TECHNICAL MEMORANDUM

ANL-CT-76-47

THE INTRUSION OF FLUID INTO THE INFLOW BRANCH OF A 180°-APPROACH MIXING TEE

by

Walter Debler

University of Michigan*

ARGONNE NATIONAL LABORATORY

U of C-AUA-USERDA

September 1976

*Work performed while Visiting Scientist at ANL Components Technology Division

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Abstract

When the flow rates in the two inlet branches of a 180°-approach mixing tee are greatly different, it is possible that the fluid with the high velocity may intrude into the conduit in which the low velocity fluid is flowing. It is shown in this report that such an intrusion should not extend over many pipe diameters.

However, if the faster flowing fluid is also the warmer, buoyancy forces may be generated through heat transfer. This in turn may lead to density stratification in what would normally be the cooler fluid's inlet conduit. An extensive eddy develops in this branch of the tee which carries warm fluid many diameters in the upstream direction of the cooler fluid. In the laboratory such an intrusion of warm fluid in the cool fluid branch yields large temperature differences between the top and bottom of the pipe. Such behaviour in prototypic systems could produce deleterious thermal stresses.*

Two mathematical models have been developed to estimate the extent of this density-driven intrusion. One is an inviscid model which incorporates two additional simplifying assumptions to give an initial estimate of the significance of the temperature difference and fluid velocity. This estimate is an initial step in an iterative procedure for a numerical solution scheme. The numerical algorithm has not been tested for computational stability. The second method which is presented is a perturbation solution for a low Reynolds number flow. The matching of the solution in two regimes will require the numerical solution of equations to determine the associated coefficients. Once this is done streamline patterns can be drawn for a variety of Froude, Reynolds, and Prandtl numbers.

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*A converse situation occurs if the faster flowing fluid is also the cooler.

Nomenclature (App. III not included)

A	Constant; also height to length ratio
C	Interfacial mixing length
D	Diameter of horizontal pipe
d	Channel width
F	Froude number
Gr	Grashof number
g	Gravitational constant
h	Cavity height
k	Thermal conductivity
K	Constant
L	Length of horizontal pipe
N	Constant
n	Reciprocal of fraction flowrate in one branch
Pr	Prandtl number
Q	Total flowrate; also (App. I), rate of heat transfer
Re	Reynolds number
T	Temperature
$\hat{T}$	$(T-T_C)/(T_H-T_C)$
t	Intermediate auxiliary variable
U	Speed
u	x component of velocity
$\hat{u}$	u/U
$\underline{V}$	Velocity vector
v	y component of velocity
$\hat{v}$	v/U

Nomenclature (Contd.)

w	Plane of ϕ and ψ coordinates
x	z-plane coordinate
y	z-plane coordinate; also (App. I), elevation
z	Plane of x and y coordinates

Greek symbols

α	Linear coefficient of thermal expansion
ζ	Vorticity in z-direction
$\hat{\zeta}$	Modified vorticity (see Eq. 4.9 ff.)
η	y/d
κ	Thermal diffusivity
ξ	x/d
ρ	Density
ϕ	Potential function
ψ	Stream function
$\hat{\psi}$	ψ/Q
$\tilde{\psi}$	$\hat{\psi}-\eta$

Subscripts

C	Cold
H	Hot
o	Reference
S	Stagnation point

The Intrusion of Fluid into the Inflow Branch of a 180°-Approach Mixing Tee

1. Introduction

The 180°-approach mixing tee may be used in future power-generating systems to mix hot and cold liquid streams to achieve the desired fluid temperature for subsequent stages of the thermodynamic cycle. Such a tee, with inlet branches horizontal, is shown in Fig. 1. Under some conditions of operation the flow of one fluid may be reduced when a large flowrate of the other fluid is wanted from the tee. Under these conditions it has been observed in tests at the Argonne National Laboratory (ANL-RDP-33, p. 3.8) that the high-flowrate fluid apparently intrudes into the low-flowrate-fluid inlet branch. This situation was concluded to be a possible occurrence when the lower surface of the low-flowrate-fluid (hot water) inlet pipe was found to be cool (consistent with the temperature of the high-flowrate-fluid, cold water), at distances of 75 diameters (D = 10 cm) upstream in the hot-water branch.

Such temperature differences between the upper and lower portions of the pipe could lead to significant thermal stresses. These stresses could affect the useful life of the pipe-tee and adjacent pieces. It is the purpose of this report to examine some of the mechanisms by which this intrusion can occur.

2. Previous Work

There does not appear to be much published information which is directly applicable to the problem at hand. A report by Fraser and Oakley[(4)] is discussed in Appendix I of this report. In their work a horizontal cavity

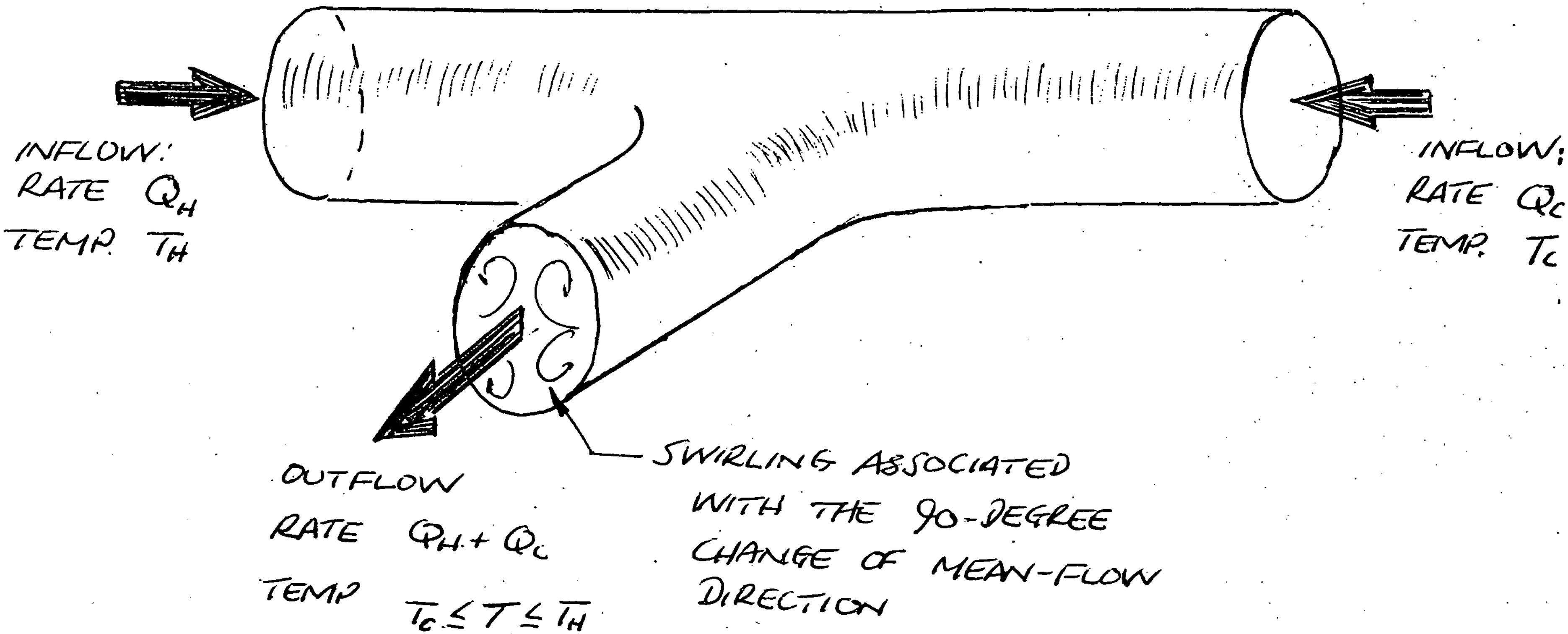

FIGURE 1. SCHEMATIC DIAGRAM OF MIXING-TEE

is differentially heated at each end and the resulting circulation is the subject of laboratory experiments and an analytical model. There is no net mass flux through the cavity as is the case in the "cold leg" of a 180°-mixing tee.*

Recent efforts by Cormack, Stone, and Leal[2] and Cormack, Leal, and Seinfeld[1] also treat a horizontal cavity which is differentially heated. They obtain extensive results for their model, which also does not have any net mass flux.

The speed of a gravity current that is advancing in an ambient fluid was analyzed by Von Karman.[7] If one adopts a coordinate system which is fixed with the front of this current, the ambient fluid would appear to flow past the intrusion. This would be similar to what is occurring in one of the legs of the mixing-tee. Von Karman obtained the speed of the fluid, for a steady phenomenon, to be

$$U = \sqrt{2\ g\ (\rho_2-\rho_1)\ (\text{depth of intrusion})/\rho_1}\ .$$

If some numerical values for possible experiments with water are used in this result one could calculate a speed of

$$U = \sqrt{2\ (32.2)\ (62.17-60.57)\ (2/12)/60.57} = 0.45\ \text{ft/sec}\ (0.137\text{m/s})$$

if the densities of water at 27 and 83°C and a intrusion height of 5 cm (pipe radius of 5 cm) are assumed. This formula by Von Karman is useful for giving an estimate of the minimum fluid speed in the conduit, U, for a known density difference $(\rho_2-\rho_1)$ in order to keep the intrusion from progressing unabated upsteam. This result is for an inviscid non-diffusive fluid, so that actually

*In this report, the low-flowrate fluid is consistently taken to be the cooler fluid, so that intrusion is always into the cold-fluid inlet pipe. However, all results are applicable as well, in converse fashion, to the case in which the hotter fluid is the low-flowrate fluid.

the speed would be less than this value.

While this information is useful it does not give any information about the length of the intrusion.

Debler[3] performed some related experiments in a channel in which the fluid had a constant, stable, vertical density gradient. He found that when the densimetric Froude number,

$$\frac{U}{(\text{depth}) \sqrt{g \text{ (density gradient)}}}$$

was less than 0.24, there existed a stagnant region of fluid at the top of the channel while fluid was flowing along the bottom into a line sink at the lower end of the tank. Above this Froude number fluid from the entire depth of the tank moved into the sink. These experiments largely confirmed the analytical predictions of Yih.[8] He showed that for values of the Froude number near $1/\pi$, 0.317, there would be a vortex in the region above the sink. This vortex would become longer as the Froude number was reduced toward $1/\pi$. The length would become infinite at $1/\pi$, at which point the analysis would cease to be valid. In this case, the model adopted by Yih was for an inviscid non-diffusive fluid.

In Sect. 4 of this report a model for an inviscid, but diffusive, fluid is developed. The solution is an iterative one with the result of the first step being similar to that of Yih's criterion.

3. <u>Analysis for Constant Density Streams</u>

Before proceeding to the mechanism which appears to play an important role, buoyancy, the contribution of momentum, or inertia, to the phenomenon will be assessed. This assessment will be made for a two-dimensional flow, even though the actual pipe tee is three-dimensional. It is believed that the essence of the problem is preserved by the model. Fig. 2 illustrates the situation. The problem is similar to several which occur in the literature

[Robertson [6]] and it can be solved directly by employing the Schwartz-Christoffel transformation. Fig. 2 also shows the diagrams which are associated with the method of solution.

A sample calculation for a flow with 1/4 of the flow in one branch (n = 4) and 3/4 in the other is included in Appendix II. The streamlines are shown in Fig. 3. The stagnation point at the boundary opposite to that in which the sink is located gives an indication of the amount of "overshoot" or intrusion that can be expected for such a situation. This position can be calculated without drawing all the streamlines (viz., Appendix II) and the results are shown in Fig. 4. These results are interesting because they show an "overshoot" of one channel width when less than 5 percent is flowing in one branch. This means that the effect of the inertia of the fluid in the high-flow branch is small in producing a lengthy intrusion region in the low-flow branch.

These calculations are related to some laboratory observations which were made by the author and K. Kasza of the Components Technology Division. During isothermal tests with a plexiglas test section, it was noted that as long as there was <u>any</u> flow in one of the in-flowing branches, the flow from the opposite branch did not appear to intrude into, or influence the flow, in the branch with the low flow. It must be added, however, that when the flow in one branch was stopped, there were large-scale, random pulsations in the supposedly stagnant branch. These motions may be due to the unstable interface between the flowing fluid, which turns through 90°, and the stagnant branch. This could result in agitating the fluid at the interface as well as producing fluid jets which penetrate into the region. An investigation into the mechanics of the flow when one branch has been completely throttled is included in a test plan prepared by K. Kasza for future experiments.

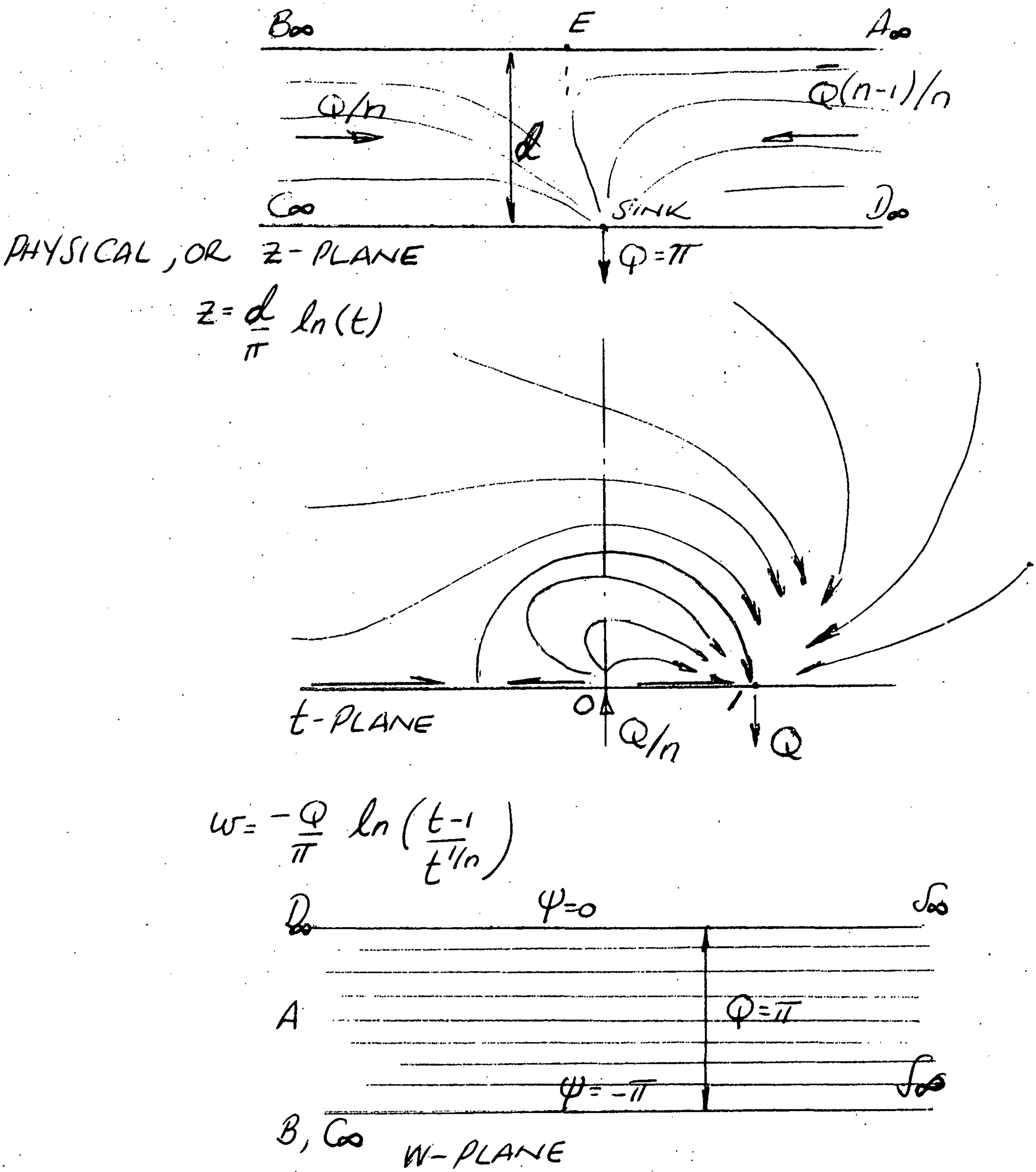

FIGURE 2.[†] TEE MODEL TRANSFORMATIONS

[†] Parameters defined in App. II

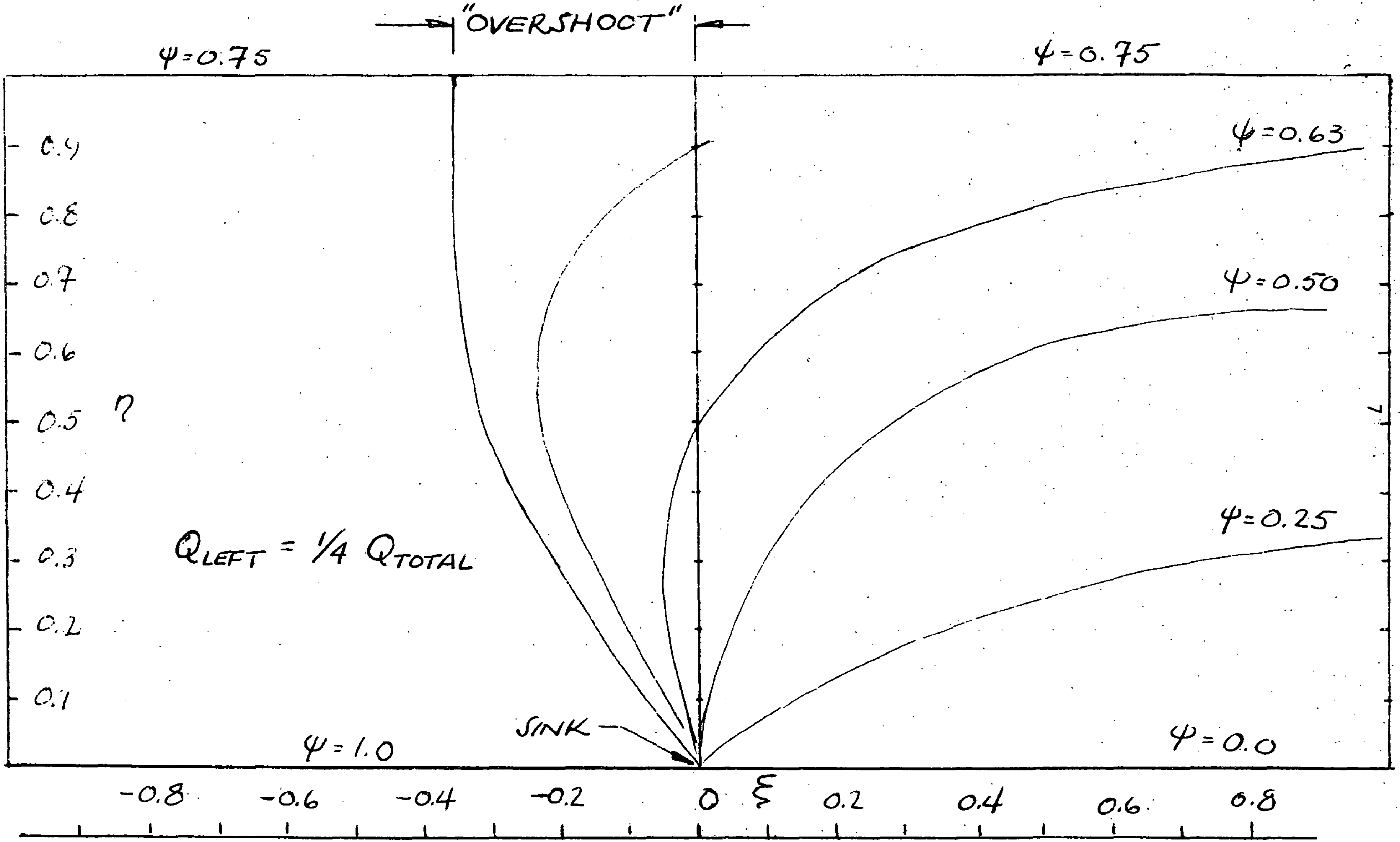

FIGURE 3. STREAMLINE PATTERN IN TWO-DIMENSIONAL CHANNEL

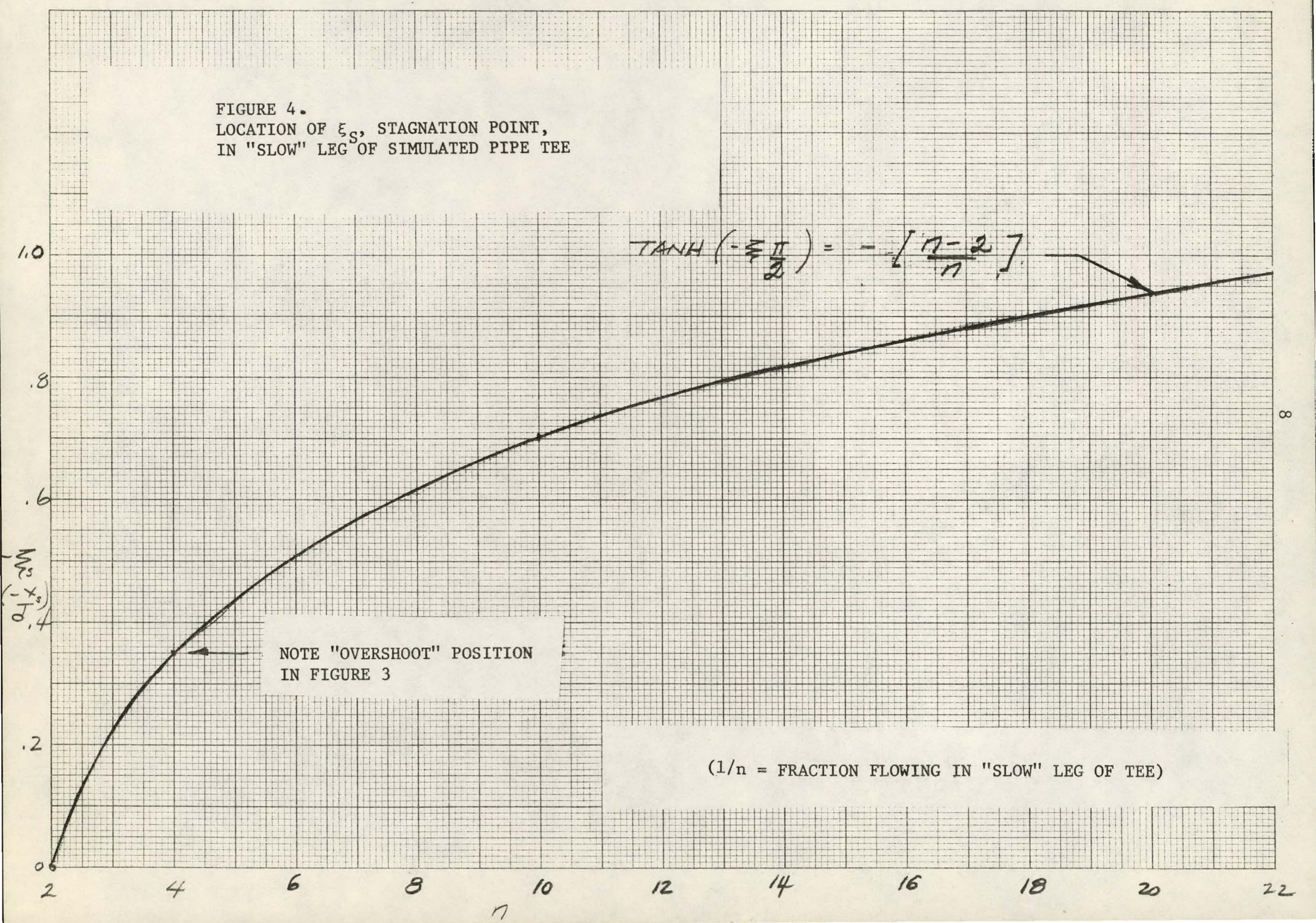
FIGURE 4.
LOCATION OF ξ_S, STAGNATION POINT, IN "SLOW" LEG OF SIMULATED PIPE TEE
$TANH\left(-\xi\frac{\pi}{2}\right) = -\left[\frac{n-2}{n}\right]$
NOTE "OVERSHOOT" POSITION IN FIGURE 3
(1/n = FRACTION FLOWING IN "SLOW" LEG OF TEE)
$-\xi_s$
$\left(\frac{-x_s}{D}\right)$
1.0
.8
.6
.4
.2
0
2
4
6
8
10
12
14
16
18
20
22
∞
n

4. Analysis with Density Differences

The present analysis adopts a model that is illustrated in Fig. 5. One sees that an impermeable boundary is assumed at x = 0 which has a temperature of T_H. It must be emphasized that the model which is illustrated in Fig. 5 has a large hot fluid flow and a small cold fluid flow. This will result in a recirculation eddy at the top of the conduit. If instead of this arrangement, one has a large flow of a cool fluid and a small flow of a warm fluid in a mixing tee, the eddy will appear at the bottom of the branch having the low flow. The interchange of T_H and T_C in Fig. 5b and the location of the sink at point C rather than point S will affect neither the details of the analysis which follows, nor the results.

The effects of inertia of the hot stream and its higher temperature could be combined to determine the extent of intrusion into the region of cool fluid. In this case, the origin for the present analysis could be made to coincide with the vertical plane through point E in Fig. 2. Then the total intrusion length, ℓ, would be the sum of ℓ, from the previous section, and ℓ_2 which follows anon.

The model that was chosen preserved what are thought to be the important features of the occurrence that has been observed:

1. A slowly moving stream at one temperature T_C combines with another, faster moving, stream with temperature T_H in the mixing region of a 180°-approach pipe-tee's junction.

2. The slower moving stream enters this junction through only a portion of the area of the conduit. (This is modeled as the line sink in Fig. 5b.) The remaining part of the conduit from which the fluid with temperature T_C issues, is occupied by the fluid from the opposite side of the tee, with temperature T_H. (This is modeled by prescribing that the plane $x = 0$, $y \neq 0$ have the temperature T_H.) Laboratory observations lend credence to this aspect

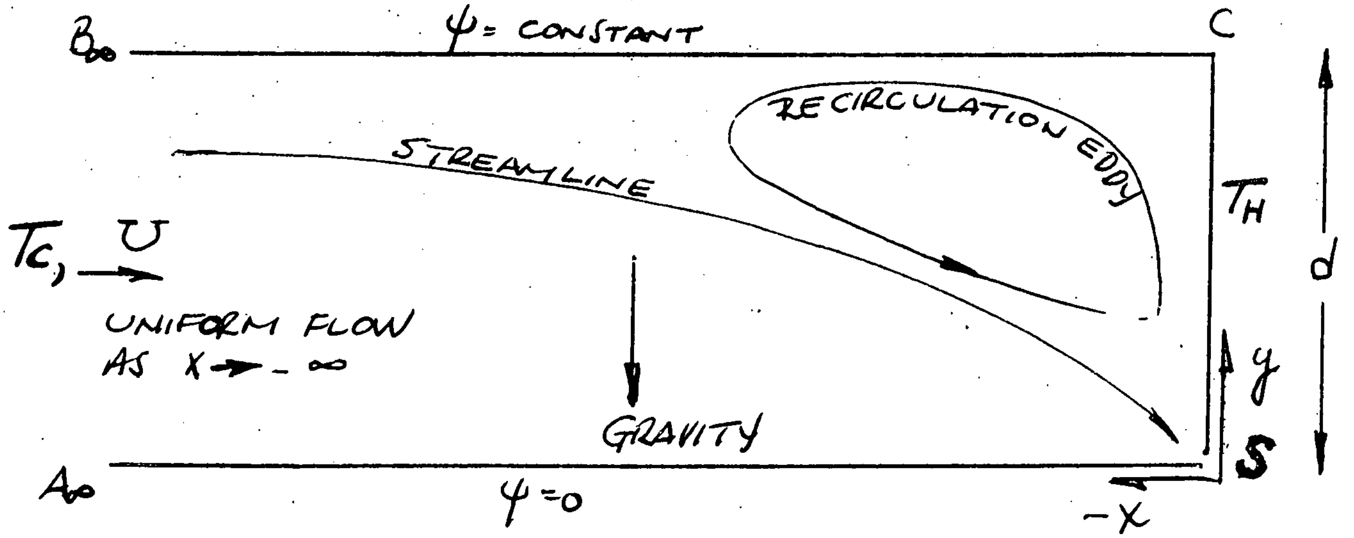

FIGURE 5b. SCHEMATIC OF FLOW MODEL TO DETERMINE CRITERION FOR CIRCULATION EDDY

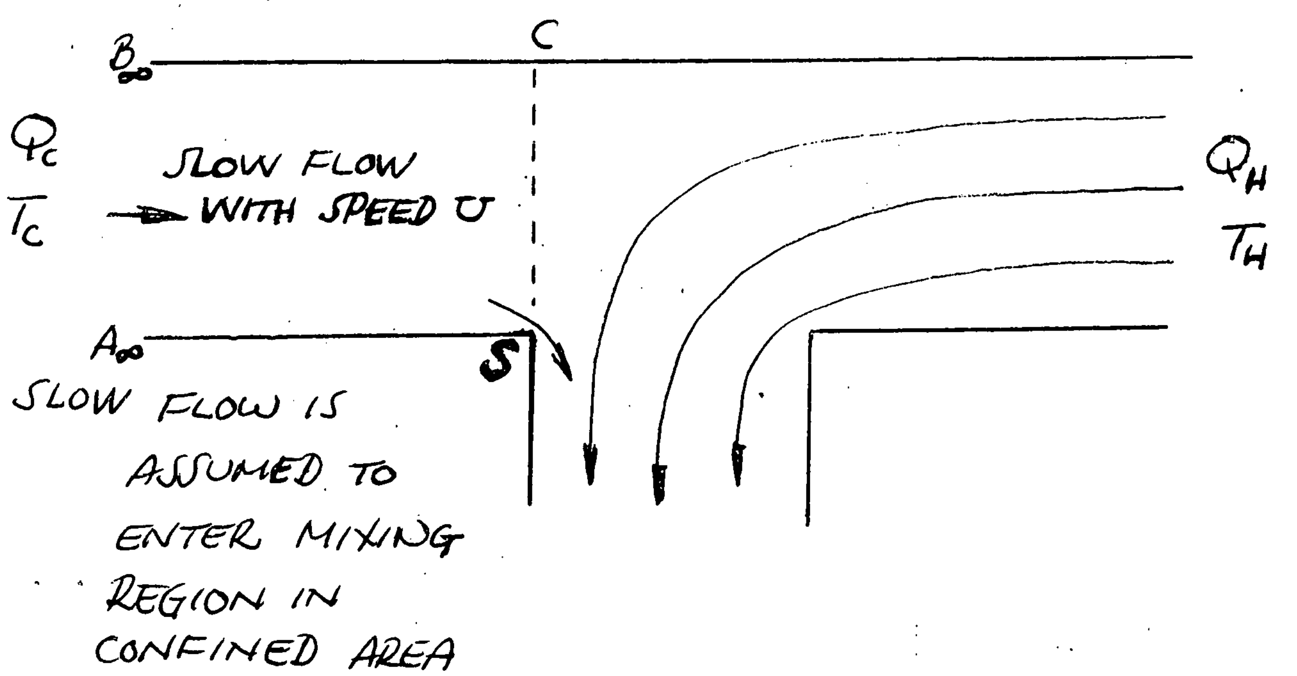

FIGURE 5a. SCHEMATIC OF MIXING-TEE WITH LOW FLOW OF COLD FLUID.

of the model.

3. The actual confluence of the two fluid streams is highly turbulent. Viscous stresses are undoubtedly minor. In the analysis which follows neither viscosity nor turbulent interactions have been taken into account. These latter effects would serve as a source of energy for the buoyant intrusion as well as to increase its size in the conduit through turbulent mixing. However, the exclusion of turbulence does not vitiate the present work because there is some evidence [Debler [3]] that the limiting criterion for the flow parameters which result can be realized in the laboratory to a large extent. Turbulent effects are evident in the test results of selective withdrawal [Debler [3]] but the actual data trends correspond well with an inviscid theory. In fact, the differences in the analytical and experimental results from those tests could well serve as a guide in applying the theoretical conclusions of this section to an actual engineering problem.

Consequently, the present view is that the major characteristics of the intrusion can be modeled if inertia and buoyancy effects are taken into account. (Appendix III contains an analysis for laminar low-Reynolds number flow.) Accordingly, the vorticity equation for an inviscid fluid in steady flow will be used. This is

$$\rho_o\left(u\frac{\partial\zeta}{\partial x} + v\,\frac{\partial\zeta}{\partial y}\right) = -\,g\,\frac{\partial\rho}{\partial x}\,, \qquad (4.1)$$

in which the Boussinesq approximation for the density, ρ, has been used. (The density is considered constant in the inertia terms but a variable in the gravitation term.) The vorticity in the z-direction is ζ and u and v are the x and y components of velocity, respectively. A linear equation of state will be assumed so that

$$\rho = \rho_o[1 - \alpha(T - T_o)], \qquad (4.2)$$

in which α is the coefficient of thermal expansion and T is the temperature. The vorticity ζ satisfies the equation

$$\Delta^2\psi = -\zeta \quad , \tag{4.3}$$

in which the stream function ψ is a consequence of the assumed two-dimensional, incompressible flow.

In Eq. 4.1 there are three unknowns, u, v, and ρ. Because the velocity components u and v can be written as $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$, Eq. 4.1 is, in fact, a relationship between the unknowns ψ and T if Eqs. 4.2 and 4.3 are also considered. The second equation which is needed to close the system is the energy equation for a steady incompressible flow

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \kappa \, \Delta^2 T \tag{4.4}$$

in which κ is the thermal diffusivity. While a solution could, in principle, be sought for ψ and T in the above equations, difficulty may be encountered because of the non-linearity of the equations.

One way of obviating this difficulty would be to linearize appropriately the equations in order to obtain a first approximation, ψ_o and T_o, to the flow. Further approximations could then be generated by an iterative scheme, if they are desired. The present method of solution begins by considering the terms on the left hand side of Eq. 4.1. Here, it will be assumed that

$$u\frac{\partial\zeta}{\partial x} >> v\frac{\partial\zeta}{\partial y} \tag{4.5}$$

The latter term appears to be small throughout most of the region since both v and $\partial\zeta/\partial y$, $\frac{\partial}{\partial y}(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})$ for a uniform, inviscid upstream flow, will be small except near x = 0. The former term in Eq. 4.5 may not be small, by comparison with the latter, because of the coefficient u, which in the present problem is relatively large everywhere, with a magnitude of the order of U, the upstream velocity. Thus, if $\partial\zeta/\partial x$ and $\partial\zeta/\partial y$ are not grossly different in magnitude, a reasonable way of including the effects of vorticity transport in the problem is to write

$$\rho_o U \frac{\partial \zeta}{\partial x} = - g \frac{\partial \rho}{\partial x} \quad (4.6)$$

This equation permits ζ to be written directly as

$$\zeta = + \frac{g\rho_o \alpha T}{\rho_o U} + f(y) \quad (4.7)$$

in which f(y) must be determined and Eq. 4.2 for ρ was employed. As $x \to \infty$ we shall assume that we have a uniform stream with velocity U and temperature T_C. Thus, ζ would be zero at large distances from the origin so that Eq. 4.7 becomes

$$0 = \frac{g\alpha T_C}{U} + f(y)$$

there, with the result that

$$\zeta = \frac{g}{U}\alpha(T - T_C) \quad (4.8)$$

everywhere.

It is convenient at this point to introduce dimensionless variables into the system of equations.

We define

$$\hat{T} = \frac{T - T_C}{T_H - T_C} \quad (4.9a,b,c)$$

$$\hat{\psi} = \frac{\psi}{Ud} = \frac{\psi}{Q}$$

and

$$(\xi, \eta) = (x/d,\ y/d)$$

These new variables imply

$$u = U\partial\hat{\psi}/\partial\eta, \quad v = - U\partial\hat{\psi}/\partial\xi$$

and

$$\zeta = - \frac{U}{d}\left(\frac{\partial^2\hat{\psi}}{\partial\eta^2} + \frac{\partial^2\hat{\psi}}{\partial\xi^2}\right) = \frac{U}{d}\hat{\zeta} \quad (4.9d)$$

Accordingly, Eq. 4.8 can be written as

$$\hat{\zeta} = \frac{gd\alpha}{U^2}(T_H - T_C)\,\hat{T}. \quad (4.10)$$

The coefficient of $\hat{T}$ is the square of the reciprocal densimetric Froude number, F, or the product of the Grashof number and the reciprocal of the square of the Reynolds number. The Froude number will be used in the discussion which follows; hence,

$$F^2 = U^2/[gd\alpha(T_H - T_C)] \tag{4.11}$$

Eq. 4.3 is transformed through the dimensionless variables into Eq. 4.9d, or

$$\Delta\hat{\psi} = - \hat{\xi} \text{ ,} \tag{4.12}$$

in which Δ is the Laplace operator in the dimensionless variables. This last equation can be combined with our result for $\hat{\xi}$, Eq. 4.10 which approximates the vorticity, because of the linearization step. The result is

$$\Delta\hat{\psi} = - \frac{\hat{T}}{F^2} \tag{4.13}$$

The dimensionless energy equation is

$$\hat{u} \frac{\partial\hat{T}}{\partial\xi} + \hat{v} \frac{\partial\hat{T}}{\partial\eta} = \frac{\kappa}{Ud} \Delta\hat{T} \tag{4.14}$$

This equation and Eq. 4.13 should be solved simultaneously for $\hat{\psi}$ and $\hat{T}$. Instead, $\hat{T}$ will be <u>approximated</u> in Eq. 4.13 for an approximate solution of $\hat{\psi}$. These values of $\hat{\psi}$ then will be used in Eq. 4.14 to determine $\hat{T}$. One could also use these values of $\hat{\psi}$ to determine u and v and use them in Eq. 4.1, as well as the values of $\hat{T}$, to find new values of ζ. The sequence of steps is now apparent; these refined values of ζ permit a refinement to the values of ψ from Eq. 4.3. This will lead to an improvement in the values of T in Eq. 4.4. The process can be repeated to obtain the degree of precision that is desired, provided that the algorithm produces a convergent process.

The initial assumption for $\hat{T}$ is $\hat{T}^o = \hat{\psi}^o$.

The reason for this choice is that if $\kappa = 0$ one knows that $\underline{V}\cdot$ grad T = 0, from Eq. 4.4, so that lines of constant temperature are streamlines. This assumed, initial value of $\hat{T}$ allows Eq. 4.13 to be written as

$$\Delta\hat{\psi}^o = - \frac{1}{F^2}\hat{\psi}^o \text{ .} \tag{4.15}$$

The boundary conditions on $\hat{\psi}$ are $\hat{\psi} = 0$ at $\eta = 0$, $\hat{\psi} = 1$ at $\eta = 1$, $\hat{\psi} = \eta$ at $\xi = -\infty$ and at $\xi = 0$, $\eta \neq 0$. The solution procedure is facilitated by setting $\tilde{\psi} = \hat{\psi} - \eta$.

Then $\tilde{\psi} = 0$ at $\eta = 0$ and $\eta = 1$,

$\tilde{\psi} = 1 - \eta$ at $\xi = 0$, $\eta \neq 0$, and

$\tilde{\psi} = 0$ at $\xi = -\infty$.

Eq. 4.15 and the boundary conditions which have been given allow a solution to be written, through the use of separation of variables, as

$$\hat{\psi}^o = \eta + \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{1}{m} \exp[(m^2\pi^2 - 1/F^2)^{1/2}\xi]\sin m\pi\eta \tag{4.16}$$

In this solution, if $F > 1/\pi$ the exponential term will decay the perturbation solution, $\tilde{\psi}$, so that the conduit flows full at large distances upstream. Under conditions of $F \leq 1/\pi$ the perturbation solution, $\tilde{\psi}$, extends to $x = -\infty$. This would imply that the conditions at $x = 0$ would be noticeable at $x = -\infty$. In the present situation, this would mean that the eddy which is associated with $\tilde{\psi}$ would transport heat from the plane $x = 0$ to $x = -\infty$. For values of $F > 1/\pi$ the distance which this heat is transported would be limited through the finite size of the eddy. It should be noted also that for $F < 1/\pi$ the solution for $\tilde{\psi}$, Eq. 4.16, also would yield waves in the upstream region of the flow. This would invalidate the assumption of uniform flow at $x = -\infty$. Hence, the result which has been obtained will be used only for $F > 1/\pi$.

It should also be observed that the boundary conditions on $\hat{\psi}$ which were observed for $\hat{\psi}^o$ are not the same as for $\hat{T}$. These are

$\hat{T} = 0$ at $\eta = 0$ but not at $\eta = 1$

$\hat{T} = 1$ at $\xi = 0$ and $\eta \neq 0$

$\hat{T} = 0$ at $\xi = -\infty$,

and $\partial\hat{T}/\partial y = 0$ at $\eta = 1$

This last condition is reasonable, no heat transfer to the wall, for a stainless-steel conduit. Moreover, it allows a change of temperature along $\eta = 1$ from $\hat{T} = 1$ at $\xi = 0$ to $\hat{T} = 0$ at $\xi = -\infty$. Hence, the initial assumption $\hat{\psi}^o = \hat{T}^o$ has an inconsistancy near $\eta = 1$. This means that while the region of the eddy has been found, the temperature distribution must await the next stage of the calculation.

The solution, Eq. 4.16, has been carried out previously by Yih[8], in a different situation. His results show streamline patterns for a variety of values of F.

The results for $\hat{\psi}$ which have been obtained at this stage of the iteration can be used to determine the temperature distribution from Eq. 4.14. Here the temperature values for the nth stage of the iteration would be obtained by solving the Poisson equation corresponding to Eq. 4.14 in which the numerical values for the left hand side of Eq. 4.14 would come from the available values of $\hat{\psi}$ and the values for the temperature derivatives from the previous iteration step.

The solution for the temperature field would give a distribution which would undoubtedly give a different effective length of the eddy, from the point of view of temperature stratification. It is not anticipated that there would be a significant change in the effective length of the eddy due to thermal diffusion; however, there could well be a noticeable increase in the effective thickness of the layer.

Until now viscous effects and turbulence have been neglected. No doubt the flows will be such that the turbulent mixing will increase the effective size of the eddy over the value given by the model under discussion. Experiments with selective withdrawal problems by Debler [3] show that the eddy size is about 10% thicker than the values predicted by equations similar to Eq. 4.16.

It is not proposed to extend the calculation scheme to higher approximations at this time. However, use of the values of T which result from the solution of Eq. 4.4, and ψ from Eq. 4.3, would allow a numerical solution for ζ from Eq. 4.1 at each point in the desired regime. These values, in turn, could be reinserted in Eq. 4.3 to refine the values of ψ. Only the question of convergency of the procedure remains. Thus, a sample calculation should prove enlightening. This iteration procedure has been programmed in Fortran for execution on an IBM 360 digital computer. Numerical experiments are planned so that the numerical stability of the algorithm can be tested.

5. Experimental Correlation with Theoretical Model

It was mentioned in Sect. 3 that the confluence of two streams in a plexiglas, 180°-approach mixing tee was observed for a variety of flowrates in the two inlet branches. As long as there was some flow in a branch there was no observable tendancy for the fluid from the second branch to intrude into the first branch.

No laboratory experiments have been conducted, until now, to test the accuracy of the criterion for significant upstream intrusion which came from the analysis of Sect. 4. There exist some data (Argonne National Laboratory report, ANL-RDP-33, p. 3.8), however, which bear upon this problem. A horizontal, 180°-approach, 10 cm dia. pipe tee was operated with a warm water flowrate in one inlet leg of 11 ℓ/min while 95 ℓ/min of colder water were flowing in the other inlet leg. The temperature difference of the fluid streams was 55°C. This would imply that the densimetric Froude number in the warm-water branch was (refer to Eq. 4.11)

$$F = \frac{(3/450)/[(1/3)^2\pi]}{\sqrt{32.2(1/3)(10^{-4})(10^2)}} = 0.27$$

Because this calculation gives a Froude number which is less than 0.317 one might expect that there would be an extensive upstream instrusion of the recirculation eddy. During the experiments that are cited in the ANL report, ANL-RDP-33, it was noted that there was a major temperature gradient, transverse to the flow direction, at a position in the horizontal pipe 60-75 diameters upstream of the pipe junction. It was surmised that an upstream intrusion of the colder water into the warm water "leg" of the test loop was responsible for this temperature differential. Thus, in this case, the prediction of upstream intrusion through the densimetric Froude number agrees with the actual occurrence in the laboratory of a phenomenon which could reasonably have been due to upstream intrusion. Clearly it would be desirable to have more information to assess the merits of the criterion of Sect. 4.

6. Conclusions

Experimental observations and a potential-flow calculation support the opinion that the intrusion of one stream into the inlet region of the second stream of a 180°-approach thermal mixing tee is not due to the inertia of the faster stream. This is true as long as there is some flow in both pipes.

The intrusion that has been observed is, no doubt, buoyancy driven. A two-dimensional, inviscid model has been developed for one branch of the pipe-tee. An assumption was made which allows an initial estimate to be given concerning the extent of the upstream intrusion. Subsequent refinement of this estimate could be achieved through detailed numerical computation.

The perturbation solution for small Reynolds number that was carried out in Appendix III should give a superior description of the flow when the numerical solutions are executed. It would be of interest to compare the result of the inviscid model and the low Reynolds number solution.

The inviscid criterion was used with the data of one experiment in which upstream intrusion was suspected. The theoretical criterion for a major upstream intrusion was satisfied.

APPENDIX I

Comments on the report, "Turbulent Free Convection Heat Transfer Rates in a Horizontal Pipe", by J. P. Fraser and D. J. Oakley (Knolls Atomic Power Laboratory report, KAPL-1494, dated 28 February 1956).

The above report presents heat transfer rates for turbulent free convection in a 3 m length of horizontal pipe that contains sodium. In addition, a formula is derived for making predictions about such flows. In this section the results of the report are compared with the work of Cormak, Stone, and Leal*(2) who have modeled the problem using a laminar model and perturbation techniques for the mathematical analysis.

The result of Fraser and Oakley's** analysis(4) gives for the rate of heat transfer, Q,

$$\frac{Q}{Q_{cond}} = \frac{8}{\pi}(AC^2)^{1/2} \frac{(Pr)(Gr)^{1/2}\ (L/D)^{3/2}}{[N + \frac{L}{D}\ (\frac{0.368}{Re^{0.2}} + 265\ C^2)]^{1/2}},$$

in which

Pr = Prandth number = ν/k,

Gr = Grashof number = $\frac{g\beta D^3(T_2 - T_3)}{\nu^2}$

$\frac{L}{D}$ = length to diameter ratio, and

Re = Reynolds number = UD/ν.

The constants A and N are free to be chosen while C is an interfacial mixing length. Finally,

$$Q_{cond} = \frac{\pi}{4}D^2\ k\ \frac{(T_2 - T_3)}{L}$$

*This paper will be subsequently referred to as C.S.L.

**This paper will be subsequently referred to as F.O.

and $T_2 - T_3$ is the temperature difference between thermocouples placed near the top and bottom of the 8-inch diameter pipe.

Because of this definition of Q_{cond}, the heat transfer, Q, is predicted to vary primarily as $(T_2 - T_3)^{3/2}$ and, if one accounts for the various L/D ratios in the complete result, the effect of L/D will be slight for large values of L/D. As L/D approaches small values, Q should vary as the square root of this parameter. This derived formula for Q is compared with the experimental results in the report and the 3/2-power variation of Q with ΔT is a reasonable representation. One set of data for the temperature distribution along the pipe is also provided and for this case there is a nearly constant temperature gradient of 7°C/m.

These results will now be compared to those from the analysis of C.S.L. This work was done for a horizontal cavity which is two-dimensional. A laminar solution was achieved. However, some extension to a turbulent flow was attempted in the paper using turbulent Prandtl numbers, etc.

The "no-slip", insulated-cavity case treated by C.S.L. gives a temperature distribution in the middle portion as

$$\frac{T - T_C}{T_H - T_C} = K_1 A x + K_1{}^2 (Gr)(Pr)A^2[(\frac{y}{h})^5/120 - (\frac{y}{h})^4/48 + (\frac{y}{h})^3/72] + K_2$$

in which A is the height to length ratio, Gr is a Grashof number based on $(T_H - T_C)$, the temperature difference between the hot and cold ends of the cavity, and K_1 and K_2 are constants. In particular

$$K_1 = 1 - 3.48\ (10^{-6})(Gr)^2(Pr)^2A^3 + \text{terms of higher order.}$$

The Nusselt number, Nu, is

$$Nu = A + 2.86 \times 10^{-6}\ (Gr)^2(Pr)^2A^3 + \text{terms of high order}$$

so that

$$Q = (\frac{T_H - T_C}{L})k\ [A + 2.86 \times 10^{-6}\ (Gr)^2(Pr)^2A^3]$$

Now the equation for the temperature distribution given by C.S.L. allows one to calculate the temperature difference between two points of different elevation, y_2 and y_3, at the same station, x, along the cavity. This would be

$$(T_2 - T_3) = (T_H - T_C)(K_1{}^2)(G_r)(P_r)A^2 \left\{ \frac{(y_2)^5 - (y_1)^5}{120\ h^5} + \ldots . \right\}$$

Becasue the Grashof number, G_r, contains $(T_H - T_C)$ it appears that $(T_2 - T_3)$ varies as $(T_H - T_C)^2$, as a first approximation, if K_1 is nearly constant. This coefficient does, in fact, depend on G_r. An estimate of G_r can be made using the data of F.O.

$$G_r \text{ (Case 3)} = 32.3 \frac{(1.45 \times 10^{-6})(38.5)(8/12)^3}{[0.29\ (10^{-3})/56]^2} = 2. \times 10^9$$

The Prandtl number is about 0.004 and $A^3 = (0.66/10)^3 = 0.0003$.

This means that K_1 can be calculated to be

$$K_1 = 1 - 3.48(10^{-6})(4 \times 10^{18})(7 \times 10^{-5})(3 \times 10^{-4})$$

$$K_1 = 1 - 180\ (10)^3$$

From this it would appear that the data of F.O. may be such that the theory of C.S.L. does not apply due to the large value of, what was intended to be, higher-order terms. An alternate conclusion appears to be that K_1 is relatively independent of $(T_2 - T_3)$ for <u>large</u> values of h/L but could depend on $(T_2 - T_3)^2$ for moderate h/L and large Grashof numbers. The latter would appear to be the case for the experiments reported by F.O. For small h/L, $T_2 - T_3 \propto (T_H - T_C)^2$ and for moderate h/L, $T_2 - T_3 \propto (T_H - T_C)^6$. This means that, since $Q \propto (T_H - T_C)^3$, $Q \propto (T_2 - T_3)^{3/2}$ for small h/L and $Q \propto (T_2 - T_3)^{1/2}$ for larger values of this ratio.

It should be noted that F.O. find that their data are reasonably correlated by 3/2 power on $(T_2 - T_3)$. Also, in their model, it is implicit that the length is very large compared to the diameter, because no end effects are included. It is such end effects which lower the exponent on $(T_2 - T_3)$

when using the approach of C.S.L.

Because the formulation of the result in C.S.L. is in terms of $T_H - T_C$, it appears to be the more useful. It would appear also that the form of their expression is appropriate to use for natural convection in long horizontal conduits. This requirement for a long conduit may also serve to control the type of flow, laminar, since surface friction will play a dominant role in the process.

The work of C.S.L. contains an informative bibliography of related work. The investigation by Gill [5] is concerned with the large Grashof number regime but it also is limited to short, and relatively full cavities. As such, the flow model does not appear to be directly applicable to stratification observed in mixing-tees.

In the related publication of Cormack, Leal and Seinfeld [2] a series of numerical experiments have been performed in which the parameter

$$(G_r)^2 A^3, \; A = \text{height/length},$$

was varied. These results indicate that as this parameter increases, ".... the numerically determined Nusselt numbers deviate considerably from the asymptotic value". Such a deviation appears already at values of $G_r{}^3A^2 = 10^5$. The data of F.O., for their Case 3, give values of this parameter of

$$(2 \times 10^9)^3(.066)^2 = (8 \times 4/9)(10^{25}).$$

In view of this result it appears that the correlation of heat transfer, Q, with temperature difference $(T_2 - T_3)$ to the 3/2-power appears to have been fortuitous in the report of F.O. It would seem that a higher power might be appropriate and the data in their figure could be equally well related by the third power on the temperature difference, rather than the 3/2 power that is associated with their brief analysis of the problem.

APPENDIX II

Mathematical Analysis for Sect. 3.

The solution of the flow in a two-dimensional channel with a sink located in one boundary and unequal flows proceeding toward the sink from each end is presented below.

The diagrams associated with the transformations to solve this problem are given in Sect. 3 of the body of this report. The mathematical statements of these transformations are as follows

$$z = x + iy = \frac{d}{\pi} \ln(t),$$

and

$$w = \phi + i\psi = -\frac{Q}{\pi} \ln\left(\frac{t - 1}{t^{1/n}}\right).$$

In these expressions

x and y are the coordinates of the problem, measured from the sink;

d is the width of the channel;

ϕ and ψ are the potential and stream functions, respectively;

t is an intermediate, auxiliary variable;

and $1/n$ is the fraction of the flow in one of the halves of the channel. The variable t can be eliminated between the two equations so that

$$\frac{\pi w}{Q} = -\ln\left\{\frac{e^{z\pi/d} - 1}{e^{z\pi/dn}}\right\}$$

The real and imaginary parts of this equation can be equated so that one will find

$$\psi = Q\left[\frac{1}{\pi}\arctan(B/A) + \frac{y}{d}(1/2 - 1/n)\right]$$

with $A = \left(\sinh\frac{\pi}{2}\frac{x}{d}\right)\left(\cos\frac{\pi}{2}\frac{y}{d}\right)$

and $B = \left(\cosh\frac{\pi}{2}\frac{x}{d}\right)\left(\sin\frac{\pi}{2}\frac{y}{d}\right)$

It is convenient to write $\frac{x}{d} = \xi$ and $\frac{y}{d} = \eta$ in what follows.

If one chooses various values for ξ and η he can readily compute ψ at these points. From these values it is possible to interpolate to obtain curves of ψ = constant. This was the procedure that was used to produce Fig. 3 in the main body of the report.

In Fig. 3 the stagnation point was determined by the array of values for ψ that was produced. The location of this stagnation point on the upper boundary can be found directly by requiring that $u = 0 = \partial\psi/\partial y$. For the problem at hand one obtains, for $\eta = 1$, the stagnation point ξ_s from the relationship

$$\tanh\left(\frac{\pi}{2}\,\xi_s\right) = -\,2\,[1/2 - 1/n].$$

Fig. 4 shows that ξ_s does not become excessively large even for moderately large values of n.

APPENDIX III

Outline for Perturbation Analysis

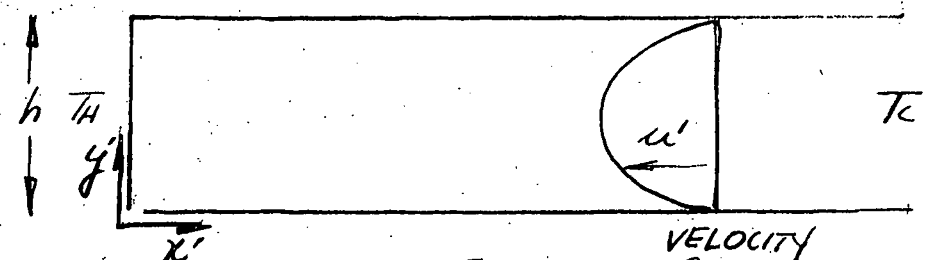

FIGURE III-1

VORTICITY EQUATION:

$$u'\frac{\partial \xi'}{\partial x'} + v'\frac{\partial \xi'}{\partial y'} = \nu \nabla^2 \xi' + g\alpha(T_H - T_C)\frac{\partial \theta}{\partial x'} \quad (1) \qquad , \quad \theta = \frac{T - T_C}{T_H - T_C} \quad (2)$$

WITH

$$\nabla^2 \psi' = -\xi'. \quad (3)$$

ENERGY EQUATION:

$$u'\frac{\partial \theta}{\partial x'} + v'\frac{\partial \theta}{\partial y'} = K \nabla^2 \theta \quad , \quad (4) \quad K = \text{THERMAL DIFFUSIVITY}$$

BOUNDARY CONDITIONS;

5) a) $\theta = 0 \quad x \to \infty \quad \psi' = \bar{U}h\left(2\left(\frac{y'}{h}\right)^3 - 3\left(\frac{y'}{h}\right)^2\right)$. $\bar{U}$ = AVERAGE VELOCITY

b) $\theta = 1 \quad x = 0, y \neq 0$ (OR AS PRESCRIBED BY A FINITE OPENING ALONG X=0)

$\psi' = -\bar{U}h$

$\partial\psi'/\partial y' = 0$

c) $\frac{\partial \theta}{\partial y'} = 0$, $\frac{\partial \psi'}{\partial x'} = 0$; $\psi' = -\bar{U}h$ ON $y' = h$

d. $\frac{\partial \theta}{\partial y} = 0$, $\frac{\partial \psi'}{\partial x'} = 0$, $\psi' = 0$ ON $y' = 0$.

USE U AND h AS CHARACTERISTIC VELOCITY AND LENGTH.

$u = u'/U$, $x = x'/h$ ETC.

THEN THE
VORTICITY EQUATION BECOMES

$$U\frac{U}{h^2}\left(u\frac{\partial\xi}{\partial x} + v\frac{\partial\xi}{\partial y}\right) = \nu\frac{U}{h}\frac{1}{h^2}\nabla^2\xi + g\alpha\frac{(T_H - T_C)}{h}\frac{\partial\Theta}{\partial x},$$

$$u\frac{\partial\xi}{\partial x} + v\frac{\partial\xi}{\partial y} = \frac{\nu}{Uh}\nabla^2\xi + g\alpha\frac{(T_H - T_C)h}{U^2}\frac{\partial\Theta}{\partial x},$$

$$= \frac{1}{\mathbb{R}}\nabla^2\xi + \frac{1}{\mathbb{F}^2}\frac{\partial\Theta}{\partial x},$$

OR

$$\mathbb{R}\left(u\frac{\partial\xi}{\partial x} + v\frac{\partial\xi}{\partial y}\right) = \nabla^2\xi + \mathbb{R}\left(\frac{1}{\mathbb{F}^2}\frac{\partial\Theta}{\partial x}\right). \quad (6)$$

ALSO $\nabla^2\psi = -\xi$. (7)

ENERGY EQUATION

$$\mathbb{P}\mathbb{R}\left(u\frac{\partial\Theta}{\partial x} + v\frac{\partial\Theta}{\partial y}\right) = \nabla^2\Theta \quad \text{WITH } \mathbb{P} = \nu/\kappa. \quad (8)$$

FOR A FAR FIELD SOLUTION FOR SMALL $\mathbb{R}$, IT
WILL BE ASSUMED THAT

$$\hat{\psi} = \hat{\psi}_0 + \mathbb{R}\hat{\psi}_1 + \mathbb{R}^2\hat{\psi}_2 + \cdots + \; ; \quad \hat{\psi}_0 = 3y^2 - 2y^3 \quad (9)$$

$$\hat{\Theta} = \hat{\Theta}_0 + \mathbb{R}\hat{\Theta}_1 + \mathbb{R}^2\hat{\Theta}_2 + \cdots + \quad \text{WITH} \quad \hat{x} = \mathbb{R}x. \quad (10), (11)$$

THE VORTICITY EQUATION BECOMES, WITH ASSUMPTIONS (11),

$$\mathbb{R}\left\{\frac{\partial\hat{\psi}}{\partial y}\mathbb{R}\frac{\partial}{\partial x}\left(-\mathbb{R}^2\frac{\partial^2\hat{\psi}}{\partial x^2}-\frac{\partial^2\hat{\psi}}{\partial y^2}\right)-\mathbb{R}\frac{\partial\hat{\psi}}{\partial x}\frac{\partial}{\partial y}\left(-\mathbb{R}^2\frac{\partial^2\hat{\psi}}{\partial x^2}-\frac{\partial^2\hat{\psi}}{\partial y^2}\right)\right\}_0 =$$

$$-\left(\frac{\partial^4\hat{\psi}}{\partial y^4}+\mathbb{R}^2\frac{\partial^4\hat{\psi}}{\partial x^2\partial y^2}+\mathbb{R}^4\frac{\partial^4\hat{\psi}}{\partial x^4}\right)+\mathbb{R}^2\left(\frac{1}{\mathbb{F}^2}\frac{\partial Q}{\partial x}\right)$$

SO THAT WITH ASSUMPTION (9) THE FOLLOWING EQUATIONS FOR THE VARIOUS POWERS OF $\mathbb{R}$ MUST BE VALID.

$\mathbb{R}^0$ $\quad \hat{\psi}_0 = 3y^2-2y^3$ SINCE $\partial^4\hat{\psi}_0/\partial y^4=0$ WITH $\hat{\psi}_0=0=\frac{\partial\hat{\psi}_0}{\partial y}\,;\,y=0$

$\hat{\psi}_0=1,\ \frac{\partial\hat{\psi}_0}{\partial y}=0\,;\,y=1$

$\mathbb{R}^1$ $\quad 0=-\frac{\partial^4\hat{\psi}_1}{\partial y^4}$ IMPLIES $\hat{\psi}_1=0$ FOR $\hat{\psi}_1=0=\frac{\partial\hat{\psi}_1}{\partial y}$ ON $y=0,1$

$\mathbb{R}^2$ $\quad 0=-\frac{\partial^4\hat{\psi}_2}{\partial y^4}+\frac{1}{\mathbb{F}^2}\frac{\partial Q_0}{\partial x} = -\frac{\partial^4\hat{\psi}_2}{\partial y^4}+\frac{1}{\mathbb{F}^2}(-C_0),\quad \frac{\partial Q_0}{\partial x}=-C_0$

$$\hat{\psi}_2=\frac{C_0}{\mathbb{F}^2}\left(y^4/24-y^3/12+y^2/24\right)$$

$\mathbb{R}^3$ $\quad 0=-\frac{\partial^4\hat{\psi}_3}{\partial y^4}-\frac{1}{\mathbb{F}^2}C_1$ OR $\hat{\psi}_3=\frac{C_1}{\mathbb{F}^2}\left(y^4/24-y^3/12+y^2/24\right)$

THE GENERAL RESULT IS, IN VIEW OF THE EXPANSION (9),

$$\hat{\psi}=3y^2-2y^3+\frac{1}{\mathbb{F}^2}\left(y^4/24-y^3/12+y^2/24\right)\left\{C_0\mathbb{R}^2+C_1\mathbb{R}^3+\cdots+\right\}.$$

OR IF $K_1 = \left(C_0 R^2 + C_1 R^3 + \cdots + \right)$,

$$\hat{\psi} = 3y^2 - 2y^3 + \frac{K_1}{F^2}\left(y^4/24 - y^3/12 + y^2/24\right). \qquad (12)$$

UNDER THE CHANGE OF HORIZONTAL SCALE $\hat{X} = RX$,
THE ENERGY EQUATION BECOMES

$$PR\left(\frac{\partial\hat{\psi}}{\partial y} R \frac{\partial\hat{\theta}}{\partial\hat{X}} - R\frac{\partial\hat{\psi}}{\partial\hat{X}}\frac{\partial\hat{\theta}}{\partial y}\right) = R^2\frac{\partial^2\hat{\theta}}{\partial\hat{X}^2} + \frac{\partial^2\hat{\theta}}{\partial y^2}.$$

THE EXPANSIONS (10) AND (11) GIVE EQUATIONS FOR THE VARIOUS POWERS OF R AS FOLLOWS.

R^0: $\frac{\partial^2\hat{\theta}_0}{\partial y^2} = 0 \quad \frac{\partial\hat{\theta}_0}{\partial y} = C_0''$ SO THAT $C_0'' = 0$

THEN $\hat{\theta}_0 = C_0' - C_0\hat{X}$

R^1: $\frac{\partial^2\hat{\theta}_1}{\partial y^2} = 0 \quad \hat{\theta}_1 = C_1' - C_1\hat{X}$

R^2: $P\frac{\partial\hat{\psi}_0}{\partial y}(-C_0) = \frac{\partial^2\hat{\theta}_2}{\partial y^2} \quad \therefore \frac{\partial\hat{\theta}_2}{\partial y} = -C_0 P\hat{\psi}_0 + C_2''$

THE CONDITION ON $\partial\hat{\theta}_2/\partial y$ AT $y=0$ WHERE $\hat{\psi}_0 = 0$ REQUIRES $C_2'' = 0$. BUT THIS REQUIRES $\frac{\partial\hat{\theta}_2}{\partial y} = 0$ OTHERWISE THE CONDITIONS AT $y=1$ CANNOT BE MET. HENCE $\hat{\theta}_2 = C_2' - C_2\hat{X}$.

R^3: $P\left(\frac{\partial\hat{\psi}_0}{\partial y}(-C_1) + \underbrace{\frac{\partial\hat{\psi}_1}{\partial y}}_{0}(-C_0)\right) = \frac{\partial^2\hat{\theta}_3}{\partial y^2}$ WITH $\hat{\psi}_1 = 0$, FROM BEFORE.

THEN

$$-\mathbb{R}\, C_1\, \hat{\psi}_0 = \frac{\partial \hat{Q}_3}{\partial y}$$. AGAIN BOUNDARY CONDITIONS ON $\frac{\partial \hat{Q}_3}{\partial y}$ REQUIRE

$$\hat{Q}_3 = C_3' - C_3 \hat{X}$$

$\mathbb{R}^4$: $$\mathbb{R}\left[\frac{\partial \hat{\psi}_0}{\partial y}(-C_2) + \cancel{\frac{\partial \hat{\psi}_1}{\partial y}}^{\,0}(-C_1) + \frac{\partial \hat{\psi}_2}{\partial y}(-C_0)\right] = \frac{\partial^2 \hat{Q}_4}{\partial y^2}$$

$$\frac{\partial \hat{Q}_4}{\partial y} = \mathbb{R}\left(\hat{\psi}_0(-C_2) + \hat{\psi}_2(-C_0)\right) + C_4''$$

THE CONDITIONS AT $y=0$ ARE MET WITH $C_4''=0$

THE CONDITIONS AT $y=1$ CAUSE

$$\hat{Q}_4 = C_4' - C_4 \hat{X},$$

SO THAT

$$\hat{Q} = \left(C_0' + \mathbb{R} C_1' + \mathbb{R}^2 C_2' \ldots\right) + K_1 \hat{X} = K_2 + K_1 \hat{X}\,. \quad (13)$$

THE NEAR FIELD SOLUTION IS OBTAINED FROM EQUATIONS (6) AND 8 WITH

$$\psi = \psi_0 + \mathbb{R}\psi_1 + \mathbb{R}^2 \psi_2 + \cdots +$$

AND

$$Q = Q_0 + \mathbb{R} Q_1 + \mathbb{R}^2 Q_2 + \cdots + .$$

$$\mathbb{R}\left[\frac{\partial \psi}{\partial y}\frac{\partial}{\partial X}\left(-\frac{\partial^2 \psi}{\partial X^2} - \frac{\partial^2 \psi}{\partial y^2}\right) - \frac{\partial \psi}{\partial X}\frac{\partial}{\partial y}\left(-\frac{\partial^2 \psi}{\partial X^2} - \frac{\partial^2 \psi}{\partial y^2}\right)\right] = -\nabla^4 \psi + \mathbb{R}\frac{1}{F^2}\frac{\partial Q}{\partial X}$$

THE ENERGY EQUATION GIVES

$$P\,R\left[\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}\right]=\nabla^2\theta$$

R^0 $\quad \nabla^2\theta_0=0$

R^1 $\quad \nabla^2\theta_1 = P\,\dfrac{\partial(\theta_0,\psi_0)}{\partial(x,y)}$

R^2: $\quad \nabla^2\theta_2 = P\left(\dfrac{\partial(\theta_0,\psi_1)}{\partial(x,y)}+\dfrac{\partial(\theta_1,\psi_0)}{\partial(x,y)}\right)$

ETC.

A SUMMARY OF THESE NEAR-FIELD SOLUTIONS, ALONG WITH THE MATCHING CONDITIONS, IS GIVEN IN TABLE I.

TABLE I

$\mathbb{R}^0$:

$$\nabla^4 \psi_0 = 0 \quad (0\text{-}1), \qquad \psi_0 \underset{x\to\infty}{=} 3y^2 + 2y^3 \quad (0\text{-}3)$$

$$\nabla^2 \theta_0 = 0 \quad (0\text{-}2), \qquad \theta_0 \underset{x\to\infty}{=} C_0' \quad (0\text{-}4)$$

$\mathbb{R}^1$

$$\frac{\partial(\zeta_0, \psi_0)}{\partial(x,y)} = -\nabla^4 \psi_1 + \frac{1}{\mathbb{F}^2}\frac{\partial \theta_0}{\partial x} \quad (1\text{-}1) \qquad \psi_1 \underset{x\to\infty}{\to} 0 \quad (1\text{-}3)$$

$$\nabla^2 \theta_1 = \mathbb{P}\left(\frac{\partial(\theta_0, \psi_0)}{\partial(x,y)}\right) \quad (1\text{-}2) \qquad \theta_1 \underset{x\to\infty}{\to} C_1' \quad (1\text{-}4)$$

$\mathbb{R}^2$

$$\left(\frac{\partial(\zeta_0, \psi_1)}{\partial(x,y)} + \frac{\partial(\zeta_1, \psi_0)}{\partial(x,y)}\right) = -\nabla^4 \psi_2 + \frac{1}{\mathbb{F}^2}\frac{\partial \theta_1}{\partial x} \quad (2\text{-}1)$$

$$\psi_2 \underset{x\to\infty}{\to} \frac{C_0}{\mathbb{F}^2}\left(\frac{y^4}{24} - \frac{y^3}{12} + \frac{y^2}{24}\right) \quad (2\text{-}3)$$

$$\nabla^2 \theta_2 = \mathbb{P}\left(\frac{\partial(\theta_0, \psi_1)}{\partial(x,y)} + \frac{\partial(\theta_1, \psi_0)}{\partial(x,y)}\right) \quad (2\text{-}2)$$

$$\theta_2 \underset{x\to\infty}{\to} C_2' - C_0 \quad (2\text{-}4)$$

IN ADDITION TO THE LIMITING CONDITIONS -- 0-2, 0-3, ETC -- THE BOUNDARY CONDITIONS AT $y=0, 1$ AND $x=0$ WITH $y \neq 0$ ARE

$$y=1 \quad \psi=1, \ \frac{\partial\psi}{\partial y}=0, \ \frac{\partial\theta}{\partial y}=0 \tag{14a}$$

SO THAT $\psi_0=1 \quad \psi_1=0=\psi_2$, ETC.

$$\partial\psi_0/\partial y = \partial\psi_1/\partial y = 0, \text{ ETC.}$$

$$\partial\theta_0/\partial y = \partial\theta_1/\partial y = 0, \text{ ETC.}$$

$$y=0 \quad \psi=0, \ \partial\psi/\partial y=0, \ \partial\theta/\partial y=0 \tag{14b}$$

SO THAT $\psi_0=0=\psi_1=\psi_2$, ETC.

$$\partial\psi_0/\partial y = \partial\psi_1/\partial y = 0, \text{ ETC}$$

$$\partial\theta_0/\partial y = \partial\theta_1/\partial y = 0, \text{ ETC}$$

$$x=0, y\neq 0 \quad \psi=1, \ \partial\psi/\partial x=0, \ \theta=1$$

SO THAT $\psi_0=1, \ \psi_1=0=\psi_2$ ETC (14c)

$$\partial\psi_0/\partial x = \partial\psi_1/\partial x = 0, \text{ ETC}$$

$$\theta_0=1, \ \theta_1=\theta_2=0, \text{ ETC.}$$

HENCE THE SOLUTION FOR ψ_0, EQUATION 0-1 IN TABLE I, BECOMES EQUIVALENT TO THE "STOKES-FLOW" SOLUTION FOR THE FLOW IN A TWO DIMENSIONAL CHANNEL. SUCH A SOLUTION, IN CLOSED FORM, MAY BE AVAILABLE WITHOUT THE KNOWLEDGE OF THE AUTHOR. A NUMERICAL SOLUTION SHOULD BE EASILY OBTAINED.

$\nabla^2\Theta_0 = 0$ FROM (0-2) & $y=1: \partial\Theta_0/\partial y = 0$

$y=0: \partial\Theta_0/\partial y = 0$

$x=0 \quad \Theta_0 = 1$

$x \to \infty \quad \Theta_0 = C_0'$

CONSIDER A SOLUTION OF THE FORM $\Theta_0 = X(x)Y(y)$

$-\frac{X''}{X} = +\frac{Y''}{Y} = -\lambda^2 \qquad Y = A\sin\lambda y + B\cos\lambda y$ SUCH THAT $0 = \sin\lambda$

$X = Ce^{-\lambda x} \qquad \lambda = n\pi$

NOW $C_0' = 0$ IS CONSISTANT WITH HAVING $\hat{\Theta}_0 \to C_0'$

FOR HAVING THE FAR FIELD TEMPERATURE APPROACH ZERO.

$$\text{HENCE, } \Theta_0 = \Sigma B_n e^{-\lambda_n x}\cos\lambda_n y \tag{15}$$

WITH $1 = \Sigma B_n \cos\lambda_n y$ TO DETERMINE THE B_n

THIS CONCLUSION GIVES THE INFORMATION NECESSARY TO SOLVE EQUATIONS (1-1) AND (1-2), IN TABLE I, FOR THE FIRST ORDER CORRECTIONS. FURTHERMORE, THE SAME CONSIDERATIONS WHICH LED TO THE CONCLUSION $C_0' = 0$ GIVE $0 = C_1' = C_2'$, ETC. ALSO THERE SHOULD BE NO TEMPERATURE GRADIENT AT $x \to \infty$ SO $0 = C_0 = C_1 = C_2$, ETC.

THESE CONSIDERATIONS PERMIT EQUATIONS (1-1) AND (1-2) TO BE WRITTEN AS

$$-\frac{\partial\psi_0}{\partial x}\frac{\partial\zeta_0}{\partial y} = -\nabla^4\psi_1 + \frac{1}{F^2}\Sigma - \lambda_n e^{-\lambda_n x}\cos\lambda_n y \tag{16}$$

AND

$$P\left[\frac{\partial\psi_0}{\partial y}\left\{\Sigma(-\lambda_n)e^{-\lambda_n x}\cos\lambda_n y\right\} - \frac{\partial\psi_0}{\partial x}\left\{\Sigma - \lambda_n e^{-\lambda_n x}\sin\lambda_n y\right\}\right] = \nabla^2\Theta_1 \tag{17}$$

THE SOLUTIONS OF EQUATION (16) AND (17) WITH THE ASSOCIATED BOUNDARY CONDITIONS FROM TABLE I (N.B. $C_i = 0 = C_i'$) SHOULD BE PERFORMED NUMERICALLY. SOME EXPERIMENTATION WILL BE NECESSARY TO DETERMINE THE LENGTH OF SECTION NEEDED, PERHAPS AS A FUNCTION OF F^2 AND $\mathcal{P}$.

ANOTHER POSSIBILITY IS TO EXTEND THE MECHANICS OF THE SOLUTION METHOD BY CONSIDERING, AT THE OUTSET, A FINITE LENGTH CONDUIT. THIS IS SHOWN SCHEMATICALLY IN FIGURE 2.

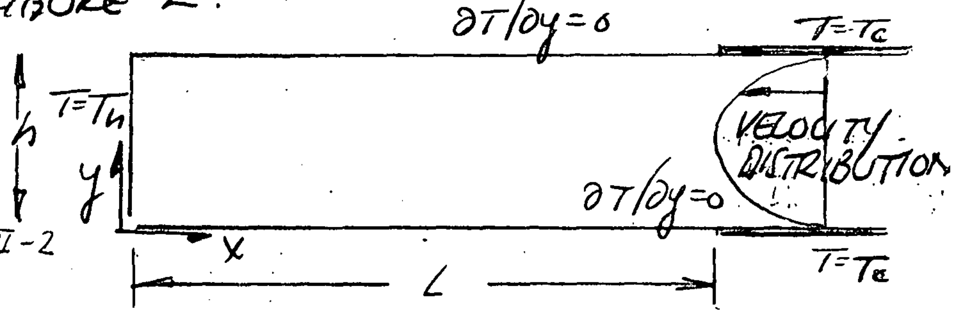

FIGURE III-2

AT X=L THERE WILL BE AN ADDITIONAL EXPANSION WHICH CAN BE MATCHED TO THE CORE SOLUTION GIVEN BY EQUATIONS (12) AND (13). IN THIS CASE ALL OF THE C_i AND C_i' NEED NOT BE ZERO. THE SOLUTIONS FOR THE STREAM FUNCTION, ψ, AND TEMPERATURE, Θ, IN THE "COLD" REGION FOLLOW CLOSELY THE FORMAT OF CORMACK, STONE, AND LEAL.[2]

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