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STABILITY ANALYSIS OF THE VON NEUMANN-RICHTMYER DIFFERENCE SCHEME WITH RATE DEPENDENT MATERIALS RELATIONS, PART 2. SUBCYCLING AND THE MALVERN RELATION

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STABILITY ANALYSIS OF THE VON NEUMANN-RICHTMYER DIFFERENCE SCHEME WITH RATE DEPENDENT MATERIAL RELATIONS, PART 2. SUBCYCLING AND THE MALVERN RELATION

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ABSTRACT

Stability criteria are developed for solving problems involving rate dependent material properties in hydrocodes such as WONDY. As severe restrictions in the allowable timestep size result for small relaxation times, subcycling has been introduced to solve this problem. That is, if the subcycle number (m) is large enough, then the timestep restriction as it exists in WONDY is sufficient for stability; this is shown herein for the case of a simple backward difference subcycling scheme for the Malvern rate dependent material relation. The problem of precisely how large m must be for a given ratio of the timestep to the relaxation time, $h = \Delta t/\tau$, was studied. Although the form of solution for m as a function of h is complicated, it can be incorporated easily into WONDY. In the

extreme cases of h, very small or large, the solution can be simply stated: if h is very small, then $m = 1$ suffices; if $h > 2$, then $m \geq h$ suffices. The fact that the solution reduces to $m \geq h$ for large h is an elegant and interesting result.

TABLE OF CONTENTS

1. INTRODUCTION

In a previous report [1] the results of stability analyses for rate dependent materials were presented. In that report it was shown that the stability condition for the WONDY (and CHART D) scheme is

$$a_3(1)\alpha^2 + 2b_3(1)\alpha \le 1 \qquad (B_1)$$

where α, $a_3(1)$, and $b_3(1)$ are given in Section 4. (WONDY is based on the VNR (von Neumann-Richtmyer) scheme.) This can be a rather severe restriction on the timestep. Moreover, WONDY is programmed to enforce (B_o) where

$$\alpha^2 + 2\frac{\lambda}{a}\alpha \le 1 \qquad (B_o)$$

and not programmed to enforce (B_1). Subcycling appears to be the best way to solve this problem.

By subcycling m times in the n^{th} cycle (the advance from t^{n-1} to t^n), it is meant that the difference equation (or equations) for the stress rate are integrated from t^{n-1} to t^n in m subcycles of timestep size $(t^n - t^{n-1})/m$. The number m is called the subcycle number. It may vary from cycle to cycle and from zone to zone.

In this report it is shown that subcycling m times causes (B_1) to modify to (B_m)

$$a_3(m)\alpha^2 + 2b_3(m)\alpha \le 1 \qquad (B_m)$$

where $a_3(m)$ and $b_3(m)$ are given in Section 4. The timestep restriction subroutine currently in WONDY enforces (B_o) and not (B_1). However, it is not necessary to reprogram WONDY to enforce (B_1) because, as is shown in Section 4, if (B_o) is enforced, then there exists an M such that if $m \ge M$ then (B_m) is satisfied.

An annotated table of contents:

Section 2 is entitled "Notation and Nomenclature"; there the symbols and terminology used are presented.

Section 3 is entitled "Lemmas"; there some facts about quadratic inequalities which are useful in Section 4 are presented.

Section 4 is entitled "Results"; the main results of this study are summarized in Results #1-7.

Section 5 is entitled "Summary and Conclusions"; it gives an overview of the report and the implications of its results.

2. NOTATION AND NOMENCLATURE

Let x_f be the motion; let μ, t, and x be the material, temporal, and spacial coordinates; $x = x_f(\mu,t)$; x is the position of material point μ at time t.

Let u_f, V_f, and ρ_f be the specific momentum, specific volume, and mass density functions; u, V, and ρ are their evaluations at (μ,t); $\rho = 1/V$; $u = \partial x/\partial t$; $V = \partial x/\partial\mu$. (The f subscript is often suppressed when the context allows little chance for confusion.)

Let E_f, $\mathcal{E}_f$, and σ_f be the specific total energy, specific internal energy, and stress functions; E, $\mathcal{E}$, and σ are their evaluations at (μ,t); $E = \mathcal{E} + \frac{1}{2} u^2$.

The conservation laws are expressed by

$$\frac{\partial\vec{U}}{\partial t} + \frac{\partial\vec{F}(\vec{U})}{\partial\mu} = 0$$

where

$$\vec{U} = \begin{pmatrix} V \\ u \\ E \end{pmatrix}$$

and

$$\vec{F}(\vec{U}) = \begin{pmatrix} -u \\ \sigma \\ u\sigma \end{pmatrix} .$$

The stress-strain relation is taken to be in the Malvern rate dependent form

$$\partial\sigma/\partial t + a^2\partial V/\partial t + (\sigma - \sigma_{eq})/\tau = 0$$

where a, σ_{eq}, and τ are the acoustic impedance, equilibrium stress, and relaxation time.

Let x_j^n be the numerical approximation to $x_f(\mu_j,t^n)$ for $j = 0,1,2...$ and $n = 0,1,2...$ and similarly define V_j^n, u_j^n, E_j^n, σ_j^n etc. Let $\Delta\mu$ and Δt be the material and temporal increments and let $\alpha = a\Delta t/\Delta\mu$ be the CFL number. Let the artificial viscosity q be given by

$$q = -\lambda \frac{\partial V}{\partial t} = -\lambda \frac{\partial u}{\partial \mu}$$

where $\lambda \geq 0$ is the coefficient of the artificial viscosity.

3. LEMMAS

In this section the roots of the quadratic

$$\lambda^2 - 2B\lambda + C = 0$$

are being considered in Lemmas 1A and 1B; the constraints on x required for the quadratic inequality

$$Ax^2 + 2Bx \le 1$$

to hold are considered in Lemma 2; and Lemma 3 gives a useful fact about the stability of matrices. The utility of these lemmas becomes evident in Section 4.

LEMMA 1A

Let $\lambda_\pm = B \pm D^{1/2}$ where $D = B^2 - C$ and B and C are real numbers and let $|\lambda|_{max} = \max(|\lambda_+|, |\lambda_-|)$.

<u>Case (a)</u>: If $D \ge 0$ and $B^2 > 1$, then

$$|\lambda|_{max} > 1 .$$

<u>Case (b)</u>: If $D \ge 0$ and $B^2 \le 1$, then

$$\left[|\lambda|_{max} \le 1 \text{ iff } 2|B| \le C + 1\right] .$$

<u>Case (c)</u>: If $D < 0$, then

$$\left[|\lambda|_{max} \le 1 \text{ iff } C \le 1\right] .$$

Moreover, the result also holds when $\le$ is replaced by $<$ inside the square bracket.

Proof Sketch: Note that if $D \ge 0$, then

$$|\lambda|_{max} = |B| + D^{1/2} .$$

Thus in case (a) one can see that $|\lambda|_{max} > 1$. In case (b)

$$|B| + D^{1/2} \leq 1$$

iff

$$D \leq 1 - 2|B| + B^2$$

iff

$$2|B| \leq 1 + C \ .$$

In case (c) note that $|\lambda|_{max} = C$. End of Proof Sketch.

LEMMA 1B

If

$$B = 1 - b \tag{1}$$

and

$$C = 1 - c \tag{2}$$

where

$$b \geq 0 \leq c \tag{3}$$

then

$$[|\lambda_\pm| \leq 1 \ \text{ iff } \ 2b + c \leq 4] \ .$$

Moreover, the result also holds when $\leq$ is replaced by $<$ inside the square brackets provided $c > 0$.

Proof Sketch: Apply Lemma 1A. The interesting case is case (b) with B negative then

$$-2B \leq C + 1$$

iff

$$2b + c \leq 4 \ . \tag{4}$$

End of Proof Sketch.

LEMMA 2

Assume A real, B positive and let

$$x' = x(B + D^{1/2}) \tag{5}$$

where $D = B^2 + A$. Consider the inequality

$$[Ax^2 + 2Bx \le 1] \ . \tag{6}$$

<u>Case (a)</u>: If $A \ge 0$, then (6) holds provided

$$[0 \le x' \le 1] \ . \tag{7}$$

<u>Case (b)</u>: If $-B^2 \le A < 0$, then (7) implies (6).

<u>Case (c)</u>: If $A < -B^2$, then (6) holds for all real x. Moreover, the result also holds if the $\le$ is replaced with $<$ inside the square brackets.

Proof Sketch: The roots of $Ax^2 + 2Bx = 1$ are given by $x_\pm = (B \pm D^{1/2})^{-1}$ if $A \neq 0$. If $A = 0$, then there is only one root $x_+ = (B + D^{1/2})^{-1}$.

<u>Case (a)</u>: In this case $D > 0$; $x_- < 0 < x_+$; and (6) holds provided $x_- \le x \le x_+$ and this is implied by (7).

<u>Case (b)</u>: In this case $D \ge 0$; $0 < x_+ \le x_-$; and (6) holds provided $x \le x_+$ and this is implied by (7).

<u>Case (c)</u>: $D < 0$; there are no real roots; therefore, $A < 0$ implies (6) holds for all real x.

End of Proof Sketch.

LEMMA 3

Let $\underset{\sim}{G}$ be a p by p matrix. If there exists a positive number τ such that the elements of $\underset{\sim}{G}(\Delta t,k)$ are bounded for $0 < \Delta t < \tau$ and for all k and if all of the eigenvalues of $\underset{\sim}{G}$, with the possible exception of one, lie in a circle inside the unit circle, then the von Neumann condition is sufficient as well as necessary for stability. That is, if there exists a constant r such that for all k and all Δt in $(0,\tau)$, $|\lambda_i| \leq r < 1$ for $i = 2,\ldots p$, then $|\lambda_1| \leq 1 + O(\Delta t)$ is necessary and sufficient for stability.

For proof see [2].

4. RESULTS

Assume that the timestep is restricted by

$$\alpha < 1 \tag{1}$$

where α is the CFL number, i.e.

$$\alpha = a\Delta t/\Delta\mu \ . \tag{2}$$

Suppose that the integration of σ from t^n to t^{n+1} is done in m subcycles. First consider the case m = 1 (i.e., no subcycling) then the equations are

$$(\dot{V})^{n+1/2}_{j+1/2} = (u^{n+1/2}_{j+1} - u^{n+1/2}_{j})/\Delta\mu \tag{3}$$

$$u^{n+1/2}_{j} = u^{n-1/2}_{j} - \Delta t(\sigma^{n}_{j+1/2} - \sigma^{n}_{j-1/2})/\Delta\mu \tag{4}$$

$$\sigma^{n+1}_{j+1/2} - \sigma_{eq} = (1 - h)(\sigma^{n}_{j+1/2} - \sigma_{eq}) - hA \tag{5}$$

where

$$h = \Delta t/\tau \tag{6}$$

and

$$A = a^2(\dot{V})^{n+1/2}_{j}\tau \ . \tag{7}$$

Eqn. (5) is just the result of a simple backward differencing of

$$\partial\sigma/\partial t + a^2\partial V/\partial t + (\sigma - \sigma_{eq})/\tau \ . \tag{8}$$

Next let's consider the case m > 1. Let

$$s^{\nu} = \sigma^{n+\nu/m}_{j+1/2} - \sigma_{eq} \tag{9}$$

for $0 \le \nu \le m$; therefore, the starting value for subcycling is

$$s^o = \sigma^n_{j+1/2} - \sigma_{eq} \, . \qquad (10)$$

Then for $0 \le \nu \le m - 1$ the simple backward difference subcycling scheme is given by

$$s^{\nu+1} = (1 - h/m)s^{\nu} - h/mA \, . \qquad (11)$$

The solution to (11) for $0 \le \nu \le m$ is

$$s^{\nu} = (1 - h/m)^{\nu}s^o + [(1 - h/m)^{\nu} - 1]A \qquad (12)$$

and in particular for $\nu = m$ we have

$$\sigma^{n+1}_{j+1/2} - \sigma_{eq} = s^m \, . \qquad (13)$$

From (12) and (13) it follows that

$$\sigma^{n+1}_{j+1/2} - \sigma_{eq} = (1 - hf(m))(\sigma^n_{j+1/2} - \sigma_{eq}) - hf(m)A \qquad (14)$$

where

$$f(m) = [1 - (1 - h/m)^m]/h \, . \qquad (15)$$

Note that

$$\lim_{m\to\infty} f(m) = g(h) \qquad (16)$$

where

$$g(h) = (1 - e^{-h})/h \, . \qquad (17)$$

Observe that $0 \le g(h) \le 1$ and also note that for all positive integer m and positive h

$$1 \geq f(m) > f(m+1) > g(h) > 0 .$$

Following the notation used in [1] and [2] let

$$w^n_{j+1/2} = -(\sigma^n_{j+1/2} - \sigma_{eq})/a \qquad (18)$$

and

$$v^{n+1}_j = u^{n+1/2}_j . \qquad (19)$$

The next step is to replace v^n_j by $v^n \xi^j$ and $w^n_{j+1/2}$ by $w^n \xi^{j+1/2}$ where

$$\xi = \exp ik\Delta\mu = \cos k\Delta\mu + i \sin k\Delta\mu .$$

Then (4) and (14) become

$$v^{n+1} = v^n + i\beta w^n \qquad (20)$$

and

$$w^{n+1} - i\beta v^{n+1} f(m) = w^n(1 - hf(m)) \qquad (21)$$

where

$$\beta = 2\alpha \sin(k\Delta\mu/2) . \qquad (22)$$

Now let

$$\vec{U}^n = \begin{bmatrix} v^n \\ w^n \end{bmatrix}$$

to get

$$\underset{\sim}{H}_1 \vec{U}^{n+1} = \underset{\sim}{H}_o \vec{U}^n$$

with

$$\underset{\sim}{H}_1 = \begin{bmatrix} 1 & 0 \\ -i\beta f(m) & 1 \end{bmatrix}$$

and

$$\underset{\sim}{H}_o = \begin{bmatrix} 1 & i\beta \\ 0 & 1 - hf(m) \end{bmatrix} .$$

It follows that

$$\overline{U}^{n+1} = \underset{\sim}{G}\, \overline{U}^n \tag{23}$$

where

$$\underset{\sim}{G} = \underset{\sim}{H}_1^{-1}\, \underset{\sim}{H}_o .$$

Specifically

$$G = \begin{bmatrix} 1 & i\beta \\ i\beta f(m) & 1 - (\beta^2 + h)f(m) \end{bmatrix} . \tag{24}$$

Note that

$$\det(\underset{\sim}{G} - \lambda \underset{\sim}{I}) = \lambda^2 - 2B_* \lambda + C_* \tag{25}$$

where

$$B_* = 1 - (\beta^2 + h)f(m)/2 \tag{26}$$

and

$$C_* = 1 - hf(m) \tag{27}$$

also let

$$D_* = B_*^2 - C_* .$$

Result #1:

If $m = 1$ or $m > h$, then a necessary and sufficient condition for the stability of $\underset{\sim}{G}$ is

$$f(m)[\alpha^2 + h/2] \leq 1 \tag{28}$$

Proof Sketch: In (26) and (27) identify the b of Lemma 1B with $(\beta^2 + h)f(m)/2$ and the c with $hf(m)$. Recall that the condition for $|\lambda_\pm| \leq 1$ was

$$2b + c \leq 4$$

and note that this is the same as

$$f(m)[\beta^2 + 2h] \leq 4$$

which if true for all β leads to (28). Thus (28) implies $|\lambda_+| \leq 1 \geq |\lambda_-|$. To use Lemma 3 it must be shown that one or the other of $|\lambda_+|$ and $|\lambda_-|$ is strictly less than one. Consider their product;

$$\lambda_+\lambda_- = C_* = 1 - hf(m) \ .$$

If $D_* \leq 0$, then $0 < C_* < 1$. If $D_* > 0$, then the case that must be considered is $f(m)(\alpha^2 + h/2) = 1$; in this case $(1 - h/m)^m = C_* = f(m)[\alpha^2 - h/2]$ and it can be seen that

$$0 < C_* < 1 \tag{29}$$

provided $m > h$. It follows that (28) implies

$$|\lambda_+\lambda_-| < 1 \ . \tag{30}$$

In case $m = 1$ then $f(m) = 1$ and (28) implies $h < 1$ and thus (30) follows. Hence, Result #1 follows from Lemmas 1B and 3. End of Proof Sketch.

<u>Result #2</u>:

If $\alpha^2 = \theta < 1$, then there exists an M such that $m \geq M$ implies

$$f(m)(\alpha^2 + h/2) \leq 1 \ . \tag{28}$$

Proof Sketch: Inequality (28) becomes

$$f(m)(\theta + h/2) \le 1 .$$

Multiply by h and observe that as $m \to \infty$ the last inequality becomes

$$0 \le H(h)$$

where

$$H(h) = h + (\theta + h/2)(e^{-h} - 1) .$$

Observe that

$$H(0) = 0$$

$$H'(h) = \frac{1}{2} + e^{-h}(1/2 - h/2 - \theta)$$

$$H'(0) = 1 - \theta$$

$$H''(h) = e^{-h}[h - 2(1 - \theta)]/2$$

therefore if H' evaluated at $h = 2(1 - \theta)$ is ≥ 0, then H is positive for all h. One can show that if $\theta < 1$, then $H'(2(1 - \theta)) > 0$ and therefore $H(h) > 0$ for all $h > 0$. End of Proof Sketch.

Remark: The reason Results #2 and #3 are of interest is because WONDY already has a timestep restriction of the form $\alpha \le .9$ built in. If we can show that the current timestep restriction routine in WONDY does not have to be altered but that for $\alpha \le .9$ all that is necessary is to choose m large enough and stability follows, then we can avoid the rewriting of the WONDY timestep restriction routine.

Now let's investigate how big m must be to satisfy (28) when $\alpha < 1$. Let's split into three cases: case (1) is with $m = 1$; case (2) is with

$0 < h < \alpha^2 + 1/2$; and case (3) is with $h \geq \alpha^2 + 1/2$. In case (1) check to see if

$$\alpha^2 + h/2 \leq 1$$

is satisfied. For example in WONDY $\alpha \leq .9$ so $\alpha^2 \leq .81$ and thus if $h \leq .38$, then $m = 1$ suffices. Thus case (1) says that for $0 \leq h \leq 2(1 - \alpha^2)$ one need not subcycle.

Case (2): $(2(1 - \alpha^2) < h < 2\alpha^2)$ Observe that for this case to be nonvacuous requires $1 < 2\alpha^2$; if not, then case (1) covers case (S). Note that

$$f(m)(\alpha^2 + h/2) = \alpha^2 + h\left(\frac{1 - \alpha^2}{2} + \frac{\alpha^2}{2m}\right) - O(h^2) \qquad (31)$$

where the $O(h^2)$ term is positive in case (2). Requiring the RHS of Eqn. (31) to be less than or equal to unity and dropping the $O(h^2)$ term leads to

$$m \geq \frac{\alpha^2}{1 - \alpha^2}\left(\frac{h}{2 - h}\right)$$

as the requirement for m in case (2). One can see that if $\alpha \leq .9$, then m need never be larger than about 5 in case (2).

Case (3): $(h \geq 2\alpha^2)$ If $\mu \geq 1$, then

$$e^{-1}(1 - 1/\mu) \leq (1 - 1/\mu)^{\mu}$$

holds. It follows that if $m \geq h$, then

$$f(m) \geq [1 - e^{-h}(1 - h/m)^h]/h \qquad (32)$$

and that (28) is satisfied provided

$$e^h[1 - h/(\alpha^2 + h/2)] \leq (1 - h/m)^h .$$

Note that the last inequality is satisfied if $m \geq h$ because then the RHS is positive and the LHS is negative in case (3). Let's summarize the foregoing cases (1), (2), and (3) in the following

Result #3:

Assume that $\alpha \leq \theta < 1$.

Case (1): If $0 \leq h \leq 2(1 - \alpha^2)$, then $m \geq 1$ satisfies (28).

Case (2): If $2(1 - \alpha^2) < h < 2\alpha^2$, then

$$m \geq \frac{\alpha^2}{1 - \alpha^2} \frac{h}{2 - h}$$

satisfies (28).

Case (3): If $h \geq 2\alpha^2$, then $m \geq h$ satisfies (28).

Proof Sketch: Was given prior to the statement of the result.

Remark: Result #3 shows that if the q coefficients are zero, then the timestep restriction routine in WONDY need not be altered; the subcycle number can be chosen large enough to satisfy the stability condition provided $\alpha < 1$; the WONDY timestep restriction insures that $\alpha \leq .9$. The next question that arises is "What if the q coefficients are not zero?" The timestep restriction in WONDY is $\alpha' \leq .9$ where $\alpha' = \alpha[\lambda/a + \sqrt{1 + (\lambda/a)^2}]$ and λ is the viscosity coefficient. Thus the question is "Does there exist a subcycle number large enough for stability provided $\alpha' \leq .9$?" The next step is to investigate this question.

The artificial viscosity q is of the form

$$q = -\lambda\Delta\mu \frac{\partial u}{\partial \mu} \qquad (34)$$

with $\lambda \geq 0$. The addition of this artificial viscous stress to σ in Eqn. (4) results in

$$u_j^{n+1/2} = u_j^{n-1/2} - \Delta t[\tilde{\sigma}_{j+1/2}^n - \tilde{\sigma}_{j-1/2}^n]/\Delta\mu \qquad (35)$$

where

$$\tilde{\sigma}_{j+1/2}^n = \sigma_{j+1/2}^n + q_{j+1/2}^{n-1/2}$$

and thus Eqn. (20) becomes

$$v^{n+1} = v^n(1 - \delta) + i\beta w^n \qquad (36)$$

where

$$\delta = 4\lambda(\Delta t/\Delta\mu)\sin^2(k\Delta\mu/2) \ . \qquad (37)$$

Then $\underset{\sim}{H}_o$ becomes

$$\underset{\sim}{H}_o = \begin{bmatrix} 1 - \delta & i\beta \\ 0 & 1 - hf(m) \end{bmatrix}$$

and $\underset{\sim}{G}$ becomes

$$\underset{\sim}{G} = \begin{bmatrix} 1 - \delta & i\beta \\ (1 - \delta)i\beta f(m) & 1 - (\beta^2 + h)f(m) \end{bmatrix}$$

and B_* becomes

$$B_* = 1 - [\delta + (\beta^2 + h)f(m)]/2 \qquad (38)$$

and C_* becomes

$$C_* = 1 - [\delta(1 - hf(m)) + hf(m)] \ . \qquad (39)$$

Identify b with $1 - B_*$ and c with $1 - C_*$ in Lemma 1B. Recall that $1 - hf(m) = (1 - h/m)^m$ therefore $1 - hf(m) \geq 0$ provided $m \geq h$. Assuming $m \geq h$, then $c \geq 0$ follows and Lemma 1B can be applied to yield

$$2(1 - B_*) + (1 - C_*) \leq 4$$

as the stability condition. In more detail this becomes

$$a_3(m)\alpha^2 + 2b_3(m)\alpha \leq 1 \tag{40}$$

where

$$a_3(m) = f(m) - \lambda\Delta\mu/(a^2\tau) \tag{41}$$

and

$$b_3(m) = \lambda/a + \Delta\mu f(m)/(4a\tau) \ . \tag{42}$$

Also let

$$d_3(m) = b_3^2(m) + a_3(m) \ .$$

Now we have the following.

Result #4:

A necessary and sufficient condition for $\underset{\sim}{G}$ to be stable is (40).

Proof Sketch: The proof parallels the proof of Result #1.

Remark: In [1] where the case $m = 1$ was considered, it was assumed that the restriction $a_3(1) \geq 0$ was imposed; Lemma 2 has been extended to cases (b) and (c) in order to take care of the case $a_3(m) < 0$ for $m = 1,2,3...$ This leads to the following.

Result #5:

Let $e_3(m) = b_3(m) + d_3(m)^{1/2}$ and $\alpha' = \alpha\, e_3(m)$.

Case (a): If $a_3(m) \geq 0$, then (40) holds provided

$$0 \leq \alpha' \leq 1 \; ; \tag{43}$$

Case (b): If $-b_3^2(m) \leq a_3(m) < 0$, then (43) implies (40);

Case (c): If $a_3(m) < -b_3^2(m)$, then (40) holds for all α.

Proof Sketch: Apply Lemma 2.

Result #6:

Let

$$z = \lambda/a + \sqrt{(\lambda/a)^2 + 1}\ .$$

If

$$\alpha z = \theta < 1 \tag{44}$$

then there exists an M such that for $m \geq M$ inequality (40) is satisfied.

Proof Sketch: Split (40) into

$$\alpha^2 + 2\,\frac{\lambda}{a}\,\alpha = \theta_1 < \theta \tag{45}$$

and

$$[f(m) - 1 - \lambda\Delta\mu/(a^2\tau)]\alpha^2 + f(m)\,\frac{h}{2} \leq 1 - \theta_1\ . \tag{46}$$

Note that (45) follows from (44). On (46), multiply by h and take the limit as $m \to \infty$ to get the requirement that

$$0 < H(h)$$

for $h > 0$ where

$$H(h) = h\left(1 + \frac{\lambda}{a}(h - 2)\alpha\right) + \left(\alpha^2 + \frac{h}{2}\right)(e^{-h} - 1) .$$

Note that

$$H(0) = 0$$

and

$$H'(h) = \frac{1}{2} + 2\frac{\lambda}{a}(h - 1)\alpha + e^{-h}\left(\frac{1}{2} - \alpha^2 - \frac{h}{2}\right)$$

and

$$H'(0) = 1 - \theta_1 > 0 .$$

Since

$$H''(h) = 2\alpha\lambda/a + e^{-h}[h - 2(1 - \alpha^2)]/2$$

it follows that if

$$h \geq 2(1 - \alpha^2)$$

then $H'(h) \geq 0$; thus if $H'(2(1 - \alpha^2)) > 0$, then $H(h) > 0$ for all $h > 0$. Let

$$G_1(\theta,z) = H'(2(1 - \alpha^2))$$

where α is replaced with θ/z. The problem is to show that $G_1(\theta,z) > 0$ for $z \geq 1$ and $\theta < 1$; note that in regard to θ the worst case is $\theta = 1$; therefore, consider $G_1(1,z)$; let $G_2(z) = G_1(1,z)$. The problem has now reduced to showing $G_2(z) > 0$ for $z \geq 1$. Let $1 - x = 1/z^2$ and $G_3(x) = G_2(z)$; then

$$G_3(x) = \frac{1}{2}(1 - e^{-2x}) + x(2x - 1) ;$$

the problem is now to show $G_3(x) \geq 0$ for $0 < x \leq 1$. Note that

$$G_3(0) = 0$$

$$G_3'(x) = e^{-2x} - 1 + 4x \quad , \quad G_3'(0) = 0$$

$$G_3''(x) = 4 - 2e^{-2x} \quad , \quad G_3''(0) = 2$$

$$G_4''(x) = 4e^{-2x} > 0 \text{ for all } x > 0$$

Therefore $G_3(x) > 0$ for all $x > 0$ and this proves that $H(h) > 0$ for all $h > 0$. End of Proof Sketch.

Now let's investigate how big m must be to satisfy (40) when (44) holds. If $\lambda = 0$, then use Result #3; for the following cases assume $\lambda > 0$.

Case (1): If $m = 1$, then $f(m) = 1$ and (40) becomes

$$-[\lambda\Delta\mu/(a^2\tau)]\alpha^2 + 2[\Delta\mu/(4a\tau)]\alpha \leq 1 - \theta_1 \qquad (47)$$

where

$$\alpha^2 + 2\frac{\lambda}{a}\alpha = \theta_1 < \theta < 1 .$$

Assume α is given; then one may evaluate θ_1 and the LHS of (47) and check to see if (47) is satisfied.

One can apply Lemma 2 to (47) to reduce the quadratic inequality to a linear inequality by identifying A with $-\lambda\Delta\mu/[(1 - \theta_1)a^2\tau]$ and B with $\Delta\mu/[(1 - \theta_1)4a\tau]$.

If (47) is not satisfied, then $m = 1$ will not suffice; to find a sufficiently large m go to case (2), (3), or (4) depending on the size of h.

Case (2): $(h < 2\alpha^2)$ Notice that (40) can be written as

$$f(m)[\alpha^2 + h/2] + \frac{\lambda}{a}(2 - h)\alpha \le 1 \ .$$

Using (31) leads to the requirement that

$$m \ge \frac{\alpha^2}{1 - \theta_1}\frac{h}{2 - h} \tag{48}$$

Case (3): $(2\alpha^2 \le h < 2)$ Using the expansion

$$f(m) = 1 - \frac{m - 1}{2m} h + \frac{(m - 1)(m - 2)}{6m^2} h^2 + O(h^3)$$

where $O(h^3)$ is negative and using Lemma 2 one arrives at the condition

$$m/h \ge b_4 + \sqrt{b_4^2 + a_4} \tag{49}$$

where

$$a_4 = 2\alpha^2/(3\eta)$$

and

$$2b_4 = (h/2 + (1 - h)\alpha^2)/\eta$$

and

$$\eta = \left(\frac{1}{2} - \frac{\alpha^2}{3}\right)h^2 + 2(1 - \theta_1)(1 - h/2) \ .$$

Case (4): $(h \ge 2)$ Using (32) leads to the requirement that

$$e^h[1 - (1 + (h - 2)\frac{\lambda}{a}\alpha)h/(\alpha^2 + h/2)] \le (1 - h/m)^h$$

which is seen to be satisfied provided $m \ge h \ge 2$.

The previous cases are summarized in the following.

Result #7:

Assume that $\alpha z = \theta < 1$ where

$$z = \frac{\lambda}{a} + \sqrt{1 + (\lambda/a)^2} .$$

Then $m \geq M$ is sufficient for the stability of $\underset{\sim}{G}$ where M is given in the following cases.

Case (1): M = 1 suffices provided (47) holds.

Case (2): If $h < 2\alpha^2$, then $M \geq \frac{\alpha^2}{1 - \theta_1} \frac{h}{2 - h}$ suffices.

Case (3): If $2\alpha^2 \leq h < 2$, then M is given by (49).

Case (4): If $h \geq 2$, then $M \geq h$ suffices.

Proof Sketch: Was given before the statement of the result.

5. SUMMARY AND CONCLUSIONS

If a stress rate relation of the Malvern form is used with the VNR scheme (in WONDY and CHARTD) without subcycling, then the timestep restriction for stability is

$$(\alpha^2 + h/2) + \frac{\lambda}{a}(2 - h)\alpha \le 1 \qquad (1)$$

and this timestep restriction can be rather severe. Recall that α is the CFL number, λ is the viscosity coefficient, $h = \Delta t/\tau$, and τ is the relaxation time. If subcycling is used, then the stability condition becomes

$$f(m)(\alpha^2 + h/2) + \frac{\lambda}{a}(2 - h)\alpha \le 1 \qquad (2)$$

where

$$f(m) = [1 - (1 - h/m)^m]/h \qquad (3)$$

and m is the number of subcycles. If

$$\alpha^2 + 2\frac{\lambda}{a}\alpha \le \theta < 1 \qquad (4)$$

(this is the timestep restriction enforced in WONDY and CHARTD with $\theta = .9$) then there exists an M such that if $m \ge M$, then (2) is satisfied. Therefore the timestep restriction subroutine currently existing in WONDY and CHARTD need not be reprogrammed to enforce (1) provided m is chosen large enough. Result #7 gives specific prescriptions for how large to choose m.

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