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Single-mode Saturation of the Bump-on-Tail Instability

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We consider a slightly unstable plasma, with only one or a few linear modes unstable. Nonlinear saturation at small amplitudes has been treated by time-asymptotic analysis[1] which is a generalization of the methods of Bogoliubov[2] and co-workers. In this paper the method is applied to instability in a collisionless plasma governed by the Vlasov equation. We consider perhaps the simplest such instability, the bump-on-tail, for a one-dimensional plasma.

A number of new difficulties arise which are not present in fluid equation cases. One is connected with the existence of the well-known Van Kampen continuum[3] of oscillating eigenmodes. In a strict sense, one never has a single neutral stable mode. Other difficulties are associated with the occurrence of indeterminate equations for the highly singular non-linear harmonics of the unstable linear mode. Further, one equation for the saturated amplitude contains the products of generalized functions and thus is not well defined. Nevertheless, we introduce a procedure which enables us to extract a finite well-defined set of conditions, and solve for the satured amplitude, frequency shift and also obtain most details of the modified electron distribution function, as well as a corrected version of the adjoint function for the so-called 1c modes[4].

We demonstrate that energy and momentum are conserved by this solution and that it is stable in normal circumstances, and we estimate the numerical value of the solution for the special case of a small bump far out on the tail of a Maxwellian. The general results of this paper are now being tested by numerical calculations, but we do indicate some general features which have been seen in earlier computations.

We also calculate the saturation by a lengthier procedure which avoids the necessity of dealing with products of generalized functions and with a partially undefined adjoint function. This result reduces essentially to the one obtained by the simpler method discussed above, and demonstrates that our method is equivalent to the ansatz that the nearly singular behavior which occurs near the resonant velocity $v = \omega/k$ can be determined by requirements on the analyticity of the higher order perturbations.

The saturated amplitude is considerably larger than one which would be estimated by simple particle-trapping considerations. To illustrate this we describe a method, which follows the growth of the unstable mode via the usual quasilinear approximation. The result predicts a reduction in growth rate to zero when the mode amplitude is of order Δ^2 (rather than $\Delta^{1/2}$, as in the method discussed above). This derivation becomes clearly invalid, however, when the growth rate drops below about half its initial value. It will be of great interest, in future experimental tests of our saturation prediction, to see whether the mode amplitude does move through this region of initial particle trapping and, if so, how it would approach its asymptotic value. Here we stress that this paper represents a speculation rather than a proven result. Our procedure is a natural extension of Bogoliubov perturbation methods to the case where singular velocity distributions tend to develop. However, while we have found what seems to be a unique final steady state, it is not at all obvious that the system can indeed get there from an initial linearly unstable state. We hope that numerical work will cast further light on this conjecture.

In essence, we consider a one-dimensional bump-on-tail distribution with finite boundaries. At a critical density N, the linear mode with the lowest phase velocity $\bar{\nu}$ lies right at the bottom of the well in velocity space and the plasma is neutral stable. The density is now increased by the fractional amount Δ and this mode moves on to the positive slope region and begins to grow. We obtain an expression for the final amplitude of this mode Γ and its frequency shift $\delta\bar{\omega}$ (from the neutral stable value $k\bar{\nu}$) in terms of Δ. This result follows from a systematic expansion of the equations to third order in $\Delta^{1/2}$ and by requiring absence of secular solutions at all stages. A tractable expression for the modified equilibrium distribution is obtained by a regularization procedure[5].

The relevant equations in the "quasilinear" approximation (i.e. neglecting the double frequency modification of the equilibrium) are:

$$|\Gamma|^2 = 12\Delta\left(\frac{\omega_p^2}{k^2}\right)^{-2} D_o^{-1}$$

$$\frac{\delta\bar{\omega}}{k} = \tilde{N}_o \Delta D_o^{-1}$$

where

$$D_o \equiv P\int \frac{\eta^{IV}}{v-\bar{\nu}}\,dv - \frac{\eta^{IV}(\bar{\nu})}{\eta^{I}(\bar{\nu})}\; P\int \frac{\eta^{I}}{v-\bar{\nu}}\,dv$$

and

$$\tilde{N}_o \equiv \frac{\eta^{IV}(\bar{\nu})}{\eta^{I}(\bar{\nu})} .$$

where ω_p^2 is the plasma frequency and $\eta \equiv -\frac{\omega_p^2}{k^2}\frac{dF^o}{dv}$

The roman numeral superscript on η denotes the number of derivatives of η with respect to v.

The full nonlinear solution, which then includes the double frequency term, is:

$$|\Gamma|^2 = 24\Delta\left(\frac{\omega_p^2}{k^2}\right)^{-2} D^{-1}$$

where

$$D \equiv \frac{1}{2} P \int \frac{\eta^{IV}}{v-\bar{v}} dv - \left[P \int \frac{\eta^{II}}{v-\bar{v}} dv \right]^2 + \pi^2[\eta^{II}(\bar{v})]^2$$

$$- \frac{1}{2} \frac{\eta^{IV}(\bar{v})}{\eta^{I}(\bar{v})} P \int \frac{\eta^{I}}{v-\bar{v}} dv + 2 \frac{\eta^{II}(\bar{v})}{\eta^{I}(\bar{v})} P \int \frac{\eta^{I}}{v-\bar{v}} dv \cdot P \int \frac{\eta^{II}}{v-\bar{v}} dv \quad .$$

and

$$\frac{\delta\bar{\omega}^2}{k} = \tilde{N}\Delta D^{-1}$$

where

$$\tilde{N} \equiv \frac{1}{2} \frac{\eta^{IV}(\bar{v})}{\eta^{I}(\bar{v})} - 2 \frac{\eta^{IV}(\bar{v})}{\eta^{I}(\bar{v})} P \int \frac{\eta^{II}}{v-\bar{v}} dv \quad .$$

A more complete description will be published shortly[6]. This method has also been used to calculate saturation of the collisionless drift instability[7], and a detailed description will be submitted for publication.

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3. N.G. van Kampen, Physica 21, 949 (1955).
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7. A. Simon and L. Gross, Bull. Am. Phys. Soc. 20, 1254 (1975).