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Similarity Solutions for Converging Shocks

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CONTENTS

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by

R. B. Lazarus and R. D. Richtmyer

ABSTRACT

This report recapitulates the known results for similarity solutions for the flow problem of a strong converging shock in spherical or cylindrical symmetry and extends that work in four ways: (1) parameters of the standard solutions are given for a large number of values of γ; (2) some new, non-analytic solutions are exhibited for relatively large values of γ; (3) the standard solutions are examined more thoroughly in the limits $\gamma \to \infty$ and $\gamma \to 1$; and (4) solutions, existing only in a narrow band of values of γ, are given for the problem of two converging shocks.

I. INTRODUCTION

As is well known,[1,2,3,4] there is a similarity solution for a shock converging on the origin in spherical or cylindrical symmetry, when that incoming shock runs with infinite Mach number into uniform material at rest and when that material obeys a gamma law equation of state $\rho e = p/(\gamma-1)$, with e the internal energy per unit mass, ρ the density, and p the pressure. The solution includes reflection of the shock at the origin, and divides space-time (r,t) into three regions, namely Region 1 ahead of the incoming shock, Region 3 behind the reflected shock, and Region 2 between the shocks (see Fig. 1-1).

Previous authors have observed that the solution is unique (given gamma and the type of symmetry) if one requires continuity of the derivatives of the flow variables throughout the interior of Region 2, and the present work includes calculations of those "standard" solutions for many values of gamma (and both symmetries), including the limiting cases $\gamma \to 1$ and $\gamma \to \infty$. But the present work also

shows that those solutions are not unique.

Even with the requirement of continuous derivatives, it is shown in Sec. VI that, for a narrow band of values for gamma near $\gamma = 2$, two new solutions exist in which Region 2 is divided by a second incoming shock, which overtakes the original shock at the origin.

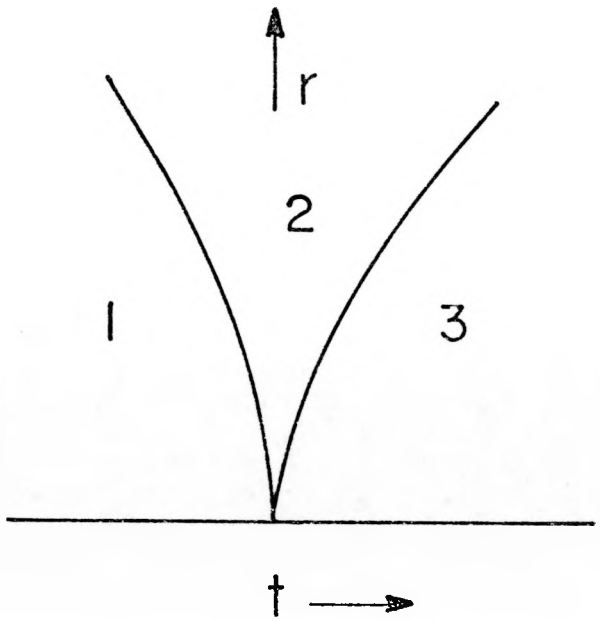

Fig. 1-1
r-t trajectories of incoming and reflected shocks.

Furthermore, as discussed in a previous report in this series,[5] for the case of a collapsing cavity, the original flow equations do not require continuity of derivatives. In particular, there is in Region 2 a limiting negative characteristic which reaches the origin concurrently with the incoming shock; jumps in the derivatives of the flow quantities can be propagated along that characteristic. Since the trajectory of that characteristic corresponds to a single value of the similarity variable, we may accept similarity solutions with jumps of derivative at that point. It is shown in Sec. III that one-parameter families of such solutions exist for gammas greater than certain critical values (in fact, the very values which are the threshold for the double-shock solution band), and that those solutions appear to be quite interesting.

Finally, it appears that these two types of new solution can be combined.

II. THE FLOW EQUATIONS

Given an inviscid fluid without heat conduction, described by its local velocity $\underline{v}(\underline{r},t)$, its density $\rho(\underline{r},t)$, its energy per unit mass $e(\underline{r},t)$, and its pressure $p(\underline{r},t)$, as given by some equation of

state $p = p(\rho,e)$, the equations governing regions of smooth flow are, in the absence of any body forces,

$$\begin{aligned} L\rho &= 0 \\ L\rho\underline{v} &= -\nabla p \\ L\rho e &= -p\nabla\cdot\underline{v}, \end{aligned} \tag{2.1}$$

where the operator L is defined by $Lf = f_t + \nabla\cdot\underline{v}f$, the subscript t denoting partial differentiation with respect to t. For a polytropic fluid, the equation of state is $p = (\gamma-1)\rho e$, and the entropy is a function only of the combination $s = p\rho^{-\gamma}$. Substituting for e in the equations above, we find that $(\underline{v}\cdot\nabla + \partial/\partial t)s = 0$, so that the entropy is indeed constant along the trajectories of fluid elements.

Introducing the new variable $c(\underline{r},t) \equiv +\sqrt{(\partial p/\partial\rho)_s} = +\sqrt{\gamma p/\rho}$, the local sound speed, we can rewrite our equations as

$$\begin{aligned} &\rho_t + \nabla\cdot(\rho\underline{v}) = 0 \\ &\underline{v}_t + \underline{v}\cdot\nabla\underline{v} + \frac{1}{\gamma\rho}\nabla(\rho c^2) = 0 \\ &c_t + \underline{v}\cdot\nabla c + (\gamma-1)c\nabla\cdot\underline{v} = 0. \end{aligned} \tag{2.2}$$

For the cases of cylindrical symmetry ($\nu=1$) or spherical symmetry ($\nu=2$), these can be written (using u for the radial fluid velocity)

$$\begin{aligned} &\rho_t + (\rho u)_r + \nu\rho u/r = 0 \\ &u_t + uu_r + \frac{1}{\gamma\rho}(\rho c^2)_r = 0 \\ &c_t + uc_r + (\gamma-1)c(u_r+\nu u/r) = 0. \end{aligned} \tag{2.3}$$

Another attractive choice of dependent variables replaces $\rho(r,t)$ by $s(\underline{r},t) = c^2\rho^{1-\gamma}/\gamma$. With the substitution $k = 2/(\gamma-1)$, this choice yields

$$\begin{aligned} &s_t + us_r = 0 \\ &(u \pm kc)_t + (u \pm c)(u \pm kc)_r = \mp\nu uc/r + c^2 s_r/\gamma(\gamma-1)s, \end{aligned} \tag{2.4}$$

displaying the equations in characteristic form.

Now, with α and κ free parameters, we try the similarity variable

$$y = \text{const.} + \log r - \alpha \log t, \tag{2.5}$$

and the substitutions

$$\begin{aligned} u(r,t) &= -\alpha r t^{-1} V(y) \\ c(r,t) &= -\alpha r t^{-1} C(y) \\ \rho(r,t) &= \rho_o r^{\kappa} R(y) \text{ or } s(r,t) = s_o\, \alpha^2 r^2 t^{-2} r^{-k(\gamma-1)} S(y). \end{aligned} \tag{2.6}$$

Using $\partial f(y)/\partial r = f'/r$ and $\partial f(y)/\partial t = -\alpha f'/t$, we can substitute these into Eq. 2.3. We find that we get common factors of ρ_0, α, r, and t; dividing these out and using the more convenient $\lambda = 1/\alpha$, we derive

$$\begin{aligned} &R' + (\kappa+\nu+1)RV + (RV)' = 0 \\ &V(\lambda+V) + V'(1+V) + C[(\kappa+2)C+2C'+C'/R]/\gamma = 0 \\ &2C(\lambda+V) + 2C'(1+V) + (\gamma-1)C((\nu+1)V+V') = 0. \end{aligned} \tag{2.7}$$

Using the variables u, c, and s, we would get

$$(1+V)S' + S\,[(2-\kappa(\gamma-1))V+2\lambda] = 0$$

and, after multiplication by $(1+V\mp C)$,

$$[(1+V)^2 - C^2](V' \pm kC') = \nu VC[C\mp(1+V)] - C^2[1\mp C/(1+V)](\kappa+k(\lambda-1))/\gamma - (1+V\mp C)(\lambda+V\pm C)(V\pm kC), \tag{2.8}$$

where of course Eq. 2.8 denotes two equations, one with all the upper signs and one with all the lower signs.

By a bit of algebra, we can get from the R, V, C equations two different expressions for $1/(1+V)$, one involving constants R'/R and $V'/(1+V)$, and the other involving constants C'/C and $V'/(1+V)$. Equating them, we can get an expression whose derivative with respect to y vanishes, leading to a constant of the motion and thus reducing our system to a system of two equations. Explicitly, the constant of motion is

$$\exp(2y/\alpha)C^2[R(1+V)]^q/R^{\gamma-1} = \text{const., with}$$
$$q = [\kappa(\gamma-1)+2(1-\alpha)/\alpha]/(\kappa+\nu+1). \tag{2.9}$$

With more algebra, we can then put our system into the form

$$V' = N_1(V,C)/D(V,C)$$
$$C' = N_2(V,C)/D(V,C), \tag{2.10}$$

where

$$D(V,C) = (1+V)^2 - C^2 \tag{2.11}$$

and

$$N_1(V,C) = -V(1+V)(\lambda+V) + C^2\left[(\nu+1)V + \frac{2\lambda-2-\kappa}{\gamma}\right]$$
$$N_2(V,C) = -1/2C[V^2(2+\nu(\gamma-1)) + V((3-\gamma)\lambda+\nu(\gamma-1)+ +\gamma+1) + 2\lambda] + C^3\left[1 + \frac{\kappa(\gamma-1)+2(\lambda-1)}{2\gamma(1+V)}\right]. \tag{2.12}$$

Since the similarity variable y does not appear explicitly in D, N_1, or N_2, our system of two ordinary first order non-linear differential equations is autonomous, and we can write it as a single equation

$$dC/dV = f(V,C). \tag{2.13}$$

An initial condition for this equation, however, is a condition on one branch of the curve $r(t) = \text{const.}|t|^{\alpha}$, corresponding to some constant value of y.

The free parameter κ allows us to handle the s = constant boundary condition of the cavity collapse problem described in Ref. 5, by taking $\kappa = -2(\lambda-1)/(\gamma-1)$, and the ρ = constant boundary condition of an infinitely strong shock, by taking $\kappa = 0$.

Since Eq. 2.13 does not contain y, it will be convenient to change similarity variable to $x = -e^{-\lambda y}$.

III. THE CONVERGING SHOCK; THE SINGULARITIES

To permit a similarity solution, any shock must be at a constant value of the similarity variable x (or y), so that the physical boundary condition along the shock trajectory $r_{shock} = r(t)$ can be a boundary condition at some x_0 for the similarity equations. Thus we must have $r_{shock} = \text{constant} \cdot t^{\alpha}$. Note that, if two or more shocks exist within one solution, they must all have that form with the same value of α, differing only through different constants of proportionality. We are interested in solutions for the range $0<\alpha<1$.

The jump conditions across a shock become, in terms of the similarity variables V, C, and R,

$$\begin{aligned} 1+V_1 &= \frac{\gamma-1}{\gamma+1}(1+V_0) + \frac{2C_0^2}{(\gamma+1)(1+V_0)} \\ C_1^2 &= C_0^2 + \frac{\gamma-1}{2}\left[(1+V_0)^2 - (1+V_1)^2\right] \\ R_1(1+V_1) &= R_0(1+V_0). \end{aligned} \tag{3.1}$$

For the initial shock converging into material at rest, let us take $t = 0$ to be the time of shock collapse, set $r_{shock} = A(-t)^{\alpha}$, and take $x = t(A/r)^{\lambda}$ as our similarity variable. Then the shock path is $x = -1$. The solution for $x < -1$, the initial and undisturbed region, is simply $u = V = 0 = c = C$, and $R(x) = 1$ (remember that we are now taking $\kappa = 0$). Then the jump conditions give us the starting values

$$V(-1) = -2/(\gamma+1)$$

$$C(-1) = +\sqrt{2\gamma(\gamma-1)}\,/(\gamma+1) \tag{3.2}$$

$$R(-1) = (\gamma+1)/(\gamma-1).$$

The solution must extend through $x = -0$, which corresponds to all of $r > 0$ at $t = -0$, and continue through positive values of x (and thus t) until we get to the reflected shock. At $x = 0$, we must have $V = C = 0$, so that u and c may be finite at finite r (see Eq. 2.6). But then the denominator in Eq. 2.10, namely $D = (1+V)^2 - C^2$, will be $+1$, whereas it starts out negative (namely $-(\gamma-1)/(\gamma+1)$). Thus it must pass through zero, but it may do so only if the numerators N_1 and N_2 vanish simultaneously. In other words, our solution of $dC/dV = N_2/N_1$ must pass through a singularity of the form $0/0$. It will not do so automatically but must be made to do so by a suitable choice of the parameter α. Specifically, we will find that a unique $\alpha(\gamma)$ (for the spherical case, and a different unique $\alpha(\gamma)$ for the cylindrical case) gives the "standard" <u>smooth</u> solutions, but that, for γ large enough, other values of α give other valid solutions. To understand the matter, we must investigate the singularities in more detail.

It should first be noted that, if we substitute $C^2 = (1+V)^2$, which is to say $D = 0$, into either N_1 or N_2 of Eq. 2.12, then the other N will vanish identically.

Substituting $C^2 = (1+V)^2$ into the first Eq. 2.12 and setting $N_1 = 0$ yields the cubic

$$0 = (1+V)[V^2 + aV + b], \tag{3.3}$$

with

$$a = 1 - \frac{(\lambda-1)(\gamma-2)}{\nu\gamma}, \qquad b = 2\frac{\lambda-1}{\nu\gamma}. \tag{3.4}$$

The solution V = -1 is irrelevant to the converging shock problem (it is the starting point for the collapsing cavity problem). The other two solutions are real when the discriminant

$$a^2-4b = 1 - 2\frac{\lambda-1}{\nu\gamma}(\gamma+2) + \frac{(\lambda-1)^2}{\nu^2\gamma^2}(\gamma-2)^2 \tag{3.5}$$

is positive. The discriminant is positive for $\lambda-1$ in the range

$$0<\lambda-1 < \frac{\nu\gamma}{(\sqrt{\gamma}+\sqrt{2})^2}, \tag{3.6}$$

and it is in that range that we will look for solutions. (The discriminant is again positive for $\lambda-1 > \nu\gamma/(\sqrt{\gamma}-\sqrt{2})^2$; this range does not seem to provide any solutions.)

Note: For the collapsing cavity problem, Eqs. 3.4 through 3.6 come out to be the same expressions, but with γ replaced by $\gamma-1$.

Observe that the two singularities are at (V,C) = (-1,0) and (0,1) when $\lambda = 1$, and move toward each other as λ increases. We will distinguish the two singularities by calling them "left" and "right" according as we choose the minus sign or the plus sign in

$$V_{sing} = 1/2(-a\pm\sqrt{a^2-4b}). \tag{3.7}$$

It will turn out that the "standard" solutions pass through the left singularity for small γ and through the right singularity for large γ. By continuity, then, there must be critical values for γ (one for spherical symmetry and one for cylindrical), for which the standard solution has λ at the top of the range given in Eq. 3.6 and passes through the coalesced singularity. It appears that "non-standard" solutions exist only for γ greater than these critical values, which are

$$\begin{aligned} \gamma_c &= 1.9092084, \text{ for } \nu = 1 \text{ (cylindrical)}, \\ \gamma_c &= 1.8697680, \text{ for } \nu = 2 \text{ (spherical)}. \end{aligned} \tag{3.8}$$

For any specific ν and γ, now, other than one of the critical pairs, let us consider solutions of Eq. 2.13 for some λ slightly displaced from the unique $\lambda(\nu,\gamma)$ which gives the "standard" solution. Consider the solution as it approaches the singularity (which will, of course, have been slightly displaced by the change in λ). If the singularity is at (V_s,C_s), say, we must have

$$\frac{dC}{dV} \approx \frac{(V- V_s)\partial N_2/\partial V + (C-C_s)\partial N_2/\partial C}{(V-V_s)\partial N_1/\partial V + (C-C_s)\partial N_1/\partial C}, \tag{3.9}$$

where the partial derivatives are evaluated at (V_s,C_s) and are simply algebraic functions of ν, γ, and λ.

The general solution of this equation is

$$[(C-C_s) - L_2(V-V_s)]^{E_2} = \text{const}\cdot[(C-C_s) - L_1(V-V_s)]^{E_1}, \tag{3.10}$$

where, with

$$R^2 = \left(\frac{\partial N_2}{\partial C_s} - \frac{\partial N_1}{\partial V_s}\right)^2 + 4\frac{\partial N_2}{\partial V_s}\frac{\partial N_1}{\partial C_s}, \tag{3.11}$$

we have

$$2L_{1,2}\partial N_1/\partial C_s = \partial N_2/\partial C_s - \partial N_1/\partial V_s \pm R \tag{3.12}$$

and

$$2E_{1,2}\partial N_1/\partial C_s = \partial N_2/\partial C_s + \partial N_1/\partial V_s \pm R. \tag{3.13}$$

For our case, it appears that R is always real and non-zero, and that L_1 and L_2 have opposite signs, in a neighborhood of the "standard" $\lambda(\nu,\gamma)$. The E's and L's are of course algebraic functions of ν, γ, and λ.

If E_1 and E_2 have opposite signs, the only solutions through the singularity are (locally) the special solutions

$$C - C_s = L_{1,2}(V - V_s). \tag{3.14}$$

For $\gamma < \gamma_c$, the standard solution is of this type. For one particular value of λ, the solution passes through the left singularity with the slope corresponding to the negative L, and, for that λ, the left singularity does indeed have E's of opposite sign. For neighboring values of λ, the solution will not pass through either the left or the right singularity.

Nor does it seem likely that, for $\gamma < \gamma_c$, there are other solutions for <u>substantially</u> different values of λ. For larger values, the E's continue to have opposite signs. For substantially smaller values of λ, the E's do have the same sign, but the left singularity moves further to the left, the positive L is less than one, and the solution hits the forbidden line C = 1+V before it can be attracted to the singularity (see Fig. 3-1).

For $\gamma > \gamma_c$, where the standard solution goes through the right singularity, we have the case where the E's have the <u>same</u> sign. In such a case, <u>all</u> solutions which come sufficiently close to (V_s, C_s)

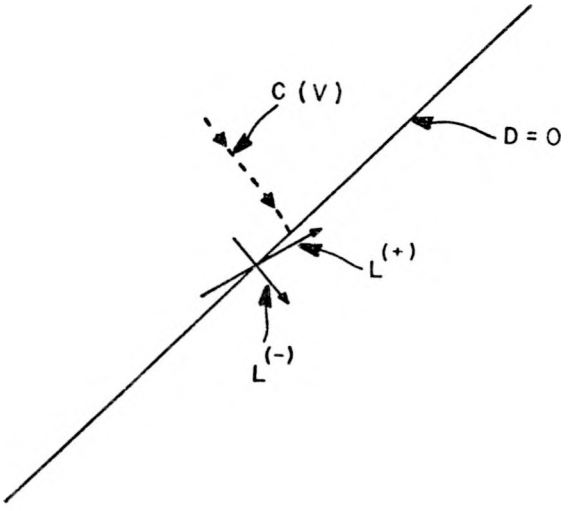

Fig. 3-1
Attractive singularity blocked above by the line D=0.

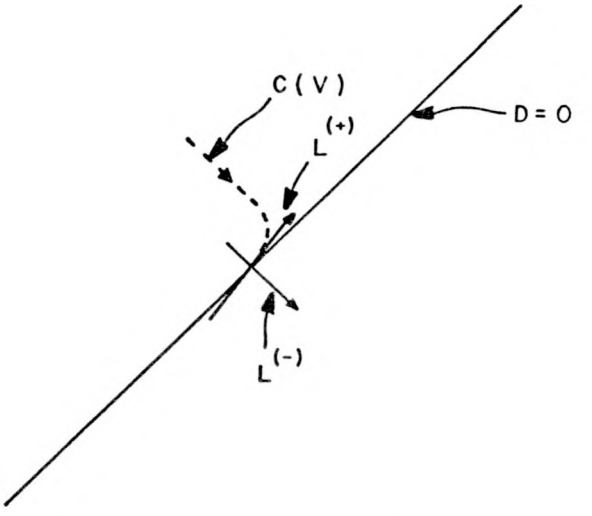

Fig. 3-2
Attractive singularity open above.

pass through the singularity, and they do so, in general, asymptotically like

$$C-C_s = L_i(V-V_s), \tag{3.15}$$

where E_i (i = 1 or 2) is the E of lesser magnitude. It turns out that that L_i is the positive L, and that the (unique) "standard" solution is precisely the special solution which goes through with negative slope (i.e., with the other L).

For the entire range $\gamma > \gamma_c$, the positive L is greater than one. To reach the singularity without first crossing C = 1+V, therefore, the solution must come in from above (see Fig. 3-2). Since the main effect of changing λ is to move the singularity (i.e., the solution curve does not change much until we approach the singularity), this means that the right singularity, with which we are here concerned, must be moved left. Thus only values of λ greater than the standard $\lambda(\nu,\gamma)$ will work. The foregoing analysis is only valid in a neighborhood of the singularity. A complete analysis will be published elsewhere.

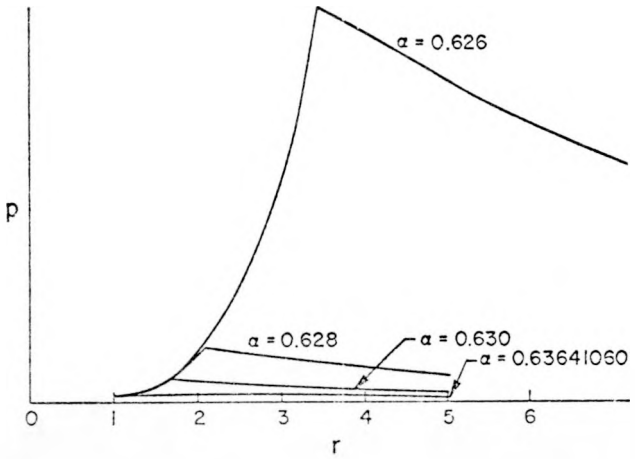

Fig. 3-3
Pressure profiles for $\gamma = 3$, spherical symmetry.

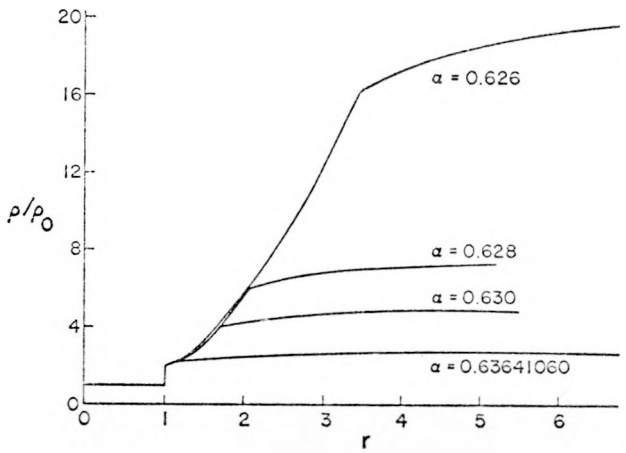

Fig. 3-4
Compression profiles for $\gamma = 3$, spherical symmetry.

Figs. 3-3 through 3-5 show the pressure, density, and velocity profiles at a time when the incoming shock is at r = 1, for the case $\gamma = 3$, $\nu = 2$. For the non-standard solutions, the corners are on the limiting characteristic and are such as to satisfy the flow equations from the left and from the right. Note: In these solutions, the curves in the V-C plane were allowed to leave the singularity in the "standard" direction. This is not necessary (see p. 10 of Ref. 5), but other solutions have not yet been studied.

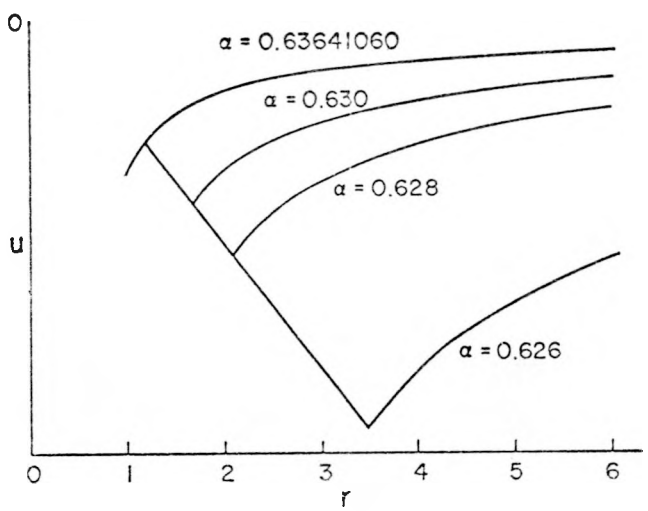

Fig. 3-5
Fluid velocity profiles for $\gamma = 3$, spherical symmetry. The downward sloping portions are distinct but too close to plot separately.

IV. THE REFLECTED SHOCK

The initial shock, which is our starting point, is at x = -1; collapse is at x = 0; and continuation to positive times is simply continuation to positive x. As one might expect on physical grounds,

it will not be possible to continue the same solution to $x = +\infty$. One expects this because large positive values of x correspond to small values of r at large positive values of t, and this region of the flow should be behind a reflected shock. As mentioned above, the trajectory of that reflected shock will have to lie on x = constant = β, say, for some $\beta > 0$.

If we can find the separate similarity solution for the region behind this reflected shock, say $\hat{V}$, $\hat{C}$, and $\hat{R}$, then the two solutions will have to satisfy the jump conditions at $x = \beta$. We will need to satisfy

$$1+\hat{V}(\beta) = \frac{\gamma-1}{\gamma+1}(1+V(\beta)) + \frac{2C^2(\beta)}{(\gamma+1)(1+V(\beta))}$$

$$\hat{C}^2(\beta) = C^2(\beta) + \frac{\gamma-1}{2}[(1+V(\beta))^2 - (1+\hat{V}(\beta))^2]. \qquad (4.1)$$

Note that the constant of motion will be a different constant on the two sides of the reflected shock, just as it is a different constant on the two sides of the initial shock.

This separate solution is needed for $\beta \leq x \leq \infty$, and the only thing we have to serve as a boundary condition is the following. We want u(r=0,t>0) to be zero, by isotropy, and we want c(r=0,t>0) to be finite. In fact, we expect u to be proportional to r, for small r, so we expect V to be constant and C to become infinite as $x \to \infty$.

The standard trick is to take a new variable $w = x^{-\sigma}$, with σ a positive number to be determined, and to try

$$\hat{V}(kw) = \hat{V}_0 + \hat{V}_1 kw + \hat{V}_2 (kw)^2 + \ldots$$

$$\hat{C}(kw) = -(kw)^{-1} + \hat{C}_1 + \hat{C}_2 kw + \ldots, \qquad (4.2)$$

where k is a free parameter (our differential equations are homogeneous in w, so that $\hat{V}(kw)$, $\hat{C}(kw)$ are solutions whenever $\hat{V}(w)$, $\hat{C}(w)$

are). Matching powers of kw, we find that, if we take

$$\lambda\sigma = 1 + \frac{(\nu+1)(\lambda-1)}{(\nu+1)\gamma - 2(\lambda-1)}$$

$$\hat{V}_0 = -2(\lambda-1)/\gamma(\nu+1), \tag{4.3}$$

then we get a solution.

If we now think of the jump conditions as an operator which can be applied to our original, incoming shock solution, for arbitrary positive x, then we have "target" functions $V_t(x)$ and $C_t(x)$, to be matched by $\hat{V}(kw)$ and $\hat{C}(kw)$. The value of x at which that match occurs is, of course, just β. If the value of kw at which the match occurs, is, say, z, then we can determine k by setting $z = k\beta^{-\sigma}$, and we have the complete solution.

V. THE LIMITS $\gamma\to\infty$ AND $\gamma\to 1$.

For $\gamma\to\infty$, we need only switch to $\tilde{V} = \gamma V$, and then we can go to the limit explicitly. The denominator D becomes simply $1-C^2$; the numerator for $\tilde{V}'$ becomes

$$\tilde{N}_1 = -\lambda\tilde{V} + C^2[(\nu+1)\tilde{V} + 2(\lambda-1)], \tag{5.1}$$

and the numerator for C' becomes

$$\tilde{N}_2 = C\cdot[C^2 - \lambda - (\nu+1-\lambda)\tilde{V}/2]. \tag{5.2}$$

These somewhat reduced equations can be integrated numerically by the methods described below for general γ.

For $\gamma\to 1$, the situation is slightly more complicated, because the singularity approaches the starting point $(V,C) = (-1,0)$. If we define $\varepsilon^2 = \gamma - 1$, then, to lowest order, our starting point is $(V_0,C_0) = (-1+\varepsilon^2/2, \varepsilon/\sqrt{2})$. The starting value for D is then $-\varepsilon^2/2$. Now if we tentatively assume that $\lambda-1$ will turn out to be of order

ε, we find that the leading terms in N_1 are

$$N_1 \approx -(\nu+1)C^2 + (1+V)^2 + (\lambda-1)(1+V), \tag{5.3}$$

and the leading terms in N_2 are

$$N_2 \approx C[C^2-(1+V)^2](\lambda+V)/(1+V), \tag{5.4}$$

considering that we must integrate from $1+V = C^2$ until $1+V = C$ at the singularity.

If we integrate $dC/dV = N_2/N_1$ holding C fixed on the right hand side, we find, consistently, that C changes only by a factor 1 - order $(\varepsilon \log \varepsilon)$, and we find, again consistently, that we must have

$$\lambda-1 = \nu\sqrt{(\gamma-1)/2}. \tag{5.5}$$

This is confirmed numerically, as well as the additional result that the Mach number of the reflected shock is $\sqrt{2/(\gamma-1)}$, independent of ν (see Table 5).

TABLE 5

BEHAVIOR OF THE SIMILARITY SOLUTIONS AS GAMMA APPROACHES UNITY

	$\frac{(1-\alpha)^2}{\gamma-1}$		$(\gamma-1)M^2$		$(\gamma-1)\beta$	
$\gamma-1$	$\nu=2$	$\nu=1$	$\nu=2$	$\nu=1$	$\nu=2$	$\nu=1$
0.1	0.4163	0.1317	1.262	1.685	1.615	1.354
0.01	0.9630	0.2755	2.117		1.986	
0.001	1.4864	0.3953	2.202	2.153	1.518	1.252
10^{-4}	1.7909	0.4587	2.102	2.073	1.215	1.109
10^{-5}	1.9216	0.4849	2.042	2.030	1.084	1.044
10^{-6}	1.9641	0.4933	2.022	2.014	1.033	1.017
0	2	1/2	2?	2?	1?	1?

VI. MULTIPLE CONVERGING SHOCKS

If there is a similarity solution corresponding to more than one incoming shock, then the shocks must have the trajectories

$$r_i = A_i(-t)^{\alpha}, \tag{6.1}$$

with $A_i = A_1 < A_2 < \ldots$ If x_i be the value of the similarity variable on which the ith shock exists, then we must have $x_i = -(A/A_i)^2$. Consider i=2.

With $D(V,C) = (1+V)^2 - C^2$, the jump conditions of Eq. 3.1 imply

$$D_2 = -D_1\left[\frac{2}{\gamma+1}\left(\frac{C_1}{1+V_1}\right)^2 + \frac{\gamma-1}{\gamma+1}\right], \tag{6.2}$$

so that D must change sign. Furthermore, since Region 2 is behind the shock, we have $R_2 > R_1 > 0$, and thus the third jump condition implies $(1+V_2)^2 < (1+V_1)^2$. But then the second jump condition implies ${C_1}^2 < {C_2}^2$, giving $D_1 > D_2$. Since D_1 and D_2 are of opposite sign, we have $D_2 < 0$, $D_1 > 0$.

Note: A "shock" existing right <u>at</u> the singularity D = 0 has Mach number unity and is not a shock at all.

Since our solution behind the initial shock starts out with D negative, we see that x_2 must be greater than the value for which the region 1 solution crosses the singularity. Thus we must have the same value of α $(=1/\lambda)$ as we have for the single shock case, since α is determined precisely by the necessity of passing through the singularity.

Rewriting the jump condition on D in the form

$$D_2 = -D_1\left[1 - \frac{2}{\gamma+1}\left(1 - \left(\frac{C_1}{1+V_1}\right)^2\right)\right], \tag{6.3}$$

and noting that $D_1 > 0$ implies $(C_1/(1+V_1))^2 < 1$, we see also that $|D_2| < D_1$, with the inequality stronger for smaller values of γ.

This means that the vector in the V-C plane connecting (V_1,C_1) to (V_2,C_2) has negative slope between -1 and 0.

When the matter is investigated numerically, it turns out that the locus of points (V_2,C_2), as x_2 ranges toward zero from the value of x corresponding to the singularity, is an arc connecting the singularity to the starting point (V_0,C_0) and lying always <u>below</u> and to the left of the original solution curve (see Fig. 6-1). When an attempt is made, however, to continue the solution from any of those points (V_2,C_2), it develops that the solution moves almost parallel to the original solution curve. Hence, the continued solution cannot pass again through the same singularity.

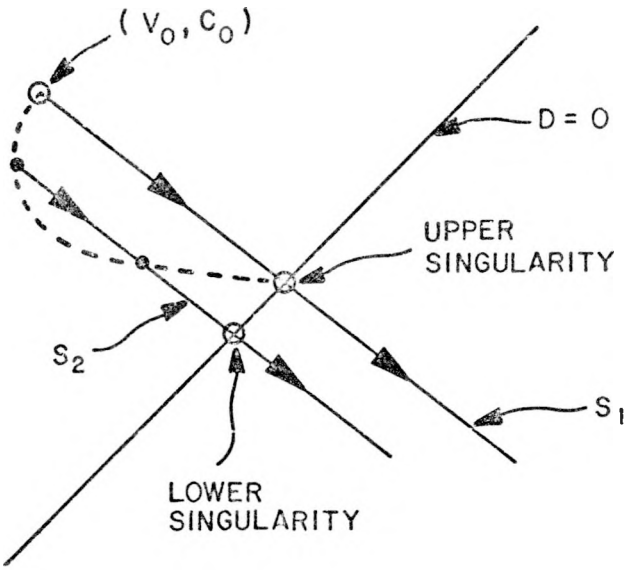

Fig. 6-1
The dashed line is the locus of points accessible from S_1 by the jump conditions.

This immediately suggests that when γ is greater than the critical value of Eq. 3.8, so that the primary solution goes through the right hand (upper) singularity, a point (V_2,C_2) can be found so that the <u>continued</u> solution will pass through the left hand (lower) singularity. This turns out indeed to be the case when γ is greater than γ_c by an amount small enough that the width of the locus (measured parallel to the 45^o line C = 1+V) is not less than the spacing between the two singularities. In fact, there will be two double shock solutions, for a band of γ values, corresponding to relatively weak and relatively strong second shocks, with the two solutions coalescing at the top of the band and then ceasing to exist as γ leaves the band.

For γ's above this band, non-standard solutions may exist with γ's sufficiently close to the upper bound of Eq. 3.6, which is to

say with the two singularities sufficiently close, to permit further solutions with two incoming shocks. A complete analysis will be published elsewhere.

The entire situation can be grasped most simply as follows. Pick values for ν and γ, with $\gamma > \gamma_c$. This determines a starting point in the V-C plane, and a one parameter family of (incomplete) solutions labeled by λ. Pick a value for λ which lets the solution pass through the right singularity and continue to the origin; call the corresponding solution curve S_1. Now that we have λ, we can locate the unused left singularity and construct the solution curve (call it S_2) which passes through it in the standard direction. Lastly, we draw in the (V_2,C_2) locus corresponding to potential second shocks. Then we have zero, one, or two double shock solutions according as that locus cuts S_2 in zero, one, or two points, because we have a physical method of jumping from solution curve S_1 to solution curve S_2. Finally, if the left singularity should have eigenvalues of the same sign, then there would by a family of S_2's, all valid.

Typical solutions are shown in Table 6-1.

TABLE 6-1
MACH NUMBERS FOR WEAK AND STRONG SECOND SHOCKS

For $\nu = 1$ ($\gamma_c = 1.9092084$)

γ	M1	M2
1.91	1.000566	1233.532
1.95	1.078474	22.34995
2.00	1.248826	8.846871
2.05	1.548093	4.701998
2.10	none	

For $\nu = 2$ ($\gamma_c = 1.8697680$)

1.87	1.000170	3508.718
1.90	1.070279	24.19837
2.00	1.848820	3.496177
2.009	2.220365	2.701936
2.01	none	

VII. CONDITIONS BEHIND THE REFLECTED SHOCK

The position of the reflected shock, as a function of time, is given by

$$r_{r.s.}(t) = A\beta^{-\alpha}t^{\alpha}, \tag{7.1}$$

where β is the value of the similarity variable x corresponding to the shock trajectory, as mentioned above, and A is the constant appearing in the trajectory of the initial shock.

For the region behind the reflected shock (inside it, geometrically speaking), it is of interest to consider the time dependence of the volume integrals of mass, internal energy, and kinetic energy, and of the "mean free path" integral of ρdr. By appealing to the original substitutions (Eq. 2.6) for u, c, and ρ, and by substituting for r the appropriate expression in terms of t and the similarity variable w (which runs from zero to $\beta^{-\sigma}$), we find the following, for given ν and γ, taking $\rho_0 = 1$.

The total mass is simply proportional to the total volume, with no other time dependence, and the integral of ρdr is simply proportional to $r_{r.s.}$. (The volume, of course, is going like $t^{(\nu+1)\alpha}$.) The total internal energy and the total kinetic energy are separately proportional to the volume times the factor $t^{-2(1-\alpha)}$. As required for physicality, α is always less than unity, so that the average values of internal and kinetic energies per unit volume decrease with time. (These results also imply that the energy densities behind the reflected shock are instantaneously infinite at collapse time. This is in accord with the fact that C(x)/x and V(x)/x remain finite at $x = -0$, so that the fluid velocity u(r,t) and the sound speed c(r,t) behind the initial shock become infinite like $r^{-(1-\alpha)/\alpha}$ at collapse.)

The various constants of proportionality are given, as functions of ν and γ, in Tables 7-1 and 7-2. I_1 and I_2 are, respectively, the internal and kinetic energies per unit volume, times $A^{-2}\beta^{2\alpha}t^{2(1-\alpha)}$. I_3 is the mass per unit volume; I_4 is the mass per unit area divided by $r_{r.s.}$.

TABLE 7-1

VARIOUS PARAMETERS OF THE STANDARD SIMILARITY SOLUTION, AS FUNCTION OF GAMMA; SPHERICAL CONVERGENCE.

ν = 2 (spherical)

γ	α	β	Mach #	$\rho_2(\beta)$	I_1	I_2	I_3	I_4
1.1	.79596980	16.1541	3.5530	333.6	1.95+6	243.33	4.8+4	3.6+4
1.2	.75714179	6.43123	2.3502	49.05	1.37+4	11.0163	1710.	1253.
1.3	.73377673	3.81021	1.9384	19.03	1031.6	2.2220	320.4	231.1
1.4	.71717450	2.68849	1.7356	10.72	197.5	.8029	114.1	81.83
1.5	.70442807	2.08773	1.61553	7.226	60.76	.3894	56.12	40.21
1.6	.69418951	1.72065	1.53621	5.479	24.73	.2243	33.30	23.91
5/3	.68837682	1.54790	1.4967	4.719	15.09	.1655	25.20	18.14
1.7	.68571652	1.47617	1.47982	4.419	12.07	.1443	22.27	16.05
1.8	.67855370	1.30320	1.43758	3.729	6.685	.1002	16.17	11.71
1.9	.67240014	1.17523	1.40470	3.251	4.058	.07355	12.45	9.061
2.0	.66704607	1.07725	1.37833	2.902	2.641	.05628	10.01	7.327
2.2	.65816533	.938224	1.33851	2.432	1.3015	.03609	7.102	5.262
2.4	.65108461	.845319	1.30973	2.134	.7399	.02521	5.489	4.115
2.6	.64530018	.779560	1.28784	1.930	.4645	.01870	4.493	3.406
2.8	.64048378	.731006	1.27056	1.783	.3138	.01453	3.835	2.936
3.0	.63641060	.693969	1.25649	1.672	.2232	.00159	3.360	2.600
3.2	.63292118	.664976	1.24482	1.587	.1663	.009571	3.024	2.358
3.4	.62989873	.641817	1.24487	1.518	.1279	.008029	2.762	2.171
3.6	.62725578	.622959	1.22641	1.463	.1008	.006819	2.550	2.023
3.8	.62492541	.607422	1.21895	1.417	.08034	.005740	2.354	1.896
4.0	.62285554	.594419	1.21246	1.3785	.06746	.005209	2.263	1.815
4.5	.61857036	.569946	1.19897	1.3056	.04302	.003673	1.960	1.622
5.0	.61522398	.552999	1.18871	1.254	.03095	.002974	1.832	1.518
5.5	.61253956	.540843	1.18033	1.217	.02222	.002258	1.665	1.419
6.0	.61033915	.531820	1.17342	1.188	.01773	.001978	1.628	1.374
6.5	.60850311	.524919	1.16776	1.165	.01348	.001544	1.503	1.308
7.0	.60694820	.519578	1.16281	1.147	.01116	.001359	1.473	1.279
8.0	.60445829	.511963	1.15480	1.120	.007824	.001033	1.398	1.227
10	.60104880	.503479	1.14368	1.087	.004273	.000615	1.263	1.149
50	.59073010	.494072	1.10718	1.0118	.3173*	.06482*	1.043	1.018
100	.58950281	.494977	1.10233	1.0055	.2859*	.05965*	.9883	.9975
∞	.58828929	.496368	1.09753	1	0.2880*	.6456*		

*: $(\gamma+1)^2 I$

TABLE 7-2

VARIOUS PARAMETERS OF THE STANDARD SIMILARITY SOLUTION, AS FUNCTIONS OF GAMMA; CYLINDRICAL CONVERGENCE.

$\nu = 1$ (cylindrical)

γ	α	β	Mach #	$\rho_2(\beta)$	I_1	I_2	I_3	I_4
1.1	.88524806	13.5364	4.10488	29.55	3.28+5	15.614	5236.	4681.
1.2	.86116303	6.09996	2.78911	11.15	6889.0	1.9614	506.1	445.0
1.4	.83532320	2.81561	2.02295	4.796	203.46	.2822	65.48	57.11
5/3	.81562490	1.69479	1.69965	2.928	21.144	.07873	19.31	16.88
1.8	.80859994	1.44082	1.61796	2.527	10.139	.05152	13.32	11.69
1.9	.80409908	1.30515	1.57247	2.316	6.4327	.03949	10.68	9.390
2.0	.80011235	1.19963	1.53602	2.154	4.3370	.03129	8.870	7.817
2.4	.78776900	.941829	1.44206	1.763	1.3299	.01528	5.270	4.694
3.0	.77566662	.763158	1.37121	1.496	.4265	.007407	3.398	3.069
3.4	.77000368	.697702	1.34349	1.399	.2485	.005175	2.828	2.575
4.0	.76363465	.634863	1.31564	1.306	.1329	.003377	2.341	2.150
5.0	.75640105	.575038	1.28751	1.219	.06190	.001948	1.917	1.783
6.0	.75156168	.540788	1.27050	1.169	.03523	.001273	1.693	1.589
10	.74182593	.483613	1.23980	1.0867	.008622	.0003962	1.325	1.281
50	.73002154	.431537	1.20756	1.0142	.6287*	.04112*	1.0736	1.0554
100	.72853594	.426147	1.20374	1.0069	.5831*	.03909*	1.0358	1.0269
∞	.727048052	.421009	1.199865	1	.5431*	.03745*		

*: $(\gamma+1)^2 I$

Another integral of possible interest is the integral, from time zero to time t, of the volume integral of a power of the temperature (or pressure or internal energy; we are dealing with a polytropic fluid) times some function of the density. One might imagine such an integral measuring the total amount, taking place up to time t, of some reactive process having such a dependence on density and temperature (assuming, of course, that the energetics of the process do not break the similarity solution). For the nth power of the temperature, we find the following rather curious result. If $2n < (\nu\alpha+\alpha+1)/(1-\alpha)$, then the integral is entirely regular and goes like $t^{\nu\alpha+\alpha+1-2n(1-\alpha)}$. But for any larger value of

n, the integral would diverge unless other effects (such as depletion of the reactants) were taken into account. For $\nu = 2$, the critical values for $n(\gamma)$ are, for example, $n(1.4) = 5.57$, $n(3) = 4.00$, $n(\infty) = 3.36$.

VIII. THE NUMERICAL INTEGRATION

All calculations were done on the Maniac II computer, using the Madcap V system. All constants and variables entering into the integration of the differential equations were carried with at least 16 decimal digits. Explicit fifth order Runge-Kutta was used, with step size controls as discussed below.

For the α search, and in fact for all the region behind the incoming shock, the independent variable used was $x = t(A/r)^{\lambda}$, and the dependent variables were $v(x) = -V(x)/x$ and $c(x) = C(x)/x$. The minus sign is historical accident; the division by x is to give nice behavior at the star point singularity $V = C = 0$. The initial value for x is -1, and integration must be continued past the unknown value $x = \beta$. An efficient method of coping with this difficulty is described below.

The equations were used in the form $dv/dx = N_1/D$, $dc/dx = N_2/D$, where now

$$D = (1-vx)^2 - (cx)^2, \tag{8.1}$$

$$N_1 = p_1[v^2(1-vx) + p_2c^2] - p_3vc^2x,$$

$$N_2 = c[v(p_5-p_{20}vx) + p_1c^2x(1 - \frac{p_2}{2(1-vx)})],$$

and the constants are

$$p_1 = 1-\alpha$$

$$p_2 = 2/\gamma$$

$$p_3 = (\nu+1)\alpha - 1 \qquad (8.2)$$

$$p_5 = 1/2\ [(\gamma+1)(1-\alpha) - \alpha\nu(\gamma-1)]$$

$$p_{20} = 1-\alpha-\alpha\nu(\gamma-1)/2.$$

For the search for the "standard" $\alpha(\nu,\gamma)$, we exploit the facts that the correct solution goes quite smoothly (in the V-C plane) from its starting point through the singularity, that the positions of the singularities are quite sensitive to the value of α, and that the solution curve for a wrong value of α does not differ much from the correct solution curve all the way up to a point where we can determine that we do indeed have a wrong value. Accordingly, an efficient iterative algorithm is to choose the next guess for α so as to move the relevant singularity on to the line connecting the initial (v,c) point to the last (v,c) point reached before the aforesaid determination. In practice, this determination was made if dV/dx changed sign or if $|dv/dx|$ became larger than three times its initial value. (When calculating the non-standard solutions discussed on pp. 10-12, the "determination" is simply suppressed.)

It is important to note that all finite numerical representations of α will be determined to be wrong if we approach the singularity with a sufficiently small step size. Conversely, almost any value for α will get us through the singularity <u>without</u> such determination if we approach with a sufficiently <u>large</u> step size. Accordingly, the step size was automatically reduced to a prescribed h_{min}, as we approached the singularity, and no further, with h_{min} chosen to give the desired accuracy. The bulk of the α search work was done with $h_{min} = 2^{-30} \approx 10^{-9}$.

The code was run in the α search mode for all desired values of (ν,γ), without continuing the solution past the singularity.

Then it was rerun with the correct α and with larger h_{min} (usually $2^{-24} \approx 6.4 \times 10^{-8}$) to get the complete solution. In this mode, it was almost always true that the solution would step smoothly through the singularity in a single h_{min} step, but this is a matter of luck. If, as happened occasionally, the code determined that a step (or a partial step within the Runge-Kutta) might accidentally land too close to the singularity, then it took a "jump" step of $8h_{min}$ and printed a notification. (Note that, as discussed elsewhere, we are passing through the singularity in an eigendirection, without change of slope.)

In continuing the solution through $x = 0$, the step size was again reduced to h_{min}, in order to permit printing out accurate values of $V(x)/x$ and $C(x)/x$ at $x = 0$.

Two problems arise, now, in connection with continuing this phase of the solution up to $x = \beta$. We must be sure to go far enough, but we do not want to waste time going too far, and we need finely spaced tables of the "target" functions defined on page 14, but only in the neighborhood of (the unknown) β. The two problems are solved as follows.

The code is given a lower bound for β, call it B_{min}; if no better information is available, then zero is the lower bound used by default, but of course we can do much better than that once we have sketched out $\beta(\nu,\gamma)$ by running a few cases. The code then saves the solution for some value of x near B_{min} and pushes ahead using large steps and saving a coarse table of the target functions. It pushes ahead until it approaches the singularity $C = -(1+V)$, which must always lie beyond β, and then finds the (approximate) reflected shock solution (see below) and an approximate value for β. Then it picks up the saved solution from near B_{min} and moves ahead with fine steps until x is safely beyond the approximate β, and, finally, gets an accurate solution for the reflected shock.

For the region behind the reflected shock, the independent variable used was $t = kx^{-\sigma}$, where k is a free parameter that cancels out of the differential equations and is used as described below, and where σ is as defined by Eq. 4.3. The dependent

variables were $v(t) = -V(x)$ and $c(t) = C(x) + 1/t$. The starting value for t was normally taken as about $2^{-25} \approx 3.2\times10^{-8}$. The differential equations were used in the form $dv/dt = M_1/(\sigma t E)$ and $dc/dt = [-1 + (1-ct)M_2/(\sigma E)]/t^2$, where

$$E = (1-ct)^2 - (1-v)^2t^2, \tag{8.3}$$

$$M_1 = v(1-v)(1-\alpha v)t^2 - p_4(1-ct)^2(v-p_6),$$

$$M_2 = (1-ct)^2(\alpha + \frac{p_{21}}{1-v}) - t^2[(1-v)(1-p_{22}v) + p_{23}v],$$

and the constants are

$$p_4 = \alpha(\nu+1)$$

$$p_6 = 2\frac{1-\alpha}{\alpha\gamma(\nu+1)} \tag{8.4}$$

$$p_{21} = (1-\alpha)/\gamma$$

$$p_{22} = \alpha(1 + 1/2\nu(\gamma-1))$$

$$p_{23} = 1/2(\gamma-1)(1-\alpha).$$

The starting value for $v(t)$ is p_6, removing the $1/t$ singularity in dv/dt. The starting value for $c(t)$ is zero; it can be determined by substitution that the starting value for M_2 is then just σ, which removes the $1/t^2$ singularity from dc/dt. A little analysis shows that $v(t)$ is even and $c(t)$ is odd, so there is in fact no $1/t$ singularity either.

The integration is carried out until $V = -v$ and $C = c - 1/t$ match the target functions. The interpolated value of x at which the match occurs is then β. If it is desired to tabulate the solution behind the reflected shock against x, which runs from β to

infinity, rather than against t, then the parameter k can be identified as $k = t_{match}\beta^{\sigma}$.

IX. THE DENSITY, AND THE MACH NUMBER OF THE REFLECTED SHOCK

Taking the initial density ρ_0 as unity, we can use the constant of the motion to find that, in the region between the incoming shock and the reflected shock (call it Region 2),

$$\rho_2(r,t) = \rho_2(x) = \frac{\gamma+1}{\gamma-1}\left[\frac{x_0}{x}\frac{C}{C_0}\left(\frac{1+V}{1+V_0}\right)^a\right]^b, \tag{9.1}$$

where $x_0 = -1$, say, $C_0 = C(x_0)$, and $V_0 = V(x_0)$, and where

$$a = \frac{1-\alpha}{\alpha(\nu+1)} \tag{9.2}$$

$$b = \frac{2\alpha(\nu+1)}{[(\nu+1)\gamma - (\nu-1)]\alpha - 2}.$$

For the region behind the reflected shock (call it Region 3), we can use the jump condition to relate the densities at β:

$$\rho_3(\beta) = \frac{(\gamma+1)M^2}{(\gamma-1)M^2 + 2}\rho_2(\beta), \tag{9.3}$$

where the Mach number, being the magnitude of the ratio of fluid speed ahead of the shock, relative to shock speed, to sound speed ahead of the shock, turns out to be simply

$$M = |(1+V_2(\beta))/C_2(\beta)|. \tag{9.4}$$

Then we can use

$$\rho_3(r,t) = \rho_3(x) = \rho_3(\beta)\left[\frac{\beta}{x}\frac{C}{C_3(\beta)}\left(\frac{1+V}{1+V_3(\beta)}\right)^a\right]^b . \qquad (9.5)$$

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