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Handbook of Bayesian Reliability Estimation Methods

by

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HANDBOOK OF BAYESIAN RELIABILITY ESTIMATION METHODS

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H. F. Martz, Jr., and R. A. Waller

ABSTRACT

Bayesian reliability estimation methods are summarized in a handbook format for convenient use by reliability practitioners. The methods given consider both attribute test data based on a binomial sampling distribution and a beta prior, as well as variables test data from an exponential sampling distribution and a gamma prior. Classical, Bayes, and empirical Bayes methods are all considered. In addition, the sample test data can arise from either an item-censored life test, either with or without the replacement of failed items as they occur, or from a time-truncated life test with replacement. Real-data examples using nuclear reactor component failure data are used to illustrate each of the methods presented.

I. INTRODUCTION

I.A. Notation and Definitions

The results summarized in this handbook rely heavily on the more complete development provided in Waller and Martz[1] and Martz and Waller.[2] However, the notation used here may depart from that of the preceding documents in order to provide consistency throughout the handbook. Some general notation is presented below, while special notation required will accompany the methods to be presented.

General Notation:

n , the number of items on test

x , the observed number of survivors of a test of duration t

t_i , the i^{th} failure time

r , the observed number of failures $(o < r < n)$

t_o , the time of test termination

$(t_1 < t_2 < \cdots < t_r)$, a sample of r ordered times-to-failure

$T = \sum_{i=1}^{r} t_i + (n - r)t_r$, the total test time when the testing is terminated at the time of the r^{th} failure and failed items are not replaced (Type II/item-censored testing without replacement)

$T = nt_r$, the total test time when the testing is terminated at the time of the r^{th} failure and failed items are replaced (Type II/item-censored testing with replacement)

$T = nt_o$, the total test time when the testing is terminated at time t_o and failed items are replaced (Type I/time-truncated testing with replacement)

ν , degrees of freedom of the chi-square distribution

ν_1, numerator degrees of freedom of the F distribution; degrees of freedom of the chi-square distribution for lower interval end points

ν_2, denominator degrees of freedom of the F distribution; degrees of freedom of the chi-square distribution for upper interval end points

$\chi^2_{\alpha;\nu}$, 100(α)th percentile of a chi-square distribution with ν degrees of freedom $[P(\chi^2 < \chi^2_{\alpha;\nu}) = \alpha]$

$F_{\alpha;\nu_1,\nu_2}$, 100(α)th percentage point of an F distribution with ν_1 numerator and ν_2 denominator degrees of freedom $[P(F > F_{\alpha;\nu_1,\nu_2}) = \alpha]$

R , reliability; the probability that an item will operate successfully for a specified length of time when used under specified conditions; the probability of failure-free operation for a specified length of time

$R(t)$, reliability; the probability that an item will operate successfully for a length of time t when used under specified conditions

PWR , pressurized water reactor

λ , failure-rate (failures/unit-time)

n_0 , beta prior distribution parameter representing "pseudo sample size"

x_0 , beta prior distribution parameter representing "pseudo number of survivors"

α_0 , gamma prior distribution shape parameter

β_0 , gamma prior distribution scale parameter

$\Gamma(x)$, gamma function

$1-\alpha$, confidence that a confidence-interval estimate contains the specified reliability parameter

$1-\gamma$, probability that a probability-interval estimate contains the specified reliability parameter

TCI, two-sided confidence interval

LCI, lower one-sided confidence interval

UCI, upper one-sided confidence interval

TBPI, two-sided Bayes probability interval

LBPI, lower one-sided Bayes probability interval

UBPI, upper one-sided Bayes probability interval

TEBPI, two-sided empirical Bayes probability interval

LEBPI, lower one-sided empirical Bayes probability interval

UEBPI, upper one-sided empirical Bayes probability interval

Note: When "0" accompanies any of the Bayes probability-interval abbreviations, it implies that the interval estimate is based entirely on the prior distribution; that is, the interval is a projected estimate before actual observed failure data become available. In References 1 and 2, such an estimate is referred to as a "no data estimate." On the other hand, if a Bayes interval estimate is derived from the posterior distribution, which includes sample test information from a sample of size n, the letter "n" will accompany the abbreviation. Such estimates are referred to as "data estimates." A "no data estimate" is the result of a prior analysis, while a "data estimate" is the result of a posterior analysis (see Definitions below).

Definitions:

Prior Information - information which exists about a reliability parameter of interest before sample test data become available. The information may be either subjective or objective.

Attribute Test Data - data in which only the survival/nonsurvival of each item on life test is recorded.

Variables Test Data - data in which the time-to-failure of each item failing during a life test is recorded.

Sample Test Data - objective reliability failure information obtained from a set of items which have been placed on life test. The data may be either attribute or variables data.

Sampling Distribution - the statistical distribution which is assumed for the sample test data.

Classical Estimation - those estimation techniques which utilize only objective sample test data in computing the estimates.

Bayesian Estimation - an estimation methodology which formally permits combining two sources of information about a reliability parameter of interest. The two sources are prior information and objective sample test data.

Empirical Bayes Estimation - an estimation methodology which is Bayesian in nature but with fewer assumptions regarding the prior distribution.

Prior Analysis - reliability estimates derived from the prior distribution before sample test data are available.

Posterior Analysis - reliability estimates derived from the posterior distribution when sample test data are available for Bayesian analysis.

Prior Distribution - a statistical distribution for the reliability parameter of interest in a Bayesian analysis. This distribution summarizes the prior information about the reliability parameter.

Posterior Distribution - the statistical distribution obtained by combining the prior distribution and the sampling distribution by means of Bayes theorem.

Confidence Interval - an interval estimator which is said to contain a reliability parameter of interest with a specified confidence.

Probability Interval - an interval estimator which contains the reliability parameter of interest with a specified probability.

Point Estimate - a single number which best estimates the reliability parameter of interest as opposed to an interval estimate.

Item-Censored Life test- a life test experiment which is terminated after a prespecified number of failures have been observed.

Time-Truncated Life test- a life test experiment which is terminated after a prespecified length of test time has elapsed.

Probability Distributions Considered:

Exponential Probability Density Function -

$$f(t) = \lambda \exp(-\lambda t) \ , \ t > 0 \ ; \ \lambda > 0$$

Gamma Prior Probability Density Function -

$$g(\lambda) = \frac{\lambda^{\alpha_o - 1} e^{-\lambda/\beta_o}}{\beta_o^{\alpha_o} \Gamma(\alpha_o)} \ , \ \lambda > 0 \ ; \ \alpha_o, \beta_o > 0$$

Binomial Distribution -

$$P(x) = \frac{n!}{(n-x)!x!} R^x (1-R)^{n-x} , \quad x = 0,1,\ldots,n; 0 \leq R \leq 1 .$$

Beta Prior Probability Density Function -

$$g(R) = \frac{\Gamma(n_o)}{\Gamma(x_o)\Gamma(n_o-x_o)} R^{x_o-1} (1-R)^{n_o-x_o-1} , \quad 0 \leq R \leq 1; \quad n_o > x_o > 0 .$$

I.B. Handbook Usage

A user of this handbook will go through a process of elimination to classify his problem of interest into one of the categories contained in Sec. III. A diagram to assist in this elimination process is presented in Figure 1.

For either Bayes or empirical Bayes methods, the information in Sec. II will be helpful in selecting a prior distribution.

I.C. Scope

The scope of the variables test data methods presented in this handbook covers both point and interval estimators of reliability and failure rates for the following three general categories of methods: (1) Classical, (2) Bayes, and (3) Empirical Bayes.

The classical estimation of reliability is limited to maximum likelihood estimation for point estimates and confidence intervals based on the chi-square distribution. The only sampling distribution considered is the exponential (constant failure rate) distribution. The sample test data are assumed to arise from either an item-censored or time-truncated life test situation. The item-censored test can be conducted either with or without the replacement of failed items while the time-truncated test assumes that failed items will be replaced as they occur.

The Bayes probability intervals are likewise given in terms of percentiles of the chi-square distribution. As discussed in

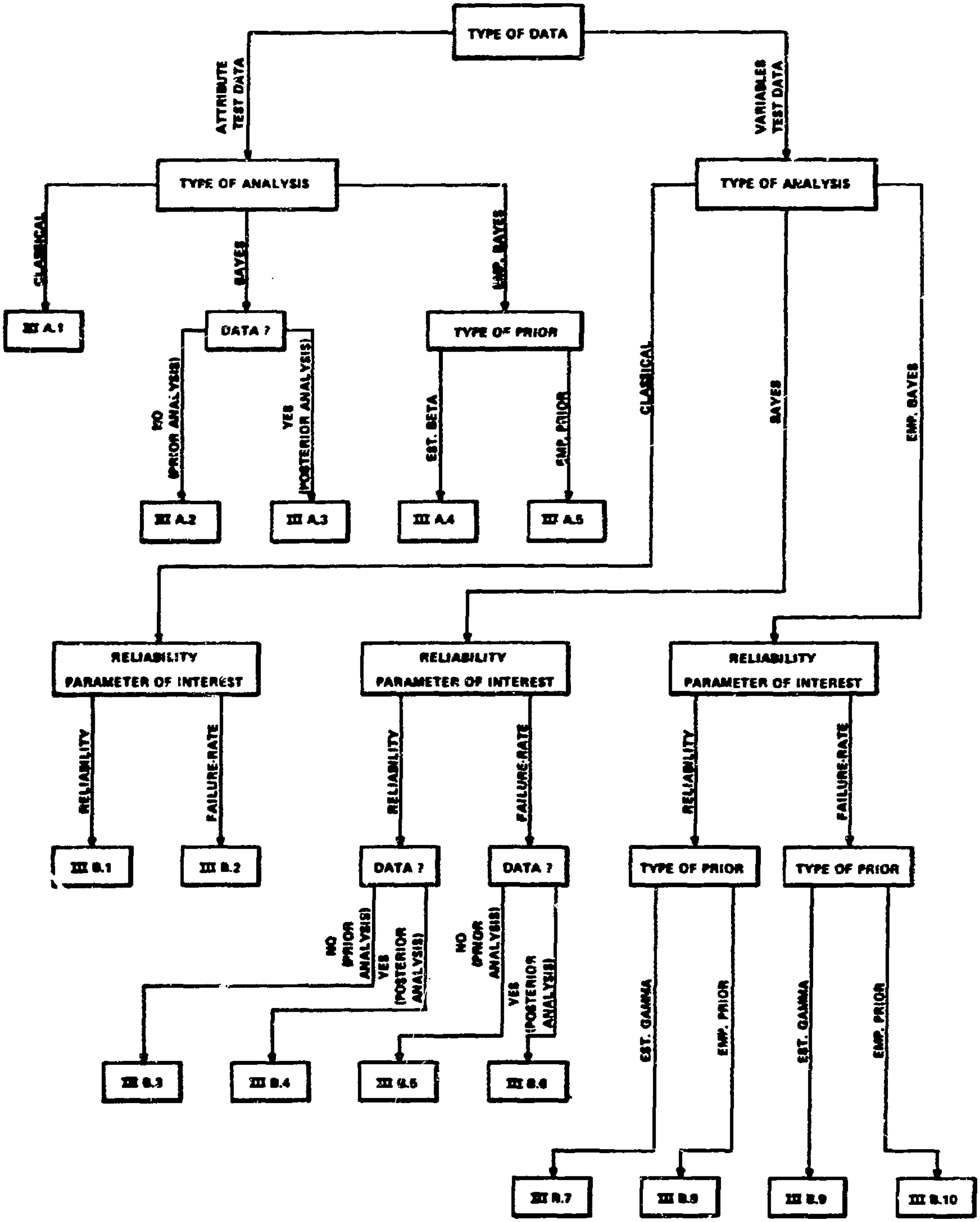

Fig. 1. A Diagram to Assist the Handbook User.

Waller and Martz,[1] a gamma prior distribution of the failure rate λ is assumed throughout Sec. III.B.

The scope of the attribute test data methods presented in this handbook covers both point and interval estimators of the probability of failure-free operation for a specified length of time (the reliability) for the following three general categories of methods: (1) Classical, (2) Bayes, and (3) Empirical Bayes.

The classical estimation of reliability is limited to maximum likelihood estimation for point estimates and confidence intervals are given in terms of percentage points of the F distribution. For reasons discussed in Martz and Waller,[2] a beta prior distribution of reliability is assumed throughout Sec. III.A. It is noted that all the interval estimates in Sec. III.A are given in terms of the F distribution. The Bayes interval estimates consider appropriate areas under a beta distribution. Since this distribution is not widely tabulated, equivalent estimates are expressed in terms of the F distribution, which is widely tabulated.

II. SELECTING A PRIOR DISTRIBUTION

II.A. General Remarks

Use of the results in this handbook requires that certain parameter values be specified. The purpose of this section is to indicate which parameters must be specified, to motivate the nature of those parameters, and to provide references to some existing procedures which may be helpful in the necessary value specifications. The discussion is divided into two parts. First, Sec. B addresses the selection of beta parameters for priors used in reliability estimation for attribute testing. Second, Sec. C presents a discussion of selecting parameters for gamma priors on failure rates in exponential time-to-failure models.

II.B. Beta Priors for Reliability Estimation in Attribute Testing Case

We suppose that R, the reliability of an item being investigated by an attribute testing experiment, is a random variable such that the prior density of R is given by the beta distribution

$$g(R) = \frac{\Gamma(n_o)}{\Gamma(x_o)\Gamma(n_o-x_o)} R^{x_o-1} (1-R)^{n_o-x_o-1} , \; 0 \leq R \leq 1; n_o > x_o > 0 .$$

The experimenter must then select values for the parameters n_o and x_o. For reference, it is convenient to think of n_o and x_o as pseudo number of trials and pseudo number of successes, respectively.

Waterman, Martz, and Waller[3] present an extensive set of tables which assist an experimenter with the translation of experiences, judgments, and beliefs into numerical choices for n_o and x_o. Those tables require an experimenter to specify values for either the 5th percentile and the mean for R, or the mean and 95th percentile for R.

II.C. Selection of Parameter Values for a Gamma Prior on Failure Rate

If we assume that the time-to-failure for an item is exponentially distributed with failure rate λ, the conjugate prior distribution for λ is the gamma density given by

$$g(\lambda) = \frac{\lambda^{\alpha_o-1} e^{-\lambda/\beta_o}}{\beta_o^{\alpha_o}\Gamma(\alpha_o)} , \; \lambda > 0; \alpha_o, \beta_o > 0 .$$

To use the Bayesian methods presented in this handbook for exponential failure data, an experimenter must specify values for the parameters α_o and β_o. We may note that the prior expected value of λ is $E(\lambda) = \alpha_o\beta_o$ and the prior variance of λ is $V(\lambda) = \alpha_o\beta_o^2$.

Thus, one method of specifying values for α_0 and β_0 is to subjectively (using all available experiences, beliefs, etc.) select values for $E(\lambda)$ and $V(\lambda)$ and specify $\alpha_0 = E^2(\lambda)/V(\lambda)$ and $\beta_0 = V(\lambda)/E(\lambda)$. That method is presented along with tables for specifying values for one of the pairs of percentiles, (5, 50), (5, 95), or (50, 95) in Martz and Horita[4]. If an experimenter wants to use reliability experience in specifying values for α_0 and β_0, then Waller, Martz, Horita, and Waterman[5] provide tables and graphs to assist in translating values for two reliability percentiles, say (5, 50), (5, 95), or (50, 95), into values for α_0 and β_0 in the gamma prior for λ.

III. POINT AND INTERVAL ESTIMATION METHODS FOR RELIABILITY AND FAILURE-RATES

III.A. Attribute Test Data

For the results in this section, we suppose that n items are placed on life test for a specified length of time and that x survivors are observed. The number of survivors follows a binomial distribution (see Martz and Waller.[2] It is desired to estimate the probability of surviving the test (the reliability or the probability of failure-free operation). The interval-estimation equations are presented in terms of percentage points of the F distribution which are given in the table in Appendix C.

III.A.1. Attribute test data: classical procedures.

a. Point estimator of R:

$$\hat{R} = x/n \ . \qquad \text{(LA-6126, Eq. 2)}$$

b. Two-sided confidence-interval estimator:

$$100(1-\alpha)\%\text{TCI}: \frac{x}{x+(n-x+1)F_{\alpha/2;2n-2x+2,2x}}$$

(LA-6126, Eq. 3)

$$\leqslant R \leqslant \frac{(x+1)F_{\alpha/2;2x+2,2n-2x}}{(n-x)+(x+1)F_{\alpha/2;2x+2,2n-2x)}} \ .$$

c. Lower one-sided confidence-interval estimator:

$$100(1-\alpha)\%\text{LCI}: R \geqslant \frac{x}{x+(n-x+1)F_{\alpha;2n-2x+2,2x}} \ .$$

(LA-6126, Eq. 4)

d. Remarks:

The results in a, b, and c above assume that the n items on test survive independently, each with the same probability R.

e. Example:

Consider Data Set 1 in Appendix A. For the Indian Point 1 reactor, the estimated pump reliability for 1972 is $\hat{R}$ = 49/50 = 0.98. A 95% two-sided confidence-interval estimate of this reliability is

$$\frac{49}{49+2(2.94)} \leqslant R < \frac{50(39.49)}{1+50(39.49)} \quad \text{or } 0.89 \leqslant R \leqslant 1.00 \ .$$

f. General references:

(i) A. H. Bowker and G. J. Lieberman, Engineering Statistics (Prentice Hall, New York, 1972) 2nd Ed.

(ii) W. J. Dixon and F. J. Massey, Jr., Introduction to Statistical Analysis (McGraw-Hill, New York, 1969) 3rd Ed (see F tables, pp. 470-485).

III.A.2. Attribute test data: Bayes with known beta prior - prior analysis.

a. Point estimator of R:

$$\tilde{R}_o = x_o/n_o \qquad \text{(LA-6126, Eq. 7)}$$

b. Two-sided probability-interval estimator:

$$100(1-\gamma)\%\text{TBPI}(0): \quad \frac{x_o}{x_o+(n_o-x_o)F_{\gamma/2;2n_o-2x_o,2x_o}} \leq R \qquad \text{(LA-6126, Eq. 8)}$$

$$\leq \frac{x_oF_{\gamma/2;2x_o,2n_o-2x_o}}{n_o-x_o+x_oF_{\gamma/2;2x_o,2n_o-2x_o}} .$$

c. Lower one-sided probability-interval estimator:

$$100(1-\gamma)\%\text{LBPI}(0): \quad R \geq \frac{x_o}{x_o+(n_o-x_o)F_{\gamma;2n_o-2x_o,2x_o}} . \qquad \text{(LA-6126, Eq. 9)}$$

d. Remarks:

No current test data is incorporated in the above estimators. They depend only on the prior parameter values n_o and x_o. For guidance in determining n_o and x_o, see Sec. II.B.1.

e. Example:

Consider Data Set 1 in Appendix A. Suppose that the PWR annual prior mean pump reliability is believed to be 0.95 (prior to the 1972 data). Further, suppose that it is believed that there is only a 5% chance that the annual pump reliability is below 0.70. From Reference (ii) below, this prior belief is consistent with a beta prior distribution with $x_o = 2.43675$ and $n_o = 2.56500$. Prior to the 1972 data, the pump reliability is estimated to be $\hat{R}_o = 2.43675/2.56500 = 0.95$ and a two-sided 95% probability-interval estimate of this reliability is

$$\frac{2.43675}{2.43675+(2.56500-2.43675)(10.01)} < R < \frac{2.43675(921.8)}{2.56500-2.43675+2.43675(921.8)} ,$$

or

$$0.65 < R < 1.00 \quad .$$

Note: Since $2n_o-2x_o = 0.2565$ and $2x_o = 4.8735$, we approximate $F_{0.025;0.2565,4.8735} \approx F_{0.025;1,5} = 10.01$ Similarly, $F_{0.025;4.8735,0.2565} \approx F_{0.025;5,1} = 921.8$ Thus, the above interval is only an approximate 95% probability-interval estimate.

f. General references:

(i) N. R. Mann, R. E. Schafer, and N. D. Singpurwalla, Methods for Statistical Analysis of Reliability and Life Data (John Wiley and Sons, New York, 1974).

(ii) M. S. Waterman, H. F. Martz, and R. A. Waller, "Fitting Beta Prior Distributions in Bayesian Reliability Analysis," Los Alamos Scientific Laboratory report LA-6395-MS (June 1976).

III.A.3. Attribute test data: Beta with known beta prior - posterior analysis.

a. Point estimator of R:

$$\tilde{R} = \frac{x + x_o}{n + n_o} \qquad \text{(LA-6126, Eq. 10)}$$

b. Two-sided probability-interval estimator:

$$100(1-\gamma)\%\text{TBPI}(n): \frac{x+x_o}{x+x_o+(n+n_o-x-x_o)F_{\gamma/2;2n+2n_o-2x-2x_o,2x+2x_o}} \leq R$$

(LA-6126, Eq. 11)

$$\leq \frac{(x+x_o)F_{\gamma/2;2x+2x_o,2n+2n_o-2x-2x_o}}{n+n_o-x-x_o+(x+x_o)F_{\gamma/2;2x+2x_o,2n+2n_o-2x-2x_o}} .$$

c. Lower one-sided probability-interval estimator:

$100(1-\gamma)\%\text{LBPI}(n)$: $R >$

$$\frac{x+x_o}{x+x_o+(n+n_o-x-x_o)F_{\gamma;2n+2n_o-2x-2x_o,2x+2x_o}} .$$

(LA-6126, Eq. 12)

d. Remarks:

The results in a,b, and c above assume that both the sample data, n and x, and the prior values, n_o and x_o, are available. The n test items are assumed to survive independently with probability R.

e. Example:

Consider Data Set 1 in Appendix A. Suppose that the PWR annual prior mean pump reliability is believed to be 0.95. Further, suppose that it is believed that there is only a 5% chance that the annual pump reliability is below 0.70. From Reference (ii) below, this prior belief is consistent with a beta prior distribution with $x_0 = 2.43675$ and $n_0 = 2.56500$. In 1972, six pump failures occurred out of 400 pumps. Thus $x=394$ and $n=400$. The updated (posterior) estimated annual pump reliability is $\tilde{R} = (394+2.43675)/(400+2.56500) = 0.98$. A two-sided 95% probability-interval estimate of this reliability is

$$\frac{394+2.43675}{394+2.43675+(400+2.56500-394-2.43675)(1.95)} \leq R$$

$$\leq \frac{(394+2.43675)(2.73)}{400+2.56500-394-2.43675+(394+2.43675)(2.73)} ,$$

or

$$0.97 \leq R \leq 0.99 \quad .$$

Note: Since $(2n+2n_0-2x-2x_0) = 12.2565$ and $(2x+2x_0) = 792.8735$, we approximate $F_{0.025;12.2565,792.8735} \approx F_{0.025;12,\infty} = 1.95$. Similarly, $F_{0.025;792.8735,2.2565} \approx F_{0.025;\infty,12} = 2.73$. Thus, the above interval is only an approximate 95% probability-interval estimate.

f. General references:

(i) N. R. Mann, R. E. Schafer, and N. D. Singpurwalla, Methods for Statistical Analysis of Reliability and Life Data (John Wiley and Sons, New York 1974).

(ii) M. S. Waterman, H. F. Martz, and R. A. Waller, "Fitting Beta Prior Distributions in Bayesian Reliability Analysis," Los Alamos Scientific Laboratory report LA-6395-MS (June, 1976).

III.A.4. Attribute test data: Empirical Bayes--estimated beta parameters.

a. Point estimators of R:

$$\tilde{R}_E = \hat{x}_o/\hat{n}_o \text{ (prior analysis),}$$ (analogous to LA-6126, Eq. 7)

$$= \frac{x+\hat{x}_o}{n+\hat{n}_o} \text{ (posterior analysis),}$$ (analogous to LA-6126, Eq. 10)

where

$$\hat{x}_o = \hat{n}_o(\Sigma\hat{R}_j)/N \text{ ,}$$ (LA-6126, Eq. 14)

and

$$\hat{n}_o = \frac{N^2(\Sigma\hat{R}_j - \Sigma\hat{R}_j^2)}{N(N\Sigma\hat{R}_j^2 - K\Sigma\hat{R}_j) - (N-K)(\Sigma\hat{R}_j)^2} \text{ ,}$$

(LA-6126, Eq. 13)

if the result is positive, otherwise

$$\hat{n}_o = \left(\frac{N-1}{N}\right) \frac{N\Sigma\hat{R}_j - (\Sigma\hat{R}_j)^2}{N\Sigma\hat{R}_j^2 - (\Sigma\hat{R}_j)^2} - 1 \text{ .}$$ (LA-6126, Eq. 15)

b. Two-sided probability-interval estimator:

$$100(1-\gamma) \text{ TEBPI(0): } \frac{\hat{x}_o}{\hat{x}_o + (\hat{n}_o - \hat{x}_o)F_{\gamma/2;2\hat{n}_o-2\hat{x}_o,2\hat{x}_o}} < R$$

(prior analysis)

(analogous to LA-6126, Eq. 8)

$$< \frac{\hat{x}_o F_{\gamma/2;2\hat{x}_o,2\hat{n}_o-2\hat{x}_o}}{\hat{n}_o - \hat{x}_o + \hat{x}_o F_{\gamma/2;2\hat{x}_o,2\hat{n}_o-2\hat{x}_o}} \text{ .}$$

$$100(1-\gamma)\%\mathrm{TEBPI}(n): \frac{x+\hat{x}_o}{x+\hat{x}_o+(n+\hat{n}_o-x-\hat{x}_o)F_{\gamma/2;2n+2\hat{n}_o-2x-2\hat{x}_o,2x+2\hat{x}_o}} \le R$$

(analogous to
LA-6126, Eq. 11)

$$\le \frac{(x+\hat{x}_o)F_{\gamma/2;2x+2\hat{x}_o,2n+2\hat{n}_o-2x-2\hat{x}_o}}{n+\hat{n}_o-x-\hat{x}_o+(x+\hat{x}_o)F_{\gamma/2;2x+2\hat{x}_o,2n+2\hat{n}_o-2x-2\hat{x}_o}}.$$

(posterior analysis)

c. Lower one-sided probability-interval estimators:

$$100(1-\gamma)\%\mathrm{LEBPI}(0):\ R \ge \frac{\hat{x}_o}{\hat{x}_o+(\hat{n}_o-\hat{x}_o)F_{\gamma;2\hat{n}_o-2\hat{x}_o,2\hat{x}_o}},$$

(prior analysis)
(analogous to
LA-6126, Eq. 9)

$$100(1-\gamma)\%\mathrm{LEBPI}(n):R \ge \frac{x+\hat{x}_o}{x+\hat{x}_o+(n+\hat{n}_o-x-\hat{x}_o)F_{\gamma;2n+2\hat{n}_o-2x-2\hat{x}_o,2x+2\hat{x}_o}}.$$

(posterior analysis)
(analogous to
LA-6126, Eq. 12)

d. Remarks:

The results in a,b, and c assume a beta prior model with unknown parameters, n_o and x_o. These parameters are estimated from past test data in which, for a series of N independent past life tests on similar test units, x_j survivors have been observed in a sample of size n_j in the j^{th} test $(j = 1,2,\ldots,N)$. It is noted that all life tests must be of approximately the same duration. In a, all summations range from j=1 to j=N, $\hat{R}_j = x_j/n_j$, and $K=\Sigma(1/n_j)$.

e. Example:

Consider Data Set 1 in Appendix A. We consider each of the eight PWRs listed to be an independent life test experiment. Now $\Sigma\hat{R}_j = 7.88$, $\Sigma\hat{R}_j^2 = 7.7648$ $(\Sigma\hat{R}_j)^2=62.0944$, $K = 0.16$ from which

$$\hat{n}_o = \frac{64(7.88-7.7648)}{8[8(7.7648)-0.16(7.88)]-(8-0.16)62.0944} = 181.15\ ,$$

and

$$\hat{x}_o = \frac{181.15(7.88)}{8} = 178.43\ .$$

Performing a prior analysis on the data in Data Set 1, the 1972 pump reliability is estimated to be $\tilde{R}_E = 178.43/181.15 = 0.98$ and a two-sided 95% probability-interval estimate is

$$\frac{178.43}{178.43+(181.15-178.43)(2.57)} \le R \le \frac{178.43(6.02)}{181.15-178.43+178.43(6.02)},$$

or

$$0.96 \le R \le 1.00.$$

<u>Note</u>: $F_{0.025;5.44,356.86} \approx F_{0.025;5,\infty} = 2.57$ and $F_{0.025;356.86,5.44} \approx F_{0.025;\infty,5} = 6.02$. Thus the above interval is only an approximate 95% probability-interval estimate.

III.A.5. Attribute test data: empirical Bayes -- estimated (empirical) prior

a. Point estimators of R:

$$\tilde{R}_E = \sum_{j=1}^{N} \hat{R}_j / N \quad \text{(prior analysis)}$$

$$= \frac{\sum_{j=1}^{N} \hat{R}_j^{\,x+1} (1-\hat{R}_j)^{n-x}}{\sum_{j=1}^{N} \hat{R}_j^{x} \; (1-\hat{R}_j)^{n-x}} \text{ .(posterior analysis)}$$

(LA-6126, Eq. 18)

b. Two-sided probability-interval estimators: (see LA-6126, Eq. 20, except that the empirical prior distribution is used instead of the empirical posterior distribution) (prior analysis)

(see LA-6126, Eq. 20) (posterior analysis)

c. Lower one-sided probability-interval estimators: (see LA-6126, Eq. 21 except that the empirical prior distribution is used instead of the empirical posterior distribution) (prior analysis)

(see LA-6126, Eq. 21) (posterior analysis)

d. Remarks:

The results in a, b, and c do not require an assumption regarding a type of prior model, such as a beta distribution. The complete prior distribution is empirically estimated from past test data in which, for a series of N independent past life tests on similar test units, x_i survivors have been observed in a sample of size n_i in the i^{th} test ($i=1,2,\ldots,N$). It is noted that all life tests must be of approximately the same duration.

The interval estimates should be used with caution. The true coverage probability is likely to be less than that desired unless the number of past data sets $\hat{R}_1,\ldots,\hat{R}_N$ is greater than 25.

e. Example:

Consider Data Set 1 in Appendix A. Suppose we desire to estimate the 1972 pump reliability of the Indian Point 1 PWR. For Indian Point 1, we have n=50 and x=49. Thus

$$\tilde{R}_E = \frac{4(1.00)^{50}(0)^1 + 3(0.98)^{50}(0.02)^1 + (0.94)^{50}(0.02)^1}{4(1.00)^{49}(0)^1 + 3(0.98)^{49}(0.02)^1 + (0.94)^{49}(0.02)^1} = 0.978.$$

f. General references:

(i) G. H. Lemon, "An Empirical Bayes Approach to Reliability," IEEE Trans. Rel. R-21, 155-158 (August 1972).

(ii) H. F. Martz, Jr., "Pooling Life-Test Data by Means of the Empirical Bayes Method," IEEE Trans. Rel. R-24, 27-30 (April 1975).

III.B. Variables Test Data--Exponential Model

The results in this section assume that the time-to-failure of an item is distributed as an exponential variable. For a general discussion of the exponential distribution, see Sec. II.A of Waller and Martz,[1] or the references listed with each subsection below. The test data are assumed to arise from either an item-censored or time-truncated experiment. The item-censored experiment can be conducted either with or without the replacement of failed items as they occur. The time-truncated experiment assumes that failed items will be replaced as they occur. In addition, for the classical estimators, time-truncated testing without replacement is also considered. It is noted here that if the data arise from field use of an item, the situation will normally be with replacement of failed items and often will be a time-truncated experiment. For the exponential model considered here, the reliability is $R(t) = \exp(-\lambda t)$. Formulas are given in this section for estimating both $R(t)$ and the failure rate λ. As mentioned earlier, for the Bayes estimators a gamma prior distribution of λ is assumed.

III.B.1. Variables test data: classical procedure --reliability estimation

a. Point estimator of R(t):

$$\hat{R}(t) = \exp(-rt/T), \qquad \text{(LA-6003, Eq. 8)}$$

where T is the total test time as defined in Sec. I.A, and r is the observed number of failures.

b. Two-sided confidence-interval estimator:

$$100(1-\alpha)\%\text{TCI:} \quad \exp\left[-t\chi^2_{1-\alpha/2;2\nu_1}/(2T)\right] \le R(t) \le \exp\left[-t\chi^2_{\alpha/2;2\nu_2}/(2T)\right]. \qquad \text{(LA-6003, Eq. 9)}$$

c. Lower one-sided confidence-interval estimator:

$$100(1-\alpha)\%\text{LCI:} \quad R(t) \ge \exp\left[-t\chi^2_{1-\alpha;2\nu_1}/(2T)\right].$$

d. Remarks:

The value assigned to ν_1 depends upon the type of life test experiment and is given by

$$\nu_1 = r \qquad \text{(item-censored life test)}$$
$$ = r+1 \qquad \text{(time-truncated life test)}$$

while $\nu_2 = r$ regardless of the type of life test experiment. It is further noted that the confidence intervals for the case of time-truncated life testing have only approximately 100(1-α)% confidence associated with them. For exact-interval estimates see Ref. (iii) below. Also, for the case of time-truncated testing without replacement,

$$T = \sum_{i=1}^{r} t_i + (n-r)t_o,$$

e. Example:

Consider Data Set 1 in Appendix A. The test data in Data Set 1 can be considered to have been obtained from a time-truncated experiment with replacement in which $n = 400$, $r = 6$, and $T = 3\ 504\ 000$. The pump reliability in PWRs for 1972 is estimated to be

$$\hat{R}(8760) = \exp[-6(8760)/3\ 504\ 000] = 0.985.$$

An approximate 95% two-sided confidence-interval estimate of this reliability is

$$\exp[-8760(26.119)/7\ 008\ 000] \leq R(8760)$$

$$\leq \exp[-8760(4.404)/7\ 008\ 000],$$

or

$$0.97 < R(8760) < 0.99 \quad .$$

f. General references:

(i) N. R. Mann, R. E. Schafer, and N. D. Singpurwalla, Methods for Statistical Analysis of Reliability and Life Data (John Wiley and Sons, New York, 1974).

(ii) I. Miller and J. E. Fruend, Probability and Statistics for Engineers (Prentice-Hall, New Jersey, 1965).

(iii) B. A. Kozlov and I. A. Ushakov, Reliability Handbook (Ed. by L. H. Koopmans and J. Rosenblatt) (Holt, Rinehart, and Winston, New York, 1970).

III.B.2. Variables test data: classical procedure -- failure-rate estimation

a. Point estimator of λ:

$$\hat{\lambda} = r/T.$$

b. Two-sided confidence-interval estimator:

$$100(1-\alpha)\%\text{TCI:} \quad \chi^2_{\alpha/2;2\nu_2}/(2T) \leq \lambda \leq \chi^2_{1-\alpha/2;2\nu_1}/(2T).$$

c. Upper one-sided confidence-interval estimator:

$$100(1-\alpha)\%\text{UCI:} \quad \lambda \leq \chi^2_{1-\alpha;2\nu_1}/(2T).$$

d. Remarks:

The value assigned to ν_1 depends on the type of experiment and is given by

$$\nu_1 = r \quad \text{(item-censored life test)}$$
$$\quad = r+1 \quad \text{(time-truncated life test),}$$

while $\nu_2 = r$ regardless of the type of life test experiment. It is further noted that the confidence intervals for the case of time-truncated life testing have only approximately $100(1-\alpha)\%$ confidence associated with them. For exact-interval estimates see Ref. (iii) below. Also, for the case of time-truncated testing without replacement,

$$T = \sum_{i=1}^{r} t_i + (n-r)t_o.$$

e. Example:

Consider Data Set 1 in Appendix A. The variables test data in Data Set 1 is considered to have been obtained from a time-truncated experiment with replacement in which n = 400, r = 6, and T = 3 504 000. Based on the 1972 data for the eight PWRs, the pump failure-rate associated with the failure to run normal mode is estimated to be

$$\hat{\lambda} = 6/3\ 504\ 000 = 1.7 \times 10^{-6} \text{ f/h}.$$

An approximate 95% two-sided confidence-interval estimate of this failure rate is

$$(4.404)/7\ 008\ 000 < \lambda < (26.119)/7\ 008\ 000,$$

or

$$6.3 \times 10^{-7} \text{ f/h} < \lambda < 3.7 \times 10^{-6} \text{ f/h} \quad .$$

f. General references:

(i) N. R. Mann, R. E. Schafer, and N. D. Singpurwalla, Methods for Statistical Analysis of Reliability and Life Data (John Wiley and Sons, New York, 1974).

(ii) I. Miller and J. E. Freund, Probability and Statistics for Engineers (Prentice-Hall, New Jersey, 1965).

(iii) B. A. Kozlov and I. A. Ushakov, Reliability Handbook (Ed. by L. H. Koopmans and J. Rosenblatt) (Holt, Rinehart, and Winston, New York, 1970).

III.B.3. Variables test data: Bayes with known gamma prior-reliability estimation -- prior analysis

a. Point estimator of R(t):

$$\tilde{R}_o(t) = (1+\beta_o t)^{-\alpha_o} . \qquad \text{(LA-6003, Eq. 17)}$$

b. Two-sided probability-interval estimator:

$$100(1-\lambda)\%\text{TBPI}(0): \quad \exp\left[-\beta_o t \chi^2_{1-\gamma/2;2\alpha_o}/2\right] \leq R(t)$$

$$\leq \exp\left[-\beta_o t \chi^2_{\gamma/2;2\alpha_o}/2\right] . \qquad \text{(LA-6003, Eq. 18)}$$

c. Lower one-sided probability-interval estimator:

$$100(1-\gamma)\%\text{LBPI}(0): \quad R(t) \geq \exp\left[-\beta_o t \chi^2_{1-\gamma;2\alpha_o}/2\right] .$$

d. Remarks:

No current test data is incorporated in the above estimators. They depend only upon the prior parameter values, α_o and β_o. For guidance in determining α_o and β_o see Sec. II.C.

e. Example:

Prior to obtaining the data in Data Set 1, suppose we desire a prior analysis of the annual pump reliability in PWRs. From past experience suppose that we subjectively assign the values 0.50 and 6.0 x 10^{-6}f/h to α_o, and β_o, respectively. The prior 1972 pump reliability is estimated to be

$$\tilde{R}_o(8760) = [1+6.0\text{x}10^{-6}(8760)]^{-0.50} = 0.97,$$

and a two-sided 95% probability-interval estimate of this reliability is

$$\ldots 0\text{x}10^{-6}(8760)(5.024)/2] \leq R(8760)$$

$$< \exp[-6.0\times10^{-6}(8760)(0.001)/2] ,$$

or

$$0.88 < R(8760) < 1.00 \quad .$$

f. General reference:

(i) N. R. Mann, R. E. Schafer, and N. D. Singpurwalla, Methods for Statistical Analysis of Reliability and Life Data (John Wiley and Sons, New York, 1974).

III.B.4. Variables test data: Bayes with known gamma prior-reliability estimation -- posterior analysis

a. Point estimator of R(t):

$$\tilde{R}_r(t) = \left(\frac{\beta_o T+1}{\beta_o T+\beta_o t+1}\right)^{r+\alpha_o} . \qquad \text{(LA-6003, Eq. 19)}$$

b. Two-sided probability-interval estimator:

$$100(1-\gamma)\%\text{TBPI}(r): \quad \exp\left[-\beta_o t\chi^2_{1-\gamma/2;2r+2\alpha_o}/(2\beta_o T+2)\right]$$

$$< R(t) < \exp\left[-\beta_o t\chi^2_{1-\gamma/2;2r+2\alpha_o}/(2\beta_o T+2)\right] .$$

(Reduces to Eq. 20 of LA-6003 when r=n.)

c. Lower one-sided probability-interval estimator:

$$100(1-\gamma)\%\text{LBPI}(r): \quad R(t) > \exp\left[-\beta_o t\chi^2_{1-\gamma;2r+2\alpha_o}/(2\beta_o T+2)\right] .$$

d. Remarks:

T in the above expressions is the total test time which is defined in Sec. I.A. The above estimators are appropriate for item-censored life testing, either with or without replacement, and for time-truncated testing with replacement.

e. Example:

Consider Data Set 1 in Appendix A. Suppose we desire a posterior analysis of the annual pump reliability of the PWRs based on the 1972 failure data reported in Data Set 1. From past experience suppose that we assign the values 0.50 and 6.0 x 10^{-6}f/h to α_o and β_o, respectively. In 1972, six pump failures occurred during a total of approximately 3 504 000 operating for the eight PWRs. The testing situation is considered to be time-truncated life testing with replacement. Thus, r = 6 and T = 400 (8760) = 3 504 000. The posterior estimated annual pump reliability is

$$\tilde{R}_6(8760) = \left[\frac{6.0\text{x}10^{-6}(3\ 504\ 000) + 1}{6.0\text{x}10^{-6}(3\ 504\ 000) + 6.0\text{x}10^{-6}(8760) + 1}\right]^{6 + 0.5}$$

$$= 0.98\ ,$$

and a two-sided 95% probability-interval estimate of the reliability is

$$\exp\left\{-(6.0\text{x}10^{-6})(8760)(24.736)/[(2)(6.0\text{x}10^{-6})(3\ 504\ 000) + 2]\right\}$$

$$< R(8760) < \exp\left\{-(6.0\text{x}10^{-6})(8760)(5.009)/[(2)(6.0\text{x}10^{-6})(3\ 504\ 000)+2]\right\}\ ,$$

or

$$0.97 < R(8760) < 0.99\ .$$

f. General reference:

(i) N. R. Mann, R. E. Schafer, and N. D. Singpurwalla, Methods for Statistical Analysis of Reliability and Life Data (John Wiley and Sons, New York, 1974).

III.B.5. Variables test data: Bayes with known gamma prior - failure-rate estimation -- prior analysis

a. Point estimator of λ:

$$\tilde{\lambda}_o = \alpha_o \beta_o .$$

b. Two-sided probability-interval estimator:

$$100(1-\gamma)\%\text{TBPI}(0): \quad \beta_o \chi^2_{\gamma/2;2\alpha_o}/2 < \lambda < \beta_o \chi^2_{1-\gamma/2;2\alpha_o}/2 .$$

c. Upper one-sided probability-interval estimator:

$$100(1-\gamma)\%\text{UBPI}(0): \quad \lambda < \beta_o \chi^2_{1-\gamma;2\alpha_o}/2 .$$

d. Remarks:

No current test data are incorporated in the above estimators. They depend only on the prior parameter values α_o and β_o. For guidance in determining α_o and β_o, see Sec. II.C.

e. Example:

Prior to obtaining the data in Data Set 1 (Appendix A), suppose we desire a prior analysis of the pump failure rate in PWRs (according to the failure to run normal mode). From past experience suppose that we assign the values 0.50 and 6.0×10^{-6}f/h to α_o and β_o, respectively. The pump failure rate is estimated to be $\tilde{\lambda}_o = 0.50\ (6.0 \times 10^{-6}\text{f/h}) = 3.0 \times 10^{-6}$f/h and an upper 95% probability-interval estimate of the failure rate is

$$\lambda < (6.0 \times 10^{-6})(3.841)/2 = 11.5 \times 10^{-6}\text{f/h}.$$

f. General reference:

(i) N. R. Mann, R. E. Schafer, and N. D. Singpurwalla, Methods for Statistical Analysis of Reliability and Life Data (John Wiley and Sons, New York, 1974)

III.B.6. **Variables test data: Bayes with known gamma prior - failure - rate estimation -- posterior analysis**

a. Point estimator of λ:

$$\tilde{\lambda}_r = \frac{\beta_o(r+\alpha_o)}{\beta_o T + 1} .$$

b. Two-sided probability-interval estimator:

$$100(1-\gamma)\%\text{TBPI}(r): \quad \beta_o \chi^2_{\gamma/2;2r+2\alpha_o}/(2\beta_o T+2) < \lambda$$

$$< \beta_o \chi^2_{1-\gamma/2;2r+2\alpha_o}/(2\beta_o T+2) .$$

c. Upper one-sided probability-interval estimator:

$$100(1-\gamma)\%\text{UBPI}(r): \quad \lambda < \beta_o \chi^2_{1-\gamma;2r+2\alpha_o}/(2\beta_o T+2) .$$

d. Remarks:

T in the above expressions is the total test time which is defined in Sec. I.A. The above estimators are appropriate for item-censored life testing either with or without replacement and for time-truncated testing with replacement.

e. Example:

Consider Data Set 1 in Appendix A. Suppose we desire a posterior analysis of the pump failure rate in PWRs (according to the failure to run normal mode) based on the 1972 failure data reported. From past experience suppose that we assign the values 0.50 and 6.0×10^{-6}f/h to α_o and β_o, respectively. In 1972, six pump failures occurred during a total of approximately 3 504 000 operating hours for the eight PWRs. The testing situation is considered to be time-truncated lifetesting with replacement. Thus r = 6 and T = 400(8760) = 3 504 000. The posterior estimated failure rate is

$$\tilde{\lambda}_6 = \frac{6.0 \times 10^{-6}(6+0.5)}{6.0 \times 10^{-6}(3\ 504\ 000) + 1} = 1.8 \times 10^{-6}\ \text{f/h},$$

and an upper 95% probability-interval estimate of the failure rate is

$$\lambda < (6.0\text{x}10^{-6})(22.362)/[2(6.0\text{x}10^{-6})(3\ 504\ 000) + 2]$$
$$= 3.0 \times 10^{-6}\text{f/h}.$$

f. General reference:

(i) N. R. Mann, R. E. Schafer, and N. D. Singpurwalla, Methods for Statistical Analysis of Reliability and Life Data (John Wiley and Sons, New York, 1974).

III.B.7. Variables test data: empirical Bayes - reliability estimation -- estimated gamma parameters

a. Point estimators of R(t):

$$\tilde{R}_E(t) = (1 + \hat{\beta}_o t)^{-\hat{\alpha}_o} \qquad \text{(prior analysis)}$$

$$= \left(\frac{\hat{\beta}_o T + 1}{\hat{\beta}_o T + \hat{\beta}_o t + 1}\right)^{r + \hat{\alpha}_o}, \text{ (posterior analysis)}$$

where

$$\hat{\alpha}_o = N\bar{\lambda}/(N_1\hat{\beta}_o),$$

and

$$\hat{\beta}_o = (N_1^2 m^2 - N_2 N\bar{\lambda}^2)/(N_1 N_2 \bar{\lambda}) .$$

In these expressions

$$N_1 = \sum\left(\frac{r_j}{r_j - 1}\right); \; N_2 = \sum\left[\frac{r_j^{\,2}}{(r_j - 1)(r_j - 2)}\right]; \; \bar{\lambda} = \Sigma\hat{\lambda}_j/N;$$

$$m^2 = \Sigma\hat{\lambda}_j^2/N \; ; \; \hat{\lambda}_j = r_j/T_j .$$

If $r_1 = r_2 \cdots = r_N = r$, these expressions simplify to become

$$\hat{\alpha}_o = \frac{(r - 1)\bar{\lambda}^2}{(r - 2)m^2 - (r - 1)\lambda^{-2}}$$

$$\hat{\beta}_o = \frac{(r - 2)m^2 - (r - 1)\bar{\lambda}^2}{r\bar{\lambda}} .$$

b. Two-sided probability-interval estimators:

$$100(1-\gamma)\%\text{TEBPI}(0): \quad \exp\left[-\hat{\beta}_o t\chi^2_{1-\gamma/2;2\hat{\alpha}_o}/2\right] < R(t)$$

$$< \exp\left[-\hat{\beta}_o t\chi^2_{\gamma/2;2\hat{\alpha}_o}/2\right]. \text{ (prior analysis)}$$

$$100(1-\gamma)\%\text{TEBPI}(r): \quad \exp\left[-\hat{\beta}_o t\chi^2_{1-\gamma/2;2r+2\hat{\alpha}_o}/(2\hat{\beta}_o T + 2)\right]$$

$$\leq R(t) \leq \exp\left[-\hat{\beta}_o t\chi^2_{\gamma/2;2r+2\hat{\alpha}_o}/(2\hat{\beta}_o T + 2)\right].$$

(posterior analysis)

c. Lower one-sided probability-interval estimators:

$$100(1-\gamma)\%\text{LEBPI}(0): \quad R(t) > \exp\left[-\hat{\beta}_o t\chi^2_{1-\gamma;2\hat{\alpha}_o}/2\right].$$

(prior analysis)

$$100(1-\gamma)\%\text{LEBPI}(r): \quad R(t) > \exp\left[-\hat{\beta}_o t\chi^2_{1-\gamma;2r+2\hat{\alpha}_o}/(2\hat{\beta}_o T + 2)\right].$$

(posterior analysis)

d. Remarks:

T in the above expressions is the total test time which is defined in Sec. I.A. The above estimators are appropriate for item-censored life testing either with or without replacement and for time-truncated testing with replacement.

The results in a,b, and c assume a gamma prior model on λ but with unknown parameters α_o and β_o. These parameters are estimated from a set of independent test results on the same or similar items. It is assumed that r_j failures are observed in total test T_j in the j^{th} experiment ($j = 1,2,\cdots,N$). Thus there exists a set of independent failure-rate estimates $\hat{\lambda}_1, \hat{\lambda}_2, \cdots, \hat{\lambda}_N$ of the item under consideration, where $\hat{\lambda}_j = r_j/T_j$. In a above, all summations range from $j = 1$ to $j = N$. It is further required that $r_j > 2$ in each related experiment.

e. Example:

Let us use the set of 13 pump failure-rate estimates in Data Set 2 (Appendix B) for estimating α_o and β_o. Unfortunately, the number of failures on which each of the failure-rate estimates is based is unknown. It will be arbitrarily assumed that each estimate is based on 10 failures. Thus $r_j = r = 10$, $j = 1,2,\cdots,13$. Now $\bar{\lambda} = 15.24 \times 10^{-6}$ and $m^2 = 1543.23 \times 10^{-12}$. Thus

$$\hat{\alpha}_o = \frac{(10 - 1)(15.24 \times 10^{-6})^2}{(10 - 2)(1543.23 \times 10^{-12}) - (10 - 1)(15.24 \times 10^{-6})^2} = 0.20$$

$$\hat{\beta}_o = \frac{(10 - 2)(1543.23 \times 10^{-12}) - (10 - 1)(15.24 \times 10^{-6})^2}{10(15.24 \times 10^{-6})}$$

$$= 67.29 \times 10^{-6} \text{f/h.}$$

Suppose we now desire a posterior analysis of the annual pump reliability of the PWRs based on the 1972 failure data reported in Data Set 1 (Appendix A). The testing situation is considered to be time-truncated testing with replacement. Thus $r = 6$ and $T = 400(8760)$ $= 3\ 504\ 000$. The posterior estimated annual pump reliability is

$$\tilde{R}_E(8760) =$$

$$\left[\frac{(67.29 \times 10^{-6})(3\ 504\ 000) + 1}{(67.29 \times 10^{-6})(3\ 504\ 000) + (67.29 \times 10^{-6})(8760) + 1}\right]^{6 + 0.20} = 0.985 .$$

A two-sided 95% probability-interval estimate of the reliability is

$$\exp\left\{-(67.29 \times 10^{-6})(8760)(23.337)/\right.$$
$$\left.[(2)(67.29 \times 10^{-6})(3\ 504\ 000) + 2]\right\}$$
$$< R(8760) < \exp\left\{-(67.29 \times 10^{-6})(8760)(4.404)/\right.$$
$$\left.[(2)(67.29 \times 10^{-6})(3\ 504\ 000) + 2]\right\},$$

or

$$0.97 < R(8760) < 0.99.$$

Note: Since $2r + 2\hat{\alpha}_0 = 12.4$, we approximate $\chi^2_{0.975;12.4} \approx \chi^2_{0.975;12} = 23.337$. Similarly, $\chi^2_{0.025;12.4}$ $\chi^2_{0.025;12} = 4.404$. Thus, the above interval estimate is only an approximate 95% probability-interval estimate.

f. General reference:

(i) N. R. Mann, R. E. Schafer, and N. D. Singpurwalla, Methods for Statistical Analysis of Reliability and Life Data (John Wiley and Sons, New York, 1974).

III.B.8. Variables test data: empirical Bayes - reliability estimation -- estimated (empirical) prior

a. Point estimators of R(t):

$$\tilde{R}_E(t) = \exp\left(-t \sum_{j=1}^{N} \hat{\lambda}_j / N\right) \qquad \text{(prior analysis)}$$

$$= \frac{\sum_{j=1}^{N} \hat{\lambda}_j^r \exp[-\hat{\lambda}_j (t + T)]}{\sum_{j=1}^{N} \hat{\lambda}_j^r \exp[-\hat{\lambda}_j T]}, \qquad \text{(posterior analysis)}$$

b. Two-sided probability-interval estimators: (see Reference (iii) below)

c. Lower one-sided probability-interval estimators: (see Reference (iii) below)

d. Remarks:

T in the expression in a is the total test time which is defined in Sec. I.A. The above estimators are appropriate for item-censored life testing either with or without replacement and for time-truncated testing with replacement.

The results in a above do not require an assumption regarding a type of prior model, such as a gamma model. Rather, the complete prior distribution is empirically estimated from a set of independent test results on the same or similar items. It is assumed that r_j failures are observed in total test time T_j in the j^{th} experiment ($j = 1,2,\cdots,N$). Thus there exists a set of independent failure-rate estimates $\hat{\lambda}_1, \hat{\lambda}_2, \cdots, \hat{\lambda}_N$ of the item under consideration, where $\hat{\lambda}_j = r_j/T_j$.

e. Example:

Let us use the set of 13 pump failure-rate estimates in Data Set 2 (Appendix B) to form a posterior estimate of the annual pump reliability in PWRs based on the 1972 failure data reported in Data Set 1 (Appendix A). The testing situation is considered to be time-truncated testing with replacement. Thus r = 6 and T = 400(8760) = 3,504,000. The posterior estimated annual pump reliability is

$$\tilde{R}_E(8760) = \frac{(1.3 \times 10^{-5})^6 \exp[-(1.3 \times 10^{-5})(8760 + 3\,504\,000)]}{(1.3 \times 10^{-5})^6 \exp[-(1.3 \times 10^{-5})(3\,504\,000)]}$$

$$\frac{+(3.0 \times 10^{-6})^6 \exp[-(3.0 \times 10^{-6})(8760 + 3\,504\,000)]}{+(3.0 \times 10^{-6})^6 \exp[-(3.0 \times 10^{-6})(3\,504\,000)]}$$

$$\frac{+ \cdots +(4.0 \times 10^{-6})^6 \exp[-(4.0 \times 10^{-6})(8760 + 3\,504\,000)]}{+ \cdots +(4.0 \times 10^{-6})^6 \exp[-(4.0 \times 10^{-6})(3\,504\,000)]}$$

$$= 0.98 \; .$$

f. General references:

(i) G. H. Lemon, "An Empirical Bayes Approach to Reliability," IEEE Trans. Rel. R-21, 155-188 (August 1972).

(ii) H. F. Martz, Jr., "Pooling Life-Test Data by Means of the Empirical Bayes Method," IEEE Trans. Rel. R-24, 27-30 (April 1975).

(iii) M. G. Lian, "Bayes and Empirical Bayes Estimation of Reliability for the Weibull Model," unpublished Ph.D. dissertation, Texas Tech University, University Microfilms, Ann Arbor, Michigan (1975).

III.B.9. Variables test data: empirical Bayes - failure-rate estimation -- estimated gamma parameters

a. Point estimators of λ:

$$\tilde{\lambda}_E = \hat{\alpha}_o \hat{\beta}_o \ , \qquad \text{(prior analysis)}$$

$$= \frac{\hat{\beta}_o (r + \hat{\alpha}_o)}{\hat{\beta}_o T + 1} \ , \qquad \text{(posterior analysis)}$$

where

$$\hat{\alpha}_o = N\bar{\lambda}/(N_1 \hat{\beta}_0) \ ,$$

and

$$\hat{\beta}_o = (N_1^2 m^2 - N_2 N \bar{\lambda}^2)/(N_1 N_2 \bar{\lambda}) \ .$$

In these expressions

$$N_1 = \sum \left(\frac{r_j}{r_j - 1} \right) \ ; \ N_2 = \sum \left[\frac{r_j^{\ 2}}{(r_j - 1)(r_j - 2)} \right] \ ; \ \bar{\lambda} = \sum \hat{\lambda}_j / N;$$

$$m^2 = \Sigma \hat{\lambda}_j^2 / N \ ; \ \hat{\lambda}_j = r_j / T_j \ .$$

If $r_1 = r_2 = \cdots = r_N = r$, these expressions simplify to become

$$\hat{\alpha}_o = \frac{(r - 1)\bar{\lambda}^2}{(r - 2)m^2 - (r - 1)\bar{\lambda}^2} \ ,$$

$$\hat{\beta}_o = \frac{(r - 2)m^2 - (r - 1)\bar{\lambda}^2}{r\bar{\lambda}} \ .$$

b. Two-sided probability-interval estimators:

$$100(1-\gamma)\%TEBPI(0): \quad \hat{\beta}_o \chi^2_{\gamma/2;2\hat{\alpha}_o}/2 < \lambda < \hat{\beta}_o \chi^2_{1-\gamma/2;2\hat{\alpha}_o}/2 .$$

(prior analysis)

$$100(1-\gamma)\%TEBPI(r): \quad \hat{\beta}_o \chi^2_{\gamma/2;2r+2\hat{\alpha}_o}/(2\hat{\beta}_o T + 2) < \lambda < \hat{\beta}_o \chi^2_{1-\gamma/2;2r+2\hat{\alpha}_o}/(2\hat{\beta}_o T + 2) .$$

(posterior analysis)

c. Upper one-sided probability-interval estimators:

$$100(1-\gamma)\%UEBPI(0): \quad \lambda < \hat{\beta}_o \chi^2_{1-\gamma;2\hat{\alpha}_o}/2 . \quad \text{(prior analysis)}$$

$$100(1-\gamma)\%UEBPI(r): \quad \lambda < \hat{\beta}_o \chi^2_{1-\gamma;2r+2\hat{\alpha}_o}/(2\hat{\beta}_o T + 2) .$$

(posterior analysis)

d. Remarks:

T in the above expressions is the total test time which is defined in Sec. I.A. The above estimators are appropriate for item-censored life testing either with or without replacement and for time-truncated testing with replacement.

The results in a,b, and c assume a gamma prior model on λ but with unknown parameters α_o and β_o. These parameters are estimated from a set of independent test results on the same or similar items. It is assumed that r_j failures are observed in total test time T_j in the j^{th} experiment $(j = 1,2,\cdots,N)$. Thus there exists a set of independent failure-rate estimates $\hat{\lambda}_1,\hat{\lambda}_2,\cdots,\hat{\lambda}_N$ of the item under consideration, where $\hat{\lambda}_j = r_j/T_j$. In a above, all summations range from j = 1 to $j = N$. It is further required that $r_j > 2$ in each related experiment.

e. Example:

Let us use the set of 13 pump failure-rate estimates in Data Set 2 (Appendix B) for estimating α_o and β_o. Unfortunately, the number of failures used to compute each failure-rate estimate is unknown. It will be arbitrarily assumed that each estimate is based upon 10 failures. Thus $r_j = r = 10$, $j = 1,2,\cdots,13$. Now $\bar{\lambda} = 15.24 \times 10^{-6}$ and $m^2 = 1543.23 \times 10^{-12}$. Thus

$$\hat{\alpha}_o = \frac{(10 - 1)(15.24 \times 10^{-6})^2}{(10 - 2)(1543.23 \times 10^{-12}) - (10 - 1)(15.24 \times 10^{-6})^2} = 0.20,$$

$$\hat{\beta}_o = \frac{(10 - 2)(1543.23 \times 10^{-12}) - (10 - 1)(15.24 \times 10^{-6})^2}{10(15.24 \times 10^{-6})},$$

$$= 67.29 \times 10^{-6} \text{f/h}.$$

Suppose we now desire a posterior estimate of the pump failure rate, for the failure to run normal mode, based on the 1972 failure data reported in Data Set 1 (Appendix A). The testing situation is considered to be time-truncated testing with replacement. Thus $r = 6$ and $T = 400(8760) = 3\ 504\ 000$. The posterior estimated failure-rate is

$$\tilde{\lambda}_E = \frac{67.29 \times 10^{-6}(6 + 0.20)}{67.29 \times 10^{-6}(3\ 504\ 000) + 1} = 1.8 \times 10^{-6} \text{f/h}.$$

and a two-sided 95% probability-interval estimate of the failure-rate is

$$(67.29 \times 10^{-6})(4.404)/[(2)(67.29 \times 10^{-6})(3\ 504\ 000) + 2]$$

$$< \lambda < (67.29 \times 10^{-6})(23.337)/[(2)(67.29 \times 10^{-6})(3\ 504\ 000) + 2],$$

or

$$6.3 \times 10^{-7} f/h < \lambda < 3.3 \times 10^{-6} f/h.$$

f. General reference:

(i) N. R. Mann, R. E. Schafer, and N. D. Singpurwalla, Methods for Statistical Analysis of Reliability and Life Data (John Wiley and Sons, New York, 1974).

III.B.10. Variables test data: empirical Bayes - failure-rate estimation - estimated (empirical) prior

a. Point estimators of λ:

$$\tilde{\lambda}_E = \sum_{j=1}^{N} \hat{\lambda}_j / N , \qquad \text{(prior analysis)}$$

$$= \frac{\sum_{j=1}^{N} \hat{\lambda}_j^{\,r+1} \exp(-\hat{\lambda}_j T)}{\sum_{j=1}^{N} \hat{\lambda}_j^{\,r} \exp(-\hat{\lambda}_j T)} . \qquad \text{(posterior analysis)}$$

b. Two-sided probability-interval estimators:

(see Reference (iii) below).

c. Upper one-sided probability-interval estimators:

(see Reference (iii) below).

d. Remarks:

T in the expression in a is the total test time which is defined in Sec. I.A. The above estimators are appropriate for item-censored life testing either with or without replacement and for time-truncated testing with replacement.

The results in a above do not require an assumption regarding a type of prior model, such as a gamma model. Rather, the complete prior distribution is empirically estimated from a set of independent failure-rate estimates $\hat{\lambda}_1, \hat{\lambda}_2, \cdots, \hat{\lambda}_N$ of the item under consideration, where $\hat{\lambda}_j = r_j / T_j$.

e. Example:

Let us use the set of 13 pump failure-rate estimates in Data Set 2 (Appendix B) to form a posterior estimate of the pump failure rate in PWRs based on the 1972 failure data reported in Data Set 1 (Appendix A). The testing situation is considered to be time-truncated testing with replacement. Thus r = 6 and T = 400(8760) = 3 504 000. The posterior estimated pump failure-rate is

$$\tilde{\lambda}_E = \frac{(1.3 \times 10^{-5})^7 \exp[-(1.3 \times 10^{-5})(3\ 504\ 000)] + (3.0 \times 10^{-6})^7 \exp[-(3.0 \times 10^{-6})(3\ 504\ 000)] + \cdots + (4.0 \times 10^{-6})^7 \exp[-(4.0 \times 10^{-6})(3\ 504\ 000)]}{(1.3 \times 10^{-5})^6 \exp[-(1.3 \times 10^{-5})(3\ 504\ 000)] + (3.0 \times 10^{-6})^6 \exp[-(3.0 \times 10^{-6})(3\ 504\ 000)] + \cdots + (4.0 \times 10^{-6})^6 \exp[-(4.0 \times 10^{-6})(3\ 504\ 000)]}$$

$$= 2.3 \times 10^{-6} \text{ f/h.}$$

f. General references:

(i) G. H. Lemon, "An Empirical Bayes Approach to Reliability," IEEE Trans. Rel. R-21, 155-158 (August 1972).

(ii) H. F. Martz, Jr., "Pooling Life-Test Data by Means of the Empirical Bayes Method," IEEE Trans. Rel R-24, 27-30 (April 1975).

(iii) M. G. Lian, "Bayes and Empirical Bayes Estimation of Reliability for the Weibull Model, unpublished Ph.D. dissertation, Texas Tech University, University Microfilms, Ann Arbor, Michigan (1975).

REFERENCES

1. R. A. Waller and H. F. Martz, Jr., "Bayesian Reliability Estimation: The State of the Art for the Time-Dependent Case." Los Alamos Scientific Laboratory report LA-6003 (October 1975).

2. H. F. Martz, Jr., and R. A. Waller, "The Basics of Bayesian Reliability Estimation from Attribute Test Data," Los Alamos Scientific Laboratory report LA-6126 (February 1976).

3. M. S. Waterman, H. F. Martz, and R. A. Waller, "Fitting Beta Prior Distributions in Bayesian Reliability Analysis," Los Alamos Scientific Laboratory report LA-6395-MS (June 1976).

4. H. F. Martz and M. Horita, "Fitting Gamma Prior Distributions in Bayesian Reliability Analysis," under preparation.

5. R. A. Waller, H. F. Martz, M. Horita, and M. S. Waterman, "Fitting Gamma Prior Distributions in Bayesian Reliability Analysis Based on Reliability Considerations," under preparation.

APPENDICES

A. Example Data Set 1

B. Example Data Set 2

C. Percentage Points of the F-Distribution

D. Percentiles of the χ^2 - Distribution

APPENDIX A

Example Data Set 1

The following data are taken from Appendix III (Failure-Data) of WASH-1400 "Reactor Safety Study", U. S. Nuclear Regulatory Commission, October 1975, pp. III 35-36. The data consist of the number of pump failures observed in 1972 in eight pressurized water reactors (PWR) in commercial operation in the U. S. The designated failure mode was failure to run normal.

i	PWR	n_i	r_i	x_i	T(h)	$\hat{R}_i$
1	Haddam Neck	50	0	50	438 000	1.00
2	Yankee Rowe	50	0	50	438 000	1.00
3	Indian Point 1	50	1	49	438 000	0.98
4	San Onofre 1	50	1	49	438 000	0.98
5	Ginna	50	0	50	438 000	1.00
6	Point Beach 1	50	0	50	438 000	1.00
7	Robinson 2	50	1	49	438 000	0.98
8	Palisades	50	3	47	438,000	0.94
	Total	400	6	394	3 504 000	7.88

Here n_i denotes the number of pumps in service and r_i represents the number of observed pump failures. Also, x_i denotes the number of survivors, T is the total test time for 1972, and $\hat{R}_i = X_i/n_i$.

APPENDIX B

Example Data Set 2

The following data are taken from Appendix III (Failure Data) of WASH-1400 "Reactor Safety Study", U. S. Nuclear Regulatory Commission, October 1972, pp. III 7-8. The data consist of failure-rate estimates of pumps for a failure to run normal mode from several published sources.

SOURCE*	FAILURE-RATE(f/h)
AVCO	1.3×10^{-5}
FARADA	3.0×10^{-6}
LMEC	1.4×10^{-4}
SRS	1.0×10^{-5}
HOLMES HN-190	3.0×10^{-6}
SHOP US-NUC	1.4×10^{-7}
BOURNE UK	2.0×10^{-6}
UNDERAKES (GERMAN)	1.0×10^{-5}
DAVIL	3.0×10^{-6}
EUROPE NUC AGENCY	1.0×10^{-6}
PUGH	3.0×10^{-6}
OTWAY	6.0×10^{-6}
PROCEEDINGS	4.0×10^{-6}

$$\bar{\lambda} = 15.24 \times 10^{-6}$$

$$S = 37.69 \times 10^{-6}$$

*Complete References Given in WASH-1400.

Here $\bar{\lambda}$ and S are the mean and standard deviation, respectively, of the 13 failure-rate estimates. Also, WASH-1400 gives the median failure-rate estimate as 3.0×10^{-5} and lower and upper bounds as 3×10^{-6} and 3×10^{-4}, respectively. These estimates are based on a log-normal distribution and coincide with the approximate 5% and 95% range end points. The interval is thus an approximate 90% probability interval.

APPENDIX C

PERCENTAGE POINTS OF THE F DISTRIBUTION

TABLE OF $F_{0.005;\nu_1,\nu_2}$

Degrees of freedom for the numerator (ν_1); degrees of freedom for the denominator (ν_2)

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9	10	12	15	20	25	30	40	60	120	
1	16211	20000	21615	22500	23056	23437	23715	23925	24091	24224	24426	24630	24836	24960	25044	25148	25253	25359	25463
2	198.5	199.0	199.2	199.2	199.3	199.3	199.4	199.4	199.4	199.4	199.4	199.4	199.4	199.5	199.5	199.5	199.5	199.5	199.5
3	55.55	49.80	47.47	46.19	45.39	44.84	44.43	44.13	43.88	43.69	43.39	43.08	42.78	42.59	42.47	42.31	42.15	41.99	41.83
4	31.33	26.28	24.26	23.15	22.46	21.97	21.62	21.35	21.14	20.97	20.70	20.44	20.17	20.00	19.89	19.75	19.61	19.47	19.33
5	22.78	18.31	16.53	15.56	14.94	14.51	14.20	13.96	13.77	13.62	13.38	13.15	12.90	12.76	12.66	12.53	12.40	12.27	12.15
6	18.63	14.54	12.92	12.03	11.46	11.07	10.79	10.57	10.39	10.25	10.03	9.81	9.59	9.45	9.36	9.24	9.12	9.00	8.88
7	16.24	12.40	10.88	10.05	9.52	9.16	8.89	8.68	8.51	8.38	8.18	7.97	7.75	7.62	7.53	7.42	7.31	7.19	7.08
8	14.69	11.04	9.60	8.81	8.30	7.95	7.69	7.50	7.34	7.21	7.01	6.81	6.61	6.48	6.40	6.29	6.18	6.06	5.95
9	13.61	10.11	8.72	7.96	7.47	7.13	6.88	6.69	6.54	6.42	6.23	6.03	5.83	5.71	5.62	5.52	5.41	5.30	5.19
10	12.83	9.43	8.08	7.34	6.87	6.54	6.30	6.12	5.97	5.85	5.66	5.47	5.27	5.15	5.07	4.97	4.86	4.75	4.64
11	12.23	8.91	7.60	6.88	6.42	6.10	5.86	5.68	5.54	5.42	5.24	5.05	4.86	4.74	4.65	4.55	4.45	4.34	4.23
12	11.75	8.51	7.23	6.52	6.07	5.76	5.52	5.35	5.20	5.09	4.91	4.72	4.53	4.41	4.33	4.23	4.12	4.01	3.91
13	11.37	8.19	6.93	6.23	5.79	5.48	5.25	5.08	4.94	4.82	4.64	4.46	4.27	4.15	4.07	3.97	3.87	3.76	3.65
14	11.06	7.92	6.68	6.00	5.56	5.26	5.03	4.86	4.72	4.60	4.43	4.25	4.06	3.94	3.86	3.76	3.66	3.55	3.44
15	10.80	7.70	6.48	5.80	5.37	5.07	4.85	4.67	4.54	4.42	4.25	4.07	3.88	3.77	3.69	3.58	3.48	3.37	3.26
16	10.58	7.51	6.30	5.64	5.21	4.91	4.69	4.52	4.38	4.27	4.10	3.92	3.73	3.62	3.54	3.44	3.33	3.22	3.11
17	10.38	7.35	6.16	5.50	5.07	4.78	4.56	4.39	4.25	4.14	3.97	3.79	3.61	3.49	3.41	3.31	3.21	3.10	2.99
18	10.22	7.21	6.03	5.37	4.96	4.66	4.44	4.28	4.14	4.03	3.86	3.68	3.50	3.38	3.30	3.20	3.10	2.99	2.87
19	10.07	7.09	5.92	5.27	4.85	4.56	4.34	4.18	4.04	3.93	3.76	3.59	3.40	3.29	3.21	3.11	3.00	2.89	2.78
20	9.94	6.99	5.82	5.17	4.76	4.47	4.26	4.09	3.96	3.85	3.68	3.50	3.32	3.20	3.12	3.02	2.92	2.81	2.69
21	9.83	6.89	5.73	5.09	4.68	4.39	4.18	4.01	3.88	3.77	3.60	3.43	3.24	3.13	3.05	2.95	2.84	2.73	2.62
22	9.73	6.81	5.65	5.02	4.61	4.32	4.11	3.94	3.81	3.70	3.54	3.36	3.18	3.06	2.98	2.88	2.77	2.66	2.55
23	9.63	6.73	5.58	4.95	4.54	4.26	4.05	3.88	3.75	3.64	3.47	3.30	3.12	3.00	2.92	2.82	2.71	2.60	2.49
24	9.55	6.66	5.52	4.89	4.49	4.20	3.99	3.83	3.69	3.59	3.42	3.25	3.06	2.95	2.87	2.77	2.66	2.55	2.43
25	9.48	6.60	5.46	4.84	4.43	4.15	3.94	3.78	3.64	3.54	3.37	3.20	3.01	2.90	2.82	2.72	2.61	2.50	2.38
26	9.41	6.54	5.41	4.79	4.38	4.10	3.89	3.73	3.60	3.49	3.33	3.15	2.97	2.85	2.77	2.67	2.56	2.45	2.33
27	9.34	6.49	5.36	4.74	4.34	4.06	3.85	3.69	3.56	3.45	3.28	3.11	2.93	2.81	2.73	2.63	2.52	2.41	2.29
28	9.28	6.44	5.32	4.70	4.30	4.02	3.81	3.65	3.52	3.41	3.25	3.07	2.89	2.77	2.69	2.59	2.48	2.37	2.25
29	9.23	6.40	5.28	4.66	4.26	3.98	3.77	3.61	3.48	3.38	3.21	3.04	2.86	2.74	2.66	2.56	2.45	2.33	2.21
30	9.18	6.35	5.24	4.62	4.23	3.95	3.74	3.58	3.45	3.34	3.18	3.01	2.82	2.71	2.63	2.52	2.42	2.30	2.18
40	8.83	6.07	4.98	4.37	3.99	3.71	3.51	3.35	3.22	3.12	2.95	2.78	2.60	2.48	2.40	2.30	2.18	2.06	1.93
60	8.49	5.79	4.73	4.14	3.76	3.49	3.29	3.13	3.01	2.90	2.74	2.57	2.39	2.27	2.19	2.08	1.96	1.83	1.69
120	8.18	5.54	4.50	3.92	3.55	3.28	3.09	2.93	2.81	2.71	2.54	2.37	2.19	2.07	1.98	1.87	1.75	1.61	1.43
∞	7.88	5.30	4.28	3.72	3.35	3.09	2.90	2.75	2.62	2.52	2.36	2.19	2.00	1.88	1.79	1.67	1.54	1.37	1.00

Example: $P(F > F_{0.005;9,15}) = P(F > 4.54) = 0.005$.

APPENDIX C (cont)

PERCENTAGE POINTS OF THE F DISTRIBUTION

TABLE OF $F_{0.01;\nu_1,\nu_2}$

Degrees of freedom for the numerator (ν_1)

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9	10	12	15	20	25	30	40	60	120	∞
1	4052	4999	5403	5625	5764	5859	5928	5981	6022	6056	6106	6157	6209	6240	6261	6287	6313	6339	6366
2	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39	99.40	99.42	99.43	99.45	99.46	99.47	99.47	99.48	99.49	99.50
3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35	27.23	27.05	26.87	26.69	26.58	26.50	26.41	26.32	26.22	26.13
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55	14.37	14.20	14.02	13.91	13.84	13.75	13.65	13.56	13.46
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.89	9.72	9.55	9.45	9.38	9.29	9.20	9.11	9.02
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.56	7.40	7.30	7.23	7.14	7.06	6.97	6.88
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.31	6.16	6.06	5.99	5.91	5.82	5.74	5.65
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.52	5.36	5.26	5.20	5.12	5.03	4.95	4.86
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	4.96	4.81	4.71	4.65	4.57	4.48	4.40	4.31
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.56	4.41	4.31	4.25	4.17	4.08	4.00	3.91
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.40	4.25	4.10	4.01	3.94	3.86	3.78	3.69	3.60
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.01	3.86	3.76	3.70	3.62	3.54	3.45	3.36
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	3.96	3.82	3.66	3.57	3.51	3.43	3.34	3.25	3.17
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03	3.94	3.80	3.66	3.51	3.41	3.35	3.27	3.18	3.09	3.01
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.52	3.37	3.28	3.21	3.13	3.05	2.96	2.87
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.55	3.41	3.26	3.16	3.10	3.02	2.93	2.84	2.75
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.46	3.31	3.16	3.07	3.00	2.92	2.83	2.75	2.65
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.37	3.23	3.08	2.98	2.92	2.84	2.75	2.66	2.57
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.30	3.15	3.00	2.91	2.84	2.76	2.67	2.58	2.49
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.23	3.09	2.94	2.84	2.78	2.69	2.61	2.52	2.42
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.17	3.03	2.88	2.79	2.72	2.64	2.55	2.46	2.36
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.12	2.98	2.83	2.73	2.67	2.58	2.50	2.40	2.31
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	3.07	2.93	2.78	2.69	2.62	2.54	2.45	2.35	2.26
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.03	2.89	2.74	2.64	2.58	2.49	2.40	2.31	2.21
25	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22	3.13	2.99	2.85	2.70	2.60	2.54	2.45	2.36	2.27	2.17
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	3.09	2.96	2.81	2.66	2.57	2.50	2.42	2.33	2.23	2.13
27	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15	3.06	2.93	2.78	2.63	2.54	2.47	2.38	2.29	2.20	2.10
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03	2.90	2.75	2.60	2.51	2.44	2.35	2.26	2.17	2.07
29	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09	3.00	2.87	2.73	2.57	2.48	2.41	2.33	2.23	2.14	2.04
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.84	2.70	2.55	2.45	2.39	2.30	2.21	2.11	2.01
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.66	2.52	2.37	2.27	2.20	2.11	2.02	1.92	1.81
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.50	2.35	2.20	2.10	2.03	1.94	1.84	1.73	1.60
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.34	2.19	2.03	1.93	1.86	1.76	1.66	1.53	1.38
∞	6.64	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.19	2.04	1.88	1.77	1.70	1.59	1.48	1.33	1.00

Degrees of freedom for the denominator (ν_2)

Example: $P(F>F_{0.01;9,15}) = P(F>3.89) = 0.01$.

APPENDIX C (cont)

PERCENTAGE POINTS OF THE F DISTRIBUTION

TABLE OF $F_{0.025;\nu_1,\nu_2}$

Degrees of freedom for the numerator (ν_1); degrees of freedom for the denominator (ν_2)

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9	10	12	15	20	25	30	40	60	120	∞
1	647.8	799.5	864.2	899.6	921.8	937.1	948.2	956.7	963.3	968.6	976.7	984.9	993.1	998.1	1001	1006	1010	1014	1018
2	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39	39.40	39.41	39.43	39.45	39.46	39.46	39.47	39.48	39.49	39.50
3	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47	14.42	14.36	14.25	14.17	14.12	14.08	14.04	13.99	13.95	13.90
4	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.75	8.66	8.56	8.50	8.46	8.41	8.36	8.31	8.26
5	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.52	6.43	6.33	6.27	6.23	6.18	6.12	6.07	6.02
6	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.37	5.27	5.17	5.11	5.07	5.01	4.96	4.90	4.85
7	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.67	4.57	4.47	4.40	4.36	4.31	4.25	4.20	4.14
8	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.20	4.10	4.00	3.94	3.89	3.84	3.78	3.73	3.67
9	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.87	3.77	3.67	3.60	3.56	3.51	3.45	3.39	3.33
10	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.62	3.52	3.42	3.35	3.31	3.26	3.20	3.14	3.08
11	6.72	5.26	4.63	4.28	4.04	3.88	3.76	3.66	3.59	3.53	3.43	3.33	3.23	3.16	3.12	3.06	3.00	2.94	2.88
12	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.28	3.18	3.07	3.01	2.96	2.91	2.85	2.79	2.73
13	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31	3.25	3.15	3.05	2.95	2.88	2.84	2.78	2.72	2.66	2.60
14	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21	3.15	3.05	2.95	2.84	2.78	2.73	2.67	2.61	2.55	2.49
15	6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	2.96	2.86	2.76	2.69	2.64	2.59	2.52	2.46	2.40
16	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	3.05	2.99	2.89	2.79	2.68	2.61	2.57	2.51	2.45	2.38	2.32
17	6.04	4.62	4.01	3.66	3.44	3.28	3.16	3.06	2.98	2.92	2.82	2.72	2.62	2.55	2.50	2.44	2.38	2.32	2.25
18	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93	2.87	2.77	2.67	2.56	2.49	2.44	2.38	2.32	2.26	2.19
19	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88	2.82	2.72	2.62	2.51	2.44	2.39	2.33	2.27	2.20	2.13
20	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84	2.77	2.68	2.57	2.46	2.40	2.35	2.29	2.22	2.16	2.09
21	5.83	4.42	3.82	3.48	3.25	3.09	2.97	2.87	2.80	2.73	2.64	2.53	2.42	2.36	2.31	2.25	2.18	2.11	2.04
22	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76	2.70	2.60	2.50	2.39	2.32	2.27	2.21	2.14	2.08	2.00
23	5.75	4.35	3.75	3.41	3.18	3.02	2.90	2.81	2.73	2.67	2.57	2.47	2.36	2.29	2.24	2.18	2.11	2.04	1.97
24	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70	2.64	2.54	2.44	2.33	2.26	2.21	2.15	2.08	2.01	1.94
25	5.69	4.29	3.69	3.35	3.13	2.97	2.85	2.75	2.68	2.61	2.51	2.41	2.30	2.23	2.18	2.12	2.05	1.98	1.91
26	5.66	4.27	3.67	3.33	3.10	2.94	2.82	2.73	2.65	2.59	2.49	2.39	2.28	2.21	2.16	2.09	2.03	1.95	1.88
27	5.63	4.24	3.65	3.31	3.08	2.92	2.80	2.71	2.63	2.57	2.47	2.36	2.25	2.18	2.13	2.07	2.00	1.93	1.85
28	5.61	4.22	3.63	3.29	3.06	2.90	2.78	2.69	2.61	2.55	2.45	2.34	2.23	2.16	2.11	2.05	1.98	1.91	1.83
29	5.59	4.20	3.61	3.27	3.04	2.88	2.76	2.67	2.59	2.53	2.43	2.32	2.21	2.14	2.09	2.03	1.96	1.89	1.81
30	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57	2.51	2.41	2.31	2.20	2.12	2.07	2.01	1.94	1.87	1.79
40	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45	2.39	2.29	2.18	2.07	1.99	1.94	1.88	1.80	1.72	1.64
60	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33	2.27	2.17	2.06	1.94	1.87	1.82	1.74	1.67	1.58	1.48
120	5.15	3.80	3.23	2.89	2.67	2.52	2.39	2.30	2.22	2.16	2.05	1.94	1.82	1.75	1.69	1.61	1.53	1.43	1.31
∞	5.03	3.69	3.12	2.79	2.57	2.41	2.29	2.19	2.12	2.05	1.95	1.83	1.71	1.63	1.57	1.49	1.39	1.27	1.00

Example: $P\{F > F_{0.025;9,15}\} = P\{F > 3.12\} = 0.025$.

APPENDIX C (cont)

PERCENTAGE POINTS OF THE F DISTRIBUTION

TABLE OF $F_{0.05;\nu_1,\nu_2}$

Degrees of freedom for the numerator (ν_1); Degrees of freedom for the denominator (ν_2)

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9	10	12	15	20	25	30	40	60	120	∞
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.9	245.9	248.0	249.3	250.1	251.1	252.2	253.3	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.46	19.46	19.47	19.48	19.49	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.63	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.52	4.50	4.46	4.43	4.40	4.37
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.83	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.40	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.11	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.89	2.86	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.73	2.70	2.66	2.62	2.58	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.60	2.57	2.53	2.49	2.45	2.41
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.50	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.41	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.34	2.31	2.27	2.22	2.18	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.28	2.25	2.20	2.16	2.11	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.23	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.18	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.14	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.07	2.04	1.99	1.95	1.90	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.02	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.00	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.97	1.94	1.89	1.84	1.79	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.94	1.90	1.85	1.80	1.75	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.92	1.88	1.84	1.79	1.73	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.89	1.85	1.81	1.75	1.70	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.88	1.84	1.79	1.74	1.68	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.78	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.69	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.60	1.55	1.50	1.43	1.35	1.26
∞	3.84	3.00	2.61	2.37	2.22	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.51	1.46	1.40	1.32	1.22	1.00

Example: $P(F > F_{0.05;9,15}) = P(F > 2.59) = 0.05$.

APPENDIX C (cont)

PERCENTAGE POINTS OF THE F DISTRIBUTION

TABLE OF $F_{0.10,\, v_1, v_2}$

Degrees of freedom for the numerator (v_1); Degrees of freedom for the denominator (v_2)

v_2 \ v_1	1	2	3	4	5	6	7	8	9	10	12	15	20	25	30	40	60	120	
1	39.86	49.50	53.59	55.83	57.24	58.20	58.91	59.44	59.86	60.19	60.71	61.22	61.74	62.05	62.26	62.53	62.79	63.06	63.32
2	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38	9.39	9.41	9.42	9.44	9.45	9.46	9.47	9.47	9.48	9.49
3	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24	5.23	5.22	5.20	5.18	5.17	5.17	5.16	5.15	5.14	5.13
4	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94	3.92	3.90	3.87	3.84	3.83	3.82	3.80	3.79	3.78	3.76
5	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32	3.30	3.27	3.24	3.21	3.19	3.17	3.16	3.14	3.12	3.11
6	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96	2.94	2.90	2.87	2.84	2.81	2.80	2.78	2.76	2.74	2.72
7	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72	2.70	2.67	2.63	2.59	2.57	2.56	2.54	2.51	2.49	2.47
8	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.56	2.54	2.50	2.46	2.42	2.40	2.38	2.36	2.34	2.32	2.29
9	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44	2.42	2.38	2.34	2.30	2.27	2.25	2.23	2.21	2.18	2.16
10	3.29	2.92	2.73	2.61	2.52	2.46	2.41	2.38	2.35	2.32	2.28	2.24	2.20	2.17	2.16	2.13	2.11	2.08	2.06
11	3.23	2.86	2.66	2.54	2.45	2.39	2.34	2.30	2.27	2.25	2.21	2.17	2.12	2.10	2.08	2.05	2.03	2.00	1.97
12	3.18	2.81	2.61	2.48	2.39	2.33	2.28	2.24	2.21	2.19	2.15	2.10	2.06	2.03	2.01	1.99	1.96	1.93	1.90
13	3.14	2.76	2.56	2.43	2.35	2.28	2.23	2.20	2.16	2.14	2.10	2.05	2.01	1.98	1.96	1.93	1.90	1.88	1.85
14	3.10	2.73	2.52	2.39	2.31	2.24	2.19	2.15	2.12	2.10	2.05	2.01	1.96	1.93	1.91	1.89	1.86	1.83	1.80
15	3.07	2.70	2.49	2.36	2.27	2.21	2.16	2.12	2.09	2.06	2.02	1.97	1.92	1.89	1.87	1.85	1.82	1.79	1.76
16	3.05	2.67	2.46	2.33	2.24	2.18	2.13	2.09	2.06	2.03	1.99	1.94	1.89	1.86	1.84	1.81	1.78	1.75	1.72
17	3.03	2.64	2.44	2.31	2.22	2.15	2.10	2.06	2.03	2.00	1.96	1.91	1.86	1.83	1.81	1.78	1.75	1.72	1.69
18	3.01	2.62	2.42	2.29	2.20	2.13	2.08	2.04	2.00	1.98	1.93	1.89	1.84	1.80	1.78	1.75	1.72	1.69	1.66
19	2.99	2.61	2.40	2.27	2.18	2.11	2.06	2.02	1.98	1.96	1.91	1.86	1.81	1.78	1.76	1.73	1.70	1.67	1.63
20	2.97	2.59	2.38	2.25	2.16	2.09	2.04	2.00	1.96	1.94	1.89	1.84	1.79	1.76	1.74	1.71	1.68	1.64	1.61
21	2.96	2.57	2.36	2.23	2.14	2.08	2.02	1.98	1.95	1.92	1.87	1.83	1.78	1.74	1.72	1.69	1.66	1.62	1.59
22	2.95	2.56	2.35	2.22	2.13	2.06	2.01	1.97	1.93	1.90	1.86	1.81	1.76	1.73	1.70	1.67	1.64	1.60	1.57
23	2.94	2.55	2.34	2.21	2.11	2.05	1.99	1.95	1.92	1.89	1.84	1.80	1.74	1.71	1.69	1.66	1.62	1.59	1.55
24	2.93	2.54	2.33	2.19	2.10	2.04	1.98	1.94	1.91	1.88	1.83	1.78	1.73	1.70	1.67	1.64	1.61	1.57	1.53
25	2.92	2.53	2.32	2.18	2.09	2.02	1.97	1.93	1.89	1.87	1.82	1.77	1.72	1.68	1.66	1.63	1.59	1.56	1.52
26	2.91	2.52	2.31	2.17	2.08	2.01	1.96	1.92	1.88	1.86	1.81	1.76	1.71	1.67	1.65	1.61	1.58	1.54	1.50
27	2.90	2.51	2.30	2.17	2.07	2.00	1.95	1.91	1.87	1.85	1.80	1.75	1.70	1.66	1.64	1.60	1.57	1.53	1.49
28	2.89	2.50	2.29	2.16	2.06	2.00	1.94	1.90	1.87	1.84	1.79	1.74	1.69	1.65	1.63	1.59	1.56	1.52	1.48
29	2.89	2.50	2.28	2.15	2.06	1.99	1.93	1.89	1.86	1.83	1.78	1.73	1.68	1.64	1.62	1.58	1.55	1.51	1.47
30	2.88	2.49	2.28	2.14	2.05	1.98	1.93	1.88	1.85	1.82	1.77	1.72	1.67	1.63	1.61	1.57	1.54	1.50	1.46
40	2.84	2.44	2.23	2.09	2.00	1.93	1.87	1.83	1.79	1.76	1.71	1.66	1.61	1.57	1.54	1.51	1.47	1.42	1.38
60	2.79	2.39	2.18	2.04	1.95	1.87	1.82	1.77	1.74	1.71	1.66	1.60	1.54	1.50	1.48	1.44	1.40	1.35	1.29
120	2.75	2.35	2.13	1.99	1.90	1.82	1.77	1.72	1.68	1.65	1.60	1.55	1.48	1.44	1.41	1.37	1.32	1.26	1.19
	2.71	2.30	2.08	1.94	1.85	1.77	1.72	1.67	1.63	1.60	1.55	1.49	1.42	1.38	1.34	1.30	1.24	1.17	1.00

Example: $P(F > F_{0.10;9,15}) = P(F > 2.09) = 0.10.$

APPENDIX D

Percentiles of the χ^2-Distribution[a]

Table of $\chi^2_{\alpha;\nu}$

ν \ α	0.005	0.025	0.050	0.950	0.975	0.995
1	0.000	0.001	0.004	3.841	5.024	7.879
2	0.010	0.051	0.103	5.991	7.378	10.597
3	0.072	0.216	0.352	7.815	9.348	12.838
4	0.207	0.484	0.711	9.488	11.143	14.860
5	0.412	0.831	1.145	11.070	12.833	16.750
6	0.676	1.237	1.635	12.592	14.449	18.548
7	0.989	1.690	2.167	14.067	16.013	20.278
8	1.344	2.180	2.733	15.507	17.535	21.955
9	1.735	2.700	3.325	16.919	19.023	23.589
10	2.156	3.247	3.940	18.307	20.483	25.188
11	2.603	3.816	4.575	19.675	21.920	26.757
12	3.074	4.404	5.226	21.026	23.337	28.300
13	3.565	5.009	5.892	22.362	24.736	29.819
14	4.075	5.629	6.571	23.685	26.119	31.319
15	4.601	6.262	7.261	24.996	27.488	32.801
16	5.142	6.908	7.962	26.296	28.845	34.267
17	5.697	7.564	8.672	27.587	30.191	35.718
18	6.265	8.231	9.390	28.869	31.526	37.156
19	6.844	8.907	10.117	30.144	32.852	38.582
20	7.434	9.591	10.851	31.410	34.170	39.997
21	8.034	10.283	11.591	32.671	35.479	41.401
22	8.643	10.982	12.338	33.924	36.781	42.796
23	9.260	11.689	13.091	35.172	38.076	44.181
24	9.886	12.401	13.848	36.415	39.364	45.559
25	10.520	13.120	14.611	37.652	40.646	46.928
26	11.160	13.844	15.379	38.885	41.923	48.290
27	11.808	14.573	16.151	40.113	43.195	49.645
28	12.461	15.308	16.928	41.337	44.461	50.993
29	13.121	16.047	17.708	42.557	45.722	52.336
30	13.787	16.791	18.493	43.773	46.979	53.672
60	35.534	40.482	43.188	79.082	83.298	91.952
100	67.328	74.222	77.929	124.342	129.561	140.169
110	75.550	82.867	86.792	135.480	140.917	151.948
120	83.852	91.573	95.705	146.567	152.211	163.648

[a]Computed by Myrle Horita.

Example: $P\{\chi^2 < \chi^2_{0.95;10}\} = P\{\chi^2 < 18.307\} = 0.95$