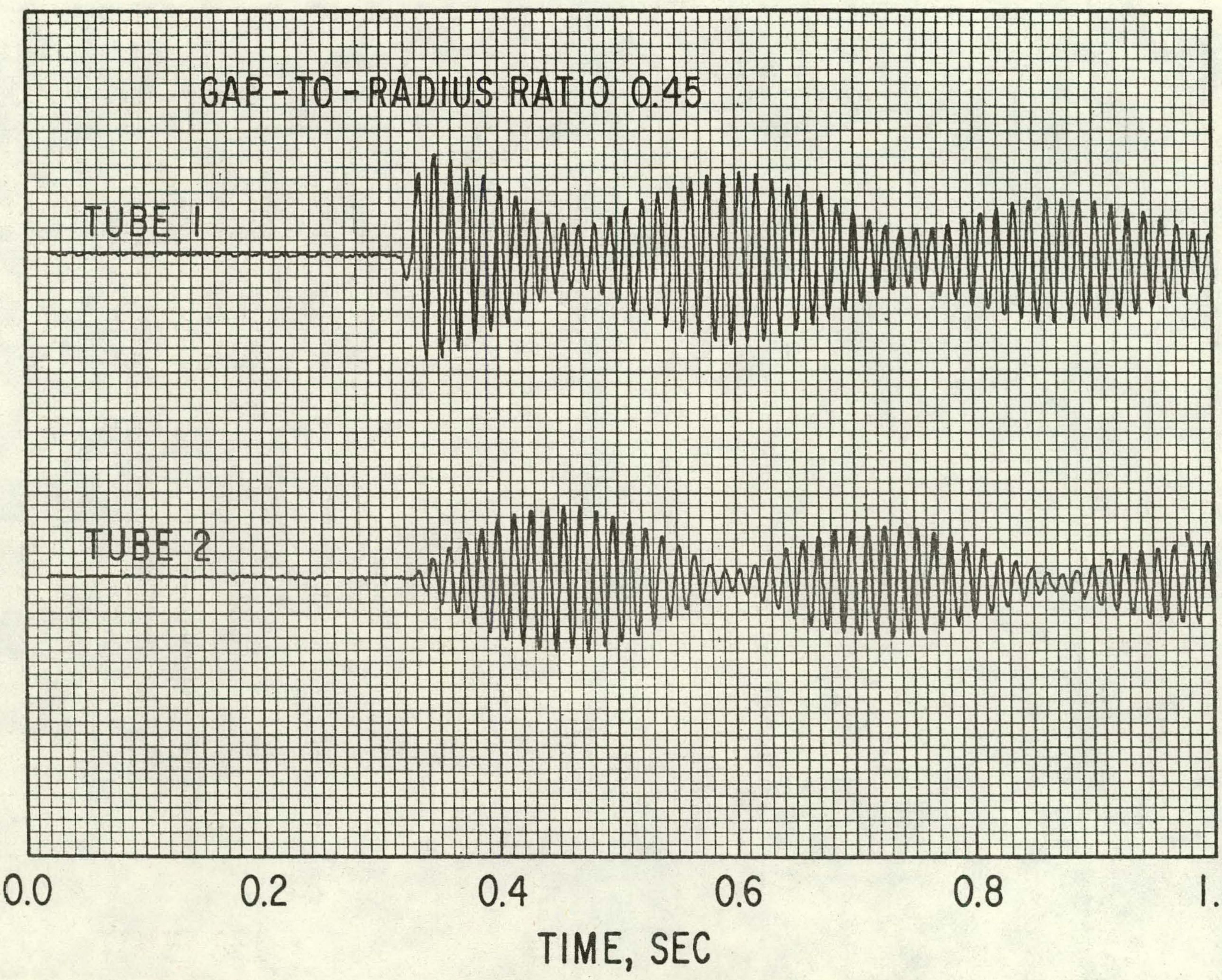

Fig. 14. Transient response for out-of-plane motion when tube 1 is excited

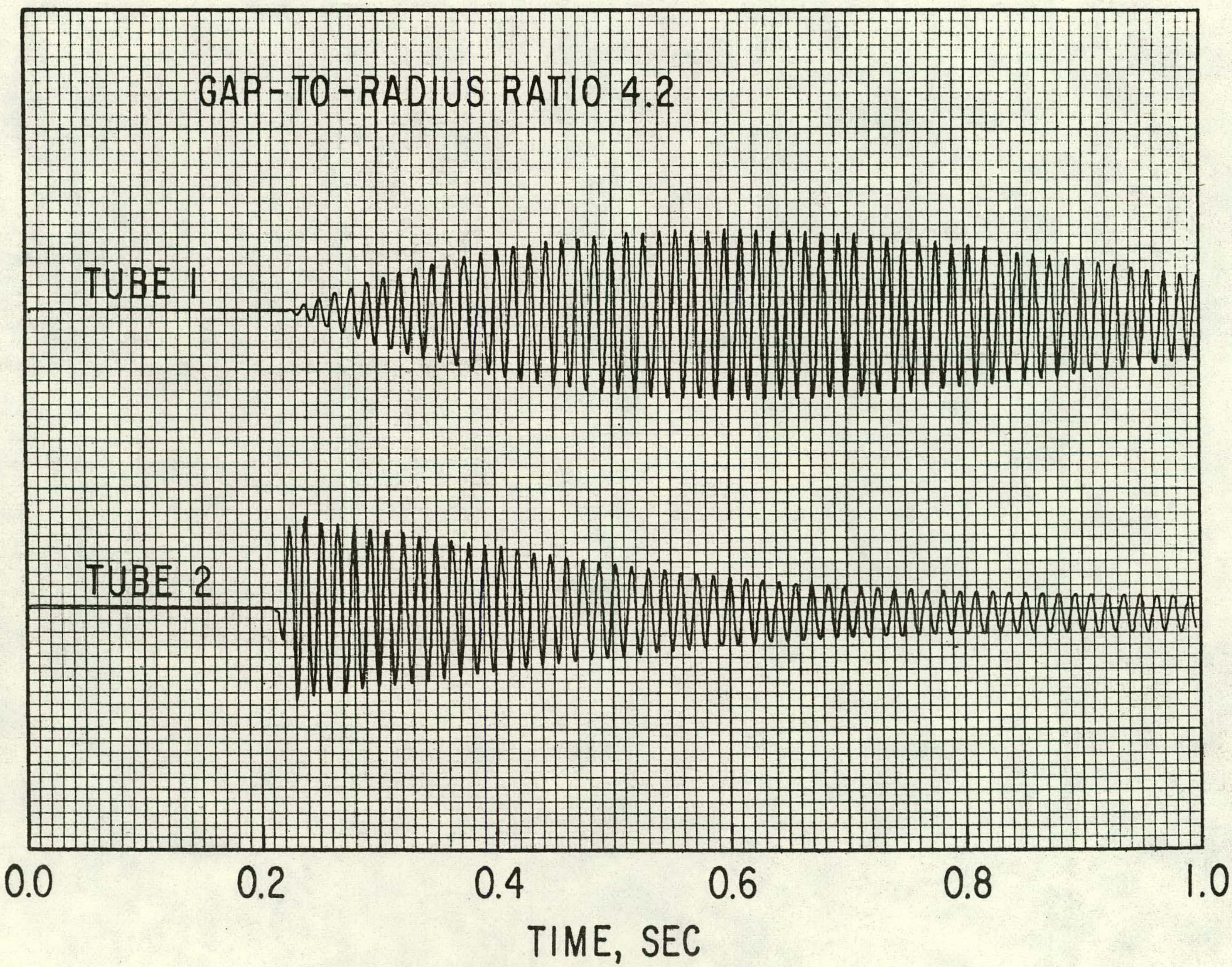

Fig. 15. Transient response for out-of-plane motion when tube 2 is excited

VI. CONCLUSIONS

A method of analysis is presented for tube banks in liquid subjected to various types of excitation including deterministic and random forces. The analysis is based on the coupled modes incorporating tube/fluid interaction. With the method of analysis, one can calculate the natural frequencies, mode shapes and tube responses. The method is being extended to include the effect of flowing fluid.

Two cantilevered tubes are tested in a water tank; natural frequencies and forced responses are measured. It is found that the analytical results and experimental data are in reasonably good agreement.

Based on the presented results, several general conclusions can be made:

(1) In a group of tubes, there are infinite number of frequency bands corresponding to the infinite number of natural frequencies in the case of a solitary tube. In each frequency band, there are 2k natural frequencies for a group of k tubes; those frequencies are distributed near the frequency of the corresponding single tube.

(2) The natural frequencies of the coupled modes for tube banks can be obtained in terms of the natural frequencies of a solitary tube and the effective added masses.

(3) In a tube bank, the values of the effective added mass become more widely spread as the number of tubes is increased.

(4) Within each frequency band, the steady state responses of a tube bank are significantly different from that of a single tube. However, in other frequency ranges, the coupling effect is small and the effect of an excitation will decay rapidly from one tube to another.

(5) Vortices can excite various coupled modes in a tube bank and the predominant mode of response may change with increasing flow velocity.

(6) Fluid jet and thrust associated with tube rupture can cause whirling motions in a tube bank, and the tubes not directly excited may have a significant amplitude.

APPENDIX A

ANALYSIS OF TWO TUBES IN A LIQUID

The mathematical model for the tubes is shown in Figure 16. M_1' and M_2' represent the concentrated masses accounting for the permanent magnets, accelerometers and aluminum mounting block. The support is considered as elastically restrained by a torsional spring and the exciting force is acting at $z = \ell$. The problem is described by the following equations of motion [8]:

$$E_1 I_1 \frac{\partial^4 u_1}{\partial z^4} + c_1 \frac{\partial u_1}{\partial t^2} + (m_1 + \bar{\mu}_1 M_1) \frac{\partial^2 u_1}{\partial t^2} - \lambda M_1 \bar{\mu}_3 \left(\frac{R_2}{\bar{R}}\right)^2 \frac{\partial^2 u_2}{\partial t^2}$$

$$= \bar{F}_1 \, \delta(z - \ell),$$

and

$$E_2 I_2 \frac{\partial^4 u_2}{\partial z^4} + c_2 \frac{\partial u_2}{\partial t} + (m_2 + \bar{\mu}_2 M_2) \frac{\partial^2 u_2}{\partial t^2} - \lambda M_2 \bar{\mu}_3 \left(\frac{R_1}{\bar{R}}\right)^2 \frac{\partial^2 u_1}{\partial t^2}$$

$$= \bar{F}_2 \, \delta(z - \ell), \tag{A.1}$$

where the index 1 denotes tube 1 and 2 denotes tube 2, z is axial coordinate, t is time, u_i $(i = 1,2)$ is tube displacement, m_i is mass per unit length of the tube, $E_i I_i$ is flexural rigidity, c_i is damping coefficient, R_i is tube radius, M_i is displaced mass of fluid by the tube, $\bar{R}$ is the distance between the centers of the two tubes, $\bar{\mu}_1$, $\bar{\mu}_2$, and $\bar{\mu}_3$ are added mass coefficients, and $\lambda = 1$ for in-plane motion and -1 for out-of-plane motion. The boundary conditions are:

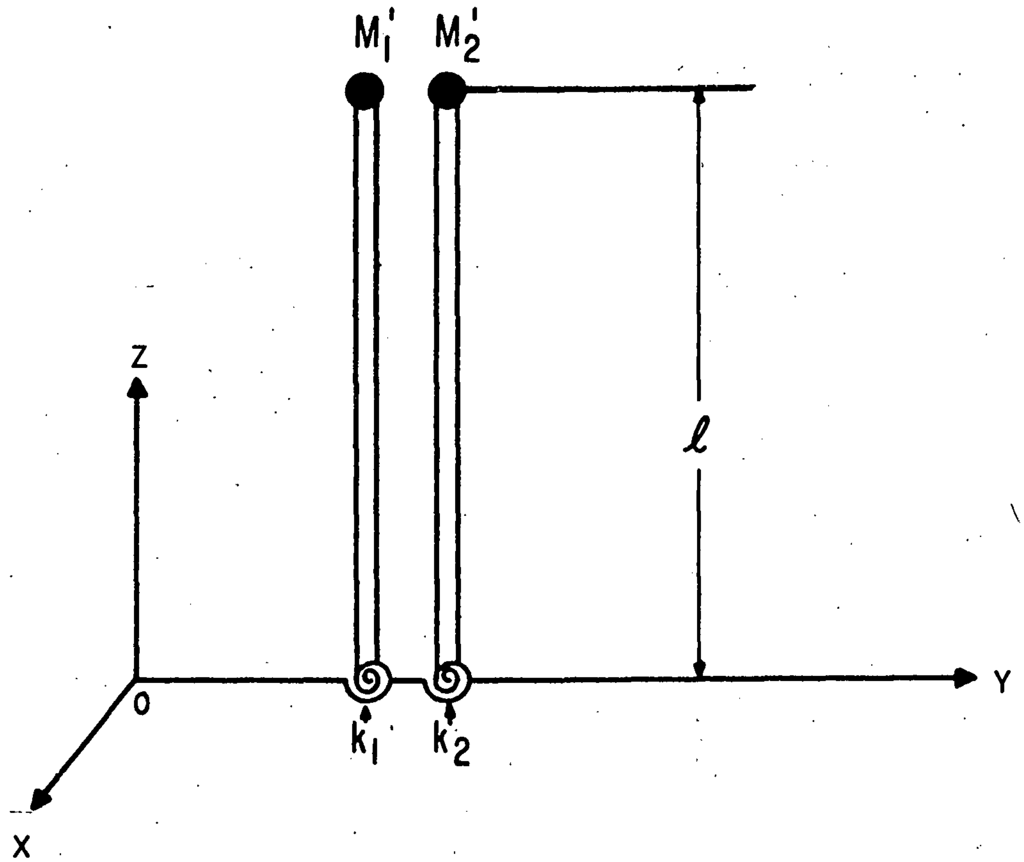

Fig. 16. Schematic of two tubes vibrating in a liquid

At $z = 0$:

$$u_1 = 0 \ , \qquad u_2 = 0 \ ,$$

$$E_1 I_1 \frac{\partial^2 u_1}{\partial z^2} = k_1 \frac{\partial u_1}{\partial z} \ , \qquad E_2 I_2 \frac{\partial^2 u_2}{\partial z^2} = k_2 \frac{\partial u_2}{\partial z} \ ;$$

and at $x = \ell$: (A.2)

$$u_1 = 0 \ , \qquad u_2 = 0 \ ,$$

$$E_1 I_1 \frac{\partial^3 u_1}{\partial z^3} = M_1' \frac{\partial^2 u_1}{\partial t^2} \ , \qquad E_2 I_2 \frac{\partial^3 u_2}{\partial z^3} = M_2' \frac{\partial^2 u_2}{\partial t^2} \ .$$

Let

$$u_i = \sum_{n=1}^{\infty} q_{ni} \, \phi_{ni} \ , \qquad i = 1, 2, \tag{A.3}$$

where ϕ_{ni} is the modal function for the tubes. Assuming the adjoint modal function is ψ_{ni}. Those modal functions, and the corresponding eigenvalue γ_{ni} and natural frequency ω_{ni} can be calculated following the method presented in Appendix B. Substituting Eq. (A.3) into (A.1) and using the biorthogonality condition yield

$$\ddot{q}_{n1} + 2\zeta_{n1} \, \omega_{n1} \, \dot{q}_{n1} + \omega_{n1}^2 \, q_{n1} - \lambda\alpha_1 \sum_m \bar{a}_{nm} \, \ddot{q}_{m2} = p_{n1} \ ,$$

and (A.4)

$$\ddot{q}_{n2} + 2\zeta_{n2} \, \omega_{n2} \, \dot{q}_{n2} + \omega_{n2}^2 \, q_{n2} - \lambda\alpha_2 \sum_m \bar{b}_{nm} \, \ddot{q}_{n1} = p_{n2} \ ,$$

where

$$\zeta_{ni} = \frac{c_i}{2\omega_{ni}(m_i + \bar{\mu}_i M_i)} \ , \qquad \omega_{ni} = \frac{\gamma_{ni}^2}{\ell^2} \sqrt{\frac{E_i I_i}{m_i + \bar{\mu}_i M_i}} \ ,$$

$$\alpha_1 = \frac{\bar{\mu}_3 M_1 R_2^2}{\bar{R}^2(m_1 + \bar{\mu}_1 M_1)} , \qquad \alpha_2 = \frac{\bar{\mu}_3 M_2 R_1^2}{\bar{R}^2(m_2 + \bar{\mu}_2 M_2)} ,$$

$$\beta_i = \frac{M_i}{m_i \ell} , \qquad p_{ni} = (1 + \beta_i)\psi_{ni}(1) \frac{\bar{F}_i}{\ell(m_i + \bar{\mu}_i M_i)} ,$$

$$\bar{a}_{nm} = \int_o^1 \phi_{m2}\psi_{n1} d\xi + \beta_1\phi_{m2}(1)\psi_{n1}(1), \quad \bar{b}_{nm} = \int_o^1 \phi_{m1}\psi_{n2} d\xi + \beta_2\phi_{m1}(1)\psi_{n2}(1) \tag{A.5}$$

Equation (A.4) consists of an infinite number of ordinary differential equations. However, typically, only a finite number of equations is taken from case to case according to the desired accuracy. As an approximation, two equations are taken in the following calculations; i.e.:

$$\ddot{q}_{11} + 2\zeta_{11}\omega_{11}\dot{q}_{11} + \omega_{11}^2 q_{11} - \lambda\alpha_1\bar{a}_{11}\ddot{q}_{12} = p_{11} ,$$

and

$$\ddot{q}_{12} + 2\zeta_{12}\omega_{12}\dot{q}_{12} + \omega_{12}^2 q_{12} - \lambda\alpha_2\bar{b}_{11}\ddot{q}_{11} = p_{12} . \tag{A.6}$$

From Eqs. (A.6), it is seen that the frequencies of uncoupled modes are given by ω_{11} and ω_{12}. The frequencies of the coupled modes can be solved easily. Those are Ω_1 and Ω_2;

$$\Omega_1 = \left\{(\omega_{11}^2 + \omega_{12}^2) - [(\omega_{11}^2 - \omega_{12}^2)^2 + 4\alpha_1\alpha_2\bar{a}_{11}\bar{b}_{11}\omega_{11}^2\omega_{12}^2]^{1/2}\right\}^{1/2} / [2(1 - \alpha_1\alpha_2\bar{a}_{11}\bar{b}_{11})]^{1/2},$$

$$\Omega_2 = \left\{(\omega_{11}^2 + \omega_{12}^2) + [(\omega_{11}^2 - \omega_{12}^2)^2 + 4\alpha_1\alpha_2\bar{a}_{11}\bar{b}_{11}\omega_{11}^2\omega_{12}^2]^{1/2}\right\}^{1/2} / [2(1 - \alpha_1\alpha_2\bar{a}_{11}\bar{b}_{11})]^{1/2}, \tag{A.7}$$

Based on Eqs. (A.3) and (A.6), the responses of the tubes to an arbitrary excitation can be calculated rather easily. The procedure is the same as that given in Reference 8.

APPENDIX B

FREE VIBRATION OF AN ELASTICALLY SUPPORTED TUBE WITH A CONCENTRATED MASS AT THE TIP

Consider a tube which is elastically supported by a torsional spring with spring constant k and a mass M' attached to the free end. The equation for free vibration is

$$EI \frac{\partial^4 u}{\partial z^4} + m \frac{\partial^2 u}{\partial t^2} = 0, \tag{B.1}$$

and the boundary conditions are

$$u = 0,\ EI \frac{\partial^2 u}{\partial z^2} = \bar{k} \frac{\partial u}{\partial z} \text{ at } z = 0,$$

and (B.2)

$$\frac{\partial^2 u}{\partial z^2} = 0,\ EI \frac{\partial^3 u}{\partial z^3} = M' \frac{\partial^2 u}{\partial t^2} \text{ at } z = \ell,$$

where EI is flexural rigidity and m is mass per unit length of the tube.

Using the standard method of separation of variables, one assumes

$$u\,(z,t) = \phi(z)\, \exp\,(i\omega t). \tag{B.3}$$

Substituting this form in Eqs. (B.1) and (B.2) results

$$\frac{d^4\phi}{d\xi^4} - \gamma^4\phi = 0, \tag{B.4}$$

$$\phi = 0,\ \frac{d^2\phi}{d\xi^2} = p\,\frac{d\phi}{d\xi} \text{ at } \xi = 0,$$

(B.5)

$$\frac{d^2\phi}{d\xi^2} = 0,\ \frac{d^3\phi}{d\xi^3} = -\gamma^4\beta\phi \quad \text{at } \xi = 1,$$

where

$$\xi = z/\ell,$$

$$\beta = \frac{M}{m\ell},$$

$$p = \frac{\bar{k}\ell}{EI},\ \gamma^4 = \frac{m\ell^4\omega^2}{EI}, \tag{B.6}$$

and ℓ is the length of the tube.

Note that the eigenvalue γ^4 appears in the last boundary condition; therefore, the eigenfunctions are not orthogonal in the ordinary sense. To overcome this difficulty, extend the space such that the space consists of two component vector $\vec{\underline{\phi}}$ (ξ) whose first component is a real function $\phi(\xi)$ and whose second component is a real number ϕ_α. Define the scalar product of two vectors $\vec{\underline{\phi}}$ and $\vec{\underline{\psi}}$ as follows:

$$< \vec{\underline{\phi}}, \vec{\underline{\psi}} > = \int_0^1 \phi(\xi)\ \psi(\xi)\ d\xi + \phi_\alpha\ \psi_\alpha. \qquad (B.7)$$

Now the eigenvalue problem of (B.4) and (B.5) can be written

$$L\ \vec{\underline{\phi}} = \begin{pmatrix} \dfrac{d^4\phi}{d\xi^4} \\ -\dfrac{d^3\phi(1)}{d\xi} \end{pmatrix} = \gamma^4\ \vec{\underline{\phi}}\ . \qquad (B.8)$$

Therefore the domain D of the operator L is the set of all function $\vec{\underline{\phi}}$ in a linear vector space which have a piecewise continuous fourth derivative, which satisfy the following equations:

$$\begin{aligned} &\vec{\underline{\phi}} : 0 \le \xi \le 1, \\ &\phi = 0,\ \frac{d^2\phi}{d\xi^2} = p\ \frac{d\phi}{\phi\xi} \text{ at } \xi = 0, \\ &\frac{d^2\phi}{d\xi^2} = 0 \text{ at } \xi = 1, \end{aligned} \qquad (B.9)$$

and

$$\phi_\alpha = \beta\phi(1).$$

Using the definition of scalar product in Eq. (B.7), one can find the adjoint system:

$$< \vec{\underline{\psi}}, L\ \vec{\underline{\phi}} > = \int_0^1 \psi\ \frac{d^4\phi}{d\xi^4}\ d\xi - \psi_\alpha\ \frac{d^3\phi(1)}{d\xi^3}$$

$$= < L^*\ \vec{\underline{\psi}}, \vec{\underline{\phi}} >. \qquad (B.10)$$

From Eq. (B.10), the adjoint operator is given by

$$L^* \vec{\bar{\psi}} = \begin{pmatrix} \frac{d^4\psi}{d\xi^4} \\ -\frac{d^3\psi(1)}{d\xi^3} \end{pmatrix} = \gamma^4 \vec{\bar{\psi}} ,$$

$$\psi = 0, \frac{d^2\psi}{d\xi^2} = p \frac{d\psi}{d\xi} \text{ at } \xi = 0, \tag{B.11}$$

$$\frac{d^2\psi}{d\xi^2} = 0 \text{ at } \xi = 1,$$

and

$$\psi_\alpha = \psi(1).$$

The eigenvectors of L and L^* subjected to the boundary conditions can be solved rather easily. The results for the original systems are as follows:

$$\phi(\xi) = \bar{\alpha}_1 \cos \gamma\xi + \bar{\alpha}_2 \sin \gamma\xi + \bar{\alpha}_3 \cosh \gamma\xi + \bar{\alpha}_4 \sinh \gamma\xi ,$$

where

$$\bar{\alpha}_2 = \frac{-2\gamma \sinh \gamma - p (\cos \gamma + \cosh \gamma)}{p (\sin \gamma + \sinh \gamma)} \bar{\alpha}_1, \tag{B.12}$$

$$\bar{\alpha}_3 = -\bar{\alpha}_1 ,$$

$$\bar{\alpha}_4 = \frac{-2\gamma \sin \gamma + p (\cos \gamma + \cosh \gamma)}{p (\sin \gamma + \sinh \gamma)} \bar{\alpha}_1,$$

and the value of eigenvalue γ is determined from the equation

$$1 + \cos \gamma \cosh \gamma + (\beta + \frac{1}{p}) (\cos \gamma \cosh \gamma - \sin \gamma \sinh \gamma)$$
$$-2 \frac{\gamma^2\beta}{p} \sin \gamma \sinh \gamma = 0. \tag{B.13}$$

Equation (B.13) has a series of solution; let the solution be γ_n (n=1, 2, 3...∞). Corresponding to each γ_n, one has the corresponding modal function ϕ_n and natural frequency ω_n. Therefore, the eigenvectors of L are

$$\vec{\underline{\bar{\phi}}}_n = \begin{pmatrix} \phi_n\ (\xi) \\ \beta\ \phi_n\ (1) \end{pmatrix}. \qquad (B.14)$$

It can easily be shown that the eigenvalues for L^* are the same as those for L and the modal function ψ_n and eigenvectors of L^* are given by

$$\psi_n\ (\xi), \qquad (B.15)$$

$$\vec{\underline{\bar{\psi}}}_n = \begin{pmatrix} \psi_n\ (\xi) \\ \psi_n\ (1) \end{pmatrix}.$$

The eigenvectors $\vec{\underline{\bar{\phi}}}_n$ and $\vec{\underline{\bar{\psi}}}_n$ possess biorthogonality conditions; i.e., the eigenvectors of L and L^* correspond to different eigenvalues are orthogonal,

$$<\vec{\underline{\bar{\phi}}}_m, \vec{\underline{\bar{\psi}}}_n> = \delta_{mn} \quad . \qquad (B.16)$$

In $\vec{\underline{\bar{\phi}}}_n$ and $\vec{\underline{\bar{\psi}}}_n$, there is a constant $\bar{\alpha}_1$. $\bar{\alpha}_1$ has been chosen such that $<\vec{\underline{\bar{\phi}}}_n, \vec{\underline{\bar{\psi}}}_n> = 1$.

ACKNOWLEDGMENT

This work was performed under the sponsorship of the Division of Reactor Development and Demonstration, U. S. Energy Research and Development Administration.

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