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The Scottish Book
A LASL Monograph

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PREFACE TO MONOGRAPH

Numerous requests for copies of this document, addressed to Los Alamos Scientific Laboratory library or to me, appear to make it worthwhile (after a lapse of some 20 yr) to reprint, with some corrections, this collection of problems.

This project was made possible through the interest and active help of Robert Krohn of the Laboratory.

It is a pleasure to give special thanks to Dr. Bill Beyer for his perspicacious review of the changes and the revised version of some formulations. Thanks are due to Martha Lee DeLanoy for editorial work.

Stan Ulam
Los Alamos, NM
May 1977

PREFACE

The enclosed collection of mathematical problems has its origin in a notebook which was started in Lwów, in Poland in 1935. If I remember correctly, it was S. Banach who suggested keeping track of some of the problems occupying the group of mathematicians there. The mathematical life was very intense in Lwów. Some of us met practically every day, informally in small groups, at all times of the day to discuss problems of common interest, communicating to each other the latest work and results. Apart from the more official meetings of the local sections of the Mathematical Society (which took place Saturday evenings, almost every week!), there were frequent informal discussions mostly held in one of the coffee houses located near the University building — one of them a coffee house named "Roma," and the other "The Scottish Coffee House." This explains the name of the collection. A large notebook was purchased by Banach and deposited with the headwaiter of the Scottish Coffee House, who, upon demand, would bring it out of some secure hiding place, leave it at the table, and after the guests departed, return it to its secret location.

Many of the problems date from years before 1935. They were discussed a great deal among the persons whose names are included in the text, and then gradually inscribed into the "book" in ink. Most of the questions proposed were supposed to have had considerable attention devoted to them before an "official" inclusion into the "book" was considered. As the reader will see, this general rule could not guarantee against an occasional question to which the answer was quite simple or even trivial.

In several instances, the problems were solved, right on the spot or within a short time, and the answers were inscribed, perhaps some time after the first formulation of the problem under question.

As most readers will realize, the city of Lwów, and with it the "Scottish Book," was fated to have a very stormy history within a few years of the book's inception. A few weeks after the outbreak of World War II, the city was occupied by the Russians. From items at the end of this collection, it is seen that some Russian mathematicians must have visited the town; they left several problems (and prizes for their solutions). The last date figuring in the book is May 31, 1941. Item Number 193 contains a rather cryptic set of numerical results, signed by Steinhaus, dealing with the distribution of the number of matches in a box! After the start of war between Germany and Russia, the city was occupied by German troops that same summer and the inscriptions ceased.

The fate of the Scottish Book during the remaining years of war is not known to me. According to Steinhaus, this document was brought back to the city of Wroclaw by Banach's son, now a physician in Poland. (Many of the surviving mathematicians from Lwów continue their work in Wroclaw. The tradition of the Scottish Book continues. Since 1945, new problems have been formulated and inscribed and a new volume is in progress.)

A general word of explanation may be in order here. I left Poland late in 1935 but, before the war, visited Lwów every summer in 1936, '37, '38, and '39. The last visit was during the summer preceding the outbreak of World War II, and I remember just a few days before I left Poland, around August 15, the conversation with Mazur on the likelihood of war. It seems that in general people were expecting another crisis like that of Munich in the preceding year, but were not prepared for the imminent world war. Mazur, in a discussion concerning such possibilities, suddenly said to me "A world war may break out. What shall we do with the Scottish Book and our joint unpublished papers? You are leaving for the United States shortly, and presumably will be safe. In case of a bombardment of the city, I shall put all the manuscripts and the Scottish Book into a case which I shall bury in the ground." We even decided upon a location of this secret hiding place; it was to be near the goal post of a football field outside the city. It is not known to me whether anything of the sort really happened. Apparently, the manuscript of the Scottish Book survived in good enough shape to have a typewritten copy made, which Professor Steinhaus sent to me last year (1956).

The existence of such a collection of problems was mentioned on several occasions, during the last 20 years, to mathematical friends in this country. I have received, since, many requests for copies of this document. It was in answer to such oral and written requests that the present translation was made. This spring in an article, "Can We Grow Geniuses in Science?", which appears in Harper's June 1957 issue, L. L. Whyte alluded to the existence of the Scottish Book. Apparently, the diffusion of this small mystery became somewhat widespread, and this provided another incentive for this translation.

Before deciding to make such an informal distribution, I consulted my teacher and friend (and senior member of the group of authors of the problems), Professor Steinhaus, about the propriety of circulating this collection. With his agreement, I have translated the original text (the original is mostly in Polish) in order to make it available through this private communication.

Even as an author or co-author of some of the problems, I have felt that the only practical and proper thing to do was to translate them verbatim. No explanations or reformulations of the problems have been made.

Many of the problems have since found their solution, some in the form of published papers (I know of some of my own problems, solutions to which were published in periodicals, among them Problem 17.1, Z. Zahorski, Fund. Math., Vol. 34, pp. 183-245 and Problem 77(a), R. H. Fox, Fund. Math., Vol. 34, pp. 278-287).

The work of following the literature in the several fields with which the problems deal would have been prohibitive for me. The time necessary for supplying the definitions or explanations of terms, all very well understood among mathematicians in Lwów, but perhaps not in current use now, would also be considerable. Some of the authors of the problems are no longer living and since one could not treat uniformly all the material, I have decided to make no changes whatsoever.

Perhaps some of the problems will still present an actual interest to mathematicians. At least the collection gives some picture of the interests of a compact mathematical group, an illustration of the mode of their work and thought; and reflects informal features of life in a very vital mathematical center. I should be grateful if the recipients of this collection were willing to point out errors, supply information about solution to problems, or indicate developments contained in recent literature in topics connected with the subjects discussed in the problems.

It is with great pleasure that I express thanks to Miss Marie Odell for help in editing the manuscript and to Dr. Milton Wing for looking over the translated manuscript.

S. Ulam
Los Alamos, NM
Fall 1957

THE SCOTTISH BOOK

1. Problem: Banach

(a) When can a metric space [possibly of type (B)] be so metrized that it will become complete and compact, and so that all the sequences converging originally, should also converge in the new metric?

(b) Can, for example, the space C_0 be so metrized?

July 17, 1935

2. Problem: Banach-Ulam

(a) Can one define, in every compact metric space E, a measure (finitely additive) so that Borel sets which are congruent should have equal measure?

(b) Suppose $E = E_1 + E_2 + \dots + E_n$, and $E_1 \cong E_2 \cong \dots \cong E_n$ and $\{E_n\}$ are disjoint; then we write $E_1 = (1/n)E$. Can it occur that $(1/n)E \cong (1/m)E$, $n \neq m$, if we assume that $(1/n)E$ are Borel sets and E is compact?

3. Theorem: Banach-Ulam

It is proved very simply that a compact set cannot be congruent to a proper subset of itself.

4. Theorem: Schreier

If $\{\xi_n\}$ is a bounded sequence, summable by the first mean to ξ, then almost every subsequence of it is also summable by the first mean to ξ.

5. Problem: Mazur

Definition: A sequence $\{\xi_n\}$ is asymptotically convergent to ξ, if there exists a subsequence of density 1 convergent to ξ. Theorem (Mazur): This notion is not equivalent in the domain of all sequences to any Töplitz method. How is it in the domain of bounded sequences?

Ad 5. (Mazur) If we call sequences, convergent in the above sense, asymptotically convergent, we have the following theorems:

(1) If a method (a_{ik}) sums all the asymptotically convergent sequences, the $a_{ik} = 0$ for $k > k_0$, $i = 1,2, \dots$ and there exist finite $\lim a_{ik}$ for $k = 1, \dots k_0$, such that the method sums all the sequences.

(2) If a method (a_{ik}) sums all the convergent sequences and every bounded sequence summable by the sequence is asymptotically convergent, then there exists a sequence of increasing

integers $\{k_n\}$ with density 1, such that for every bounded sequence $\{\xi_n\}$ summable by this method, the sequence $\{\xi_{k_n}\}$ is convergent.

From (1) it follows that there does not exist a permanent method summing all the asymptotically convergent sequences; from (2) it follows that a permanent method summing all bounded asymptotically convergent sequences must also sum some other bounded sequences.

July 22, 1935

6. Problem: Mazur-Orlicz
(Prize: Bottle of wine, S. Mazur)

Is a matrix, finite in each row and invertible (in a one-to-one way), equivalent to a normal matrix?

7. Problem: Mazur-Banach

Are two convex, closed, infinite-dimensional convex subsets of a Banach space [of type (B)] always homeomorphic?

8. Problem: Mazur
(Prize: Five small beers, S. Mazur)

(a) Is every series summable by the first mean representable as a Cauchy product of two converging series? Or else, equivalently,

(b) Can one find for each convergent sequence $\{z_n\}$ two convergent sequences $\{x_n\}$, $\{y_n\}$ so that

$$z_n = \frac{x_1 y_n + x_2 y_{n-1} + \dots + x_n y_1}{n}.$$

9. Problem: Mazur-Orlicz

Theorem: Ulam

If E is a class of sets, each finite, each of which contains at most n elements, and such that every $n + 1$ of these sets have a common element, then there exists an element common to all sets of E.

10. Theorem: Banach-Mazur

Let H be an arbitrary abstract set and E the set of all real valued functions defined on H. The sequence $x_n(t) \to x(t)$ (such that $t \subset H$, $x_n, x \subset E$) if $\lim x_n(t) = x(t)$ for each $t \in H$. Theorem: Each linear functional $f(x)$ defined in E is of the form

$$f(x) = \sum_{i=1}^{n} \alpha_i x(t_i)$$

where α_i and t_i do not depend on x.

10.1. Theorem: Mazur, Auerbach, Ulam, Banach (Problem: Mazur)

If $\{K_n\}$ $n = 1,2,\ldots$ is a sequence of convex bodies, each of diameter $\leq a$ and the sum of their volumes is $\leq b$, then there exists a cube with the diameter $c = f(a,b)$ such that one can put all the given bodies in it disjointly.
A Corollary: One kilogram of potatoes can be put into a finite sack.
Problem: Determine the function $c = f(a,b)$.

11. Problem: Banach-Ulam

Assume that there is a measure defined in the space of all integers. This measure is finitely additive and any single point has measure zero. Let us extend this measure to product spaces over the set of integers (finite or infinite products) in such a way that the measure of a subproduct equals the numerical product of the measures of its projections.

(a) Is the set of all sequences convergent to infinity measurable?
(b) Is the set of all pairs (x,y) where x,y are relatively prime measurable?
(c) **Theorem: Schreier** The set of all pairs (x,y) where $x < y$ is nonmeasurable.

Remark: A set is not measurable if a measure can be defined in it in at least two different ways and still satisfy conditions above.

12. Problem: Banach

A surface S is homeomorphic to the surface of a sphere and it has:
(a) a tangent plane everywhere
(b) a continuously varying tangent plane.

Is S equivalent to the surface of a geometric sphere? (That is to say, does there exist a homeomorphism of the whole space which transforms the given surface S into the surface of the sphere?)

13. Problem: Ulam

Let E be the class of all subsets of the set of integers. Two subsets $K_1, K_2 \subset E$ are called equivalent or $K_1 \equiv K_2$ if $K_1 - K_2$ and $K_2 - K_1$ are at most finite sets. There is given a function F(K) defined for all $K \subset E$; its range is contained in E and

$$F(K_1 + K_2) \equiv F(K_1) + F(K_2)$$

$$F(\text{compl. } K) \equiv \text{compl. } F(K)$$

Question: Does there exist a function f(x) [x and f(x) natural integers] such that

$$f(K) \equiv F(K)?$$

14. Problem: Schauder-Mazur

Let $f(x_1 \ldots x_n)$ be a function defined in the cube K_n. Let us suppose that f possesses almost everywhere all the partial derivatives up to the r^{th} order and the derivatives up to the order $r-1$ are absolutely continuous on almost every straight line parallel to any axis. All the partial derivatives (up to the order r) $\in L^p$, $p > 1$.

Does there exist a sequence of polynomials $\{w_i\}$ which converge in the mean in the p^{th} power to f and in all partial derivatives up to the order r?

For $r = 1$ this was settled positively by the authors. An analogous problem for domains other than K_n.

15. Problem: Schauder

Let $f(x_1 \ldots x_n)$ be a function defined in K_n, i.e., in the n-dimensional cube. Does there exist for every n a $p_n \geq 2$, such that if $f \in L^{p_n}$ then there exists a function $u(x_1 \ldots x_n)$ continuous in K_n:

(a) Vanishing on the boundary of K_n,

(b) Possessing first derivatives on almost every line parallel to the axes and absolutely continuous.

(c) Possessing almost everywhere second partial derivatives ($\in L^{p_n}$) and satisfying the equation: $\Delta u = f$.

The author proved that for $n = 2,3$; $p_n = 2$. Mazur observed that for $n = 4$, p_n cannot be equal to 2.

For which n does there exist a $p_n > 2$?

15.1. Problem: Mazur-Orlicz

(Prize: Two small beers, S. Mazur)

Is a space E, of type (F), for which there exists a sphere K which is bounded, necessarily of type (B)? (A sphere is bounded if and only if $x_n \in K$, and the numbers $t_n \to 0$, then $t_n x_n \to 0$.)

Ad 15.1. Negative answer: It suffices to take for E the space of numerical sequences

$$x = \sum_{n=1}^{\infty} \xi_n$$

such that

$$\sum_{n=1}^{\infty} |\xi_n|^p < \infty; \quad 0 < p < 1,$$

with ordinary operations, and

$$\|x\| = \left(\sum_{n=1}^{\infty} |\xi_n|^p \right)^{1/p}$$

instead of the space (l^p) one can also take (L^p) which consists of real valued functions $x = x(t)$ in $<0,1>$, measurable, and such that

$$\int_0^1 |x(t)|^p \, dt < \infty,$$

with ordinary algebraic operations and

$$\|x\| = \left[\int_0^1 |x(t)|^p \, dt\right]^{1/p}.$$

May 1, 1937, Mazur

16. Problem: Ulam

To find a Lebesgue measure in the space of all measurable functions satisfying the following conditions:

If $\{H_n\}$ are measurable sets contained on the line $\{x = x_n\}$, then the set of all measurable functions $f(x)$, satisfying the condition $f(x_n) \in H_n$ has a measure equal to $|H_1| \cdot |H_2| \ldots$ where $|H_n|$ denotes the measure of the set H_n.

17. Problem: Ulam

To prove a converse of Poisson's theorem; that is: Given a sequence of urns containing white balls (1) and black ones (0), with unknown compositions $\{p_n\}$ and given also the result x_n of drawing from each urn in turn, prove that with probability 1,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i = p$$

implies that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} p_i = p.$$

17.1. Problem: Ulam

Let f be a continuous function defined for all $0 \leq x \leq 1$. Does there exist a perfect set of points C and an analytic function ϕ so that for all points of the set C we have $f \equiv \phi$?

18. Problem: Ulam

Let a steady current flow through a curve in space which is closed and knotted: Does there exist a line of force which is also knotted (knotted = nonequivalent through any homeomorphism of the whole space R_3 with the circumference of a circle)?

19. Problem: Ulam

Is a solid of uniform density which will float in water in every position a sphere?

20. Problem: Ulam

Consider one-to-one and continuous transformations of the plane of the form

$$x' = x; \quad y' = f(x,y)$$

and

$$y' = y; \quad x' = g(x,y)$$

and also transformations which result from composing the above a finite number of times. Can every homeomorphic transformation be approximated by such?
(Analogous problem for the n-dimensional space)

20.1. Problem: Mazur-Orlicz

For every integer n determine the smallest natural integer, k_n, with the following property: If $f(x_1, \ldots x_n)$ is an irreducible polynominal, there exist points

$$(x_{11}, \ldots x_{1n}), \ldots (x_{k_n 1}, \ldots x_{k_n n})$$

such that

$$f(\lambda_1 x_{11} + \ldots + \lambda_{k_n} x_{k_n 1}, \ldots \lambda_1 x_{1n} + \ldots + \lambda_{k_n} x_{k_n n}),$$

considered as a polynominal of the variables $\lambda_1 \ldots \lambda_{k_n}$ is irreducible. Is the sequence k_n bounded? ($x_{11} \ldots x_n$ and $\lambda_1 \ldots \lambda_{k_n}$ are real or complex variables).

21. Problem: Ulam

Can one make from the disc, $x^2 + y^2 \leq 1$, the surface of a torus using transformations with arbitrary small counterimages? (That is to say, for every $\epsilon > 0$ there should exist a transformation called f(p) of the disc into the torus, such that if $|p_1 - p_2| \geq \epsilon$ then $f(p_1) \neq f(p_2)$.

22. Problem: Ulam-Schreier

Is every set z of real numbers a Borel set with respect to sets G which are additive groups of real numbers? (Can any set z be obtained through the operations Σ performed countably many times and through operations of forming differences of sets, starting with sets G such that if x,y belong to the set G, then $x - y$ also belongs to G?)

23. Problem: Schauder

Definition (a): A function defined in a certain n-dimensional region is called monotonic in this region if, in every subregion, it assumes its maximum and minimum on the boundary. A function is called a saddle function if, after subtracting an arbitrary linear function, it is always monotonic. Definition (b): Let C be a plane region; c = a Jordan curve which is its boundary; K = a space curve over c with one-to-one projection. (That is to say, two different points of K have different projections on c). I shall say that the curve K satisfies the triangle condition with a constant $\triangle$, if the steepness of the plane defined by any three different points of K is always $\leq \triangle$. By the steepness of a plane $z = ax + by + c$, we mean the number $\sqrt{a^2 + b^2}$.

Rado (and later J. von Neumann) proved this theorem: The surface (function), defined in a convex region C which is continuous $z = f(x,y)$, and is a saddle function and whose boundary curve satisfies the triangle condition with the constant $\triangle$, satisfies a Lipschitz condition with the same constant $\triangle$; that is to say, for any two points (x_1,y_1) and $(x_2,y_2) \in R$ we have:

$$|f(x_1,y_1) - f(x_2,y_2)| = \triangle\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Problem A: What can one say if the boundary curve is assumed to be merely continuous? For example, is a Lipschitz condition satisfied in every closed domain contained entirely in the interior of C?

Problem B: Can one prove anything analogous to Rado's theorem for functions of a greater number of variables $(n > 2)$?

24. Problem: Mazur

(Prize: Two small beers, S. Mazur)

In a space E of type (B), there is given an additive functional F(x) with the following property: If x(t) is a continuous function in $0 \leq t \leq 1$ with values in E, then F[x(t)] is a measurable function. Is F(x) continuous?

25. Problem: Schauder

Recently, the theory of integral equations was generalized for singular integral equations; that is to say, in which the integral expression $\int K(s,t)\, g(t)\, dt$ is considered as an improper integral in the sense of Cauchy. Under certain additional assumptions, the three well-known theorems of Fredholm (for equations with fixed limits) are also valid. In the sense of the theory of operations, equations of this type are probably not totally continuous in the corresponding spaces of type (B).

Problem: Find a new class of linear operations $F(x)$, which contains as special cases the integral equations of the above type (improper) and for which Fredholm theorems do not hold any more. The equations are of type: $y = x + F(x)$.

26. Problem: Mazur-Orlicz
(Prize: One small beer, S. Mazur)

Let E be a space of type $\{F_0\}$ and $\{F_n(x)\}$ a sequence of linear functionals in E converging to zero uniformly in every bounded set $R \subset E$. Is the sequence then convergent to zero uniformly in a certain neighborhood of zero? [E is of a type $\{F_0\}$ means that E is a space of type (F) with the following condition: If $x_n \in E$, $X_n \to 0$ and the number series

$$\sum_{n=1}^{\infty} |t_n|$$

is convergent, then the series

$$\sum_{n=1}^{\infty} t_n x_n$$

is convergent. $R \subset E$ is a bounded region if $x_n \in R$ and if the numbers $t_n \to 0$, then $t_n x_n \to 0$.]

Ad 26. The answer is negative. M. Eidelheit, June 4, 1938.

27. Problem: Mazur-Orlicz
(Prize: Five small beers, S. Mazur)

Let E be a complex space of type (B); $F(x)$, $G(x)$ complex polynomials defined in E. Let us assume that there exist elements $x_n \in E$ such that $|x_n| \leq 1$ and $F(x_n) \to 0$, $G(x_n) \to 0$. Does there exist then an element x_0 such that $F(x_0) = 0$, $G(x_0) = 0$?

Ad 27. The answer is positive. If there is no $x_0 \in E$, such that $F(x_0) = 0$, $G(x_0) = 0$, then there exist complex polynomials $\phi(x)$, $\psi(x)$ in E with the property that

$$F(x)\,\phi(x) + G(x)\,\psi(x) \equiv 1.$$

Mazur-Orlicz, April 4, 1939

28. Problem: Mazur
(Prize: Bottle of wine, S. Mazur)

Let

$$\sum_{n=1}^{\infty} a_n$$

be a series of real terms and let us denote by R the set of all numbers a for which there exists a series differing only by the order of terms from

$$\sum_{n=1}^{\infty} a_n,$$

summable by the method of the first mean to a. Is it true that if the set R contains more than one number but not all the numbers, then it must consist of all numbers of a certain progression $\alpha x + \beta (x = \pm 1, \pm 2, \ldots$ and $0)$?

The same question for other methods of summation. [It is known that
(1) There exists a series

$$\sum_{n=1}^{\infty} a_n$$

such that R consists of all terms of a sequence given in advance $\alpha x + \beta (x = 0, \pm 1, \pm 2, \ldots)$; (2) If the sequence $\{a_n\}$ is bounded, then R consists of either one number or contains all the numbers — the first case occurs only when the series a_n is absolutely convergent.]

29. Problem: Ulam

Is the group H_n of all homeomorphisms of the surface of an n-dimensional sphere simple? (In the following sense: the component of identity does not contain a nontrivial normal subgroup.) It is known (Schreier-Ulam) that the theorem holds for $n = 2$ and the component of identity of H_n does not contain any closed, normal proper subgroups for any n.

30. Problem: Ulam

Two elements a and b of a group H are equivalent if there exists $h \in H$ such that there is a relation $a = hbh^{-1}$. Two pairs of elements: a', a'' and b', b'' are called simultaneously equivalent if there exists $h \in H$ such that we have $a' = hb'h^{-1}$ and $a'' = hb''h^{-1}$.
Question: For which groups does it suffice for simultaneous equivalence of two pairs of elements a', a'' and b', b'' that every combination of the elements a' and a'' be equivalent to the corresponding combination of the elements b' and b''. (The necessity of this condition is obvious.)

31. Problem: Ulam

In a metric group which is complete and compact, is the set of elements, equivalent to a given one, always of first category? Does this theorem hold under the additional assumptions that the group is connected or simple?

June 18, 1936

Ad 31. Banach-Mazur: Counterexample: $S_a = e^{ix} \to e^{i(x+a)}$, $T_b = e^{ix} \to e^{-i(x+b)}$.

June 18, 1936

32. Problem: Ulam

Let G be a compact metric group (the group operation we shall denote by $\times$). Does there exist for every $\epsilon > 0$ a finite number of elements of the group: $a_1, a_2, \ldots a_N$ for which we can define a group operation (denoted by the symbol $\circ$) so that with respect to this operation the given finite system forms a group and:

(1) $(a_i \times a_j, a_i \circ a_j) < \epsilon$ i,j = 1,2,... N [(a,b) denotes the distance between the elements a,b].

(2) The inverses of the element a_i(i = 1,2, ... N) with respect to the two operations are distant from each other by less than ϵ?

33. Problem: Ulam

Two sequences of sets of real numbers A_n and B_n are called equivalent if there exists an arbitrary function f mapping the set of all numbers into itself in a one-to-one way and such that $f(A_n) = B_n$.

Questions:

(a) Is every sequence A_n of projective sets equivalent to a certain sequence of Borel sets?

(b) Is every sequence of measurable sets — in the sense of Lebesgue — equivalent to a sequence of Borel sets? Can one prove that there exists a sequence not equivalent to any sequence of sets which is Lebesgue measurable?

Ad 33. August 1, 1935. There exist sequences of projective sets and sequences of measurable sets nonequivalent to sequences of Borel sets. (Communicated by Mr. Szpilrajn, who obtained additional results concerning this notion of equivalence of sequences of sets.) (Fund. Math. **26**)

Ulam.

34. Problem: Ulam

A class K of sets of real numbers has the following properties:

(1) The class K contains all sets measurable in the sense of Lebesgue.

(2) If $A \in K$ and $B \in K$ then $A - B \in K$.

(3) If $A_n \in K$ then $\Sigma A_n \in K$.

(4) Suppose the whole space is decomposed into a noncountable number of sets A_ξ all disjoint, each noncountable and each belonging to K, then there exists in the class K a set which contains exactly one element from each of the sets A_ξ.

Question: Is the class K the class of all subsets of our space?

35. Problem: Ulam

Is the projective Hilbert space (that is to say, the set of all diameters of the unit sphere in Hilbert space metrized by the Hausdorff formula) homeomorphic to the Hilbert space itself?

36. Problem: Ulam

Can one transform continuously the full sphere of a Hilbert space on its boundary in such a way that the transformation should be identity on the boundary?

Ad 36. There exists a transformation with the required property — given by Tychonoff.

37. Problem: Ulam

A class of sets K is called a ring if, whenever $A \in K$, $B \in K$, then both $(A+B)$ and $(A-B) \in K$. Two rings of sets K and L are isomorphic if one can make correspond to every set of the ring K, in a one-to-one fashion, a set of the ring L so that the sum of sets goes over into the sum, the difference into the difference, and the counterimage contains all the sets of the ring K.
Questions:

(a) How many nonisomorphic rings of sets of real numbers exist?
(b) How many nonisomorphic rings of sets of integers exist?
(c) Is the ring of projective sets isomorphic to the ring of Borel sets?

Analogous questions for rings in the sense of countable addition, i.e., countable summation of sets which belong to K also belongs to K.

38. Problem: Ulam

Let there be given N elements (persons). To each element we attach K others among the given N at random (these are friends of a given person). What is the probability P_{KN} that from every element one can get to every other element through a chain of mutual friends? (The relation of friendship is not necessarily symmetric!) Find $\lim_{n \to \infty} P_{KN}$. (0 or 1?)

39. Problem: Auerbach

The absolute value of a number x satisfies the following conditions:

(1) $\phi(x) \geq 0$, $\phi(x) \not\equiv 0,1$
(2) $\phi(x+y) \geq \phi(x) + \phi(y)$
(3) $\phi(xy) = \phi(x)\phi(y)$.

The only continuous functions satisfying these conditions are: $\phi(x) = |x|^{\alpha}$, where α is constant and $0 < \alpha \leq 1$. Do there exist discontinuous functions with the above properties?

Ad 39. This follows from Lebesgue's theorem [See for example, E. Kamke, Zur Definition der affinen Abbildung, Jahresb. d.D.M.V., **36** (1927): There exists a complex function of a complex variable $w = f(z)$ discontinuous and such that: $f(z_1 + z_2) = f(z_1) + f(z_2)$, $f(z_1 z_2) = f(z_1)f(z_2)$; $\phi(x) = |f(x)|$ satisfies (1), (2), (3), and is discontinuous.]

S. Mazur, April 10, 1937

40. Problem: Banach-Ulam

Can one define a completely additive measure function for all the projective sets on the interval (0,1) which, for Borel sets, coincides with Lebesgue measure? In particular, can one define such a measure on the ring of sets of the sets P(A)? (projective) All this with the additional requirement that congruent sets should have equal measure.

July 26, 1935

41. Problem: Mazur

Does there exist a 3-dimensional space of type (B) with the property that every 2-dimensional space of type (B) is isometric to a subspace of it? This is equivalent to the question: Does there exist in the 3-dimensional Euclidian space a convex surface W which has a center 0 with the property that every convex curve with a center is affine to a plane section of W through 0? More generally, given an integer $k \geq 2$, does there exist an integer i and an i-dimensional space of type (B) such that every k-dimensional space of type (B) is isometric to a subspace of it; given k, determine the smallest i.

42. Problem: Ulam

To every convex, closed set X, contained in a sphere K in Euclidean space, there is assigned another convex, closed set f(X), contained in K, in a continuous manner (in the sense of the Hausdorff metric); does there exist a fixed point, that is to say, a closed convex X_0 such that $f(X_0) = X_0$?

Theorem: Mazur

Let E be the class of convex closed sets contained in a sphere K with the properties:

(1) If $A \in K$, $B \in K$, then also $\lambda A + (1 - \lambda)B \in K$, for $0 \leq \lambda \leq 1$ [$\lambda A + (1 - \lambda)B$ denotes the set of points $\lambda x + (1 - \lambda)y$ for $x \in A$ and $y \in B$];

(2) If $A_n \in K$ and the sequence $\{A_n\}$ converges to A, then $A \in K$.

Suppose that f(x) is a continuous function in K whose values belong to K; then there exists a fixed point; that is, an $X_0 \in K$ such that $f(X_0) = X_0$. Examples of such a class E are, for instance: The class of all closed, convex sets contained in K with the diameter not greater than a given number $\phi > 0$.

43. Definition of a Certain Game: Mazur

Given is a set E of real numbers. A game between two players A and B is defined as follows: A selects an arbitrary interval d_1, B then selects an arbitrary segment (interval) d_2 contained in d_1; then A in his turn selects an arbitrary segment d_3 contained in d_2 and so on. A wins if the intersection $d_1, d_2, \ldots, d_n \ldots$ contains a point of the set E; otherwise, he loses. If E is a complement of

a set of first category, there exists a method through which A can win; if E is a set of first category, there exists a method through which B will win.
Problem: (Prize: One bottle of wine, S. Mazur) It is true that there exists a method of winning for the player A only for those sets E whose complement is, in a certain interval, of first category; similarly, does a method of win exist for B if E is a set of first category?

Ad 43. Mazur's conjecture is true.

S. Banach, August 4, 1935

A Modification of Mazur's Game

(1) (Ulam) There is given a set of real numbers E. The players A and B give in turn the digits 0 or 1. E wins if the number formed by these digits in a given order (in the binary system) belongs to E. For which E does there exist a method of win for the player A (player B)?

(2) (Banach) There is given a set of real numbers E. The two players A and B give in turn real numbers which are positive and such that a player gives always a number smaller than the last one given. The player A wins if the sum of the given series of numbers is an element of the set E. The same question as for (1).

44. Problem: H. Steinhaus

A continuous function $z = f(x,y)$ represents a surface such that through every point of it there exist two straight lines contained completely in the surface. Prove that the surface is then a hyperbolic paraboloid. Do the same without assuming continuity of f.

Ad 44. July 30, 1935. This problem was solved affirmatively by Banach — also without assuming continuity. The proof is based upon the remark: Any two straight lines on this surface either intersect or else their projections on the plane xy are parallel.

45. Problem: Banach

Let G be a metric group which is complete and non-Abelian; $U_1(x)$, $U_2(x)$, ... $U_n(x)$ multiplicative operations defined in G and with values belonging to G. Prove that if the operation $U(x) = U_1(x)U_2(x)...U_n(x)$ is of a Baire class, then it must be continuous. This statement is true for $n = 2$.

46. Problem: Banach

Is the sphere in a space of type (B) unicoherent? (That is to say, is, in every decomposition of it into continua A, B, the intersection AB connected?)

Ad 46. An affirmative answer to Prof. Banach's problem follows from the following theorem of Borsuk: In every space which is connected, locally connected, complete and unicoherent, there exists a simple closed curve which is a retract. In general linear spaces, in which the multiplication is continuous, an affirmative answer to Prof. Banach's problem follows from my theorems in Fund. Math., Vol. 26, p. 61.

I. Eilenberg.

47. Problem: Banach

Can every permutation of a matrix $\{a_{ik}\}$ $i,k = 1,2,\ldots\infty$ be obtained by composing a finite number of permutations in such a way that the rows go over into rows and columns into columns (*Vide* Problem 20: Ulam).

48. Problem: Mazur-Banach

Let E be a set of real numbers which is countable, closed, and bounded. W is the set of all continuous real valued functions defined on E. Is the space W [if we define the norm of a function $f \in W$ as follows: $||f|| = \max_{x \in E} f(x)$] isomorphic to the space c of all convergent sequences?

Ad 48. The answer is affirmative, Mazur, February 15, 1939. (The solution is unpublished.)

49. Problem: Mazur-Banach

Does there exist a space E of type (B) with the property (W) which is universal for all spaces of type (B) with the property (W)? One should investigate, this question for the following properties (W):

(1) The space is separable and weakly compact (that is, from every bounded sequence one can select a subsequence weakly convergent to an element).

(2) The space contains a base (countable).

(3) The adjoint space is separable.

The space E is universal isometrically (or isomorphically) for spaces of a given class K if every space of this class is isometric (or isomorphic) to a linear subspace of the space E.

50. Problem: Banach

Prove that the integral of Denjoy is a Baire functional in the space M (that is to say, in the space of measurable functions).

51. Problem: Mazur

(a) Is a set of functions, measurable in $<0,1>$ with the property that every two functions of the set are orthogonal, at most countable? (I do not assume that the functions are square-integrable!)

(b) Analogous question for sequences: Is the set of sequences with the property that any two sequences $\{\xi_n\}$, $\{\eta_n\}$ of this set are orthogonal, that is

$$\sum_{n=1}^{\infty} \xi_n \eta_n = 0$$

at most countable?

Ad 51. Solved by Mazurkiewicz.

52. Problem: Banach

Show that the class of functions which are continuous and defined in the interval (0,1) and which have everywhere a derivative, does not form a Borel set in the space C of all continuous functions in (0,1). One can show that it is not a set F_σ and also it is the complement of an analytic set.

Ad 52. Solved by Mazurkiewicz.

53. Problem: Banach

A surface element C (i.e., a one-to-one continuous image of a disc) has the following property: For every $\epsilon > 0$ one can find $\eta > 0$ such that any two points of C with a distance less than η can be connected by an arc contained in C with a length less than ϵ. Show that C has a finite area and almost everywhere a tangent plane.

Ad 53. There exists a surface element C of the form $z = f(x,y)$, $0 \leq x,y \leq 1$ satisfying above conditions but without possessing a finite area.

Mazur, August 1, 1935

54. Problem: Schauder

(a) A convex closed, compact set H is transformed by a continuous mapping U(x) on a part of itself. H is contained in a space of type (F). Does there exist a fixed point of the transformation?

(b) Solve the same problem for arbitrary linear topological spaces or such spaces in which there exist arbitrarily small convex neighborhoods.

[A solution exists for spaces of type (F_0); in the more general theorem H need not be compact; only U(H) is assumed compact].

55. Problem: Mazur

There is given, in an n-dimensional space E — or, more generally — in the space of type (B), a polynomial W(x) bounded in an ϵ-neighborhood of a certain nonbounded set $R \subset E$ (an ϵ-neighborhood of a set R is the set of all points which are distant by less than ϵ from R). Does there exist a polynomial V(x) and a polynomial of first degree $\phi(x)$ such that

(1) $W(x) = V(\phi(x))$;

(2) The set $\phi(R)$, that is to say, the image of the set R under the mapping $\phi(x)$ is bounded?

Ad 55. In the case of Euclidean spaces, a solution for a finite system of polynomials: There exists a linear substitution with the determinant $\neq 0$ under which all the polynomials of the given system go over into polynomials depending on a smaller number of the given variables. (Studia Math. 5).

56. Problem: Mazur-Orlicz

In a space E of type (B) there is given a functional F(x) of degree m and discontinuous. [F is of degree m means that for x_0, $h_0 \in E$ there exist numbers $a_0, \ldots a_m$ such that $F(x_0 + th_0) = a_0 + ta_1 + \ldots. + t^m a_m$ for rational t.] Do there then exist points $x_n \in E$ such that $x_n \to 0$ and $|F(x_n + x)| \to +\infty$ or even only

$$\overline{\lim_{n\to\infty}} |F(x_n + x)| = +\infty$$

for all $x \in E$? Not solved even for finite-dimensional spaces E.

57. Problem: Ruziewicz

Given are two functions w(h) and $\phi(h)$, decreasing with $|h|$ to 0, and satisfying the conditions

$$\lim_{h\to 0} \frac{w(h)}{|h|} = \infty,$$

and

$$\lim_{h\to 0} \frac{w(h)}{\phi(h)} = \infty.$$

Does there exist a function satisfying the conditions:

(1) $|f(x + h) - f(x)| < w(h)$;

(2) $\overline{\lim_{h\to 0}} \left|\frac{f(x + h) - f(x)}{\phi(h)}\right| = \infty$?

58. Problem: Ruziewicz

A set E_1 (of real numbers) precedes the set E_2, which we denote by E_1pE_2, if:

(1) E_1 is of a lower homoie class than $E_2 (E_1 < E_2)$,

(2) There does not exist a set E_3 so that $E_1 < E_3 < E_2$.

(a) Do there exist sets A,B,C and $\{A_n\}$, $n = 1,2,\ldots N$, $(N > 1)$, such that ApBpC and $ApA_1pA_2p\ldots pA_npC$?

(Remark: For n = 2 such sets exist; cf. Fund. Math., **15**, p. 95.)

(b) Do there exist sets A,B,C and $\{A_n\}$, $n = 1,2,\ldots$ ad inf. such that ApBpC and $ApA_1pA_2pA_3p\ldots$ ad inf., and $A_n < C$ for $n = 1,2,3,\ldots$?

59. Problem: Ruziewicz

Can one decompose a square into a finite number of squares all different?

60. Problem: Ruziewicz

Can one, for every $\epsilon > 0$, represent the surface of a sphere as a sum of a finite number of regions which are smaller in diameter than ϵ, closed, connected, congruent, and have no interior point in common? We assume that the boundaries of these sets are: (a) polygons, (b) curves of finite length, (c) sets of measure zero.

61. Problem: Steinhaus (cf. Problem 44, July 30, 1935)

(a) Determine the surfaces $z = f(x,y)$ such that in each of their points there intersect two plane curves congruent to each other.

(b) Determine the surface $z = f(x,y)$ such that in each point there intersect two plane curves congruent to one of them (for every point the same curve).

Ad 61. All surfaces of revolution have this property; whether these are the only ones, is not known.

July 21, 1935, Ruziewicz

62. Problem: Mazur-Ulam

In a group G there are given groups G_n $n = 1,2,\ldots$ ad inf. with the following properties: $G = G_1 + G_2 + \ldots + G_n + \ldots$, $G_n \subset G_{n+1}$, G_n is isomorphic to G_1. Is G isomorphic to G_1?

Ad 62. As R. Baer remarked, the answer is trivial: G_n the group of numbers with the denominator n, $G = \Sigma G_n \equiv$ the group of rational numbers.

63. Problem: Mazur-Ulam

The set E of elements of a group G we call a base if E spans a group which is identical with G, but no proper subset of the set E has this property. If there is a base in a group G, does there exist a base for every subgroup H of it?

64. Problem: Mazur

In a space E of type (B) there are given two convex bodies A and B and their distance from each other is positive. (A convex body is a convex set which is closed, bounded and possesses interior points.) Does there exist a hyperplane H which separates the two bodies, A,B? That is to say, it has the property that one of the bodies lies on one, the other on the other side of this hyperplane. [Hyperplane means a set of all points x satisfying the equation $F(x) - c = 0$, where $F(x)$ is a linear functional $\neq 0$, and c a constant.]

Ad 64. The theorem is true even when the two bodies are not disjoint, but do not have common interior points.

January 11, 1936, Eidelheit

65. Problem: Mazur

In a space E of type (B) there is given a convex set W, containing 0 and nowhere dense. Is the smallest convex set containing W, symmetric with respect to 0 (that is to say, the set generated by elements $x-y$, where $x \in W$, $y \in W$) also nowhere dense?

Ad 65. False, the set W composed of functions which are nondecreasing, in the space (C) of all continuous functions is convex and nowhere dense. It contains 0; the convex set containing W and symmetric with respect to zero contains all functions of bounded variations and is not nowhere dense.

Mazur

66. Problem: Mazur

The real-valued function $z = f(x,y)$ of real variables x,y possesses the first partial derivatives $\partial f/\partial x$, $\partial f/\partial y$ and the second partial derivatives $\partial^2 f/\partial x^2$, $\partial^2 f/\partial y^2$. Do there exist then almost everywhere the mixed second partial derivatives $\partial^2 f/\partial x\partial y$, $\partial^2 f/\partial y\partial x$? According to a remark by Prof. Schauder, this theorem is true with the following additional assumptions: The derivatives $\partial f/\partial x$, $\partial f/\partial y$ are absolutely continuous in the sense of Tonelli, and the derivatives $\partial^2 f/\partial x^2$, $\partial^2 f/\partial y^2$ are square integrable. An analogous question for n variables.

67. Problem: Banach

(A modification of Mazur's game, August 1, 1935)

We call a half of the set E [in symbols, (1/2)E] an arbitrary subset $H \subset E$ such that the sets $E, H, E-H$ are of equal power.

(1) Two players A and B give in turn sets E_i $i = 1,2,\ldots$ ad inf. so that $E_i = (1/2)E_{i-1}$ $i = 1,2,\ldots$ where E_0 is a given abstract set. The player A wins if the product $E_1E_2\ldots E_iE_{i+1}\ldots$ is vacuous.

(2) The game, similar to one above, with the assumption that $E_i = 1/2\,[E_0 - E_1 - \ldots - E_{i-1}]$ $i = 2,3,\ldots$ ad inf., and $E_1 = (1/2)E_0$. The player A wins if $E_1 + E_2 + \ldots = E_0$.

Is there a method of win for the player A? If E_0 is of power cofinal with $\aleph_0$, then the player A has a method of win. Is it only in this case? In particular, solve the problem if E_0 is the set of real numbers.

Ad 67. There exists a method of play which will guarantee that the product of the sets is vacuous. The solution was given by J. Schreier, August 24, 1935.

68. Problem: Ulam

There is given an n-dimensional manifold R with the property that every section of its boundary by hyperplane of n-1 dimensions gives an n-2 dimensional closed surface (a set homeomorphic to a surface of the sphere of this dimensionally). Prove that R is a convex set. This question was settled affirmatively for n = 3 by Schreier. (That is to say, a manifold contained in E_3, such that every section by a plane gives a single simple closed curve, must be convex.)

69. Problem: Mazur-Ulam

The problem of characterizing the spaces of type (B) among metric spaces: There is given a complete metric space E with the following properties:

(1) If $p,q \in E$, there exists exactly one $x \in E$, such that x is a metric center of the couple (p,q);

(2) If $p,q \in E$, there exists exactly one $x \in E$, such that q is a metric center of the couple (p,x).

Is the space E isometric to a certain space of type (B)? [Every space of type (B) has the properties of (1) and (2).]

Definition of a metric center of a couple of points (p,q): We take the set of all points $x \in E$ such that $\overline{px} + \overline{xq} = \overline{pq}$; we denote it by R. By R_1 we denote the set of all points $r \in R$ such that $\overline{rx} \leq d(R)/2$ for every $x \in R$, where d(R) is the diameter of the set R; we denote by R_{n+1} the set of all points $r \in R_n$ such that $\overline{rx} \leq d(R_n)/2$ for all $x \in R_n$. One can show that the intersection $R_1R_2\ldots R_n\ldots$ contains at most one point; if such a point exists, we call it the metric center of the pair (p,q).

Ad 69. The answer is negative.

S. Mazur, December 21, 1936

70. Problem: Ulam

Prove the following lemma: Let $f_1(x_1,x_2,\ldots,x_n; t_1,t_2,\ldots,t_r)$ $0 \leq x_i \leq 1$; $0 \leq t_j \leq 1$, $i = 1\ldots n$; $j = 1\ldots r$ be a polynomial with variables x_i and t_j real valued and vanishing identically at the point X = $(0,0\ldots 0; t\ldots t_r)$; ϵ a positive number. There exists then a polynomial f_2 in the same variables and constants K and ρ both positive and independent from ϵ (both K and ρ = 1?) such that the following conditions are satisfied.

(1) $f_1(x_1,\ldots x_n; t_1,\ldots t_r) - f_2(x_1,\ldots x_n; t_1,\ldots t_r) < \epsilon$

(2) The derivatives with respect to the variables x at the point $x_i = 0$, $i = 1\ldots n$ imitate the behavior of the polynomial; that is to say, if T' and T'' denote two sets of variables $t_1'\ldots t_r'$ and $t_1''\ldots t_r''$ so that $|f_2(x_1\ldots x_n; T') - f_2(x_1\ldots x_n; T'')| < \epsilon$ then we have for every i:

$$\left| \left[\frac{\partial f_2\,(x_1 \ldots x_n; t_1' \ldots t_r')}{\partial x_i}\right]_{x_1 = x_2 = \ldots x_n = 0} - \left[\frac{\partial f_2\,(x_1 \ldots x_n; t_1'' \ldots t_r'')}{\partial x_i}\right]_{x_1 = x_2 = \ldots x_n = 0} \right| < K\epsilon$$

(3) The derivatives with respect to at least one of the variables x at the point $x_1 = x_2 = \ldots = 0$ are *essentially* different from zero. That is to say, there exist points T* and T** such that

$$\left| \left[\frac{\partial f_2(x_1 \ldots x_n; T^*)}{\partial x_i}\right]_{x_1 = x_2 = \ldots = 0} - \left[\frac{\partial f_2(x_1 \ldots x_n; T^{**})}{\partial x_i}\right]_{x_1 = x_2 = \ldots = 0} \right| > \rho$$

From an affirmative solution, i.e., from this lemma, there would follow an affirmative answer to Hilbert's problem concerning introduction of analytic parameters in n-parameter groups. (The problem was solved for compact groups by von Neumann.)

71. Problem: Ulam

Find all the permutations f(n) of the sequence of natural integers which have the property that if $\{n_k\}$ is an arbitrary sequence of integers with a density α, then the sequence $f(n_k)$ has also a density α, in the set of all integers.

72. Problem: Mazur

Let E be a space of type (F) with the following property: If $Z \subset E$ is a compact set, then the smallest convex set containing Z is also compact. Is E then a space of type (F_0)? [See Problem 26 for a definition of (F_0).]

73. Problem: Mazur-Orlicz

Let c_n be the smallest number with the property that if $F(x_1, \ldots x_n)$ is an arbitrary symmetric n-linear operator [in a space of type (B) and with values in such a space], then

$$\max_{\|x_i\| \leqslant 1,\, i = 1,2,\ldots n} |F(x_1, \ldots x_n)| \leqslant c_n \max_{\|x\| \leqslant 1} |F(x_1, \ldots x_n)|.$$

It is known (Mr. Banach) that c_n exists. One can show that the number c_n satisfies the inequalities

$$\frac{n^n}{n!} \leqslant c_n \leqslant \frac{1}{n!} \sum_{k=1}^{n} \binom{n}{k} \cdot k^n$$

Is $c_n = n^n/n!$? (Mazur-Orlicz)

74. Problem: Mazur-Orlicz

Given is a polynomial

$$W(t_1, \ldots t_n) = \sum_{k_1 + \ldots + k_n = n} a_{k_1 \ldots k_n} t^{k_1} \ldots t^{k_n}$$

in real variables $t_1, \dots t_n$, homogeneous and of order n; let us assume that $|W(t_1, \dots t_n)| \leq 1$ for all $t_1, \dots t_n$ such that $|t_1| + \dots + |t_n| \leq 1$. Do we then have

$$\left|a_{k_1 \dots k_n}\right| \leqslant \frac{n^n}{k_1! \dots k_n!} \quad ?$$

75. Problem: Mazur

In the Euclidean n-dimensional space E, or, more generally, in a space of type (B) there is given a polynomial W(x). α is a number $\neq 0$. If a polynomial W(x) is bounded in an ϵ-neighborhood of a certain set $R \subset E$ is it then bounded in a δ-neighborhood of the set αR (which is the set composed of elements αx for $x \in R$)? (See Problem 55.)

Ad 75. From the solution of Problem 55, it follows that the theorem is true in the case of a Euclidean space. (Mazur)

76. Problem: Mazur

Given is (in the 3-dimensional Euclidean space) a convex surface W and point 0 in its interior. Consider the set V of all points P defined by the property that the length of the interval P0 is equal to the area of the plane section of W through 0 and perpendicular to this interval. Is the set V convex?

77. Problem: Ulam

(Prize for (a): One bottle of wine, S. Eilenberg)

(a) Let A and B be two topological spaces such that the spaces A^2 and B^2 are homeomorphic. Is then the space A homeomorphic to the space B?

(b) Let A and B be two metric spaces such that A^2 is isometric to B^2. Is A isometric with B?

(c) Let A and B be two abstract groups such that A^2 and B^2 form isometric groups. Is A isomorphic with B?

[We understand by A^2 resp. B^2 the set of ordered pairs of elements of the set A (or B).] A topology [or, in the Problem (c), the group operation] in such sets is defined in the obvious manner. A metric in such a space, when the original space is metric, is defined, for example, in the "Euclidean" manner: by the square root of the sum of squares of the distances between projections.

78. Problem: Steinhaus

Find all surfaces with the following property: Through every point of the surface there lie two curves congruent respectively to two given curves A and B. Compare Problem 61. (Such a surface is, for example, a cylinder: the curves A and B are here a circle and a straight line).

August 2, 1935

79. Problem: Mazur-Orlicz

A polynomial $y = U(x)$ maps, in a one-to-one fashion, a space X of type (B) onto a space Y of type (B); the inverse of this mapping $x = U^{-1}(y)$ is also a polynomial. Is the polynomial $y = U(x)$ of first degree? Not decided even in the case when X and Y are a Euclidean plane; in this case, the question is given for a one-to-one mapping $t' = \phi(t,s)$, $s' = \psi(t,s)$ of a plane into itself where $\phi(t,s)$, $\psi(t,s)$ are polynomials; the inverse mapping has also the form $t = \Phi(t',s')$, $s = \Psi(t',s')$ where $\Phi(t',s')$, $\Psi(t',s')$ are polynomials. Is the mapping the affine; that is to say, of the form $t' = a_1t + b_1s + c_1$, $s' = a_2t + b_2s + c_2$ where $a_1b_2 - a_2b_1 \neq 0$?

Ad 79. Trivial. In the Euclidean space:

$$y_1 = x_1 + k$$
$$y_2 = x_2 + \phi_2(x_1)$$
$$y_3 = x_3 + \phi_3(x_1,x_2)$$
$$\cdots\cdots$$
$$y_n = x_n + \phi_n(x_1 \ldots x_{n-1})$$

where k is an arbitrary constant and $\phi_2 \ldots \phi_n$ are arbitrary polynomials in their variables. The inverse mapping is obvious at once.

80. Problem: Mazur

Let E be a complete metric space; we denote by E^∞ a complete metric space formed by the set of all sequences $\{e_n\}$ of elements of E. By a distance between two such sequences $\{e'_n\}$, $\{e''_n\}$ we understand the number

$$\sum_{n=1}^{\infty} 2^{-n} \frac{(e'_n, e''_n)}{1 + (e'_n, e''_n)}$$

[For $e', e'' \in E$ we denote by (e',e'') the distance between the elements e', e"]. If R is a given set contained in E, then we denote by R_δ the set of all sequences $\{r_n\}$ of elements of R, and by R_σ the set of all sequences $\{r_n\}$ of elements of R such that $r_n = r_0$ almost always; r_0 is a fixed element of R. Is it true that: If the set R is an F_σ set but not closed, then R_δ is an $F_{\sigma\delta}$ set but not an F_σ; if the set R is an $F_{\sigma\delta}$ but not an F_σ, then R_σ is an $F_{\sigma\delta\sigma}$ but not an $F_{\sigma\delta}$; more generally, if R is an $F_{2\xi+1}$ but not an $F_{2\xi}$, and if R_δ is an $F_{2\xi+2}$ but not an $F_{2\xi+1}$, then R_σ is an $F_{2\xi+3}$ but not an $F_{2\xi+2}$ ($F_0 = F$, $F_1 = F_\sigma$, $F_2 = F_{\sigma\delta}$, $F_3 = F_{\sigma\delta\sigma}, \ldots$)? Investigate in particular the case when the space E is compact or of type (B) or of type (F).

81. Problem: Steinhaus
(Compare Problems 44 and 61)

A parabolic hyperboloid and a plane are composed, in *two* ways, of curves which are congruent (AA; BB), straight lines and parabolas. Do there exist other surfaces of this kind? Are they composed of (AB), (CD)? Is it true that such surfaces, namely, all surfaces having in each point two intersecting curves congruent to A and B respectively (exeptis excipiendis), are necessarily of the form $z = f(x) + g(y)$? (The plane, sphere, and circular cylinder are considered trivial.)

August 6, 1935

82. Definition: Steinhaus

f(t) is independent (in the sense of correlation) from $y_1(t), y_2(t), \ldots y_n(t)$ $(0 \leq t \leq 1)$, if, for every function of n-variables $F(y_1, y_2, \ldots y_n)$ and for every 4-tuple of numbers $\alpha_1\beta_1\alpha_2\beta_2$ the sets which are defined as follows: $A = E_t(\alpha_1 \leq f(t) \leq \beta_1)$, $B = E_t(\alpha_2 \leq F(y_1(t), y_2(t), \ldots y_n(t)) \leq \beta_2)$ have the property that $|AB| = |A| \cdot |B|$.

Problems:

(1) Is a set of functions mutually independent (that is to say, each independent of all the other n) at most countable?

(2) Does a system like that have to be complete and orthogonal, or only complete?

August 6, 1935

Remark: The notion of independence introduced above is what natural scientists call "complete lack of correlation." (Their definitions are, however, not too precise.)

Ad 82. Under the assumption that the functions which are independent are integrable, together with their ℓ^{th} power, we have the following relation:

$$\int_0^1 y_{k_1}(t)\, y_{k_2}(t) \ldots y_{k_\ell}(t)dt = \prod_{i=1}^{\ell} \int_0^1 y_{k_i}(t)dt.$$

It follows immediately that the system $\{\phi_i(t)\}$ where $\phi_i(t) = y_i(t) - \int_0^1 y_i(t)dt$ is orthogonal. If we assume that $y_i(t) \in L$, then the system is "lacunary" (and therefore cannot be complete). Lacunarity follows in this case from the relation

$$\int_0^1 \left| \sum_{i=1}^{n} \phi_i(t) \right| dt \geqslant \sqrt{M \sum_{i=1}^{n} \phi_i^2(t)dt}$$

where M does not depend on n nor does it depend on the sequence

$$\left\{ \int_0^1 \phi_i^2(t)dt \right\}.$$

October 12, 1935

83. Problem: Auerbach

We assume about a continuous function f(x) that it satisfies at every point the condition

$$\overline{\lim_{h \to 0}} \left| \frac{f(x + h) - f(x)}{h^a} \right| < M$$

(M a constant, $0 < a < 1$, a constant). Does the function f(x) satisfy a Hölder condition? (It is easy to prove that in every interval of a certain dense set of intervals, Hölder condition holds with the exponent a and with the same constant.)

Ad 83. The answer is negative. We define the function f(x) as a triangular one in intervals $1/n - x_n$, $1/n + x_n$ with the height $1/n$.

Marcinkiewicz.

84. Problem: Auerbach

One assumes about a convex surface in the 3-dimensional space that all its plane sections by planes going through a fixed point 0 inside the surface are projectively equivalent. Is this surface an ellipsoid?

85. Problem: Banach

(a) Does there exist a sequence of measurable functions $\{\phi_n(t)\}$ $(0 \leq t \leq 1)$, belonging to L^2, orthogonal, normed, complete and such that the development of every polynomial is divergent almost everywhere?

(b) The same question if, instead of polynomials, we consider analytic functions for $0 \leq t \leq 1$.

One can prove that the answer to Question (a) is an affirmative one if we admit only polynomials of degree less than n (n arbitrarily given ahead of time).

86. Problem: Banach

Given a sequence of functions $\{\phi_n(t)\}$ orthogonal, normed, measurable, and uniformly bounded; can one always complete it, using functions with the same bound, to a sequence which is orthogonal, normed, and complete? Consider the case when infinitely many functions are necessary for completion.

87. Problem: Banach

Let $y = U(x)$ be an operation which is continuous and satisfies a Lipschitz condition. The operation is defined in $L^\beta (\beta \geq 1)$ and its image is also contained in L^β. We assume that for a certain $\alpha > \beta$ there exists a constant M_α such that if $x \in L^\alpha$ then $U(x) \in L^\alpha$ and $||U(x)||_\alpha \leq M_\alpha ||x||_\alpha$. Show that for every γ such that $\beta < \gamma < \alpha$ there exists an M_γ with the property: If $x \in L^\gamma$ then $U(x) \in L^\gamma$ and $||U(x)||_\gamma \leq M_\gamma ||x||_\gamma$. This theorem is true under additional assumptions: U is a linear operation (follows from a theorem of M. Riesz). Banach showed that the theorem is true if $\alpha = \infty$.

$$\left[\|x(t)\|_\gamma = \left\{ \int_0^1 |x(t)|^\gamma dt \right\}^{1/\gamma} \right].$$

88. Problem: Mazur

Given is a sequence of numbers (a_n) with the property that for every bounded sequence (x_n) the series $|a_1x_1+a_2x_2+ \dots +a_nx_n+ \dots| + |a_2x_1+a_3x_2+ \dots +a_{n+1}x_n+ \dots| + \dots + |a_mx_1+a_{m+1}x_2+ \dots +a_{m+n-1}x_n+ \dots| + \dots$ converges. Is the series

$$\sum_{n=1}^{\infty} n|a_n|$$

convergent?
Remark: If sequences of numbers $(a_{1n}),(a_{2n}),\dots (a_{mn})$ are given with the property that for every bounded sequence of numbers (x_n) the series $|a_{11}x_1+a_{12}x_2+ \dots +a_{1n}x_n+ \dots| + |a_{21}x_1+a_{22}x_2+ \dots +a_{2n}x_n+ \dots| + \dots + |a_{m1}x_1+a_{m2}x_2+ \dots +a_{mn}x_n+ \dots| + \dots$ converges; then, according to a remark by Mr. Banach, the series

$$\sum_{m=1}^{\infty} \left(|a_{m1}| + |a_{m2}| + \dots + |a_{mn}| + \dots\right)$$

can diverge.

89. Problem: Mazur

Let W be a convex body, located in the space (L^2), and such that its boundary W_b does not contain any interval; let $x_n \in W$ $(n = 1,2,\dots)$ $x_0 \in W_b$ and in addition let the sequence (x_n) converge weakly to x_0. Does then the sequence (x_n) converge strongly to x_0? It is known that this statement is true in the case where W is a sphere. Examine this problem for the case of other spaces.

90. Problem: Ulam-Auerbach

It is known that every semisimple group of Lie (e.g., the projective group in n-variables) contains four elements generating a dense subgroup. Can one lower the number 4?

91. Problem: Mazur

A convex body W with a center is given, in the n-dimensional Euclidean space. It is affine to its conjugate body. Is W then an ellipsoid? The answer is negative in the case when n is an even number; for odd n the problem is not solved. It is equivalent to this: If a space of type (B) of n-dimensions is isometric to its conjugate space, is it then isometric to the Euclidean space?

92. Problem: Mazur

Given is a bounded sequence of numbers (s_n). There exist sequences of numbers (ℓ_n) with the property that:

(1) $\ell_n > 0$ $(n = 1,2,\dots)$;

(2) $\ell_1 + \ell_2 + \dots = \infty$;

(3) The sequence

$$\left(\frac{\ell_1 s_1 + \dots + \ell_n s_n}{\ell_1 + \dots + \ell_n}\right)$$

converges.

Do there exist sequences (ℓ_n) which, in addition to the properties (1), (2), and (3), satisfy the condition:

(4a) The sequence (ℓ_n) is fully monotonic; that is, all the differences $\triangle_n^1 = \ell_n - \ell_{n+1}$, $\triangle_n^2 = \triangle_n^1 - \triangle_{n+1}^1$, ... are nonnegative; or, only the condition:

(4b) The sequence (ℓ_n) is nonincreasing. If two sequences are given (ℓ_n'), (ℓ_n'') which satisfy the conditions (1), (2), (3), (4a) or merely (1), (2), (3), (4b), then can the limits

$$\lim_{n\to\infty} \frac{\ell_1' s_1 + \dots + \ell_n' s_n}{\ell_1' + \dots + \ell_n'}$$

and the limit

$$\lim_{n\to\infty} \frac{\ell_1'' s_1 + \dots + \ell_n'' s_n}{\ell_1'' + \dots + \ell_n''}$$

be different?

Ad 92. There exist sequences (ℓ_n'), (ℓ_n'') satisfying the conditions (1), (2), (3), (4b) such that for a certain bounded sequence s_n composed for 0's and 1's, the two limits

$$\lim_{n\to\infty} \frac{\ell_1' s_1 + \dots + \ell_n' s_n}{\ell_1' + \dots + \ell_n'}$$

and

$$\lim_{n\to\infty} \frac{\ell_1'' s_1 + \dots + \ell_n'' s_n}{\ell_1'' + \dots + \ell_n''}$$

exist but are different.

Mazur, August 10, 1935

93. Problem: Mazur

Let R be a plane set. The system of functions $x = f(t)$, $y = g(t)$ $(0 \leq t \leq 1)$ is called a parametric description of the set R, if the set of points $(f(t), g(t))$ is identical with R. Assume that for a given set R there exists a parametric description $x = f_1(t)$, $y = g_1(t)$ for which the functions $f_1(t)$, $g_1(t)$ are continuous and there also exists another parametric description $x = f_2(t)$, $y = g_2(t)$ where the functions $f_2(t)$, $g_2(t)$ are of bounded variation; does there exist a parametric description of R $x = f(t)$, $y = g(t)$ so that the functions $f(t)$, $g(t)$ are simultaneously continuous and of bounded variation? Assume that for a given set R there exist parametric descriptions $x = f(t)$, $y = g(t)$ for which the functions are of bounded variation and continuous—for every such description, we determine the length $d(f(t), g(t))$ of the set R and we take the lower bound of the numbers $d(f(t), g(t))$ denoted by d; does there exist a parametric description of R $x = f_0(t)$, $y = g_0(t)$ also with functions of bounded variation and continuous and such that $d(f_0(t), g_0(t)) = d$? The same problem in the case of the n-dimensional Euclidean space.

Ad 93.* The theorem is true; we can represent R by ξ functions $x = f_\xi(t)$, $y = g_\xi(t)$, continuous and of bounded variation, in such a way that the length of the curve (by Jordan's definitions) is at most twice the Carathéodory measure of R.

A. J. Ward, March 23, 1937

94. Problem: Z. Lomnicki-Ulam

Let $\lim_{n\to\infty} k_n/n = f < 1$, where always $k_n < n$. Prove that

$$\lim_{n\to\infty} \int_0^P \int\int\ldots\int x_1 \ldots x_{k_n} (1 - x_{k_n}) \ldots (1 - x_n)dx_1 \ldots dx_n dp = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \begin{matrix} P < f \\ P \geqslant f \end{matrix}$$

$$\begin{Bmatrix} x_1 + \ldots + x_n = np \\ 0 \leqslant x_i \leqslant 1, i = 1, \ldots n \end{Bmatrix}.$$

Compare Problem 17.

Ad 94. This conjecture was proved by S. Bochner in April, 1936 — he even gave the order of convergence. A paper on this topic will appear in Annals of Math.

S. Ulam, 1936

95. Problem: Schreier-Ulam

Is a group R of real numbers (under addition) isomorphically contained in a group S_∞ of all permutations of the sequence of natural integers?

Ad 95. The answer is affirmative.

Schreier-Ulam, November 1935

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*Written in English in original manuscript.

96. Problem: Ulam

Can the group S_∞ of all permutations of integers be so metrized that the group operation (composition of permutations) is a continuous function and the set S_∞ becomes, under this metric, a compact space? (locally compact?)

Ad 96. One cannot metrize in a compact way.

Schreier-Ulam, November 1935

97. Definition: Kuratowski-Ulam

Two sets (spaces) A and B are called quasi-homeomorphic if, for every ϵ there exists a continuous mapping f_ϵ of the space A onto the space B such that the counterimages are smaller than ϵ; that is to say, from $|x' - x''| > \epsilon$ it follows that $f(x') \neq f(x'')$ and, conversely, a continuous mapping g_ϵ with counterimages smaller than ϵ of the space B onto the space A.
Problem: Are two manifolds (topological spaces such that every point has a neighborhood homeomorphic to the n-dimensional Euclidean sphere) which are quasi-homeomorphic, of necessity homeomorphic?

98. Problem: Schreier-Ulam

Does there exist a finite number of analytic transformations of the n-dimensional sphere into itself, $f_1, \ldots f_n$, such that by composing these transformations a finite number of times, one can approximate arbitrarily any continuous transformation of the sphere into itself? How is it for one-to-one transformations? (Analytic means here — differentiable any number of times.)

99. Problem: Ulam

By product sets in the unit square, we understand the sets of all pairs (x,y) where x belongs to a given A, y to a given set B. Do there exist sets which cannot be obtained through the operations of forming countable sums and differences of sets starting from product sets? Do there exist non-projective sets with respect to product subsets?

100. Problem: Ulam-Banach

Let Z be a closed set contained in the surface of the n-dimensional sphere. Does there exist a sequence of homeomorphic mappings of the surface of the sphere onto itself, converging to a mapping of the surface onto Z?

Ad 100. For n = 2, affirmative answer by Borsuk.

101. Problem: Ulam

The group U of permutations of the sequence of integers is called infinitely transitive if it has the following property: If A and B are two sets of integers, both infinite and such that their complements to the set of all integers are also infinite, then there exists in the group U an element (permutation) such that f(A) = B. Is the group U identical with the group S_∞ of all permutations?

102. Problem: Ulam

(a) Let ϵ be a positive number; p and q two points of the unit square. In the first case let the point p be fixed and q wander at random. In the other case, assume that both points move at random. Is the probability of approach of the two points p and q within a distance $\leq \epsilon$ of each other, after n steps, greater in the first case than in the second?

(b) Let a,b denote two rotations of a circle of radius 1 through angles a,b. Let ϵ be a positive number. We define a set of pairs $E^1_\epsilon(a,b)$ as follows: Two rotations a,b belong to it if (na − b) mod 2π is smaller than ϵ earlier than (na − nb) mod 2π; that is to say, for smaller n than is the case for (na − nb) mod 2π. We denote by $E^2_\epsilon(a,b)$ the complement of the set of pairs E^1_ϵ (a,b) with respect to the set E of all pairs. Which of the two sets $E^1_\epsilon(a,b)$, $E^2_\epsilon(a,b)$ has greater measure? (Show that asymptotically these sets have equal measures.)

103. Problem: Schreier-Ulam

Does there exist a separable group S, universal for all locally compact groups? (That is, a group such that every locally compact group should be continuously isomorphic with a subgroup of it?) The authors deduced from J. von Neumann's representation of compact groups the existence of a compact group, universal for all compact groups.

104. Problem: Schauder

Let f(x,y,z,p,q) denote a function of five variables possessing a sufficient number of derivatives and satisfying the inequality: $f > M(|p|^{z+a} + |q|^{z+a})$; M constant, $a > 0$.

One has to find a minimum of the integral where $z = z(x,y)$, $p = z_x$, and $q = z_y$:

$$\iint_\Omega f(x,y,z,p,q)\,dxdy \tag{1}$$

(The region Ω should be sufficiently regular), among all z which possess all the first, possibly also the second, continuous derivatives, and which assume the same values on the boundary. One may assume that the given boundary value has a given number of derivatives with respect to the arc length of the boundary curve. Equation (1) is assumed regular. A similar condition for free boundary conditions. Prove the existence of a function, minimizing in a given class. (Regular problem: $f_{pp}f_{qq} - 4f_{pq} > 0$)

105. Problem: Schauder

The question is to find a system of functions $x(u,v)$, $y(u,v)$, $z(u,v)$ minimizing the parametric variational problem

$$\int_k \cdots \int f(x,y,z,X,Y,Z)\, du dv \qquad X = \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \quad \text{, etc.} \tag{2}$$

corresponding to Problem 104. It is allowed to change the class of admissible functions; these could be, for example, functions which are absolutely continuous in the sense of Tonelli. If not, then the problem is not solved. Mazur and Schauder solved Eq. (2) in the case when f does not contain x,y,z explicitly (even without any conditions analogous to those in Problem 104) but only within the class of functions absolutely continuous in the sense of Tonelli. Even this case (x,y,z does not appear) was not solved for functions $x(u,v) \ldots z(u,v)$ sufficiently regular.

106. Problem: Banach

(Prize: One bottle of wine, S. Banach)

Let

$$\sum_{i=1}^{\infty} x_i$$

be a series [x_i are elements of a space of type (B)] with the property that under a certain ordering of its terms the sum $= y_0$, under some other ordering, equals y_1. Prove that for every real number ℓ there exists an ordering of the given series such that the sum of it will be: $\ell y_0 + (1-\ell)y_1$. In particular, consider the case where x_i are continuous functions defined on the interval $(0,1)$. The convergence according to norm means uniform convergence.

Ad 106. It does not hold in the space L^2 and also not in C. We define, for every n, 2^n functions $f_{n,i}(x)$ as follows:

$$f_{n,i}(x) = 1, \quad \frac{i-1}{2^n} < x < \frac{i}{2^n} \quad \text{if } i = 2^n$$

$$f_{n,i}(x) = -1, \quad \frac{i-1}{2^n} < x < \frac{i}{2^n} \quad \text{if } i \neq 2^n$$

$$f_{n,i}(x) = 0 \quad \text{otherwise.}$$

Consider the orderings:

$$0 \equiv f_{1,1}+f_{1,3}+f_{1,2}+f_{1,4}+f_{2,1}+f_{2,2^2+1}+f_{2,2}+f_{2,2^2+2}+f_{2,3}+f_{2,2^2+3}+f_{2,4}+f_{2,2^2+4}+ \ldots$$
$$1 \equiv f_{1,1}+f_{1,2}+f_{1,3}+f_{2,2^2+1}+f_{2,2^2+2}+f_{1,4}+f_{2,2^2+3}+f_{2,2^2+4}+ \ldots$$

Since the $f_{i,k}$ assume integer values, one cannot order the series in such a way that it converges in L^2 to $0 < \ell < 1$.

Marcinkiewicz?

107. Problem: Sternbach

Does there exist a fixed point for every continuous mapping of a bounded, plane continuum E, which does not cut the plane, into part of itself? The same for homeomorphic mappings of E into all of itself.

108. Problem: Banach-Mazur-Ulam

Let E be a space of type (B) which has a base and H a set everywhere dense in E.
(1) Does there exist a base whose terms belong to H?
(2) The same question under the additional assumption that the set H is linear.

Ad 108. Affirmative answer (Krein).

109. Problem: Mazur-Ulam

Given are n functions of the real variable: $f_1, \dots f_n$. Denote by $R(f_1 \dots f_n)$ the set of all functions obtained from the given functions through rational operations, (expressions of the form

$$\left.\frac{\sum a_{k_1 \dots k_n} f_1^{k_1} \dots f_n^{k_n}}{\sum b_{k_1 \dots k_n} f_1^{k_1} \dots f_n^{k_n}}\right) .$$

Does there exist, in the set R, a function f such that its indefinite integral does not belong to the set R?

An analogous question in the case where we include in the set R all the functions obtained by *composing* functions belonging to R.

October 16, 1935

Ad 109. An affirmative answer for the first question was found by Docents, Dr. S. Kaczmarz and Dr. A. Turowicz.

March 1938

110. Problem: Ulam
(Prize: One bottle of wine, S. Ulam)

Let M be a given manifold. Does there exist a numerical constant K such that every continuous mapping f of the manifold M into part of itself which satisfies the condition $|f^n x - x| < K$ for $n = 1,2,\dots$ [where f^n denotes the n^{th} iteration of the image $f(x)$] possesses a fixed point: $f(x_0) = x_0$? (By a manifold, we mean a set such that the neighborhood of every point is homeomorphic to the n-dimensional Euclidean sphere.) The same under more general assumptions about M (general continuum?) October 1, 1935.

Ad 110. An affirmative answer in the case where M is a locally contractable 2-dimensional continuum.

March 1936

J. von Neumann observed that from the n-dimensional theorem an affirmative answer would follow for Hilbert's problem concerning the introduction of analytic parameters in n-parameter groups.

March 1936

111. Problem: Schreier

Does there exist a noncountable group with the property that every countable sequence of elements of this group is contained in a subgroup which has a finite number of generators? In particular, do the groups S_∞, and the group of all homeomorphisms of the interval have this property?

112. Problem: Schreier

Is an automorphism of a group G, which transforms every element into an equivalent one of necessity an inner automorphism?

113. Problem: Schreier

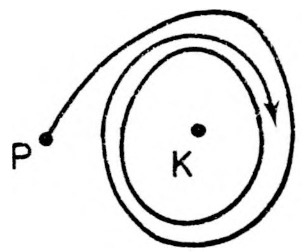
Let C denote the space of continuous functions of a real variable (under uniform convergence in every bounded interval); let F(f) denote an operation which is continuous, which has an inverse which maps C onto itself, and such that it maps the composition of two functions f(g) into the composition of F(f) and F(g).

Is F(f) of the form $F(f(t)) = hfh^{-1}(t)$, where h is a continuous function strictly monotonic in this interval $(-\infty, +\infty)$ and

$$\lim_{t\to -\infty} h(t) = -\infty, \quad \lim_{t\to +\infty} h(t) = +\infty?$$

114. Problem: Auerbach-Ulam

The circumference of a circle can be approximated by a one-to-one continuous image of a half line p in an *essential* manner; that is to say, the Abbildungsgrad of the transformation obtained by central projecting of the line into the circumference, is equal to $+\infty$ and the approximated circle is the set of points of condensation.



Is it possible to approximate analogously the surface of a sphere in the 3-dimensional space by a one-to-one continuous image of a plane?

115. Problem: Ulam

Does there exist a homeomorphism of the Euclidean space R_n with the following property? There exists a point p for which the sequence of points $h^n(p)$ is everywhere dense in the whole space. Can one even demand that all points except one should have this property? For a plane such a homeomorphism (with the desired property only for certain points) was found by Besicovitch.

116. Problem: Schreier-Ulam

Let G be a compact group. It is known that almost every (in the sense of Haar measure) couple of elements $\phi, \psi \in G$ generates in G an everywhere dense subgroup. Let there be given a sequence $\{c_n\}$ of zero's and 1's. Let us put $f_n = \phi$ if $c_n = 0$, $f_n = \psi$ if $c_n = 1$. Prove that for almost every pair ϕ, ψ and almost every sequence $\{c_n\}$ the sequence f_1, $f_1(f_2)$, $f_1(f_2 f_3)$,... is everywhere dense in G. Investigate whether this sequence is *uniformly dense*; that is, for every region $V \subset G$ we should have $\lim q_n/n$ = measure of V, if q_n denotes the number of the elements $f_1, f_1 f_2, \ldots f_1 f_2 \ldots f_n$, which fell into V. Investigate also whether an analogous theorem holds for similar sequences of images of a point p obtained with the aid of two transformations $\Phi(p)$ and $\Psi(p)$, which are *strongly transitive* mappings of the space S into itself preserving measure.

117. Problem: Fréchet*

One considers a Jordan curve which has a tangent (oriented) at every point. Does there exist at least one parametric representation of this curve where the coordinates are differentiable functions of the parameter and where the derivatives of the three coordinates do not vanish simultaneously?

Ad 117.** In general, no; but we can represent the curve of functions of a parameter t in such a way that dx/dt, dy/dt, dz/dt exist (and are not all zero), except for a set N of values of t, such that m(N) = 0 and also the set of points of the curve, corresponding to N, has Carathéodory measure zero. A. J. Ward, Fund. Math. **28**.

March 23, 1937

118. Problem: Fréchet*

Let $\triangle(n)$ be the greatest of absolute values of determinants of order n whose terms are equal to ± 1. Does there exist a simple analytic expression of $\triangle(n)$ as a function of n; or, more simply, determine an analytic asymptotic expression for $\triangle(n)$.

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*Written in French in the original manuscript.
**Written in English in the original manuscript.

119. Problem: Orlicz

Does there exist an orthogonal system composed of functions uniformly bounded and having the property possessed by the Haar system; that is to say, such that the development of every continuous function in this system is uniformly convergent?

120. Problem: Orlicz

Let x^{n_i} be a sequence of powers with interger exponents on the interval (a,b) and

$$\sum_{i=1}^{\infty} \frac{1}{n_i} = +\infty.$$

Give the order of approximation of a function satisfying a Hölder condition by polynomials:

$$\sum_{i=1}^{\infty} a_i x^{n_i}.$$

121. Problem: Orlicz

Give an example of a trigonometric series

$$\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

everywhere divergent and such that

$$\sum_{n=1}^{\infty} \left(a_n^{2+\epsilon} + b_n^{2+\epsilon}\right) < +\infty$$

for every $\epsilon > 0$.

122. Problem: Mazur-Orlicz

Does there exist in every space of type (B) of infinitely many dimensions, a series which is unconditionally convergent but not absolutely? (A series

$$\sum_{n=1}^{\infty} x_n$$

is called unconditionally convergent if it converges under every ordering of its terms, absolutely convergent, if the series

$$\sum_{n=1}^{\infty} \|x_n\|$$

converges.)

123. Problem: Steinhaus

Given are three sets A_1, A_2, A_3 located in the 3-dimensional Euclidean space and with finite Lebesgue measure. Does there exist a plane cutting each of the three sets A_1, A_2, A_3 into two parts of equal measure? The same for n sets in the n-dimensional space.

Ad 123. Solution in "Z Topologii," Mathesis Polska 1936.

124. Problem: Marcinkiewicz

What can one say about uniqueness for the integral equation

$$\int_0^1 y(t)\, f(x-t)dt = 0 \quad 0 \leqslant x \leqslant 1 \ ? \tag{3}$$

I know that if the sequence of integrals $f_k(x) = \int_0^x f_{k-1}(x)\, dx$ $f_0 = f$, $k = 1,2,\ldots$ is complete in L^2 then the only solution of Eq. (3) is $y \equiv 0$. This is the case also if f is of bounded variation and $f(0) \neq 0$. Finally, if Eq. (3) possesses even one nonzero solution, y, then every (iterated) integral of y also satisfies this equation.

I conjecture that if $f(0) \neq 0$ and f is continuous then Eq. (3) has only the solution $y \equiv 0$.

125. Problem: Infeld

(Originating in physics)

We shall say that a decent function of two variables $f(x,y)$ satisfies the condition A if there exists a function $y = \phi(x)$ such that

$$\left.\begin{aligned} &xf_x + yf_y = 0 && (1) \\ &4f_x f_y = 1 && (2) \end{aligned}\right\} A$$

for $y = \phi(x)$. [We see that $\phi(x)$ exists for $f(x/y)$ and for $f = ax + by$, if $ab = 1/4$.] Do we have the criterion: For every function $f(x,y)$ satisfying A there exists $F(x/y)$ such that $F(x/\phi(x)) = f(x, \phi(x))$ (with the exception of the case of $f = ax + by$).

126. Problem: M. Kac

If:

(1) $\int_0^1 f(x)dx = 0$

(2) $\int_0^1 f^2(x)dx = \infty$,
show that

$$\lim_{n\to\infty}\left[\int_0^1 \exp\left(i\frac{f(x)}{\sqrt{n}}\right) dx\right]^n = 0 .$$

(It is known that if $\int_0^1 f^2(x)dx = A$, then the above limit $= e^{-1/2)}$.

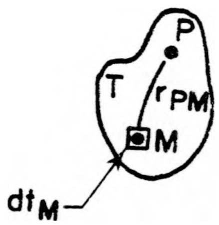
Ad 126. Solved affirmatively by A. Khintchin; it will appear in the fourth communiqué on dependent functions in Studia Math., Vol 6 or 7.

127. Problem: Kuratowski

Is it true that in every 0-dimensional metric space (in the sense of Menger-Urysohn) that every closed set is an intersection of a sequence of sets which are simultaneously closed and open? (The answer is affirmative for metric separable spaces.)

128. Problem: Nikliborc

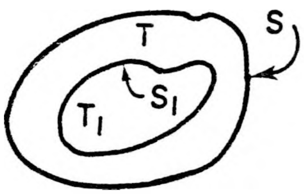
There is given, in a 3-dimensional space, a solid T which unicoherent and homogeneous. Let $V(P) = \int_T dt_M/r_{PM}$



Assume that V(P) is a polynomial in P in all of T + S. Show that T is an ellipsoid. It is known that if this polynomial is of second degree then the theorem is true.

129. Problem: Nikliborc

There are given two closed spaces S and S_1, each homeomorphic to the surface of the sphere and constituting the boundary of a solid T.



Suppose that $V(P) = \int_T dt_M/r_{PM}$ is a constant in T_1 (the solid bounded by S_1). Prove that S and S_1 are homothetic ellipsoids. It is known that if S and S_1 are homothetic then they are ellipsoids.

130. Problem: Kaczmarz

Let $\{f_n(t)\}$ be a system of uniformly bounded, orthogonal, lacunary functions. Does there exist a constant $\gamma > 0$, such that for every finite system of numbers $c_1, c_2, \ldots c_n$ we have:

$$\max \left| c_1 f_1(t) + \ldots + c_n f_n(t) \right| \geqslant \gamma \sum_{k=1}^{n} |c_k| \,.$$

Remark: The system is lacunary if, for every $p > 2$ there exists a constant M_p, such that

$$\sqrt[p]{\int_0^1 |c_1 f_1 + \ldots + c_n f_n|^p \, dt} \leqslant M_p \left(\sum_1^n c_k^2 \right)^{1/2}.$$

131. Problem: A. Zygmund

Given is a function f(x), continuous (for simplicity), and such that

$$\overline{\lim_{h \to 0}} \left| \int_h^1 \frac{f(x+t) - f(x)}{t} \, dt \right| < \infty, \text{ for } x \in E, \ |E| > 0.$$

Is it true that the integral

$$\int_0^1 \frac{f(x+t) - f(x)}{t} \, dt$$

may not exist almost everywhere in E? Similarly, for other Dini integrals?

132. Problem: W. Sierpinski

Does there exist a Baire function F(x,y) (of two real variables) such that for every function f(x,y) there exists a function $\phi(x)$ of one real variable (depending on the function f) for which $f(x,y) = F(\phi(x), \phi(y))$ for all real x and y.

February 25, 1936

133. Problem: Eilenberg

There is given, in a metric space E, a family of sets which are open and closed, covering the space E. Find a family of sets which are simultaneously open and closed and disjoint, covering the space E and smaller than the preceding family.
Remarks:

(1) A family of sets K is smaller than the family K_1 if every set of the family K is contained in a certain set of the family K_1.

(2) The problem includes the Problem 127 of Prof. Kuratowski.

(3) For separable spaces, the solution is trivial.

134. Problem: Eilenberg

Is the Cartesian product $K_1 \times K_2$ of two indecomposable continua, K_1 and K_2, of necessity an indecomposable continuum?

135. Problem: Eilenberg

Is a non-unicoherence of a locally connected continuum an invariant of locally homeomorphic mappings?

136. Problem: Eilenberg

Can an interior mapping (that is to say, one such that open sets go over into open sets) increase the dimension?
Remark: This question occupied R. Baer, who obtained some partial results.

Ad 136. A. Kolmogoroff, Annals of Math. **38** (1937), pp 36-38 gave an example of a continuous, interior mapping which increased the dimension from 1 to 2.

B. Knaster

137. Problem: Eilenberg

Given is a continuous mapping f of a compact space X, such that $\dim X > \dim f(X) > 0$. Does there exist a closed set $Y \subset X$ such that $\dim Y < \dim f(Y)$? In particular, does there exist, for every continuous mapping of a square into the interval, a closed 0-dimensional set whose image consists of a certain interval? We assume about the set X that it has the same dimension in every one of its points.

138. Theorem: Eilenberg

(a) Any set compact and convex, located in a linear space of type (B_0) is an absolute retract.

(b) A set compact and convex, in the sense of Wilson, is an absolute retract. [A set $Y \subset X$ is a retract with respect to X if there exists a continuous function $f \in Y^X$ such that $f(y) = y$ for $y \in Y$.

We call a compact space an absolute retract if it is a retract in every space which is metric, separable, containing it.] Absolute retracts have the fixed point property: (*vide* K. Borsuk, Fund. Math. 17.) A set X is convex in the sense of Wilson if, for every $x,y \in X$ and $0 \leq t \leq 1$ there exists one and only one point $z \in X$ such that $\zeta(x,z) = t\,\zeta(x,y)$; $\zeta(z,y) = (1 - t)\,\zeta(x,y)$.

May 17, 1936

139. Problem: Ulam

Is every one-to-one continuous mapping of the Euclidean space into itself equivalent to a mapping which brings sets of measure 0 into sets of measure 0?

140. Theorems: von Neumann

(a) A compact group of transformations to the Euclidean space is equivalent to a group of transformations carrying sets of measure 0 into sets of measure 0.

(b) Let f_n be a sequence of one-to-one mappings of Euclidean space. There exists a homeomorphism h such that the mappings hfh^{-1} carry sets of measure 0 into sets of measure 0.

140. Problem: Ulam

Two mappings (not necessarily one-to-one) f and g of a set E into part of itself are called equivalent if there exists a one-to-one mapping, h of the E into itself such that the set $f = hgh^{-1}$. What are necessary and sufficient conditions for the existence of such h?

141. Theorem: Ulam

In the group M of one-to-one measurable transformations of the circumference of a circle into itself, two transformations which are rotations through different irrational angles *are not* equivalent. An analogous theorem holds for the group of transformations of the surface of the n-dimensional sphere into itself.

142. Problem: Ulam. Theorem: Garrett Birkhoff

For every abstract group G there exists a set Z and a subset X contained in the square of the set Z: $X \subset Z^2$, such that the group G is isomorphic to a group of all one-to-one transformations f of the set Z into itself, under which the mapping $(x,y) \to (f(x), f(y))$ carries the set X into itself.

143. Problem: Mazur

Let K denote the class of functions of two integer-valued variables x,y such that:

(1) The functions $x,y,0,x+1,xy$ belong to K;

(2) If the functions $a(x,y)$, $b(x,y)$, $c(x,y)$ belong to K, then the function $f(x,y) = c(a(x,y), b(x,y))$ also belongs to K;

(3) If the function $a(x,y)$ belongs to K, then the function $f(x,y)$ for which $f(0,y) = 1$, $(f(x+1,y) = a(x,f(x,y))$ belongs to K. Does the class K contain the function

$$d(x,y) = \begin{cases} 1 \text{ for } x \neq y \\ 0 \text{ for } x = y \end{cases} ?$$

144. Problem: Mazur-Ulam

Let K denote a sphere in a separable space of type (B). Does there exist a one-to-one mapping of K into the interval $0 \leq x \leq 1$ under which the image of every open set in K is a set of positive measure?

145. Problem: Ulam

Given is a countable sequence of sets A_n. Find necessary and sufficient conditions for the possibility of introduction of a countably additive measure $m(A_n)$ such that $m(\Sigma A_n) = 1$, $m(p) = 0$; (p) denotes a set composed of a single point. [Possibly a stronger condition: $m(A_p) = 0$ for a certain given subsequence p_k]. We demand that the measure should be defined for each of the sets of a Borel ring of sets over the sequence A_n.

146. Problem: Ulam

It is known that in sets of positive measure there exist points of density 1 [that is to say, points with the property that the ratio of the length of intervals to the measure of the part of the set contained in these intervals tends to 1 (if the length of the interval converges to 0)]. Can one determine the speed of convergence of this ratio for almost all points of the set?

147. Theorem: Auerbach-Mazur
(Problem: Mazur)

Suppose that a billiard ball issues, under the angle 45°, from a corner of a rectangular table with a rational ratio of the sides. After a finite number of reflections from the cushion will it come to one of the remaining three corners?

September 4, 1936

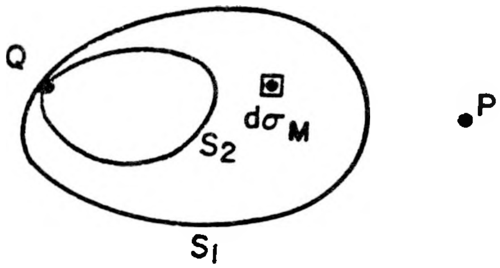
148. Theorem: Auerbach

Let $P(x_1 \ldots x_n)$ denote a polynomial with real coefficients. Consider the set of points defined by the equation $P(x_1 \ldots x_n) = 0$. A necessary and sufficient condition for this set not to cut the Euclidean (real) space is: All the irreducible factors of the polynomial P in the real domain should be always nonnegative or always nonpositive.

1936

149. Theorem: Nikliborc

Let S_1 and S_2 denote two closed and convex surfaces tangent at a point Q. Let S_2 be contained in the domain whose boundary is S_1:



Let

$$V_k(P) = \int_{S_k} \frac{d\sigma_M}{r_{PM}} \qquad k = 1,2$$

Theorem: $V_1(Q) > V_2(Q)$.

150. Problem: Nikliborc

Let S denote a closed surface and f(M) a continuous function defined on S. Let $V(P) = \int_S f(M) (1/r_{PM})\, d\sigma_M$. Let us assume that the plane π has the property: If P_1 and P_2 denote two arbitrary points of space, located outside a sufficiently large sphere and symmetrically with respect to the plane π then $V(P_1) = V(P_2)$. Prove that

(1) The plane π is a plane of symmetry for the surface S.

(2) In points of symmetry M_1 and M_2 belonging to S we have $f(M_1) = f(M_2)$.

151. Problem: Wavre*

(Prize: A "fondue" in Geneva)

Does there exist a harmonic function defined in a region which contains a cube in its interior, which vanishes on all the edges of the cube? One does not consider $f \equiv 0$.

November 6, 1936

Ad 151. Does there exist an algebraic function f(z) holomorphic in every point of a curve traced on a surface of Riemann and such that one has

$$\int_{\ell}^{z} \frac{f(z)}{z - x}\, dz = 0 \qquad f(z) \not\equiv 0,$$

*Written in French in original manuscript.

the point x being contained in a certain domain? The curve ℓ will be open. One should find f(z) and ℓ.
(Prize: A "fondant" in Lwów)

152. Problem: Steinhaus

A disc of radius 1 covers at least two points with integer coordinates (x,y) and at most 5. If we translate this disc through vectors nw (n = 1,2,3...), where w has both coordinates irrational and their ratio is irrational, then the numbers 2,3,4 repeat infinitely many times. What is the frequency of these events for $n \to \infty$?
Does it exist?
(For computation of the frequency: 100 grammes of caviar)
(For proof of existence of frequency: A small beer)
(For counterexample: A demitasse)

November 6, 1936

153. Problem: Mazur

Given is a continuous function f(x,y) defined for $0 \leq x,y \leq 1$ and the number $\epsilon > 0$; do there exist numbers $a_1, \dots a_n$, $b_1, \dots b_n$, $c_1, \dots c_n$ with the property that

$$\left| f(x,y) - \sum_{k=1}^{n} c_k f(a_k, y) f(x, b_k) \right| \leqslant \epsilon$$

in the interval $0 \leq x,y \leq 1$?
(Prize: A live goose, Mazur)
Remark: The theorem is true under the additional assumption that the function f(x,y) possesses a continuous first derivative with respect to x or y.

November 6, 1936

154. Problem: Mazur

Let $\{\phi_n(t)\}$ be an orthogonal system composed of continuous functions and closed in C.
(a) If $f(t) \sim a_1 \phi_1(t) + a_2 \phi_2(t) + \dots$ is the development of a given continuous function f(t) and $n_1, n_2 \dots$ denote the successive indices for which $a_{n_1} \neq 0$, $a_{n_2} \neq 0$, ... can one approximate f(t) uniformly by the linear combinations of the functions $\phi_{n_1}(t)$, $\phi_{n_2}(t), \dots$?
(b) Does there exist a linear summation method M such that the development of every continuous function f(t) into the system $\{\phi_n(t)\}$ is uniformly summable by the method M to f(t)?

September 15, 1936

155. Problem: Mazur-Sternbach

Given are two spaces X,Y of type (B). $y = U(x)$ is a one-to-one mapping of the space X onto the whole space Y with the following property: For every $x_0 \in X$ there exists an $\epsilon > 0$ such that the mapping $y = U(x)$, considered for x belonging to the sphere with the center x_0 and radius r, is an isometric mapping. Is the mapping $y = U(x)$ an isometric transformation?

This theorem is true if U^{-1} is continuous. This is the case, in particular, when Y has a finite number of dimensions or else the following property: If $||y_1 + y_2|| = ||y_1|| + ||y_2||$, $y_1 \neq 0$, then $y_2 = \lambda y_1$, $\lambda \geq 0$.

November 18, 1936

156. Problem: Ward*

A surface $x = f(u,v)$, $y = g(u,v)$, $z = h(u,v)$, f,g,h being continuous functions, has at each point a tangent plane in the geometric sense; also, to each point of the surface corresponds only one pair of values of u,v. Does there exist a representation of the surface by functions $x = f_1(u,v)$, $y = g_1(u,v)$, $z = h_1(u,v)$, in such a manner that the partial derivatives exist and the Jacobians $\partial(f_2,f_3)/\partial(u,v)$, $\partial(f_3,f_1)/\partial(u,v)$, $\partial(f_1,f_2)/\partial(u,v)$ are not all zero, except for a set N of values of u,v such that the corresponding set of points of the surface has surface measure (in Carathéodory's sense) zero? [Let (x,y,z) be a point of a surface S, and P a plane through (x,y,z). Then if, for every $\epsilon > 0$, there exists a sphere $K(\epsilon)$ of center (x,y,z), such that the line joining (x,y,z) to any other point of $S \cdot K(\epsilon)$ always makes an angle of less than ϵ with P, we say that P is the tangent plane to S at (x,y,z).]

March 23, 1937

157. Problem: Ward*

f(x) is a real function of a real variable, which is approximately continuous. At each point x, the upper right-hand approximate derivative of f(x) (that is

$$\overline{\lim_{h \to 0^+}} \frac{f(x+h) - f(x)}{h},$$

neglecting any set of values of h which has zero density at $h = 0$) is positive. Is f(x) monotone increasing?
(Lunch at the "Dorothy", Cambridge)

March 23, 1937

*Written in English in original manuscript.

158. Problem: Stöilow*

Construct an analytic function f(z) continuous in a domain D admitting there a perfect discontinuum set P of singularities such that f(P) is a discontinuous set. Such function would permit one to form a "quasi-linear" function; that is to say, one which has the following properties:

(1) The function is continuous and univalent in the whole plane z.

(2) The function tends toward ∞ for $|z| \to \infty$.

(3) The function has a perfect set of singularities.

See: Stöilow, "Remarques sur les fonctions analytiques continues dans un domaine où elles admettant un ensemble parfait discontinu de singularités," Bulletin de la Societe Roumaine de Mathem. (1936) **38**, 117-120.

May 1, 1937

159. Problem: Ruziewicz

Let Φ denote the set of all continuous functions defined in (0,1), $f(0) = 0$, $0 \le f(x) \le 1$ for $0 \le x \le 1$. Let

$$P(x) = \sum_{n=0}^{\infty} a_n x^n$$

be a power series and let $P_k(x)$ denote the k^{th} partial sum.

Does there exist a power series P(x) with the following property; for every $\epsilon > 0$ there exists $N(\epsilon)$ such that for every function $f \in \Phi$ there exist $n \le N$, so that $|f(x) - P_n(x)| < \epsilon$?

May 22, 1937

Ad 159. In this formulation the answer is negative. Since

$$\max_{0 \leqslant x \leqslant 1} |\sin^2 2^n \pi x - \sin^2 2^m \pi x| = 1 \quad (m \neq n),$$

there does not exist a function which approximates both $\sin^2 2^n\pi x$ and $\sin^2 2^m\pi x$ with a precision $\le 1/3$ in the interval <0,1>. If the requested universal N(1/2) existed, then by taking

$$\sin^2 2\pi x,\ \sin^2 2^2\pi x, \ldots \sin^2\{2^{N(1/3)+1}\pi x\},\ \sin^2\{2^{N(1/2)+1}\ 2\pi x\},$$

we would conclude on the basis of Problem 159, that for a certain $k \le N(1/3)$, the polynomial $P_k(x)$ would approximate simultaneously $\sin^2 2^n\pi x$ and $\sin^2 2^m\pi x$ for $n \neq m$ with precision $\le 1/3$. (Sternbach)

Let Φ denote an arbitrary set of continuous functions defined in (0,1), $f(0) = 0$ $|f(x)| \le N$; in order that the set Φ should have the requested property, it is necessary and sufficient that the functions of the set Φ be equicontinuous. (Mazur)

June 10, 1936

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*Written in French in original manuscript.

160. Problem: Mazur

Let G denote a metric group.

(1) Let the group G be complete and have the property that for every $\epsilon > 0$, every element $a \subset G$ has a representation $a = a_1 a_2 \ldots a_n$, where $(a_k, e) < \epsilon$. Is the group G connected in the sense of Hausdorff? (That is to say, cannot be represented as a sum of two disjoint, closed sets $\neq 0$?)

(2) If a group G is connected in the sense of Hausdorff, is it then arcwise connected?

June 10, 1937

161. Theorem: M. Kac

Let r_n be a sequence of integers such that

$$\lim_{n\to\infty}\left(r_n - \sum_{k=1}^{n-1} r_k\right) = \infty.$$

One has then

$$\lim_{n\to\infty}\left| \underset{0 \leqslant x \leqslant 1}{E} \left\{ a < \frac{\sin 2\pi r_1 x + \ldots + \sin 2\pi r_n x}{\sqrt{n}} < b \right\} \right| = \frac{1}{\sqrt{\pi}} \int_a^b e^{-y^2}\, dy$$

(One can put, for example, $r_n = 2^{n}$.)
Problem: Is the theorem true for $r_n = 2^n$?

June 10, 1937

162. Problem: H. Steinhaus

We assume that $f(x)$ is measurable (L), periodic, $f(x+1) = f(x)$ and $f(x) = +1$ or -1. Do we have, almost everywhere,

$$\limsup_{n\to\infty} f(nx) = +1, \quad \liminf_{n\to\infty} f(nx) = -1\ ?$$

More generally: If $f_n(x)$ are measurable, uniformly bounded, and $f_n(x + 1/n) \equiv f_n(x)$, do we have then

$$\limsup_{n\to\infty} f_n(x) = \text{constant almost everywhere?}$$

$$\liminf_{n\to\infty} f_n(x) = \text{constant almost everywhere?}$$

(Dinner at "George's")

July 3, 1937

Ad 162. A more general theorem, formulated by Professor Banach, is true: If f(x) is an arbitrary measurable function with period 1 then one has almost everywhere the relations:

$$\overline{\lim_{n\to\infty}}\ f(nx) = \underset{0\leqslant x\leqslant 1}{\text{essential upper bound}}$$

$$\underline{\lim_{n\to\infty}}\ f(nx) = \underset{0\leqslant x\leqslant 1}{\text{essential lower bound.}}$$

M. Eidelheit, October 16, 1937

163. Problem: J. von Neumann*

Given is a completely additive and multiplicative Boolean algebra B. That is to say:

(1) B is a partially ordered set with the relation $a \subset b$.

(2) Every set $S \subset B$ has the least upper (greatest lower) bound $\Sigma\,(S)$ $(\Pi\,(S))$. [We write: $\Sigma\,(a,b) = a + b$, $\Pi\,(a,b) = ab$, $\Sigma\,(B) = 1$, $\Pi\,(B) = 0$.]

(3) We have a general "distributivity law" $(a + b)c = ac + bc$.

(4) Every element $a \in B$ has an (according to (3), unique) "inversion" in a: $a + (-a) = 0$, $a(-a) = 1$.

A measure in B is a numerical function:

$$1.\ \mu(a)\begin{cases} = 0 \text{ for } a = 0 \\ > 0 \text{ for } a \neq 0 \end{cases}$$

2. $a_i \in B$ $(i = 1,2,\ldots)$ $a_i a_j = 0$ for $i \neq j$ imply $\mu(\Sigma_i\,(a_i)) = \Sigma_i\,\mu(a_i)$.

Obviously, one has to determine:

(5) If $S \subset B$, $(a,b \in S,\ a \neq b) \to ab = 0$, then S is at most countable.

Question: When does there exist a "measure" in B?

Remarks: As one verifies without difficulty, the following "generalized distributivity" law is necessary!

(6) Let $a_1^i \leq a_2^i \leq \ldots$ for $i = 1,2,\ldots$, then we have

$$\prod_i\Big(\sum_j\big(a_j^i\big)\Big) = \sum_{j(i)}\Big(\prod_i\big(a_{j(i)}^i\big)\Big)$$

without the assumption that $a_1^i \leq a_2^i \leq \ldots$ characterizes, according to Tarski, the "atomic" Boolean algebras.

(1) to (5) do not imply (6). Counterexample: The Boolean algebra of Borel sets, modulo sets of first category. Example for (1) to (5): Measurable sets (or Borel sets modulo sets of measure 0 when one employes Lebesgue measure). Is (5), (6) sufficient?

(Prize: A bottle of whiskey of measure >0.)

July 4, 1937

*Written in German in original manuscript.

164. Problem: Ulam

Let a finite number of points, including 0 and 1, be given on the interval (0,1); a number $\epsilon > 0$, and a transformation of this finite set into itself; T, with the following property: For every point p, $|p,T(p)| > \epsilon$. Let us call "a permissible step" passing from the point p to T(p) or to one of the two neighbors (points nearest from the left or from the right side) of the point T(p).
Question: Does there exist a universal constant k such that there exists a point p_0 from which, in a number of allowed steps $E(k/\epsilon)$ one can reach a point q which is distant from p_0 by at least 1/3?

165. Problem: Ulam

Let p_n be a sequence of rational points in the n-dimensional unit sphere. The first N points p_1 ... p_N are transformed on N points (also located in the same sphere) q_1 ... q_N all different. We define a transformation on the points $p_n, n > N$ by induction as follows: Assume that the transformation is defined for all points p_ν, $\nu > N$ and their images are all different. This mapping has a certain Lipschitz constant L_{n-1}. The Lipschitz constant of the inverse mapping we denote by L'_{n-1}. We define the mapping at the point p_n so that the sum of the constants $L_n + L'_n$ should be minimum. (In the case where we have several points satisfying this postulate we select one of them arbitrarily.)
Question: Is the sequence $\{L_n + L'_n\}$ bounded?
(Prize: Two bottles of wine.)

166. Problem: Ulam

Let M be a topological manifold, f a real-valued continuous function defined on M. We denote by G_f^M the group of all homeomorphic mappings T of M onto itself such that $f(T(p)) = f(p)$ for all $p \in M$.
Question: If N is a manifold *not* homeomorphic to M, does there exist such f_0, that G_f^M is not isomorphic to any G_f^N?

167. Problem: Ulam

Let S denote the surface of the unit sphere in Hilbert space. Let f_1 ... f_n be a finite system of real-valued, continuous functions defined on S. Let T be a continuous transformation of S into part of itself. Does there exist a point p_0, such that $F_\nu(T(p_0)) = f_\nu(p_0)$ $\nu = 1, \ldots n$.

168. Problem: Ulam

Does there exist a sequence of sets A_n such that the smallest class of sets containing these, and closed with respect to the operation of complementation and countable sums, contains all the analytic sets (on the interval)?
(Prize: Two bottles of beer.)

169. Problem: E. Szpilrajn

Does there exist an additive function $\mu(\epsilon)$, equal for congruent sets, defined for all plane sets, and which is an extension of the linear measure of Carathéodory? $(0 \leq \mu(\epsilon) \leq +\infty)$?

170. Problem: Szpilrajn

Is every plane set, all of whose homeomorphic plane images are measurable Lebesgue (L), measurable absolutely? [That is to say, measurable with respect to every Carathéodory function ("Massfunction").]

This is true for linear sets; for plane sets an analogous theorem is true if one replaces homeomorphisms by generalized homeomorphisms in the sense of Mr. Kuratowski.

Ad 170. Affirmative answer follows from an unpublished result of von Neumann.

171. Problem: J. Schreier - S. Ulam

Let T(A) denote the set of all mappings of a set A into itself. An operation is defined for pairs of elements of the set T: U(f,g) = h for all h ∈ T(A). (U(f,g) ≢ 1).

Assumptions:

(1) U(f,g) is associative; that is, (U(f,U(g,h)) = U(U(f,g),h).

(1) U(f,g) is invariant with respect to permutations of the underlying set; i.e., if p is a permutation of the set A, then $U(p^{-1}fp,p^{-1}gp) = p^{-1}U(f,g)p$.

Theorem: U(f,g) = f(g) (composition).

172. Problem: M. Eidelheit

A space E of type (B) has the property

(a) if the weak closure of an arbitrary set of linear functionals is weakly closed. [A sequence of linear functionals $f_n(x)$ converges weakly to f(x) if $f_n(x) \to f(x)$ for every x.]

The space E of type (B) has the property

(b) if every sequence of linear functionals weakly convergent converges weakly as a sequence of elements in the conjugate space $\overline{E}$.

Question: Does every separable space of type (B) which has property (a) also possess property (b)?

June 4, 1938

173. Problem: M. Eidelheit

Let A denote a set of all linear operations mapping a given space of type (B) into itself. Is the set of operations in A which have continuous inverses dense in A (under the usual norm)?

July 23, 1938

174. Problem: M. Eidelheit

Let $U(x)$ be a linear operation defined in a space of type (B_0) mapping this space into itself and such that the operation $x - \lambda U(x)$ has inverses for sufficiently small λ. Can we then have

$$(x - \lambda U)^{-1} = x + \lambda U(x) + \lambda^2 U(U(x)) + \dots ?$$

July 23, 1938

175. Problem: Borsuk

(a) Is the product (Cartesian) of the Hilbert cube Q_w with the curve which is shaped like the letter T, homeomorphic with Q_w?
(b) Is the product space of an infinite sequence of letters T homeomorphic to Q_w?

August 10, 1938

176.

In a ring of type (B) (normed, complete linear ring with the norm satisfying the condition: $|xy| \leq |x||y|$) containing a unit element there is given an element a possessing an inverse a^{-1}. Question: Does there exist a sequence of polynomials $c_0^{(n)}I + c_1^{(n)}a + \dots + c_m^{(n)}a^n$ converging to a^{-1}? (I = the unit element, c are numbers.)

M. Eidelheit, September 12, 1938

Ad 176. Answer is negative. Example: The ring of linear operations $U(x)$ of the space (C) into itself $U(x) = x(t^2)$, $0 \leq t \leq 1$.

M. Eidelheit, November 11, 1938

177. Problem: M. Kac

What are the conditions which a function $\Phi(x,y)$, must satisfy in order that for every pair of Hermitian matrices A and B the matrix $\Phi(A,B)$ be "positive definite"?

September 11, 1938

178. Problem: M. Kac

Let

$$\phi(x,y) = \frac{1}{\frac{1}{x} + \frac{1}{y} - 1}$$

Prove that if

$$\phi\left(\int_{-\infty}^{+\infty} e^{i\xi x}\, d\sigma_1(x), \int_{-\infty}^{+\infty} e^{i\xi x}\, d\sigma_2(x)\right) = \frac{1}{1+\xi^2},$$

then

$$\sigma_1(x) = \alpha_1 e^{-\beta_1 |x|} \text{ and } \sigma_2(x) = \alpha_2 e^{-\beta_2 |x|}$$

(This is analogous to Cramer's theorem that if

$$\int_{-\infty}^{+\infty} e^{i\xi x}\, d\sigma_1(x) \times \int_{-\infty}^{+\infty} e^{i\xi x}\, d\sigma_2(x) = e^{-\xi^2/4}$$

then $\sigma_1(x)$ and $\sigma_2(x)$ are of the form $e^{-\beta_1 \xi^2}$ and $e^{-\beta_2 \xi^2}$.)

September 11, 1938

179. Problem: Offord*

If $a_0, a_1, \ldots a_n$ are any real or complex numbers and if $\epsilon_\nu = \pm 1$ $\nu = 1,2,\ldots n$; then the following theorem is true:

$$|a_0 + \epsilon_1 a_1 + \epsilon_2 a_2 + \ldots + \epsilon_n a_n| = \min_{0 \leqslant \nu \leqslant n} |a_\nu|$$

except for a ratio at most $A/n^{1/4}$ of the 2^n sums.
Problems:

(i) To find a short proof of this result.

(ii) When the a's are all equal to 1 the size of the exceptional set is $(A/\sqrt{n})2^n$. Is this the right upper bound whatever the numbers a_ν?

January 1, 1939

180. Problem: Kampé de Fériet**

Let v(t,E) be a stationary random function (in the sense of E. Slutsky, A. Khinchine): (E a random event)

$$\left.\begin{array}{ll} \overline{v(t,E)} = 0 & \overline{v(t,E)^2} = \text{Constant} \\ \overline{v(t,E),\, v(t+h,E)} = \text{function of } h \text{ alone} & \end{array}\right\} \begin{array}{l}\text{for all } t \\ -\infty < t < +\infty.\end{array}$$

*Written in English in original manuscript.
**Written in French in original manuscript.

Does there exist a random variable A which, with uniform probability, assumes every value α between 0 and 1,

$$\text{Prob}\,[A < \alpha] = \alpha \qquad 0 \leq \alpha \leq 1$$

such that

(1) $E = \phi(\alpha)$

(2) $v[t_1, \phi(\alpha)]$ and $v[t_2, \phi(\alpha)]$ are two independent functions (in the sense of H. Steinhaus) for every couple t_1, t_2 $(t_1 \neq t_2)$?

May 16, 1939.

181. Problem: H. Steinhaus

Find a continuous function (or perhaps an analytic one) f(x), positive and such that one has

$$\sum_{n=-\infty}^{\infty} f(x+n) \equiv 1$$

(identically in x in the interval $-\infty < x < +\infty$); examine whether $(1/\sqrt{\pi})e^{-x^2}$ is such a function; or else, prove the impossibility; or else, prove uniqueness.

Ad 181. The function $(1/\sqrt{\pi})e^{-x^2}$ does not have the property—this follows from the sign of the second derivative for $x = 0$ of the expression

$$\sum_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}}\, e^{-(x+n)^2}.$$

H. Steinhaus

We take a function g(x) positive, continuous, and such that

$$\sum_{n=-\infty}^{+\infty} g(x+n) = f(x) < +\infty$$

in the interval $(-\infty, +\infty)$ for example, $g(x) = e^{-x}$ and the function $g(x)/f(x)$ satisfies the conditions.

S. Mazur

December 1, 1939

182. Problem: B. Knaster

The disc cannot be decomposed into chords (not reducing to a single point), but a sphere can be so decomposed (noneffectively). Give an effective decomposition of a sphere into chords. The same for the n-dimensional sphere for "chords" of dimension $k \leq n - 2$.
(Prize: Small light beer)

December 31, 1939

183. Problem: Bogolubow*

Given is a compact, connected and locally connected group of transformations of the n-dimensional Euclidean space. Prove (or give a counterexample) that one can introduce in this space such coordinates that the transformations of the group will be linear.
(Prize: A flask of brandy)

February 8, 1940

184. Problem: S. Saks

A subharmonic function ϕ has everywhere partial derivatives $\partial^2\phi/\partial x^2$, $\partial^2\phi/\partial y^2$. Is it true that everywhere $\Delta\phi \geq 0$?
Remark: It is obvious immediately that $\Delta\phi \geq 0$ at all points of continuity of $\partial^2\phi/\partial x^2$, $\partial^2\phi/\partial y^2$, therefore on an everywhere dense set.
(Prize: One kilo of bacon)

February 8, 1940

185. Problem: S. Saks

Is it true that for every continuous surface $z = f(x,y)$ $(0 \leq x \leq 1, 0 \leq y \leq 1)$ the surface area is equal to

$$\lim_{h\to 0} \int_0^1\int_0^1 \left[\frac{f(x+h) - f(x,y)}{h}\right]^2 + \left[\frac{f(x,y+h) - f(x,y)}{h}\right]^2 dxdy$$

Remark: The theorem is true for curves [for surfaces it was given by L. C. Young, but the proof (cf. S. Saks, *Theory of the Integral,* 1937) contains an essential error].

186. Problem: S. Banach

Does there exist a sequence $\{\phi_i(t)\}$ of functions, orthogonal, normed, and complete in the interval $(0 \leq t \leq 1)$ with the property that for every continuous function $f(t)$, $0 \leq t \leq 1$ (not identically zero) the development

*Written in French in original manuscript.

$$\sum_{i=1}^{\infty} \phi_i(t) \int_0^1 f(t)\, \phi_i(t)dt$$

is in almost every point unbounded?

March 21, 1940

187. Problem: P. Alexandroff*

(1) Let P be a mutilated polyhedron (that is to say, in its simplest decomposition one has deleted a certain number of simpleces of arbitrary dimensions) contained in R^n. $R^n - P$ is then also a mutilated polyhedron. We understand by the Betti group of this polyhedron the usual Betti group in the sense of Vietoris. The duality law of Alexander is then true for mutilated polyhedra. Prove that if $P \subset R^n$ is a mutilated *topological* polyhedron (that is to say, a topological image of a mutilated polyhedron) the duality theorem of Alexander still holds.

(2) Prove (or refute) the theorem: For every Hausdorff space which is bicompact, the inductive definition of dimension is equivalent to the definition given with the aid of coverings (Uberdeckungen).

(3) Prove (or refute) the impossibility of an interior continuous transformation of a cube with p-dimensions into a cube of q dimensions for $p < q$.

P. Alexandroff, April 19, 1940

188. Exercise: S. Sobolew

One has proved the existence of a Cauchy problem

$$u\Big|_{x_n = 0} = \phi_0(x_1, \ldots x_{n-1})$$

$$\frac{\partial u}{\partial x_n}\Big|_{x_n = 0} = \phi_1(x_1, \ldots x_{n-1}),$$

for the quasi-linear partial differential equation of the form

$$\sum_{i=1}^{n} \sum_{i=1}^{n} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = F$$

of the hyperbolic tube (where A_{ij} and F depend on $x_1, \ldots, x_n$, u, $\partial u/\partial x_1 \ldots \partial u/\partial x_n$), if the function ϕ_0 possesses square integrable derivatives up to the order $[n/2] + 3$ and function ϕ_1 partial

*Written in French in original manuscript.

derivatives up to the order $\lfloor n/2 \rfloor + 2$ (also square integrable). We assume in addition that the derivatives of functions A_{ij} and F with respect to $\partial u/\partial x_i$ and to u are continuous.

For the nonlinear equation of the general form

$$\phi\left(x_1 \ldots x_n, u, \frac{\partial u}{\partial x_1}, \ldots \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1^2} \ldots \frac{\partial^2 u}{\partial x_n^2}\right) = 0$$

one can easily show the existence of a solution if only ϕ_0 has derivatives up to the order $\lfloor n/2 \rfloor + 4$, and ϕ_1 up to the order $\lfloor n/2 \rfloor + 3$, square integrable. One should: Construct an example of such an equation and such boundary conditions having derivatives of the order less by 1, square integrable, such that the solution would not exist, or else lower the number of derivatives necessary for the existence of solution, to the number necessary in the case of quasi-linear equations. (This latter number cannot be lowered any more as shown by known examples.)
(For solution of the problem: A bottle of wine)

April 20, 1940

188.1. A Problem

Let $z(x,y)$ be a function absolutely continuous on every straight line parallel to the axes of the coordinate system, in the square $0 \leq x, y \leq 1$ let $f(t)$ and $g(t)$ be two absolutely continuous functions in $0 \leq t \leq 1$ with values also in $(0,1)$. Is the function $t = z(f(t),g(t))$ also absolutely continuous? If not, then perhaps under the additional assumptions that

$$\int_0^1\int_0^1 \left|\frac{\partial z}{\partial x}\right|^p dxdy < \infty, \quad \int_0^1\int_0^1 \left|\frac{\partial z}{\partial v}\right|^p dxdy < \infty, \text{ where } p > 1.$$

February 22, 1940, M. Eidelheit

189. Problem: A. F. Fermant*

Let $w = f(z)$ be a regular function in the circle $|z| < 1$, $f(0) = 0$, $f'(0) = 1$. We shall call the "principal star" of this function the following one-leafed star-like domain: On the leaf of the Riemann surface corresponding to the function $w = f(z)$ to which the point $w = f(z) = 0$ belongs, we take the biggest one-leafed region belonging to the surface.

Prove the theorem: The "principal star" of the function $w = f(z)$ contains a circle of a radius not less than an absolute constant $B^\star$ (generalization of a theorem of A. Block).

190. Problem: L. Lusternik*

Let there be given in the Hilbert space L_2 an additive functional $f(x)$ defined on a part of L_2, and a self-adjoint operator A. If f is linear, then it is an element of L_2 and $Af = f(Ax)$. Let us ex-

*Written in Russian in original manuscript.

tend the operation A over all additive functionals f by the formula: $Af = f(Ax)$. If there is a point of the continuous spectrum of A, then we can find an infinite set of additive functionals f, not identically equal to zero, for which $(A - \lambda E)f \equiv 0$; that is, $f(Ax - \lambda x) \equiv 0$.

These f(x) can be considered as, so to say, "ideal" associated elements for the points of the continuous spectrum since the properties of the continuous spectrum are reflected on the structure of the sets of the ideal associated elements.
(For the solution: A bottle of champagne to the solver, L. Lusternik, Lwów)

September 4, 1940

191. Problem: E. Szpilrajn

Auxiliary definitions: I call *measure* every nonnegative, completely additive set function defined on a certain completely additive class of sets K, subsets of a fixed set χ and such that $\mu(\chi) = 1$. The measure μ is *convex* (according to M. Fréchet: "sans singularités"), if for every set A such that $\mu(A) > 0$ there exists a set $B \subset A$, such that $\mu(A) > \mu(B) > 0$. The measure μ is *separable* if there exists a countable class $D \subset K$, such that for every $\eta > 0$ and every $M \in K$ there exists $L \in D$, such that $\mu[(M - L) + (L - M)] < \eta$. The class K is a class of sets stochastically independent with respect to μ if $\mu(A_1A_2\ldots A_n) = \mu(A_1)\mu(A_2)\ldots\mu(A_n)$ for every disjoint sequence $\{A_k\}$ of sets belonging to K.
Definition of a base: The class $B \subset K$ is called a base of a measure μ if

(1) B is a class of sets stochastically independent with respect to μ and;

(2) All sets of the class K can be approximated, up to sets of measure 0, by sets of the smallest countably additive class of sets containing B.

Remarks: Let B_n denote the set of numbers from the interval $<0,1>$, whose n^{th} binary digit = 1. The sequence $\{B_n\}$ is a base for the Lebesgue measure in the interval $<0,1>$. It follows easily that every convex, separable measure has a base. In the known examples of nonseparable measures, there also exists a base.
Problem: Does every convex measure possess a base?

Lwów, April 1941

192. Definitions:

(1) A topological space T has the property (S) (of Suslin) if every family of disjoint sets, open in T, is at most countable.

(2) A space T has property (K) (of Knaster) if every noncountable family of sets, open in T, contains a noncountable subfamily of sets which have elements common to each other.
Remarks:

(1) One sees at once that the condition (K) implies (S) and, in the domain of metric spaces, each is equivalent to separability.

(2) B. Knaster proved in April 1941 that, in the domain of continuous, ordered sets, the property (K) is equivalent to separability. The problem of Suslin is therefore equivalent to the question whether, for ordered continuous sets, the property (S) implies the property (K).
Problem: B. Knaster and E. Szpilrajn: Does there exist a topological space (in the sense of Hausdorff, or, in a weaker sense, e.g., spaces of Kolmogoroff) with the property (S) and not satisfying the property (K)?

(3) According to Remark (2), a negative answer would give a solution of the problem of Suslin. Problem: E. Szpilrajn. Is the property (S) an invariant of the operation of Cartesian product of two factors?

(4) One can show that if this is so, then this property is also an invariant of the Cartesian product of any number (even noncountably many) factors.

(5) E. Szpilrajn proved in May 1941 that the property (K) is an invariant of the Cartesian product for any number of factors and B. Lance and M. Wiszik verified that if one space possesses property (S), and another space has property (K), then their Cartesian product has also property (S).

Lwów, May 1941

193.

The "expected" number of matches: 7
The "median" number of matches: 9
Probability that $x \leq 9 \rightarrow 0.68$
Probability that $x \leq 18 \rightarrow 0.45$
Probability that $x \leq 27 \rightarrow 0.997$
"The probable" number of matches: 6
The probability that $x \leq 6$ is 0.5
(Two boxes with five matches)
(The exact solution requires lengthy computations.)

Hugo Steinhaus, May 31, 1941

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