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# TRANSIENT ANALYSIS OF A TWO-DIMENSIONAL PLATE BY NUMERICAL AND ANALYTICAL TECHNIQUES(I)

F. R. Norwood(II), D. L. Hicks(III) and M. M. Madsen(III)


## Summary

An exact analytical solution is used for comparison with three wavecodes. The computer wavecodes employed in the comparison were HONDO, TOODY, and THREEDY. HONDO is a two-dimensional code based on the Galerkin finite element method in a time differenced form; TOODY is a two-dimensional code based on the von Neumann-Richtmyer, artificial viscosity, finite difference method; and THREEDY is a three-dimensional code based on the operator splitting method. Referring the error to a dimensionless $L_1$ norm, at a point in the plate, and for two transit times the errors found were as follows: Implicit THREEDY, .1315; TOODY, .0800; HONDO, .0671; explicit THREEDY, .0569.


## 1. Introduction

In recent years, several computer codes have been developed to analyze wave propagation problems in linear and non-linear materials. This code development has been directed to problems for which the exact solution would be extremely difficult to find. Although wavecodes have been used to obtain approximations to the solution of complicated and important problems, there are many unanswered questions about their accuracy.

The present work presents comparisons of wavecode generated approximations with an exact solution to a two-dimensional wave propagation problem in an elastic plate. The exact solution was obtained in [1] by using integral transforms and Cagniard's method of inversion. The approximate solutions were found by using the wavecodes HONDO, TOODY, and THREEDY. HONDO [2] uses the finite element method of Galerkin; TOODY [3] relies on the artificial viscosity method of von Neumann and Richtmyer [4]; and THREEDY [5] employs the operator splitting method of Bagrinovski and Godunov [6] to separate three-dimensional mechanics problems into one-dimensional problems which are solved numerically by a modification of the conservative difference method of Lax, Wendroff, and Richtmyer [7]. HONDO and TOODY are wavecodes for two-dimensional

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II. Computational Physics and Mechanics Division II - 5166, Sandia Laboratories, Albuquerque, New Mexico, 87115.

III. Computational Physics and Mechanics Division I - 5162, Sandia Laboratories, Albuquerque, New Mexico, 87115.

problems, while THREEDY is for three-dimensional problems, with explicit and implicit options, where the implicit option avoids the CFL (Courant, Friedrichs, and Lewy) constraint. The explicit version of THREEDY was modified into the version THREEDY-MOC by replacing its boundary value algorithm with the algorithm MOC based on the method of characteristics (see [8]). MOC, together with the method of Lax, Wendroff, and Richtmyer in its one-dimensional form, made THREEDY-MOC highly accurate for the test problem (see [9,10] for further discussion).

## 2. Statement of the Problem

The sample problem selected for the comparison may be stated as follows: In a rectangular coordinate system, consider an elastic plate confined to $0 \le Y \le h$ and extending to infinity in the X and Z directions. (Herein X, Y, Z are Lagrangean and x, y, z are Eulerian coordinates.) A load is suddenly applied over a portion of the surface (Fig. 1), at time $t = 0$, normal to the surface $Y = 0$, and the resulting motion in the interior of the plate is determined. The problem will be assumed to be independent of Z and thus the boundary conditions are given by

$$\Sigma_{XY}(X,h,t) = \Sigma_{YY}(X,h,t) = \Sigma_{XY}(X,0,t) = 0$$

and

$$\Sigma_{YY}(X,0,t) = \mu PH(t)H(x) ,$$

where $\mu$ is the shear modulus, H is the Heaviside unit step function, P is a load scaling constant, and $\Sigma$ is the first Piola-Kirchoff stress operator (taken positive in compression).

The solution to this problem by wavecodes corresponds to the solution of the conservation laws of continuum mechanics when the constitutive equation is the Lagrangean linearized elasticity relation. Only TOODY

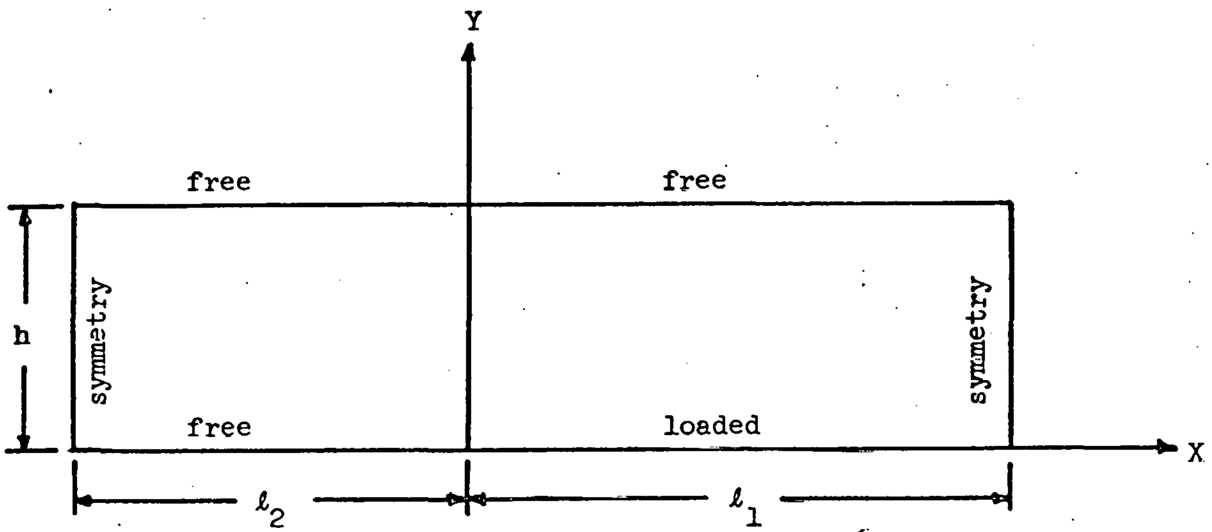


Figure 1. Plate Problem Computer Model with $X = \ell_1$ and $X = -\ell_2$ as Symmetry Boundaries.

does not have this relation as one of its material law options, but TOODY's constitutive equation approaches this relation as the strains go to zero. The three codes were run with zero initial conditions for the velocity vector and the stress operator, using 20 or 21 zones or nodes through the thickness of the plate.

## 3. Governing Equations

Let $\underset{\sim}{x}$, $\underset{\sim}{X}$, and t be the spatial, material and temporal coordinates. Let $\underset{\sim}{x}_{Xt}$ be the motion function $\underset{\sim}{x} = \underset{\sim}{x}_{Xt}(X,t)$. In addition, let $\underset{\sim}{u}_{Xt}$, $V_{Xt}$ and $\rho_{Xt}$ be the specific momentum, specific volume, and mass density functions of $\underset{\sim}{X}$ and t, with $\underset{\sim}{u}$, V, and $\rho$ their evaluations at $(\underset{\sim}{X},t)$; that is, $\rho = 1/V$, $\underset{\sim}{u} = \partial x/\partial t$. If $t^o$ is the initial time of interest and $V^o = V(\underset{\sim}{X},t^o)$, then, in terms of the material deformation gradient $\underset{\sim}{\mathfrak{F}} = \partial \underset{\sim}{x}/\partial \underset{\sim}{X}$, one finds that $V/V^o = \det(\underset{\sim}{\mathfrak{F}})$.

Define $E_{Xt}$ and $\mathcal{E}_{Xt}$ to be the specific total energy and specific internal energy functions, respectively, with E and $\mathcal{E}$ their evaluations at $(\underset{\sim}{X},t)$. Then it follows that $E = \mathcal{E} + \underset{\sim}{u}^T \underset{\sim}{u}/2$, where the superscript T denotes the transpose. (The Xt subscripts are often suppressed to simplify the notation.)

In the Lagrangean divergence form, the conservation laws may be written as follows. The conservation of volume is given by

$$\frac{\partial(\rho^o V)}{\partial t} = \frac{\partial(\underset{\sim}{u}^T \operatorname{cof} \underset{\sim}{\mathfrak{F}}_I)}{\partial X_I} \qquad (1)$$

where cof $\underset{\sim}{\mathfrak{F}}_I$ is the I-th column of the cofactor matrix cof $(\underset{\sim}{\mathfrak{F}})$. The conservation of momentum is expressed by

$$\frac{\partial(\rho^o u_i)}{\partial t} = - \frac{\partial \Sigma_{iI}}{\partial X_I} \, . \qquad (2)$$

The conservation of energy may be written as

$$\frac{\partial(\rho^o E)}{\partial t} = - \frac{\partial(u_i \, \Sigma_{iI})}{\partial X_I} \, . \qquad (3)$$

The material law in the test problem treated here is the Lagrangean linearized elasticity relation and is given by

$$\underset{\sim}{\Sigma} = \lambda \mathrm{Tr}(\underset{\sim}{\ell})\underset{\sim}{I} + 2\mu\underset{\sim}{\ell} \, , \qquad (4)$$

where $\lambda$ and $\mu$ are constants and $\underset{\sim}{\ell}$ is the Lagrangean infinitesimal strain measure defined by

$$\underset{\sim}{\ell} = \underset{\sim}{I} - (\underset{\sim}{\mathfrak{F}} + \mathfrak{F}^T)/2 \, .$$

Note that, if (4) is substituted into (2), then the linear partial differential equations of elasticity result.

It seems appropriate at this point to indicate some differences between TOODY and THREEDY. THREEDY differences the conservative form of the conservation of volume (eqn. 1), while TOODY computes volume from grid point positions. Also, THREEDY differences the conservative form of the conservation of momentum (eqn. 2), while TOODY differences the mixed Lagrangean-Eulerian equation for $\underset{\sim}{u}$ given by

$$\rho \left.\frac{\partial u_i}{\partial t}\right|_X = - \left.\frac{\partial \sigma_{ij}}{\partial x_j}\right|_t ,$$

where $\underset{\sim}{\sigma}$ is the Cauchy stress operator. In addition, THREEDY differences the conservative form of the conservation of energy (eqn. 3), but TOODY differences the mixed Lagrangean-Eulerian equation for $\mathcal{E}$

$$\rho \frac{\partial \mathcal{E}}{\partial t} = -\sigma_{ij} D_{ij} ,$$

where $\underset{\sim}{D}$ is the stretch measure. Another difference is that THREEDY has options for stress-strain relations between the various well known stress operators (Cauchy, Piola-Kirchoff) and strain measures (Euler-Almansi, Lagrange-Green, Lagrangean infinitesimal, Eulerian infinitesimal), and TOODY has one basic form for its stress-strain relation expressed by

$$\dot{\underset{\sim}{\sigma}} = \underset{\sim}{W}\,\underset{\sim}{\sigma} - \underset{\sim}{\sigma}\,\underset{\sim}{W} + \lambda \mathrm{Tr}(\underset{\sim}{D})\underset{\sim}{I} + 2\mu \underset{\sim}{D} , \tag{5}$$

where $\underset{\sim}{W}$ is the spin measure.

The Lagrangean linearized elasticity relation cannot be cast into the preceding form. However, as the strains go to zero, TOODY's stress-strain relation approaches the Lagrangean linearized elasticity relation. The TOODY relation (5) is frame-indifferent and conserves angular momentum; the Lagrangean linearized elasticity relation (4) does not satisfy those conditions; however, (4) results in a system of linear PDE's whose exact solution can be obtained by linear transform techniques. HONDO, just as THREEDY, has the Lagrangean linearized elasticity relation as one of its options. In addition, HONDO uses the Galerkin finite element method in a time differenced form, where the linear elements are the piecewise linear basis functions. It was found that TOODY's scaled output did not change significantly below $P = 10^{-2}$. Therefore, the comparisons were made with TOODY and THREEDY having $P = 10^{-3}$ and HONDO had $P = .8 \times 10^{-3}$.

## 4. The Test Problem

From eqns. (2) and (4) it follows that the equation to be solved is

$$\rho^o \frac{\partial^2 x_i}{\partial t^2} = (\lambda + 2\mu) \frac{\partial}{\partial x_i}\left(\frac{\partial x_\alpha}{\partial X_\alpha}\right) + \mu \frac{\partial^2 x_i}{\partial X_\beta\, \partial X_\beta} , \tag{6}$$

which is the familiar equation of linear elasticity theory. The Helmholtz decomposition yields wave equations for the dilatational and shear waves. The problem posed by eqn. (6) and the boundary conditions of Section 2 was solved by Norwood [1] using integral transforms. This problem was modelled for the wavecodes by imposing symmetry boundaries as shown in Figure 1.

The point X = h/2, Y = h/2 will be used for the comparison. At this point the value $u_y$, the y-component of velocity will be saved. If $\ell_1$ and $\ell_2$ are long enough, then the solution for the geometry of Figure 1 will agree with the exact solution up to time t. In the present work, $\ell_1$ and $\ell_2$ are set so that there is agreement for more than two dilatation transit times. A dilatation transit time is the time it takes a dilatational wave to travel the thickness of the plate; that is, the transit time $t_1$ is given by

$$t_1 = h/C_1 ,$$

where $C_1$ is the dilatational wave speed.

The exact solution at time $t^n$, $u_y^e(t^n)$ will be compared with the calculated solutions $u_y^n$ by the non-dimensionalized $L_1$ norm defined as

$$\text{Error} = \frac{\sum_{n=0} \left| u_y^e(t^n) - u_y^n \right| \Delta t^n}{\sum_{n=0}^{N} \left| u_y^e(t^n) \right| \Delta t^n} .$$

where $\Delta t^n = t^n - t^{n-1}$. The initial and boundary data for the calculations are given by:

$$1 = \rho^o = \lambda = \mu$$

$$\underset{\sim}{u}^o = \underset{\sim}{0}$$

and

$$\underset{\sim}{\Sigma}^o = \underset{\sim}{0}$$

where $\underset{\sim}{u}^o$ is the initial velocity vector and $\underset{\sim}{\Sigma}$ is the initial stress operator. The codes were set up with 21 zones through the thickness of the plate (therefore h = 21) and 63 along the length with $\ell_1 = 37$ (loaded) and $\ell_2 = 26$ (not loaded).

Consider a point $\underset{\sim}{X}$ on the loaded surface and away from the edge of the load. When the load is applied, a plane shock wave starts traveling from $\underset{\sim}{X}$ into the interior of the plate. The instant the load $\mu P$ is applied the value of V jumps from $V^o$ to $V^*$, the value of $\underset{\sim}{u}$ jumps from $\underset{\sim}{u}^o$ to $\underset{\sim}{u}^*$, the value of the traction $\underset{\sim}{\tau}$ (in the Y-direction) jumps from

$\underset{\sim}{\tau}^o$ to $\underset{\sim}{\tau}^*$, and the value of the longitudinal strain $\varepsilon$ jumps from $\varepsilon^o$ to $\varepsilon^*$ where in general

$$\varepsilon = 1 - V/V^o = 1 - \frac{\partial y}{\partial Y}$$

because there is strain only in one direction (Y). The shock impedance is given by $a^o$, with

$$a^o = [(\lambda + 2\mu)\rho^o]^{1/2}$$

where $a^o$ is the dilatational acoustic impedance. From the shock jump relations

$$a^o[V^* - V^o] = -[u_y^* - u_y^o]$$

$$a^o[u_y^* - u_y^o] = [\tau_y^* - \tau_y^o]$$

where $\tau_y^* = \mu P$ and $\tau_y^o = 0$ we get

$$\varepsilon^* = P/(\lambda + 2\mu)$$

because $\varepsilon^o = 0$. In the case at hand, when $\lambda$ and $\mu$ are unity, then the instantaneous strain is P/3. The problem has been run with loads of P = 1.0, 0.1, .01, and .001 to see what effect the grid distortion has on accuracy of the velocity profiles. When P = .001, the grid distortion after two transit times is not noticeable on a grid plot. However, when P = 1.0, there is significant grid distortion after two transit times. This means that as P increases the frequency of grid rezoning should be increased.

## 5. Conclusions and Recommendations

For the comparison of HONDO, TOODY, and THREEDY to the exact solution, Poisson's ratio $\nu$ (NU in Figure 2) was selected as 0.25. For this value of $\nu$, the implicit version of THREEDY had an error of .1315; TOODY had an error of .0800; HONDO had an error of .0671; and the explicit version of THREEDY had an error of .0569. THREEDY-MOC produced the best calculation with an error of .0119. The comparison of THREEDY-MOC with the exact solution is shown in Figure 2.

On the basis of these results it is recommended that the MOC boundary value algorithm be incorporated into THREEDY. Also, it is recommended that TOODY be so modified as to include the option of running the set of test problems with the Lagrangean linearized elasticity relation. There are several features in TOODY whose effects could be tested against the subset of problems for which there is a known solution. This subset should be increased to encompass problems involving periodic geometry, boundary values, and initial data corresponding to the geometry, values, and data to be used in running a wavecode.

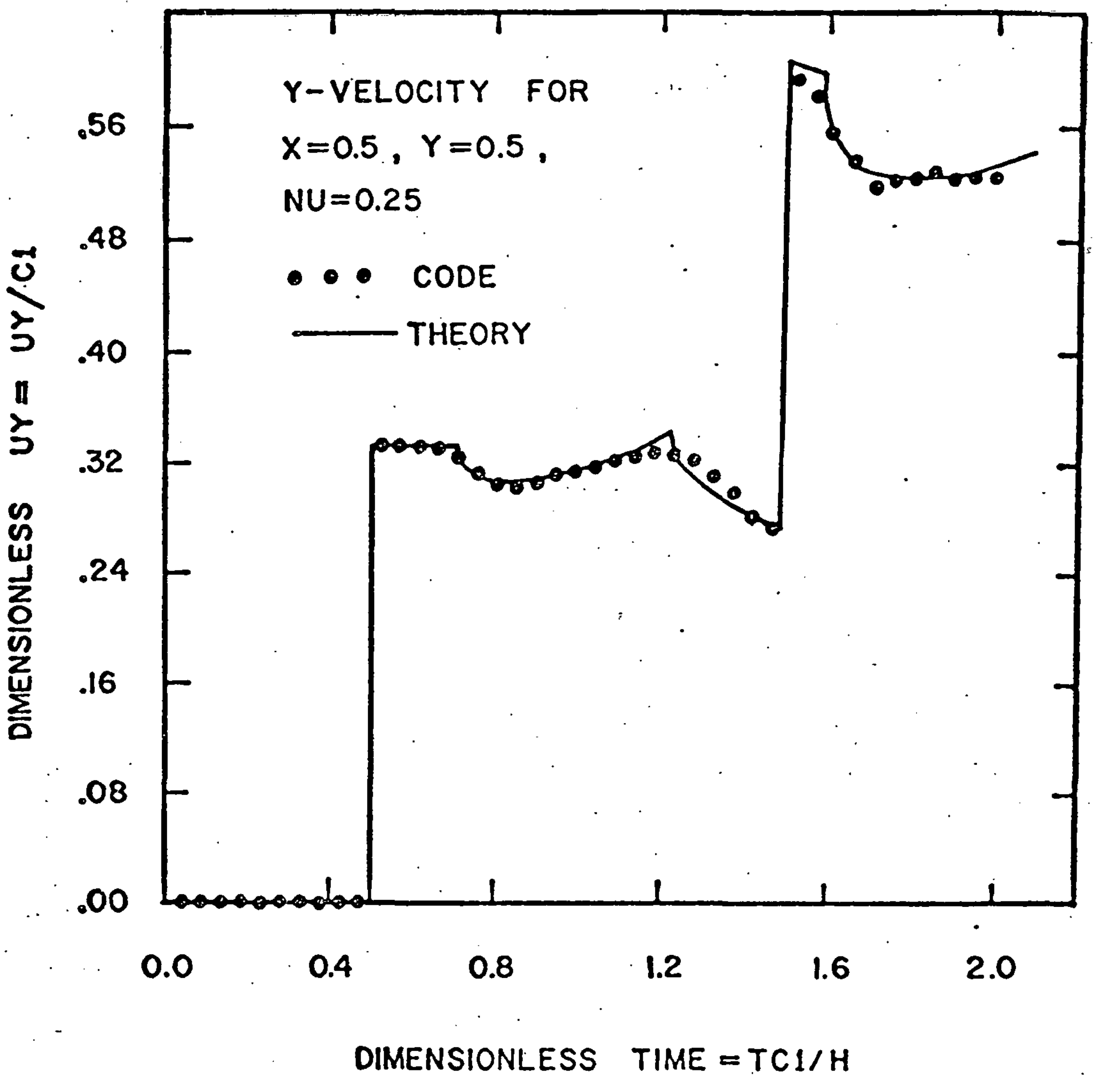


Figure 2. Comparison of THREEDY-MOC with Exact Solution.

Finally, it is recommended that THREEDY continue to be based on operator splitting. As seen in the test problem results presented here, the operator splitting approach can yield accuracies just as good as the nonsplit finite difference or finite element approaches, and, in addition, operator splitting schemes can be made more efficient than their nonsplit counterparts as pointed out in [11] and [12].

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