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A CDC6600 Subroutine for Normal Random Variables

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A CDC6600 SUBROUTINE FOR NORMAL RANDOM VARIABLES

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ABSTRACT

A value y for a uniform variable on (0,1) is generated and a table of 96 percent points for the (0,1) normal distribution is interpolated for a value of the normal variable x(0,1) on $0.02 \le y \le 0.98$. For the tails, the inverse normal is computed by a rational Chebyshev approximation in an appropriate variable. Then $X = x\sigma + \mu$ gives the $X(\mu,\sigma)$ variable.

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Introduction

This document provides the analytical basis for a CDC6600 subroutine for values of a normally distributed random variable $X(\mu,\sigma)$, where μ and σ are the mean and standard deviation of X. Analytically, X values are obtained as solutions of the equation

$$F(x) = y \tag{1}$$

where y is a value of a uniformly distributed variable and F is the cummulative normal distribution with mean μ and standard deviation σ.

RVNORM(RMU,SIG) for Values of a Normal Random Variable $X(\mu,\sigma)$

The methods employed below use the normal distribution F for a (0,1) normal variable x. $X(\mu,\sigma)$ is then computed by means of

$$X = \sigma x + \mu \quad .$$

First a value y, $0 < y < 1$, for a uniform variable is obtained from the systems random number generator RANF. The scheme is to invert (1) over 96% of the range, $0.02 \leq y \leq 0.98$, by a table look-up of percent points followed by linear interpolation. For the other 4%, $0 < y < 0.02$ or $0.98 < y < 1$ we use a low accuracy rational Chebyshev approximation of Hastings [2] which is as nearly consistent with the accuracy of the linear interpolation as we can get. The percent point table, labeled PCT for y = 0.01 (0.01) 0.51, was obtained from (1) using subroutine NORM described in [1]. Only 51 values are needed because y is always reduced to the interval $0 < y \leq 0.5$ by making use of the symmetry of F about $x = 0$, $F(x) = 1 - F(-x)$, putting (1) in the form

$$F(\pm x) = \begin{cases} y, & 0 < y \leq 0.5 \\ 1-y & 0.5 < y < 1 \end{cases} \quad .$$

This combination of table lookup, linear interpolation and Chebyshev approximation were selected primarily for speed of execution with an accuracy consistent with most statistical applications.

Error Analysis for Linear Interpolation

The error $E(y)$ in the interpolation

$$x = \frac{x_2 - x_1}{y_2 - y_1}(y - y_1) + x_1 + E(y), \qquad y_1 \le y \le y_2 \tag{2}$$

is given by

$$E(y) = \frac{(y-y_1)(y-y_2)}{2}\frac{d^2x}{dy^2}(\xi), \qquad y_1 < \xi < y_2 ,$$

and the relative error $E(y)/x$ can be bounded by

$$\left|\frac{E(y)}{x}\right| \le \frac{h^2}{8} \max_{y_1 \le y \le y_2} \left|\frac{d^2x}{dy^2}\right|$$

where

$$h = y_2 - y_1 = 0.01$$

$$y = F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$$

$$\frac{dx}{dy} = \frac{1}{F'(x)} = \sqrt{2\pi}\, e^{x^2/2} \tag{3}$$

$$\frac{d^2x}{dy^2} = x\sqrt{2\pi}\, e^{x^2/2}\frac{dx}{dy} = x(2\pi)\, e^{x^2}$$

$$\frac{d^3x}{dy^3} = (2\pi)^{3/2}(2x^2+1)e^{3x^2/2} ,$$

$$\frac{d^nx}{dy^n}(0.5) \ge 0 , \qquad n=1,2,\cdots .$$

Then,

$$\left|\frac{E(y)}{x}\right| \le \frac{\pi}{4} h^2 \left|\frac{x_2}{x_1}\right| e^{x_2^2} \quad , \qquad \text{where } |x_2| > |x_1| > 0$$

and x_1, x_2 corresponds to y_1, y_2, respectively. For $y_2 = 0.98$, $x_2 = 2.0537$, $y_1 = 0.97$, $x_1 = 1.8808$ this gives a relative error bound,

$$\left|\frac{E(y)}{x}\right| \le 0.0058 \quad \text{or} \quad 0.58\% \ .$$

For x close to zero (or y close to 0.5) the accuracy is higher since

$$\left|\frac{E(y)}{x}\right| \le \frac{(y-0.50)}{x}\frac{(y-0.51)}{2} \max_{0\le x\le x_1} \left|\frac{d^2x}{dy^2}\right| , \qquad 0.50 \le y \le 0.51$$

$$\left|\frac{E(y)}{x}\right| \le \frac{h}{2}\left[2\pi(0.02507)e^{(0.02507)^2}\right] \le 0.00079 \quad \text{or} \quad 0.079\%$$

where $\frac{y-0.50}{x} \le 1$ by virtue of the expansion

$$x = (y-0.50) + \frac{(2\pi)^{3/2}}{3!}(y-0.50)^3 + \cdots$$

for $y \ge 0.50$ and the results in (3).

Error Analysis for the Inverse Normal Approximation

In [2], the inverse of $1-y = F(-x)$, $0 < y \le 0.5$ is given by the rational Chebyshev approximation

$$x = w - \left[\frac{a_1 + a_2 w}{1 + b_1 w + b_2 w^2}\right] + E(y), \qquad w = \sqrt{-2 \ln y}$$

$$a_1 = 2.30753 \ , \qquad b_1 = 0.99229$$
$$a_2 = 0.27061 \ , \qquad b_2 = 0.04481$$

with a uniform error bound $|E(y)| \le 0.003$. In order to get the largest relative error, we want $|x|$ to be as small as possible. Therefore, at the upper range where $y = 0.02$ and $x = -2.0537$ we get

$$\left|\frac{E(y)}{x}\right| \leq \frac{0.003}{2.0537} = 0.0015 \quad \text{or} \quad 0.15\% \ .$$

Thus, the overall error can be expected to be no more than 0.6%, occurring near the end of the linearly interpolated range.

Testing

In order to pick up any gross errors, 3000 random numbers $\hat{y}$ were generated by RANF and X(0,1) was computed for each $\hat{y}$. Then $\hat{y}$ was recomputed by $\hat{y} = F(X)$ using the library subroutine FNORM (labeled NORM in [1]) for the cummulative normal F. The results showed errors consistent with expected values.

In addition, x was computed for each of the y values 0.001 (0.001) 0.999 and compared with the inverse normal from the relations

$$x = \begin{cases} -\sqrt{2} \ \text{ierfc}\ (2y) & 0 < y \leq 0.5 \\ \sqrt{2} \ \text{ierfc}\ (2(1-y)) & 0.5 < y < 1 \end{cases} \ .$$

where ierfc(x) is the inverse coerror function [1]. A maximum relative error of 0.37% was computed near the ends of the interpolatory interval at $y = 0.025$ and $y = 0.975$. Relative errors near $x = 0$ were on the order of 0.002%. Relative errors for $y < 0.02$ were on the order of 0.077%. These errors are consistent with the error analysis above.

References

[1] Amos, D. E. and Daniel, S. L., "CDC6600 Codes for the Error Function, Cummulative Normal and Related Functions," SC-DR-72-0918, December 1972.

[2] Hastings, C., Jr., "Approximations for Digital Computers," Princeton University Press, Princeton, NJ, 1955.

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