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LOCAL OPERATOR PRODUCTS IN GAUGE THEORIES II

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ABSTRACT

We study the operator product expansion of two gauge invariant currents. We discuss the operators that may appear in the expansion. We study the hadronic matrix elements of gauge variant operators that may appear in the operator product expansion and show that in the physical applications contemplated so far their hadronic physical matrix elements either vanish or cannot be isolated from strong interactions. We also study the gauge dependence of the Green's functions product of two (weak or e.m.) currents and show that it corresponds to the gauge dependence of the additional renormalization counterterms due to weak and electromagnetic interactions.

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I. INTRODUCTION

In this paper, we shall discuss the operator product expansion [1] of two gauge invariant operators (in particular two currents) in detail in an unbroken nonabelian gauge theory [2] using the results of ref.[1] [3]. We shall keep our discussion general, without restricting ourselves to a specific application (though we shall discuss some of them briefly). The main topics we shall deal with are the following: (i) We shall discuss the operators that may appear in the operator product expansion of two gauge invariant operators to all orders. (ii) We shall discuss the 'quark-gluon' matrix elements as well as the hadronic matrix elements of the gauge variant operators. (iii) We shall explain how the gauge independence of the observable quantities works. (iv) We shall explain which operators need be taken into account in the specific physical applications [4-7] that have been contemplated thus far.

In sec. II, we shall briefly outline the theory we shall be working with. It is the unbroken nonabelian theory of quarks and gluons for strong interactions [4] coupled to a spontaneously broken gauge theory of weak and electromagnetic interactions [2]. We state the nature of the processes under consideration. For them, the lowest order in the weak and the electromagnetic interactions the relevant amplitudes can be written in terms of a product of gauge invariant currents. We shall also recapitulate the results of I.

In sec. III, we shall discuss briefly the renormalization of a product of an arbitrary number of gauge invariant operators and show the

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gauge independence of such a product at nonexceptional momenta. We shall briefly explain there why this discussion is relevant. This discussion is a relatively straightforward generalization of previous discussions [3, 8-10] for the case of one gauge invariant operator and can be given in a compact way using an observation due to Dixon [11] .

In sec. IV, we shall discuss the case for a product of two currents. We shall write down the WT (Ward-Tahanishi) identity satisfied by the Green's functions of this product. We shall then analyze this WT identity for the two gluon Green's function in detail and show how the Green's function can be expanded. This together with the WT identity will allow us to separate the Green's functions of the current product into two pieces, one piece of which has an operator expansion in terms of a known set of operators. Information about the other piece can be directly obtained by analyzing the gauge dependence of the matrix elements[2] of the product of two currents.

We shall perform this analysis in sec. V and show that the entire gauge dependence of the matrix elements of the product of two currents can be isolated in terms of the gauge dependence of the "known" piece [i.e. in the form of the gauge dependence of their coefficients and matrix elements[2]] . This means that the other "unknown" piece has gauge independent matrix elements and therefore must be expressible in terms of a linearly independent set of local operators whose matrix elements[2] are gauge independent and a set whose matrix elements all vanish. Both of these sets were found in I. Thus in sec. V we shall know

the complete set of operators in terms of which this operator product may be expanded, and information on the gauge (in)dependence of Wilson coefficients of the linearly independent operators.

In sec. VI, we shall turn to the question of how the gauge independence of the physical matrix elements of the current product works. In this section we shall be considering the gauge independence of the so-called quark gluon matrix elements in perturbation theory. Such a statement has to be interpreted as a <u>limiting statement</u> as explained in sec. II of I. In view of the fact that such matrix elements may be infrared divergent and meaningless in perturbation theory, such a discussion may only have a heuristic value.

In sec. VII, we shall consider the hadronic matrix elements[3] of the current product and the operators appearing in the operator product expansion. We shall make use of the fact that a gauge invariant local operator with the right quantum numbers can be used as an interpolating field for a physical hadron. Then under certain reasonable assumptions,[4] one can discuss directly the gauge independence of the hadronic matrix elements. Here the results stated/proved about the renormalization and the gauge independence of an arbitrary number of gauge independent operators in sec. III find a direct application.

The amplitudes for physical processes of interest can be conveniently grouped into two classes. The first class consists of diagrams in which two currents are connected <u>directly</u> via a single internal gauge (or unphysical higgs') bosons. In all other cases, the net four-momentum

Q transferred across the two currents is necessarily uncertain on account of the necessarily uncertain momenta of the external lines. In such a physical process, one can only observe the matrix elements of an operator product averaged over Q with a smooth distribution function on f(Q). Such averaged over matrix elements (either quark-gluon or as shown in sec. VII, hadronic) of all operators in the expansion other than the class I gauge invariant operators vanish (in absence of additional gauge invariant scurces only). This is sufficient to explain gauge independence of such amplitudes. [As explained in sec. VI, the external line uncertainty has to be brought in to explain why the gauge dependence of S-matrix[5] at exceptional momenta is physically irrelevant] .

The other case is due to the exchange of a virtual weak or electromagnetic guage boson with no additional external lines attached to it. Hence, in this case, $Q \equiv 0$ [i.e. with no uncertainty in Q]. However, here there is no way to observe the matrix element of a product of two currents with a specific momentum q entering via one of the currents. One can only observe one combination of the form $\int d^n q\, D_{\mu\nu}(q) < |j_\mu(q)\, j_\nu(-q)>$. In this case, one must isolate the effect of additional renormalizations due to the weak and electromagnetic interactions. Furthermore, only the Lorentz invariant operators appear in the operator product expansion of the above quantity. In this case, isolating renormalization counterterm effectively removes all the class II, III, IV operators as far as the quark-gluon matrix elements are concerned

i.e. such matrix elements of the difference between all the class II, III, IV operators in the expansion and the class II, III operators in the new counterterms vanish. Similarly, one can show that the hadronic matrix elements of the class II, III, IV operators at $Q \neq 0$ vanish and at $Q = 0$, the Lorentz invariant operators of these classes have their hadronic matrix elements proportional to the corresponding S-matrix elements,[6,7] [In particular they vanish if the S-matrix elements vanish.] the proportionality constant being independent of the external momenta. Thus hadronic matrix elements of these operators cannot be distinguished from the corresponding strong interaction S-matrix elements. This is an analogue of the wavefunction renormalization for external lines in the case of the quark gluon matrix elements. Thus in these cases also these operators cannot be measured since we can only observe the total effect of strong, weak and electromagnetic interactions.

This however, is not true when one is considering matrix elements of these operators in presence of additional gauge-invariant sources which can inject a variable (and measurable) four-momentum into the amplitude and/or which carries free Lorentz indices. [As opposed to the gauge invariant interpolating fields which must be considered at a fixed value of $p^2 = m^2_{had}$] . In these cases the effect of class II gauge invariant operators can be observed.

Finally, let us make several comments. In this paper we shall not prove operator product expansion but rather assume that it holds. [The operator product expansion for a product of two elementary fields in dimensionally regularized gauge theories with minimally

subtracted normal products has been proven recently [12] .] Furthermore we can only analyze (or prove) the operator product expansion for Green's function in perturbation theory. Such an analysis is not invalidated by the fact that such perturbation expansion of Green's functions of the elementary fields of the theory have no direct physical relevance. However, in applying these results, provable in each order in the perturbation expansion, to a physical situation one has to make an assumption that such results hold in form[8] in the exact (i.e. when summed to all orders) theory.

We note that our analysis is done by perturbing around the "false vacuum" [13] . In other words we have not taken into account the instanton solutions [14] that should contribute to the Euclidean functional integral for the Green's functions of operators. It is expected [13] that these will not contribute to the leading terms [most singular] in the operator product expansions and will be a special case of a nonleading correction[9]. Under these circumstances, the analysis presented here will be complete for the leading terms in the operator product expansion.

II. THE THEORY

[A] Preliminary

We shall work in a renormalizable theory of strong, weak and electromagnetic interactions. The theory of strong interactions is the same as that discussed for example by Gross, Wilczek, and Georgi and Politzer [4] . In brief, it is a nonabelian unbroken gauge theory with a gauge group[10] $G_s = SU(L)$, L being the number of colors which contain (massless gluons interacting with quark multiplets. All L quarks with the same flavor belong to a fundamental representation of SU(L). In addition the theory of strong interactions possesses approximate chiral (global) SU(N) x SU(N) symmetry (N being the number of flavors) which is a symmetry of transformations on the N quarks of the same color but different flavors. The weak and electromagnetic interactions are then incorporated in a renormalizable manner by assigning quarks of the same color to representations of weak and electromagnetic gauge group G_w. Thus the total gauge group is $G_s \otimes G_w$ and the global transformations of G_w are a subgroup of U(N) x U(N). Thus the gauge-invariant lagrangian density has the structure[11]

$$\mathcal{L}_0[A,\psi,\bar{\psi}] = -\frac{1}{4}F^{\alpha}_{\mu\nu}F^{\alpha\mu\nu} + \sum_{\ell=1}^{L}\sum_{i=1}^{N}\bar{\psi}_i^{\,j}[\not{D}^{j\ell}_{in} - m_i\delta_{j\ell}\delta_{in}]\psi^{\ell}_{n} \qquad (2.1)$$

where the upper (lower) index on ψ denotes color (flavor). And[12]

$$F^{\alpha}_{\mu\nu} = \partial_\mu A^{\alpha}_{\nu} - \partial_\nu A^{\alpha}_{\mu} + g_0 f_s^{\alpha\beta\gamma} A^{\beta}_{\mu} A^{\gamma}_{\nu}$$

$$D^{j\ell}_{\mu\,in} = \partial_\mu \delta^{j\ell}\delta_{in} - i\, g_0 \delta_{im}(T^{\alpha}_s)^{j\ell} A^{\alpha}_{\mu} - i\, e_\alpha (T^{\alpha}_w)_{im}\delta^{j\ell} W^{\alpha}_{\mu}$$

$T^{\alpha}_{s(w)}$ are representations of $G_{s(w)}$.

$$[T^{\alpha}T^{\beta}] = f^{\alpha\beta\gamma}T^{\gamma} \quad ,$$

The rest of the notations and definitions are as those in I.

It will be necessary to think of the weak and electromagnetic processes in the above discussed framework of a renormalizable theory of strong, weak and electromagnetic interactions rather than think of them as corrections to renormalizable theory of strong interactions to the extent that in the former case one knows precisely the renormalization effect due to these interactions in the form of renormalization counterterms in terms of known operators. The processes of our interest will be the lowest order weak and electromagnetic processes. Since the flavor gauge bosons couple to hadrons via quark flavor currents only, clearly, such processes can be expressed in terms of the Green's functions of a product of these flavor currents (and their divergences, if the process under consideration is a "weak" process, via which unphysical higgs' scalars couple to hadrons). Finally, we note that the theory of strong interactions has a charge conjugation symmetry and the vector currents have odd charge conjugation. The (strong) two gluon or two ghost functions of such quantities vanish on account of this charge conjugation property and the fact that the only tensor with two group indices is $\delta_{\alpha\beta}$.

[B] Results of I

We shall state the results obtained in I. Consider a set of local functionals of gauge fields, matter fields (collectively denoted by A) and ghost fields (c, $\bar{c}$) $\{ F[A, c, \bar{c}; g_0, \eta_0] \}$ such that

(i) F is invariant under global gauge transformations,

(ii) F has equal number of ghosts and antighosts in each term,

(iii) The total power of all fields--power of $g_0 = 2$ for each term.

(I) Then the subset of operators for which its Green's functions renormalized and evaluated off mass shell (as discussed in I) satisfy

$$\lim_{\mu^2 \to 0^+} \langle F[A, c, \bar{c}, g_0, \eta_0] \rangle_{\mu^2} = 0$$

is the set of linear combinations of

(i) Class II operators whose all tree matrix elements vanish. They are characterized by an "F-function" which has the property that[13]

$$J_k \left. \frac{\delta F_k}{\delta A_j} \right|_{A=c=\bar{c}=0} \Phi_j[J] \Big|_{J=J(\text{phy})} = 0$$

(ii) Class III operators[14] $\left[\text{Their form is } \frac{\delta s}{\delta c_\alpha} X_\alpha[A, c, \bar{c}] \right]$

(iii) "Class IV" operators[15] defined by [16]

$$f_\beta X_\beta[A, c, \bar{c}] \cdot \bar{c}_\alpha \mathcal{G}_0 X_\alpha - \bar{c}_\beta \frac{\delta X_{\beta x}}{\delta \bar{c}_\alpha} f_\alpha$$

We shall call this the set S_0.[15]

(II) The set of operators whose truncated renormalized physical Green's functions satisfy

$$\lim_{\mu^2 \to 0^+} \frac{\partial}{\partial \eta} \langle F[A, c, \bar{c}; g_0 \eta_0] \rangle_{\mu^2} = 0$$

and for at least one matrix element

$$\lim_{\mu^2 \to 0^+} \langle F[A, c, \bar{c}; g_0 \eta_0] \rangle_{\mu^2} \neq 0$$

is the set of linear combinations of the following linearly independent operators

(i) All class I operators,

(ii) A subset of class II operators such that (a) no linear combination of them belongs to the set so discussed above and (b) In its F_k-function, when $k \equiv (\alpha, \mu, x_k)$ refers to a gauge field, the Lorentz index μ is carried by the gauge field[17] [after contracting all the dummy indices] .

We shall call the above set S. One can, of course, add to any number of S an arbitrary linear combination of operators in S_0 discussed above.

III. OPERATOR PRODUCT OF GAUGE INVARIANT OPERATORS

We shall briefly explain why the study of the renormalization and gauge dependence of Green's functions of an arbitrary number of gauge invariant operators is of interest:

(i) Given a gauge theory, interactions external to the theory couple to the theory via a coupling to a local gauge invariant quantity (perhaps modified by operators of the form of their renormalization counter terms) such as currents, energy momentum tensor. Thus one needs to know how to renormalize the Green's functions of product of these local gauge invariant operators.

(ii) In actual applications, one uses operator product expansion of two gauge invariant operators. Before one can understand what operators appear in the expansion one has to study the gauge dependence properties of Green's functions of these product of local operators.

(iii) One may be interested in considering the renormalized matrix elements of a gauge invariant physical quantity O(x) between actual hadronic states which are supposed to be bound states of quarks. These can be generated up to overall constant factors by considering a Green's function of O(x) together with gauge invariant interpolating local fields for hadrons with correct quantum numbers. For example, a matrix element $\langle \pi | O(x) | \pi \rangle$ may be generated by considering $\langle \bar{\psi}_{(z)} \gamma_5 \tau \psi_{(z)} \, O(x) \, \bar{\psi}(w) \gamma_5 \tau \psi(w) \rangle_0$. One must therefore know how to renormalize such quantities in order to use them. We shall indeed use them in considering gauge independence of the matrix elements of various field operators

between hadronic states.

The main discussion of the renormalization of a product of an arbitrary number of gauge invariant operators is a straightforward generalization of methods used in refs. [8-10]. Given the results in ref. 8, a compact way of expressing and generalizing the results to this case is the one given by Dixon [11]. His discussion can be extended in a straightforward manner to the case of an arbitrary number of insertions of a gauge invariant operator. The main difference in this case is that the nilpotent operator $\mathcal{G}'$ appearing in the WT identity[18] now has the structure $\mathcal{G}' = \mathcal{G}'^{(0)} + \sum_{(i)} N_{(i)} \mathcal{G}'^{(1)}_{(i)}$ and the counterterms are series in powers of all sources $N_{(i)}$.[9]

The discussion of the gauge independence of the matrix elements of a product of n gauge invariant operator is a straightforward generalization of the discussion given in I except at exceptional momenta cases. [If operators $O_1 \ldots O_n$ are inserted at momenta $q_1 \ldots q_n$ respectively, the exceptional momentum cases are the ones in which at least one of the partial sums satisfies $(\sum q_i) = 0$.] Such a case for two currents is of physical interest and will be treated extensively in the next two sections.

[A] Renormalization of a product of n gauge invariant operators

Consider n gauge invariant operators $O^{(1)}_{x_1} \ldots O^{(n)}_{x_n}$. The unrenormalized but dimensionally regularized Green's functions with an arbitrary number of insertions of each at arbitrary momenta can be obtained from the generating functional

$$W[j, \xi, \bar{\xi}, \kappa, \ell, N^{(i)}] = \int [d A dc d\bar{c}] \exp i\{ \tilde{S}[A, c, \bar{c}, \kappa, \ell; g_0 \eta_0] + N^{(i)}_x O^{(i)}_x [A, g_0] - \tfrac{1}{2} f^2_\alpha [A] + j_i A_i + \bar{\xi}_\alpha c_\alpha + \bar{c}_\alpha \xi_\alpha \}. \quad (3.1)$$

The corresponding renormalized Green's functions can be obtained from

$$W^R[J, \zeta, \bar{\zeta}, K, L, N^{(i)}] = \int [d A dc d\bar{c}] \exp i \{ \tilde{S}[A, c, \bar{c}, \kappa, \ell; g_0, \eta_0] + N^{(i)}_x O^{(i)}_x [A, g_0] - \tfrac{1}{2} f_\alpha [A^2] + \text{C. T.} + \text{source terms} \} \quad (3.2)$$

where the right hand side is to be expressed in terms of renormalized quantities; and the C. T. stand for all the renormalization counter terms which are local polynomials in A, c, $\kappa_i - (\partial\bar{c})_i$, ℓ, $N^{(i)}$ with divergent coefficients (as $\epsilon = n - 4 \to 0$).

Dixon's observation can be restated to include insertions of an arbitrary number of gauge invariant operators and is that there exists a canonical transformation[20]

$$A_i = \frac{\delta \mathcal{F}}{\delta \kappa_i} \qquad \kappa'_i - \partial^{\alpha'}_i \bar{c}_\alpha = \frac{\delta \mathcal{F}}{\delta A'_i}$$

$$-c_\alpha = \frac{\delta \mathcal{F}}{\delta \ell_\alpha} \qquad -\ell'_\alpha = \frac{\delta \mathcal{F}}{\delta c'_\alpha} \quad (3.3)$$

where, $\mathcal{F}[A'; c', \bar{c}', \kappa', \ell, N; g_0, \eta_0, \epsilon, \hbar]$ can be chosen to be a local functional of its arguments, such that

$$\tilde{S}[A, c, \bar{c}, \kappa, \ell] - \sum_i N^{(i)}_x O^{(i)}_x [A] + \text{C. T.} \quad (3.4)$$

$$= \tilde{S}[A', c', \bar{c}', \kappa', \ell'] - \sum_i N^{(i)}_x O^{(i)}_x [A'] + O^{GI}[A', N]$$

where $O^{GI}[A', N]$ is a sum over local gauge invariant functionals of A' and N with divergent coefficients. We shall find this result useful

in explaining the gauge independence of the renormalized matrix elements of n gauge invariant operators.

[C] Gauge independence of a product of an arbitrary number of gauge invariant operators.

Here, we shall consider this topic only for the case when the gauge invariant operators are inserted at "non exceptional momenta". Here, it means that if $q_1 \ldots q_n$ denote the momenta at which the n gauge invariant operators are inserted, they satisfy, for <u>every</u> partial sum $\sum q_i$, $(\sum q_i) \neq 0$. [It includes $(\sum_{i=1}^{n} q_i) \neq 0$.] In this case the proof of gauge invariance is a straightforward extension of the proof of gauge invariance for the case of one gauge invariant operator. We shall give it here for the sake of completeness.

We consider the generating functional for the Green's functions [which is equal to W^R of Eq. (3.2)]

$$W[j, \xi, \bar{\xi}, \kappa, \ell, N^{(1)} \ldots N^{(n)}] = \int [dAdcd\bar{c}] \exp i\{S_{eff} + j_i A_i + \sum_{(i)} N^{(i)}_{x_i} O^{(i)}_{x_i} + C.T.\} \tag{3.5}$$

where C.T. are a sum over terms each containing a product of p ($1 \leq p \leq n$) distinct N-sources. They contain gauge invariant operators and gauge invariant operators of the form discussed at the end of previous subsection. One then performs the field transformation on A_k and c_α given by Eq. (3.3). One cannot, of course, perform the transformations on the sources κ and ℓ as they are external sources in W. As explained in the previous subsection, the net effect of such a transformation on the exponent in W is

(i) To transform away the class II and class III counterterms.

(ii) Replace (κ, ℓ) by $\kappa'[\kappa, \ell, A', c', \bar{c}]$ and $\ell'[\kappa, \ell, A', c', \bar{c}]$ as given by the relations of Eq. (3.3).

(iii) Produce new terms from the transformations on the source terms and the gauge fixing term.

The exponent, thus becomes

$$\mathcal{L}_0[A':g_0,\eta_0] + \frac{\delta \mathcal{F}}{\delta A'_i} D^\alpha_i[A'] c'_\alpha - \frac{1}{2} g_0 \frac{\delta \mathcal{F}}{\delta c'_\alpha} f_{\alpha\beta\gamma} c'_\beta c'_\gamma$$
$$- \frac{1}{2}\eta_0 (\partial^\alpha_i \frac{\delta \mathcal{F}}{\delta \kappa_i})^2 + j_i \frac{\delta \mathcal{F}}{\delta \kappa_i} + \ldots + \sum_i N^{(i)} O^i[A'] + O^{GI}[N, A'] \tag{3.6}$$

where $\mathcal{F} \equiv \mathcal{F}[A', c, \bar{c}; \kappa, \ell, N; g_0, \eta_0, \epsilon, \lambda]$ with

$$\partial_i^\alpha \frac{\delta \mathcal{F}}{\delta \kappa_i} = \frac{\delta \mathcal{F}}{\delta \bar{c}_\alpha} \quad . \tag{3.7}$$

and the dots indicate terms coming from changing the ghost source terms. We, thus have at $\xi = \bar{\xi} = 0$, (dropping primes on A', c', $\bar{c}'$)

$$W[j, \kappa, \ell, N] = \int [dAdcd\bar{c}] \exp i\{\mathcal{L}_0[A; g_0, \eta_0] - \frac{1}{2}\eta_0 (\frac{\delta \mathcal{F}}{\delta c_\alpha})^2 - \mathcal{L}_0 \mathcal{F}$$
$$+ j_i \frac{\delta \mathcal{F}}{\delta \kappa_i} + G.I. \text{ terms}\} \quad . \tag{3.8}$$

This quantity is discussed in the appendix A and it is shown there that it has the structure [see Eq. (A.7)]

$$W[j, \kappa, \ell, N] = j_k \int [dAdcd\bar{c}] \; X_k \; [A, c, \bar{c}, \kappa, \ell, N]$$
$$\exp i\{S_{eff}[A, c, \bar{c}, \kappa, \ell] + j_i A_i + \ldots\} \tag{3.9}$$
$$+ \text{"Gauge invariant" terms.}$$

where X_k is at least linear in N. When the gauge invariant operators are inserted at nonexceptional momenta, the above quantity vanishes[21] identically at $J = J^{(phy)}$ [either because of an absence of pole in the external $(\text{momentum})^2$ or because of $\partial_i^\alpha J_i^{(phy)} = 0$].Therefore, one may

as well omit these counterterms for the discussion of the physical matrix elements at non-exceptional momenta.

Then as in the Eq. (3.28) of I, one obtains the net change in the matrix elements of $T[O^1_{x_1} \dots O^n_{x_n}]$ under a change in η. They consist of a connected part of (up to a factor)

$$< j_i \, D^\alpha_i M^{-1}_{\alpha\beta} f_\beta \{O^{(1)} \dots O^{(n)} + \text{gauge-invariant C. T.}\} >_j \tag{3.10}$$

and the change on account of the change in the wavefunction renormalization. At non-exceptional momenta, the only contributions to it survive in physical matrix elements are of the form of Fig. (2a) of I and they are cancelled on shell exactly by the change in the wavefunction renormalization provided one does on shell renormalization.[21]

Further one notes that if one is considering the gauge invariance of the Green's functions of the product <u>without external lines</u> [$j \equiv 0$], then the restriction of nonexceptional momenta can be dropped since obviously the terms on the r.h.s. of Eq. (3.9) and in (3.10) vanish identically at $j = 0$. Thus the Green's function $< 0 | T(O^{(1)} \dots O^{(n)}) | 0 >$ invariant for any momenta entering via the n operators.

We shall deal with the gauge invariance of a product of two gauge invariant operators at exceptional momenta ($q_1 = -q_2$) extensively in the next two sections.

One can extend the discussion of the renormalization of n gauge invariant operators/ to the case of one class II (or III) operator and n gauge invariant operators. This discussion will be useful in Sec. VII. To this end one considers the action

$$\tilde{S}'[A, c, \bar{c}, k, \ell, N, M] = S' - \tfrac{1}{2} f_\alpha^{\,2}$$

$$= \tilde{S}[A, c, \bar{c}, \kappa, \ell] + \sum_i N^{(i)}_x O^{(i)}_x[A] + \text{C. T.}[A, c, \bar{c}, \kappa, \ell, N]$$

$$+ M_x \mathcal{G}''F_x[A, c, \bar{c}, \kappa, \ell, N] \tag{3.11}$$

where C. T. denotes the counterterms to renormalize the (products of) n gauge invariant operators. $O'_x = \mathcal{G}' F[A, c, \bar{c}, \kappa, \ell]$ is a class II (III) operator [where $\mathcal{G}'$ was defined in Ref. 8]. $\mathcal{G}''$ is defined by

$$\mathcal{G}'' = \left[\frac{\delta\tilde{S}'}{\delta A_i}\frac{\delta}{\delta\kappa_i} + \frac{\delta\tilde{S}'}{\delta c_\alpha}\frac{\delta}{\delta\ell_\alpha} + \frac{\delta\tilde{S}'}{\delta\kappa_i}\frac{\delta}{\delta A_i} + \frac{\delta\tilde{S}'}{\delta\ell_\alpha}\frac{\delta}{\delta c_\alpha}\right]_{M=0} \tag{3.12}$$

and satisfies

$$\mathcal{G}''^2 = 0 \quad . \tag{3.13}$$

We also have

$$\frac{\delta\tilde{S}'}{\delta A_i}\frac{\delta\tilde{S}'}{\delta\kappa_i} + \frac{\delta\tilde{S}'}{\delta c_\alpha}\frac{\delta\tilde{S}'}{\delta\ell_\alpha} = 0 + O(M^2) \tag{3.14}$$

Now one can explicitly evaluate $<\mathcal{G}''F_x>_{j\xi,\bar{\xi}k,\ell,N}$ as done for example in ref. 9 (the second paper). The result is

$$<\mathcal{G}''F_x>_{n=0} = \left(\frac{\delta\Gamma'_0}{\delta\phi_i}\frac{\delta}{\delta\kappa_i} + \frac{\delta\Gamma'}{\delta\Omega_\alpha}\frac{\delta}{\delta\ell_\alpha} + \frac{\delta\Gamma'}{\delta\kappa_i}\frac{\delta}{\delta\varphi_i} + \frac{\delta\Gamma'}{\delta\ell_\alpha}\frac{\delta}{\delta\Omega_\alpha}\right)_{M=0} <F_x>_{M=0} . \tag{3.15}$$

Using this equation together with a discussion entirely parallel to the corresponding discussion in Ref. 9, we shall be able to conclude that the renormalization counterterms for $\mathcal{G}''F_x$ to all orders (at $M = 0$) are of the same form i.e. a local $\tilde{F}'_x$ can be chosen such that $<\mathcal{G}''\tilde{F}_x>$ is a finite functional when expressed in terms of renormalized quantities $\Phi, \omega, \bar{\omega}, K, L, N, g, \eta$ and such that $\{\tilde{F}_x\}_0 = F_x$. [It should be noted that $\mathcal{G}''$ contains explicit divergences.]

In the following we shall for the sake of concreteness use the minimal subtraction scheme of 't Hooft and Veltman. We shall write

$$\{\mathcal{G}''F_x\}_0 = O'_x + O(N)$$
$$= O'_x + \sum_i N_y^{(i)} \frac{\delta O_y^{(i)}}{\delta A_n} \frac{\delta F}{\delta k_n} \quad . \qquad (3.16)$$

Now the local operators O'_x and $\sum N_y^{(i)} \frac{\delta O_y^{(i)}}{\delta A_n} \frac{\delta F}{\delta k_n}$ can be separately renormalized using the minimal subtraction scheme. [To do this, e.g. one would compute $<O'_x>$ as a functional of $\Phi, \omega, \bar{\omega}, K, L, N, g, \eta$ in the one loop approximation and add to renormalization counterterms so as to cancel just the pole part of it. One would then include these counterterms in computing Green's functions to the next order as explained, for example, in 1]. Let $(C.T.)'[A, c, \bar{c}, k, \ell, N, \epsilon]$ be the renormalization counterterm for O'_x and $(C.T.)''$ be the total renormalization counterterms for $\sum_i N_y^{(i)} \frac{\delta O^{(i)}}{\delta A_n} \frac{\delta F}{\delta k_n}$ [$(C.T.)'$ contains terms independent of N while $(C.T.)''$ does not.] We shall now prove that

$$<\mathcal{G}''F_x>^R = <\mathcal{G}''F_x> = <O'_x>^R + \left\langle \sum_i N_y^{(i)} \frac{\delta O_y^{(i)}}{\delta A_n} \frac{\delta F}{\delta k_n} \right\rangle^R$$
$$\equiv <O'_x + (C.T.)'> + \left\langle \sum_i N_y^{(i)} \frac{\delta O_y^{(i)}}{\delta A_n} \frac{\delta F}{\delta k_n} + (C.T.)'' \right\rangle . \qquad (3.17)$$

Indeed, we shall prove the following algebraic identity which is stronger than the one made above

$$\mathcal{G}''\tilde{F}_x = O'_x + (C.T.)' + \sum N_y^{(i)} \frac{\delta O_y^{(i)}}{\delta A_n} \frac{\delta F}{\delta k_n} + (C.T.)'' \quad . \qquad (3.18)$$

To prove the above identity, let us assume that the above identity is valid up to a^{n-1} terms in the loop expansion in the sense that when both sides

are expanded in terms of a and the <u>bare</u> quantities, they agree up to a^{n-1} terms. We then have to show that the a^n terms in that relation agree. This is indeed obvious if one goes back to the proof (not given here) for the statement that $\tilde{F}$ can be chosen so that $<\mathcal{G}''\tilde{F}>$ is finite. Clearly if one takes both sides of this equation up to a^{n-1} terms and evaluates the n loop divergence in them [e.g. by expressing $<\mathcal{G}''\tilde{F}>$ in terms of a and the renormalized quantities] it will correspond to the new a^n counterterms to be subtracted from both sides as the a^n counterterms. The question is precisely whether the counterterms so computed for $<\mathcal{G}''\tilde{F}>$ can be incorporated by modifying $\tilde{F}$. This is always possible (as the proof referred to above says) provided one is willing to subtract additional terms of $O(a^{n+1})$ and higher that are divergent. This is not in contradiction to the minimal subtraction scheme since such terms simply become a part of the (n+1) loop counterterms. [Indeed if one does not add them now, they will arise in higher orders] Thus it is possible to generate the minimally subtracted Green's function of $\mathcal{G}''\tilde{F}$, O'_x, $\sum_i N_y^{(i)} \frac{\delta O^{(i)}}{\delta A_n} \frac{\delta F}{\delta k_n}$ from quantities that satisfy Eq. ( ). Hence etc.

Next we shall show that in absence of external sources,

$$<\mathcal{G}''F_x>_0^R = <\mathcal{G}''\tilde{F}_x>_0 = 0 \quad . \qquad (3.19)$$

The proof is identical to the derivation of the form for the Green's functions of a class II operator given in Sec. V of ref. 8. In short in the <u>absence of external sources</u>, the operator $\mathcal{G}''F_x$ can be transformed away by a field transformation.

As a result one has at $j = \xi = \bar{\xi} = 0$,

$$\langle O'_x \rangle^R_N = \sum_i N^{(i)}_y \left\langle \frac{\delta O^{(i)}}{\delta A_n} \frac{\delta F}{\delta A_n} \right\rangle^R_N \quad . \tag{3.20}$$

We shall use this result in Sec. VII.

IV. ANALYSIS OF THE WT IDENTITY

It was shown in the last section that the renormalized Green's functions of the operator product satisfy the equation

$$\lim_{\mu^2 \to 0^-} \frac{\partial}{\partial \eta} \left\langle \left| \tilde{\mathcal{J}}^a_\mu (q + \tfrac{Q}{2}) \; \tilde{\mathcal{J}}^b_\nu (-q + \tfrac{Q}{2}) \right| \right\rangle_{\mu^2, \eta} \doteq 0 \tag{4.1}$$

$$0 \neq q \neq \frac{Q}{2} \neq 0$$

where the above Green's function is

$$\int e^{i(q+Q)\cdot x} \; e^{i(-q+Q)\cdot y} \left\langle \left| T\left(\mathcal{J}^a_\mu(x) \, \mathcal{J}^b_\nu(y)\right) \right| \right\rangle d^n x \, d^n y$$

Now we shall assume that it is possible to express the Green's functions of $T(\mathcal{J}^a_\mu(x) \; \mathcal{J}^b_\nu(y))$ in terms of an operator product expansion $\sum C^{(i)}(x-y) \; O^{(i)}\left(\frac{x+y}{2}\right)$ where $O^{(i)}$ refers to minimally subtracted normal product operators. $O^{(i)}$ can be converted into the operators renormalized according to our renormalization convention as explained in I [$O^{(i)}$ will then be a linear combination of such operators]. Then, we will have, in momentum space,

$$\left\langle \tilde{\mathcal{J}}^a_\mu (q + \tfrac{Q}{2}) \; \tilde{\mathcal{J}}^b_\nu (-q + \tfrac{Q}{2}) \right\rangle_{\mu^2, \eta} = \sum C_i(q, \mu^2, n^2) \left\langle O_i [Q] \right\rangle^R_{\mu^2, \eta} \tag{4.2}$$

We shall assume that O_i's in the Eq. (4.2) are linearly independent. Then Eq. (4.1) implies that only the following operators O_i's can appear on the r.h.s. of Eq. (4.2):

(i) A set of operators satisfying ($Q \neq 0$)

$$\lim_{\mu^2 \to 0^+} \left\langle O_i [Q] \right\rangle_{\mu^2, \eta} = 0$$

(ii) A set of operators whose matrix elements are linearly independent [in the sense that $\sum d_i \langle O_i[Q]\rangle_{\mu^2,\eta} = O(\mu^2) \Rightarrow d_i = O(\mu^2)$ for each i] and satisfy

$$\lim_{\mu^2 \to 0^+} \frac{\partial}{\partial\eta} \langle O_i[Q]\rangle_{\mu^2,\eta} = 0 \qquad Q \neq 0$$

with

$$\lim_{\mu^2 \to 0^+} \frac{\partial}{\partial\eta} C_i(\mu^2) = 0$$

(iii) An arbitrary set of local operators with $C_i(\mu^2) = O(\mu^2)$.

The contribution of this set of operators can always be made as small as one wants by letting $\mu^2 \to 0$.

The set of operators in (i) and (ii) are just the set of operators of the four classes discussed in Sec. II-B. [Since $Q \neq 0$, only class I operators satisfy condition (ii) above.] Since this set is closed under renormalization, we may express the sum over these operators in terms of the sum over the minimally subtracted operators.

In the following, we shall analyze the WT identity for the Green's functions of the product of two currents. We shall separate a set of operators from the current product. In the next section, we show that the entire gauge dependence of the Green's functions of the product of two currents near mass shell can be isolated in terms of the gauge dependence of the matrix elements of these operators and their coefficient functions. It is clear that among the operators of class II, III and IV only the class II operators can contribute to this gauge dependence since only they have nonvanishing matrix elements. It is further easy to show that, in order

that a class II operator have a non-vanishing matrix element, it must have a non-vanishing proper vertex with two external gluons or quarks in the tree approximation. Further, from the discussion given in I, if the two point function of a class II operator is identical to that of some gauge invariant operator, its matrix elements are gauge independent. We shall therefore need to isolate only the class II operators with non-vanishing gauge dependent matrix element(s) and this can therefore be done by considering the Green's functions with two external lines.

We shall now write down the WT identities satisfied by these Green's functions and then the related proper vertices using the fact that $\mathcal{J}$ are gauge invariant quantities.

As usual, we consider the generating functional [Summation integration convention resumed $\mathcal{J}_\mu^{(k)}(x) \to \mathcal{J}_i$]

$$W[j, \xi, \bar{\xi}, \kappa, \ell, N] = \int [dA dc d\bar{c}] \exp i\{S[A, c, \bar{c}, \kappa, \ell] + j_i A_i + \bar{\xi}_\alpha c_\alpha + \bar{c}_\alpha \xi_\alpha + N_i \mathcal{J}_i\} \tag{4.3}$$

Since $\mathcal{J}_i$ is a gauge invariant quantity, the corresponding generating

functional for proper vertices, Γ_0 satisfies the renormalized WT identity

$$\frac{\delta\Gamma_0}{\delta K_i}\frac{\delta\Gamma_0}{\delta\Phi_i}+\frac{\delta\Gamma_0}{\delta\omega_\alpha}\frac{\delta\Gamma_0}{\delta L_\alpha}=0$$

Differentiating this WT identity with respect to N_i and N_j and setting $N = 0$ we obtain the WT identity similar to Eq. (2.3)

$$\left(\frac{\delta\Gamma_0}{\delta K_m}\frac{\delta}{\delta\Phi_m}+\frac{\delta\Gamma_0}{\delta\Phi_m}\frac{\delta}{\delta K_m}+\frac{\delta\Gamma_0}{\delta L_\alpha}\frac{\delta}{\delta\omega_\alpha}+\frac{\delta\Gamma_0}{\delta\omega_\alpha}\frac{\delta}{\delta L_\alpha}\right)\Gamma_{0,N_iN_j}$$
$$+\left\{\frac{\delta\Gamma_{0,N_i}}{\delta K_m}\frac{\delta\Gamma_{0,N_j}}{\delta\Phi_m}+\frac{\delta\Gamma_{0,N_i}}{\delta L_\alpha}\frac{\delta\Gamma_{0,N_j}}{\delta\omega_\alpha}+(i\longleftrightarrow j)\right\}=0 \qquad (4.4)$$

where

$$\Gamma_{0,N_iN_j}=\left.\frac{\delta^2\Gamma_0}{\delta N_i\delta N_j}\right|_{N=0}\quad \Gamma_{0,N_i}=\left.\frac{\delta\Gamma_0}{\delta N_i}\right|_{N=0} \qquad (4.5)$$

and other derivatives of Γ_0 in the above equation are taken at $N = 0$. Now differentiate the above equation with respect to ω_α and set $\omega = \bar{\omega} = K = L = 0$. We then obtain

$$\left(\frac{\delta^2\Gamma_0}{\delta\omega_\alpha\delta K_m}\frac{\delta}{\delta\Phi_m}+\frac{\delta\Gamma_0}{\delta\Phi_m}\frac{\delta}{\delta\omega_\alpha\delta K_m}\right)\Gamma_{0,N_iN_j}$$
$$=\frac{\delta^2\Gamma_{0,N_i}}{\delta\omega_\alpha\delta K_m}\frac{\delta\Gamma_{0,N_j}}{\delta\Phi_m}+(i\longleftrightarrow j) \quad . \qquad (4.6)$$

We next consider the terms that are linear in Φ. On the r.h.s. such terms are

$$\left\{\frac{\delta^2\Gamma_{0,N_i}}{\delta\omega_\alpha\delta K_m}\frac{\delta^2\Gamma_{0,N_j}}{\delta\Phi_m\delta\Phi_p}\right\}_{\Phi=\omega=\bar{\omega}=K=L=0}\Phi_p+(i\longleftrightarrow j) \qquad (4.7)$$

But

$$\left\{\frac{\delta^2\Gamma_{0,N_i}}{\delta\omega_\alpha\delta K_m}\right\}_{\Phi=\omega=\bar{\omega}=K=L=0}$$

which is the 1PI part of the Green's function with one insertion of $\mathcal{J}_i$ and $t^\alpha_{ij}c_\alpha A_j$ each and one external line vanishes identically, as discussed in sec. II, for the baryon vector current on account of charge conjugation and for any vector or axial vector current which is not a singlet under the flavor group SU(N). Thus the WT for a proper two point function of $\mathcal{J}_i\mathcal{J}_j$ reads,

$$p_\alpha\Gamma^{(2)\mu\nu}_{\alpha\beta}(p,q,-q+Q,-p-Q)=Q^{(2)\mu\nu}_\sigma(p,q,-q+Q,-p-Q)$$
$$\{g_{\sigma\beta}(p+Q)^2-(p+Q)_\sigma(p+Q)_\beta\} \qquad (4.8)$$

where $\Gamma^{(2)\mu\nu}_{\alpha\beta}(p,q,-q+Q,-p-Q)$ and $Q^{(2)\mu\nu}_\sigma(p,q,-q+Q-p-Q)$ are, up to factors, fourier transforms of

$$\left.\frac{\delta^2\Gamma_{0,N_iN_j}}{\delta\Phi_m\delta\Phi_p}\right|_{\Phi=\omega=\bar{\omega}=K=L=0}\qquad \left.\frac{\delta^2\Gamma_{0,N_iN_j}}{\delta\omega_\alpha\delta K_m}\right|_{\Phi=\omega=\bar{\omega}=K=L=0}$$

when p refers to a gauge boson and we have noted that

$$\left[\frac{\partial^2\Gamma_0}{\delta\omega_\alpha\delta K_m}\right]_{\Phi=\omega=\bar{\omega}=K=L=0}\sim\partial^\alpha_m$$

on account of its Lorentz transformation property. Since we already have considered the gauge independence of a product of gauge invariant operators at non-exceptional momenta in the preceding section, we only have to consider it here for the exceptional momentum case i.e. $Q \equiv 0$. In this case there are three independent Lorentz invariants, viz. p^2, $2p\cdot q$ and q^2, on which $\Gamma^{(2)}$ and $Q^{(2)}$ can depend on. We note from the analytic structure of these two point 1PI functions that $p^2 = -\mu^2 < 0$ (where μ^2 is a number which can be taken arbitrarily close to, but is

never zero) and $p\cdot q = 0$ is a nonsingular point for $\Gamma^{(2)}$ and $Q^{(2)}$. We can expand these functions around this point in the domain[22] $-2\mu^2 \leq p^2 < 0$ $|2p\cdot q| < |q^2| - 2\mu^2$. One thus writes,[23]

$$\Gamma^{(2)\mu\nu}_{\lambda\sigma}(p,q,-q,-p) = \sum_i S^{(i)}_{\lambda\sigma\mu\nu}(p,q)\, f^{(i)}\left[\frac{p^2+\mu^2}{\mu^2}, \frac{2p\cdot q}{q^2}, \frac{m^2}{q^2}, \frac{\mu^2}{q^2}, \frac{1}{q^2}\right]$$

$$Q^{(2)\mu\nu}_{\lambda}(p,q,-q,-p) = \sum_i T^{(i)}_{\lambda\mu\nu}(p,q)\, f^{(i)}\left[\frac{p^2+\mu^2}{\mu^2}, \frac{2p\cdot q}{q^2}, \frac{m^2}{q^2}, \frac{\mu^2}{q^2}, \frac{1}{q^2}\right] \quad (4.9)$$

where $f^{(i)}$ are series in powers of $\frac{1}{q^2}$, $\frac{p^2+\mu^2}{\mu^2}$ and $\frac{2p\cdot q}{q^2}$ with coefficients that are functions of $\frac{\mu^2}{q^2}, \frac{m^2}{q^2}$. $S^{(i)}$ and $T^{(i)}$ are linearly independent polynomials in p and q that must satisfy

$$p_\lambda S_{\mu\nu,\lambda\sigma}(p,q) = T_{\lambda\mu\nu}(p,q)(p^2 g_{\lambda\sigma} - p_\lambda p_\sigma) \quad (4.10)$$

to be in accord with Eq. (4.6). This is the equation we have to solve to know what sort of terms are present in $\Gamma^{(2)}$. We note that the above equation, being a polynomial equation, must hold for each power of q separately. Thus differentiating the equation with respect to q n times $(n = 0, 1, \ldots)$ and setting $q = 0$, we obtain[24]

$$p_\lambda S^{(n)}_{\mu\nu,\lambda\sigma}{}^{\alpha_1\cdots\alpha_n}(p) = T_{\lambda\mu\nu}{}^{\alpha_1\cdots\alpha_n}(p)(p^2 g_{\lambda\sigma} - p_\lambda p_\sigma) \qquad n = 0, 1, \ldots \quad (4.11)$$

But this is precisely the equation satisfied by the local part of the proper vertex with 2 external lines and with a gauge invariant operator $O_{\mu\nu\alpha_1\cdots\alpha_n}$ inserted at zero momentum. It was discussed in detail in Sec. V of I and it was shown there that the most general solution to the above equation has the form[25]

$$S^{(n)}_{\mu\nu,\lambda\sigma}{}^{\alpha_1\cdots\alpha_n}(p) = F^{(n)}_{\mu\nu\lambda\tau}{}^{\alpha_1\cdots\alpha_n}(p)(g_{\tau\sigma}p^2 - p_\tau p_\sigma) \pm (\lambda \longleftrightarrow \sigma) \quad (4.12)$$

(the homogeneous equation being the Eq. (4.11) with its r.h.s. replaced by zero) provided $S^{(n)}$ was at least quadratic in p. It was further remarked there that the first term on the r.h.s. of Eq. (4.12) is a two point function of an operator of class II at zero momentum. We are here interested in the coefficient functions of these class II operators only.

We have thus been able to show that the two gluon function can be expressed as

$$\Gamma^{(2)}_{\lambda\sigma}(p,q,-q,-p) = \sum_i C^{(i)}{}_{\tau_1\cdots\tau_n}(q)\left\{\tilde{F}^{\tau_1\cdots\tau_n}_{\mu\nu\lambda\xi}(p)(p^2 g_{\xi\sigma} - p_\xi p_\sigma) \pm (\lambda \longleftrightarrow \sigma)\right\}$$

$$+ \text{ other terms} \quad (4.13)$$

where $C^{(i)}$ are determined once $\Gamma^{(2)}$ is given. Here, we have expanded $f^{(i)}$ in the powers of its arguments $\frac{p^2+\mu^2}{\mu^2}$, $\frac{2p\cdot q}{q^2}$ and have absorbed the powers of p in $\tilde{F}$. Thus in the coordinate space, the two gluon function is expressible as the two point function of

$$\sum_i C^{(i)}\left\{O^{II}_{(i)}[A, c, \bar{c}]\right\}_0 + \text{remainder} \quad (4.14)$$

$C^{(i)}$ can be determined in each order. A similar analysis can be done for the two fermion Green's functions. (The main difference is that there are now one particle reducible diagrams.) In the end, we can write

$$Z_{0,N_iN_j}[\Phi,\omega,\bar{\omega}] = \sum C^{(i)}\left\langle O^{II}_{(i)}[A, c, \bar{c}]\right\rangle^R + \text{remainder} \quad (4.15)$$

where C_i has the loop expansion

$$C^{(i)} = \sum a^n C^{(i,n)} \tag{4.16}$$

Note that we have expressed the right-hand side of the above expression in terms of renormalized operators. The remainders in Eqs. (4.15) and (4.14) differ by terms whose two gluon functions for example vanish at $p^2 = -\mu^2$.

V. THE GAUGE DEPENDENCE OF THE GREEN'S FUNCTIONS OF A PRODUCT OF TWO CURRENTS

As in Eq. (3.28) of I, the gauge variation of the connected renormalized Green's functions of $T(\mathcal{J}_i \mathcal{J}_j)$ is obtained by considering

$$Z^{-1} \frac{\partial \eta_0(g,\eta)}{\partial \eta} j_k < D_k^\alpha c_\alpha \bar{c}_\beta f_\beta \mathcal{J}_i \mathcal{J}_j >_j^{conn} \tag{5.1}$$

and adding to it the change on account of the change in the external line wavefunction renormalization which is given by

$$Z^{-1} \frac{\partial \eta_0(g,\eta)}{\partial \eta} Z_{(k)}^{-\frac{1}{2}} \frac{\partial Z_{(k)}^{\frac{1}{2}}}{\partial \eta_0} j_k < A_k \mathcal{J}_i \mathcal{J}_j >_j^{conn} . \tag{5.2}$$

Of course, the above objects are to be expressed in terms of renormalized quantities and then a contribution of up to a specific number of loops is to be taken.

Consider the diagrams in (5.1) that survive at $J = J^{(phy)}$. Among them are the diagrams that have the structure

$$Z^{-1} \frac{\partial \eta_0(g,\eta)}{\partial \eta} j_k \left\{ \frac{\delta}{\delta j_p} < D_k^\alpha c_\alpha \bar{c}_\beta f_\beta >_j \right\}_{j=0} \Gamma^{(2)}_{pq} \frac{\delta}{\delta j_q} < \mathcal{J}_i \mathcal{J}_j >_j^{conn} \tag{5.3}$$

where $\Gamma^{(2)}_{pq}$ is the unrenormalized inverse propagator at $j = 0$. (These diagrams look like fig. (2.a) of I.) These are cancelled at $J = J^{(phy)}$ by the term (5.2) (in the limiting sense explained in I).

Let us consider the remaining terms that survive at $J = J^{(phy)}$ with $q_1 = -q_2 \neq 0$. They can be grouped into two groups.

$$Z^{-1} \frac{\partial \eta_0(g,\eta)}{\partial \eta} j_k \left[\left\{ \frac{\delta}{\delta j_p} < D_k^\alpha c_\alpha \bar{c}_\beta f_\beta \mathcal{J}_i >_j^{conn} \right\}_{j=0} \Gamma^{(2)}_{pq} \left\{ \frac{\delta^2}{\delta j_q \delta j_s} < \mathcal{J}_j >_{j=0} \right\} \Gamma^{(2)}_{sq} + (i \longleftrightarrow j) \right.$$
$$\left. + \left\{ \frac{\delta}{\delta j_s} < D_k^\alpha c_\alpha \bar{c}_\beta f_\beta \; i \; j >^{conn} \right\}_{j=0} \Gamma^{(2)}_{sq} \right] \varphi_q[j] \tag{5.4}$$

$$Z^{-1}\frac{\partial\eta_0(g,\eta)}{\partial\eta}\, j_k < t^{\alpha}_{k\ell} A_\ell c_\alpha \; i\, j\bar{c}_\eta >^{\text{proper}}_{j=0} G^{-1}_{\eta\beta} < c_\beta f_\gamma \bar{c}_\gamma >_j \tag{5.5}$$

where $G^{-1}_{\eta\beta}$ is the unrenormalized inverse ghost propagator.

We shall now show that at $J = J^{(\text{phy})}$, the fourier transform of (5.4) with respect to $(x_i - x_j)$ is equivalent to the following expansion

$$\sum_{i,n} \frac{\partial C^{(i)}}{\partial\eta}\, \tau_1 \cdots \tau_n(q) \left\langle O^{(\text{II})\,\tau_1\cdots\tau_n}_{(i)\,\nu}[A, c, \bar{c}] \right\rangle^R_J \tag{5.6}$$

when all the external momenta p_i satisfy $|2p_i \cdot q| < |q^2|$ and that in the same region, the corresponding fourier transform of expression (5.5) has the expansion at $J = J^{(\text{phy})}$

$$\sum_{i,n} C^{(i)}_{\tau_1\cdots\tau_n}(q) \frac{\partial}{\partial\eta} \left\langle O^{(\text{II})\,\tau_1\cdots\tau_n}_{(i)\,\mu\nu}[A, c, \bar{c}] \right\rangle^R_J . \tag{5.7}$$

Here $C^{(i)}$ refer to those appearing in (4. ;).

Before giving the proofs of these statements let us note its consequence. These imply that

$$\frac{\partial}{\partial\eta} Z_{,N_i N_j}[J^{(\text{phy})}] = \frac{\partial}{\partial\eta}\left\{\sum_{i,n} c^{(i)}_{\tau_1\cdots\tau_n} \left\langle O^{(\text{II})\,\tau_1\cdots\tau_n}_{(i)\,\mu\nu} \right\rangle^R_J\right\} \tag{5.8}$$

with the consequence that the entire gauge dependence of $\Gamma_{0,N_i N_j}$ at $\Phi = \Phi^{(\text{phy})}$ is isolated in terms that these operators and their expansion coefficients that appear in the expansion given by Eq. (4.23). [Therefore the "remainder" in that equation has the property that its physical matrix elements are gauge independent. So that if we were to expand

$$\text{"The remainder"} = \sum_{m,n} E^{\tau_1\cdots\tau_n}_{(m)}(q)\, O^{\tau_1\cdots\tau_n}_{(m)\,\mu\nu}[A, c, \bar{c}] \tag{5.9}$$

we could immediately recognize what the linearly independent $O_{(m)}$'s must be using the result of I.]

Let us recall that we determined the coefficient functions C's by considering the renormalized two point function (truncated) of $T(\mathcal{J}_i \mathcal{J}_j)$ and expanding its fourier transform in powers of $\frac{2p\cdot q}{q^2}$, $\frac{p^2+\mu^2}{q^2}$ etc. and identifying the coefficient functions C(q) as the coefficients of the tree two point functions of various class II operators (which are polynomials in p). This is to be done in successive loop approximations.

We therefore consider the η-variation of this two point function at an arbitrary momentum. Let us first consider the Green's function with two external gauge bosons. The contributions to its η variations are

$$\text{(i)}\quad Z^{-1}\frac{\partial\eta_0(g,\eta)}{\partial\eta}\, j_k \left\{\frac{\delta}{\delta j_p} < D^{\alpha}_k c_\alpha \bar{c}_\beta f_\beta >_j\right\}_{j=0} \Gamma^{(2)}_{pq} \frac{\delta^2}{\delta j_s \delta j_q} < \mathcal{J}_i \mathcal{J}_j >^{\text{conn}}_j j_s \tag{5.10}$$

$$\text{(ii)}\quad Z^{-1}\frac{\partial\eta_0(g,n)}{\partial\eta}\; {}^{-\frac{1}{2}}_{(k)} \frac{\partial Z^{\frac{1}{2}}_{(k)}}{\partial\eta_0}\, j_k < A_k \mathcal{J}_i \mathcal{J}_j A_m >^{\text{conn}} j_m . \tag{5.11}$$

(iii) The contribution of (5.4) evaluated for the special case of the two point function by putting $\Gamma^{(2)}_{sq}\omega_q[j] \to j_s$.

(iv) The contribution of (5.5) for the two point function which is

$$Z^{-1}\frac{\partial\eta_0(g,\eta)}{\partial\eta}\, j_k < t^{\alpha}_{k\ell} A_\ell c_\alpha \bar{c}_\eta \mathcal{J}_i \mathcal{J}_j >^{\text{proper}}_{j=0} G^{-1(2)}_{\eta\beta} \left\{\frac{\delta}{\delta j_s} < c_\beta \bar{c}_\gamma f_\gamma >_j\right\}_{j=0} j_s . \tag{5.12}$$

In the first term there is a contribution of the form

$$Z^{-1}\frac{\partial\eta_0(g,\eta)}{\partial\eta}\, j_s j_k\; D^{\alpha}_k c_\alpha \bar{c}_\eta\; {}_{j=0} G^{-1(2)}_{\eta\beta} \left\{\frac{\delta}{\delta j_p} < c_\beta \bar{c}_\gamma f_\gamma >_j\right\}_{j=0} \Gamma^{(2)}_{pq} \frac{\delta^2}{\delta j_q \delta j_s} < \mathcal{J}_i \mathcal{J}_j >_j . \tag{5.13}$$

Noting

$$< D^{\alpha}_k c_\alpha \bar{c}_\eta >_{j=0} = \left\{\frac{\delta^2 Z}{\delta\xi_\eta\, \delta\kappa_k}\right\} \tag{5.14}$$

with all the sources zero, it is easy to show that

$$< D^{\alpha}_{k} c_{\alpha} \bar{c}_{\eta} >_{j=0} = -i \left(\frac{1}{\partial^2}\right)_{\eta\xi} \partial^{\xi}_{k} \frac{1}{\sqrt{\eta}} \tag{5.15}$$

at the renormalization point $p^2 = -\mu^2$. Further knowing the diagram that contributes to Z we can show that at the renormalization point

$$\left[\frac{\delta}{\delta j_p} < c_\beta \bar{c}_\gamma f_\gamma >_j\right]_{j=0} G^{-1(2)}_{\eta\beta} = \sqrt{\eta}\, \partial^{\eta}_{p} Z^{\frac{1}{2}} \quad . \tag{5.16}$$

Using these inputs, it is easy to see that the expression (5.13) at $p^2 = -\mu^2$ is merely the change in the two point function on account of the change with η of the longitudinal parts of the external line propagator. Therefore, in truncated two point function, their contribution is $O(p^2 + \mu^2)$ only.

The remaining contribution from the first term viz

$$Z^{-1} \frac{\partial \eta_0(g,\eta)}{\partial \eta} j_k \left[\frac{\delta}{\delta j_p} < t^{\alpha}_{k\ell} c_\alpha \bar{c}_\beta f_\beta >^{\text{proper}}\right] \Gamma^{(2)}_{pq} \left[\frac{\delta^2}{\delta j_q \delta j_s} < \mathcal{J} j >_j\right]_{j=0} j_s \tag{5.17}$$

is cancelled at the renormalization point by the contribution (ii). Hence they add up to $O(p^2 - \mu^2)$.

Thus the only contributions that may contribute to $O(1)$ are those of (5.4) and (5.5) evaluated for the special case of the two point function. This being a change with respect to η of the two point function, has the same general structure and analytic properties as the two point function. One further notes that this change when truncated has the general structure

$$\bar{\Gamma}^{(2)}_{ij} F_{jk} + (i \longleftrightarrow k) \tag{5.18}$$

where $\bar{\Gamma}^{(2)}_{ij}$ is the renormalized inverse propagator at $J = 0$ and $F_{j,k}$ is a function (non polynomial in general) of p^2, $p \cdot q$, q^2. We can now expand this change in the two point function as we expanded the two point function. Here, one also expands the fourier transform of $\Gamma^{(2)}_{ij}$ in powers of $(p^2 + \mu^2)$ after taking $\mathcal{L}^{(2)}_{ij}$ common. The expansion of the expression (5.4) for

the case of two gluon truncated renormalized Green's function will have the structure

$$\delta^{(3)}\Gamma^{(2)}_{\mu\nu,\lambda\sigma}(p,q,-q,-p) = \sum_{i,n} B^{(i)}_{\tau_1 \cdots \tau_n}(q) \left\{ F^{(n)\,\tau_1 \cdots \tau_n}_{\mu\nu\lambda\xi}(p) \left[g_{F\sigma} p^2 - \left(1 - \frac{1}{\eta}\right) p_\xi p_\sigma \right] + (\lambda \longleftrightarrow \sigma) \right\} \tag{5.20}$$

We consider next the contribution (iv) namely that of the expression (5.5) for the case of the two point function. Noting that it contains a factor

$$\frac{\delta}{\delta j_s} < c_\beta \bar{c}_\gamma f_\gamma >\Big|_{j=0} \tag{5.21}$$

which from Lorentz covariance and the ghost Feynman rules is proportional to ∂^{β}_{s}. This in its Fourier transform it is proportional to p_σ or p_λ. Thus $\delta^{(3)}\Gamma^{(2)}$ of Eq. (5.20) is the only contribution which (i) survives at $p^2 = -\mu^2$ (ii) has a componant which has the structure of the two point function of a class II operator. These terms are

$$\delta\Gamma^{(2)}_{\mu\nu,\lambda\sigma} = \sum_{i,n} B^{(i)}_{\tau_1 \cdots \tau_n}(q) \left\{ F^{(n)\,\tau_1 \cdots \tau_n}_{\mu\nu\lambda\xi}(p) \left(g_{\xi\sigma} p^2 - p_\xi p_\sigma\right) \pm (\lambda \longleftrightarrow \sigma) \right\}$$

$$+ \text{ terms that vanish at } p^2 = -\mu^2 \tag{5.22}$$

$$+ \text{ terms that have other structures } + O(\mu^2) .$$

Thus comparing this expression with that for $\Gamma^{(2)}_{\mu\nu,\lambda\sigma}$ given by ( ), we can read off the η-variation of the coefficient functions for those class II operators whose two point functions do not vanish at $p^2 = -\mu^2$ and are not of $O(\mu^2)$, viz,

$$\eta \frac{\partial}{\partial \eta} C^{(i)}_{\tau_1 \cdots \tau_n}(q) = B^{(i)}_{\tau_1 \cdots \tau_n}(q) \quad . \tag{5.23}$$

We further note that on account of the structure of the renormalized matrix elements of a class II operator [see Eq. (5.22) for details] , if its two point function vanished at $p^2=-\mu^2$ or is of $O(\mu^2)$, its matrix elements vanish [or are $O(\mu^2)$] to all orders. Thus the above identification is valid for all operators whose matrix elements are linearly independent (in particularly nonvanishing).

Now consider the contribution (5.4) for renormalized Green's functions with an arbitrary number of external lines. At $J = J^{(phy)}$ it is

$$J_k \sum B^{(m)}_{\tau_1 \cdots \tau_n}(q)\, \tilde{F}^{(m)\,\tau_1 \cdots \tau_n}_{ijk,\,\ell}\, \Phi_k[\,J\,]\,\Big|_{J=J^{(phy)}} \tag{5.24}$$

where $\tilde{F}^{(m)}_{ijk,\,\ell}$ is the Fourier transform of $F^{(m)\,\tau_1 \cdots \tau_n}_{\mu\nu\lambda\sigma}(p)$. This, clearly, has the structure of

$$\sum_m B^{(m)}_{\tau_1 \cdots \tau_n}(q)\, \Big\langle O^{(II)\,(m)}_{J,\,\tau_1 \cdots \tau_n} \Big\rangle^R_{J\,(phy)} \quad . \tag{5.25}$$

Now using the identification of Eq. ( ) and noting that it is valid for all operators O^{II} whose matrix elements do not vanish, we arrive at the result of (5.6).

Next consider the contribution of (5.5) in more details. We note that the quantity appearing in it viz.

$$\begin{aligned} &< t^{\alpha}_{k\ell} A_\ell c_\alpha \mathcal{J}_i \mathcal{J}_j \bar{c}_\eta >^{proper}_{j=0} \\ &\qquad \equiv \frac{\delta \Gamma_{0,\,N_i N_j}}{\delta \bar{\Omega}_\eta\, \delta \kappa_k}\Bigg|_{\kappa=\Omega=\bar{\Omega}=\ell=\varphi=0} \\ &\qquad \equiv \frac{\delta \Gamma_{0,\,N_i N_j}}{\delta \bar{\omega}_\eta\, \delta \kappa_k}\Bigg|_{\kappa=\omega=\bar{\omega}=L=\Phi=0} \end{aligned} \tag{5.26}$$

is precisely the quantity that appears in the WT identity of Eq. (4.6). The Fourier transform of this quantity is $Q_{\mu\nu,\,\lambda}(p, q, -q, -p)$ upto factors. Indeed we have the expansion for Q

$$Q_{\mu\nu\lambda}(p, q, -q, -p) = \sum_{i,n} C^{(i)}_{\tau_1 \cdots \tau_n}(q)\, \tilde{Q}^{(i)\,\tau_1 \cdots \tau_n}_{\mu\nu\lambda}(p) \tag{5.27}$$

and for Γ

$$\Gamma^{(2)}_{\mu\nu,\,\lambda\sigma}(p, q, -q-p) = \sum_{i,n} C^{(i)}_{\tau_1 \cdots \tau_n}(q) \Big\{ F^{(i)\,\tau_1 \cdots \tau_n}_{\mu\nu,\,\lambda\xi}(p)\,[\,p^2 g_{\xi\sigma} - p_\xi p_\sigma\,] \pm (\lambda \longleftrightarrow \sigma) \Big\} + \cdots \tag{5.28}$$

and according to the WT identity of Eq. (4.6),

$$p\; F^{(i)\,\tau_1 \cdots \tau_n}_{\mu\nu,\,\lambda\xi} = n\, \tilde{Q}^{(i)\,\tau_1 \cdots \tau_n}_{\mu\nu\xi}(p) \quad . \tag{5.29}$$

In coordinate space this can be expressed as

$$Q^{\alpha\,\tau_1 \cdots \tau_n}_{\mu\nu k} = \partial^\alpha_m\, F^{\tau_1 \cdots \tau_n}_{\mu\nu,\,km} \tag{5.30}$$

in obvious notations. This is equivalent to

$$c_\alpha\, Q^{\alpha\;\;\tau_1 \cdots \tau_n}_{ijk} = \mathcal{G}(F_{ijkm} A_m)\Big|_{A=0} \tag{5.31}$$

where $\mathcal{G}$ was defined in Eq. (4.28) of ref. 8 and it does <u>not</u> act on i and j as they do not carry <u>color</u> index in them. Thus, the contribution of Eq. (5.5) has the form, at $J = J^{(phy)}$

$$\sum_{m,n} C^{(m)}_{\tau_1 \cdots \tau_n}(q)\, Z^{-1} \frac{\partial \eta_0(g,\eta)}{\partial \eta}\, J_k \Big\langle Q^{\alpha(m)\,\tau_1 \cdots \tau_n}_{\mu\nu k}[\,A\,]\, c_\alpha \bar{c}_\beta f_\beta \Big\rangle_J \tag{5.32}$$

but referring to the formula for the gauge variation of class II operators in sec. V of I, we obtain that this contribution at $J = J^{(phy)}$ equal

$$\sum C^{(m)}_{\tau_1 \cdots \tau_n}(q)\, \eta\, \frac{\partial}{\partial \eta} \Big\langle O^{(II)\,(m)\,\tau_1 \cdots \tau_n}[\,A, c, \bar{c}\,] \Big\rangle^R_J \tag{5.33}$$

where $O^{(II)\,(m)\,\tau_1 \cdots \tau_m}_{\mu\nu}$ has its 'F-function' equal to $F^{\tau_1 \cdots \tau_n}_{\mu\nu,\,km}$.

A similar analysis can be carried out for the 2 fermion truncated Green's function of $\mathcal{J}_i \mathcal{J}_j$ and then the results of (5.6) and (5.7) will be completely established.

Thus we have succeeded in expressing

$$Z_{0,N_iN_j}[J,\zeta,\bar{\zeta}] = \sum C^{(m)}_{\tau_1\ldots\tau_n}(q) \left\langle O^{II(m)}_{\tau_1\ldots\tau_n}[A,c,\bar{c}]\right\rangle^R_{J,\zeta,\bar{\zeta}} + \langle \text{Remainder}\rangle_{J,\zeta,\bar{\zeta}} \tag{5.34}$$

with property that

$$\eta \frac{\partial}{\partial\eta} \langle \text{Remainder}\rangle_{J,\zeta,\bar{\zeta}}\Big|_{\substack{J=J(\text{phy})\\ \zeta=\bar{\zeta}=0}} = O(\mu^2) \quad . \tag{5.35}$$

VI. GAUGE-INDEPENDENCE - QUARK GLUON MATRIX ELEMENTS

As briefly discussed in the introduction, processes of interest fall into two categories. A process of lowest order in e^2 has two vertices where the weak or electromagnetic gauge bosons interact with hadronic constituents. The amplitude under consideration can always be written in terms of the Green's functions of the time ordered products defined in Sec. II of I. Imagine now cutting the Feynman diagram at the two vertices where the weak or electromagnetic gauge bosons interact with quarks. This will separate the amplitude into two pieces. One of it will contain all the hadronic (quark, gluons or physical hadrons) external lines. Now there are two possibilities. Either the other part does not have any external lines or it has external lines which must be leptons or other flavor gauge bosons. The former case corresponds only to the case of the <u>exchange of a virtual gauge boson.</u> In this case two statements follow immediately: (i) The net momentum injected via the two vertices into the hadronic part of the diagram is <u>exactly zero</u>, i.e., one is dealing with the case of an operator product at an exceptional momentum ($q_1 = -q_2 = q$). (ii) Being a virtual exchange

one can only observe the sum over all q weighted with the propagator of the boson. In the latter case, when there are external lines, the momenta of the external lines are necessarily uncertain. Let $\{p_i\}$ denote the momenta of the external lines and let

$a_\mu^{(i)}(p_i)$ denote the wavefunction for $p_{i\mu}$. Then the μth component $= Q_\mu = \sum_i (p_i)_\mu$ has the distribution function:

$$\int \left\{ \prod_{i,\mu} a_\mu^{(i)}(p)\, d\, p_{i\mu} \right\} \delta(Q_\mu - \sum_i p_{i\mu}) = f(Q_\mu - \langle Q_\mu \rangle) \qquad (6.1)$$

Assuming that $a_\mu^i(p)$ are smooth functions of p uncorrelated for different i's (except for to yield $\langle Q_\mu \rangle = \sum_i \langle p_{i\mu} \rangle$), it is reasonable that $f(Q - \langle Q_\mu \rangle)$ is a smooth function of $Q - \langle Q \rangle_\mu$. [For our purpose, it will be sufficient that smoothness here implies only an absence of δ-function singularity in $f(Q - \langle Q \rangle)$.] Indeed for the case of two external lines $(i = 1, 2)$ and $\langle Q \rangle = 0$, one can prove easily to be the case. Thus, in this sort of situation, one cannot observe the matrix elements of the operator product at a specific value of Q, but only the value averaged over Q with a smooth function. An example of this case is given below.

In deep inelastic scattering, what one directly observes is the square of the amplitude for $e(k_1)\, p(k_2) \to e(p_1)\, X_n(p_n)$ (summed over n). While the above discussion does not directly apply to this case, an analogous discussion holds. Here one must use wave packets for the electron and the proton. Let them be $a(\vec{k}_1)$ $b(\vec{k}_2)$ respectively. Then the amplitude under consideration has the structure $(q_1 = k_1 - p_1)$

$$\int d^3k_1\, d^3k_2\, a(\vec{k}_1)\, b(\vec{k}_2)\, f_n(k_1, k_2, q_1)\, \delta^4(k_1 + k_2 - k' - p_n) \qquad (6.2)$$

and its square summed over all n has the structure $q_2 = k'_1 - p_1$

$$\int d^3k_1\, d^3k_2 d^3k'_1 d^3k'_2\, a^*(\vec{k}'_1) b^*(\vec{k}'_2)\, a(\vec{k}_1)\, b(\vec{k}_2\, \delta^4(k_1 + k_2 - k'_1 - k'_2)$$
$$\sum_n f_n(k_1, k_3, q_1)\, f_n^*(k'_1, k'_2, q_2)\, \delta^4(k_1 + k_2 - p_1 - p_n) \quad . \qquad (6.3)$$

It is the $\sum_n$ term to which one applies the OPE. Note that here, the net momentum transferred across the two currents is $q_1 - q_2 = k_1 - k'_1 \equiv Q$. Again, while $\langle Q \rangle = 0$, the above quantity involves an integration over $\vec{k}_1$, $\vec{k}'_1$ such that $\vec{k}_1 - \vec{k}'_1$ is not zero always. In fact, again what one is observing is a quantity averaged over $\vec{Q}$ with a smooth distribution function

$$\int d^3k_1 d^3k_2\, a^*(\vec{k}_1 - \vec{Q})\, b^*(\vec{k}_2 - \vec{Q})\, a(\vec{k}_1)\, b(\vec{k}_2) \quad . \qquad (6.4)$$

Now class II operators have the property that their renormalized matrix elements at a momentum insertion Q vanish for $Q \neq 0$ and are nonsingular at $Q = 0$. Therefore their matrix elements averaged over Q with a distribution function for Q "smooth" around $Q = 0$ vanish identically. Thus only the gauge invariant operators of class I contribute to the operator product expansion relevant to such processes.

Before proceeding to the other case, let us make some remarks. It may seem artificial to have to bring in the uncertainty for the external

momenta to expla in gauge invariance. In view of this let us make the following two observations:

(i) Gauge independence of the S-matrix or that of the matrix elements of a gauge invariant operator are subject to a similar exception, i.e., these quantities are not wholy gauge independent for a set of exceptional momenta given by $(\sum p_i)^2 = 0$ where $(\sum p_i)$ is any partial sum of the external momenta. In these cases, the relevant matrix element has a singularity (a pole and a cut). The gauge dependent part of these quantities at exceptional momenta have a weaker singularity however. So that with due account of the external line momentum distribution, the contribution of these gauge dependent parts vanishes. Thus, the uncertainty for the external momenta has to be involved in the discussion of the gauge invariance for these quantities also.

(ii) Since every measurement on the momenta of the external particles has to be made in a finite region of space, they are bound to be uncertain always.

[B] Now, we shall consider the other case in which is an exchange of a virtual weak or electromagnetic boson. As remarked earlier, here one can observe the matrix elements of operators at $Q \equiv 0$ exactly. [Basically this is so because a virtual particle can be emitted and absorbed anywhere in space time and in the sum of all such processes, the net momentum transfer across the two vertices of the gauge boson is exactly zero, as opposed to the case when there are non hadronic external lines on the other

side of the two gauge boson vertices. Matrix elements at $Q = 0$ cannot be and should not be obtained as a limit for $Q \neq 0$; because this is an exceptional momentum case.] We shall show that when a due account of the renormalizations too lowest order in e due to these weak and electromagnetic interactions is taken, the only operators that survive in operator product expansion are the operators of class I and operators whose matrix elements vanish [to $0(\mu^2)$.] It is, here, that it is necessary to think in terms of a renormalizable theory of strong, weak and electromagneaic interactions rather than thinking of these as external perturbations onto a renormalizable theory of strong interactions only. Also, it is, here, that the remark that one can only observe one combination $\int D_{\mu\nu}(q) < j_\mu(q) j_\nu(-q)>$ becomes relevant.

To compute the change in the wavefunction renormalization to $O(e^2)$ for quarks, gluons and ghosts, we must consider the corresponding proper two point function

$$\Gamma^{(2)}(p) = \int d^n q\, D_{\mu\nu}(q) \times \{ \text{corresponding truncated connected Green's function of } T(\mathcal{J}_\mu^{(k)}(x)\mathcal{J}_\nu^{(\ell)}(y))\}\, e^{iq.(x-y)} \tag{6.4}$$

These Green's functions are the quantities which we expanded in a specific domain of the Lorentz invariants. Let us denote their fourier transforms symbolically as $Z^{\mu\nu}(p, q, -q, -p)$ [where Z will carry the additional Lorentz

or Dirac indices depending on which two point function we are considering.] Now to obtain the wave function renormalization constant, we must evaluate this object at $p^2 = m^2 - \mu^2$ ($m = 0$ for gluons and ghosts) according to the renormalization prescription of Sec. IID of I. Let us keep aside the case of the Fermions. [We shall indeed be able to bypass the discussion of that case.] Since for gluons, $p^2 = -\mu^2 < 0$, let us choose a Lorentz frame in which $p = (0, \vec{p})$. Let us further perform a Wick rotation on the momenta p and q (under which p is unaffected). Then we will learn that for $p = (0, \vec{p})$,

$$\Gamma^{(2)}(p) = i \int d^n q_\varepsilon \; Z^{\mu\nu}(p, q, -q-p)\, D_{\mu\nu}(q) \tag{6.5}$$

where $q = (i\, q^0_\varepsilon, \vec{q}_\varepsilon)$, $q_\varepsilon \equiv (q^0_\varepsilon, \vec{q}_\varepsilon)$ and of course the integration range of $q_{\varepsilon\mu}$ is $(-\infty, \infty)$. Now for $p^2 = -\mu^2$ with μ^2 very small, $\vec{p}^{\,2}$ is very small and hence

$$2p\cdot q = 2\vec{p}\cdot\vec{q} \le 2|\vec{p}|\,|\vec{q}| = 2\mu|\vec{q}| < |\vec{q}_\varepsilon|^2 \le q_{\varepsilon 0}^2 + |\vec{q}_\varepsilon|^2 = -q^2 \tag{6.6}$$

for $|\vec{q}_\varepsilon| > 2\mu$. Thus, if the function $Z_{\mu\nu}$ is expanded as

$$Z_{\mu\nu}(p, q, -q, -p) = \sum_\nu X^{(i)}_{\mu\nu}(p, q)\, f^{(i)}(p^2,\ p\cdot q, q^2) \tag{6.7}$$

where $X^{(i)}_{\mu\nu}$ are independent polynomials in p, q; the domain of the integration variables viz:

$$-\infty \le q_{\varepsilon\nu} \le \infty\ ;\ q_{\varepsilon\nu} \text{ real}$$

corresponds to a domain of the arguments of f_i belongs to the set of values

$p^2, p\cdot q, q^2$ contained in

$$\{p^2 = -\mu^2;\ 2p\cdot q, q^2 \ni |2p\cdot q| < |q^2|\} \cup \{p^2 = -\mu^2;\ p\cdot q,\ q^2 \quad q_{\varepsilon\nu} \le 2\mu\} \tag{6.8}$$

Now, the volume of the phase space for the second set of q's, is $0(\mu^4)$. In this phase space $Z_{\mu\nu}$ can at worst behave like $\frac{1}{\mu^2}(\ell n^2)^n$ (because of the pole $\frac{1}{(p+q)^2}$ in the 1-particle reducible diagrams. Hence the contribution from this part of the phase space is $0(\mu^2)$. Now the first domain of Lorentz invariants is a subdomain of the one in which we expanded the function $Z_{\mu\nu}$ in powers of $\frac{p^2-\mu^2}{\mu^2}, \frac{2p\cdot q}{q^2}$ this being $\left|\frac{p^2+\mu^2}{\mu^2}\right| < 1$, $\left|\frac{2p\cdot q}{q^2}\right| < 1$. Therefore, we may substitute the expansion for $Z_{\mu\nu}(p, q, -q, -p)$ of Eq. (4.11) in the expression for $\Gamma^{(2)}(p)$. And thus,

$$\Gamma_{\lambda\sigma}(p)\Big|_{p^2=-\mu^2} \sim \int_{|q_{\varepsilon\nu}|>2\mu} d^n q_\varepsilon \sum_{i,n} C^{(i)}_{\tau_1 \cdots \tau_n}(q) \left\{ F^{(i)\tau_1\cdots\tau_n}_{\mu\nu,\lambda\xi}(p)(p^2 g_{\xi\sigma} - p_\xi p_\sigma) \pm (\lambda \leftrightarrow \sigma) \right\} D^{\mu\nu}(q)\Big|_{p^2=-\mu^2} + 0(\mu^2) \tag{6.9}$$

Now, let

$$\int_{|q_{\varepsilon\nu}|>2\mu} d^n q_\varepsilon \; C^{(i)}_{\tau_1\cdots\tau_n}(q)\; D^{\mu\nu}(q) = T^{(i)\mu\nu}_{\tau_1\cdots\tau_n}(\mu^2) \tag{6.10}$$

where T is a momentum independent tensor contructed out of basic Lorentz tensors. [Note: $T^{(i)}$ may contain terms $0(\mu^{-2})$, $0(\mu^{-4})$ etc.] Thus,

$$\Gamma_{\lambda\sigma}(p) = \sum_i \left\{ F^{(i)\tau_1\cdots\tau_n}_{\mu\nu,\lambda\xi}(p)\, T^{(i)\mu\nu}_{\tau_1\cdots\tau_n}(\mu^2)(p^2 g_{\xi\sigma} - p_\xi p_\sigma) \pm (\lambda \leftrightarrow \sigma) \right\}_{p^2=-\mu^2} + 0(\mu^2) \tag{6.11}$$

Now, from Lorentz covariance,

$$F^{(i)\tau_1 \cdots \tau_n}_{\mu\nu,\lambda\xi}(p)\, T^{(i)\mu\nu}_{\tau_1 \cdots \tau_n}(\mu^2) = g_{\lambda\xi}\, \tilde{F}^{(i)}(p^2) + p_\xi p_\lambda\, \bar{F}^{(i)}(p^2) \tag{6.12}$$

where $\tilde{F}^{(i)}$ and $\bar{F}^{(i)}$ are polynomials in p^2. They will, in particular, contain $\tilde{F}^{(0)} = C = \text{const}$. Thus, substituting Eq. (6.12) into Eq. (6.11), we get, at $p^2 = -\mu^2$,

$$\Gamma^{(2)}_{\lambda\sigma}(p)\Big|_{p^2=-\mu^2} = 2\sum_i \tilde{F}^{(i)}(-\mu^2)\,(p^2 g_{\sigma\lambda} - p_{\sigma\lambda}) \tag{6.13}$$

[We note that, in general, the above sum receives contribution from all i's and not just the i that corresponds to the class two operator $\left(\frac{\delta\tilde{S}}{\delta A_i} A_i + \cdots\right)$, whose coefficient generally corresponds to wavefunction renormalization constant.] The above statement of Eq. (6.13) holds to each order in the perturbation series. Of course, to use Eq. (6.13), we must start from the lowest order for $\Gamma^{(2)}_{\lambda\sigma}$, introduce the wavefunction renormalizations (and coupling constant renormalizations) to that order and subtract their contribution while computing $\Gamma^{(2)}_{\lambda\sigma}$ in the next order, before one can determine the wavefunction renormalization constant to the next order [which, then, will not be $2\sum_i F^{(i)}(\mu^2)$. Thus $e^2\left(\frac{\partial Z}{\partial e^2}\right)_n \neq -2\sum_i F_i(\mu^2)$ generally]. However, it is clear that the total effect of all the renormalization counter terms up to, say, n loop approximation has to be such that

at $p^2 = -\mu^2$, the total renormalization counterterm which is

$$e^2\left(\frac{\partial Z}{\partial e^2}\right)_{e^2=0}\left\{\frac{\delta\tilde{S}}{\delta A_i} A_i - (\partial\bar{c}!_i \mathcal{G}_0 A_i\right\} + e^2\left(\frac{\partial\tilde{Z}}{\partial e^2}\right)_{e^2=0} \bar{c}_\alpha M_{\alpha\beta} c_\beta$$
$$+ e^2\left(\frac{\partial Z_\psi}{\partial e^2}\right)_{e^2=0}\left\{\frac{\delta\tilde{s}}{\delta\psi_\alpha}\psi_\alpha + \bar{\psi}_\alpha\frac{\delta\tilde{s}}{\delta\psi_\alpha}\right\} + e^2\frac{\partial}{\partial e^2}\left(\frac{X}{Z^{\frac{1}{2}}\tilde{Z}}\right)\frac{\partial S_{eff}}{2mg_0}$$
$$+ \cdots\cdots \tag{6.14}$$

must have its two gluon proper vertex at $p^2 = -\mu^2$ equal to

$$-2\sum_i F_i(\mu^2)\,(p^2 g_{\lambda\sigma} - p_\lambda p_\sigma)\,. \tag{6.15}$$

It is well-known that the above counterterms can be expressed entirely in terms of the Lorentz invariant dimension four class I, II and III operators.

Now consider again the two point function $\Gamma^{(2)}_{\lambda\sigma}(p)$. The contribibution to its transverse part comes from <u>both</u> the class I and class II operators in the expansion. We separate $\sum_i \tilde{F}$ in the two corresponding parts

$$\Gamma^{(2)}_{\lambda\sigma}(p)\Big|_{p^2=-\mu^2} = \left[2\sum_i \tilde{F}^{(I)}_i(-\mu^2) + 2\sum_i \tilde{F}^{(II)}_i(-\mu^2)\right] \times (p^2 g_{\sigma\lambda} - p_\sigma p_\lambda) \tag{6.16}$$

Now consider the n-point function of the class II operators appearing in the OPE. Analogous discussion will show that the relevant matrix elements of the class II operators are generated by,

$$\sum_i F_i[-\mu^2] \quad J_k \Phi_k[J]\Big|_{J=J^{phy}} \quad . \tag{6.17}$$

which is precisely of the form of the matrix elements of class II operators in (6.14). It is then easy to see that since all the gauge dependence of $T(j_\mu(x)j_\nu(0))$ was isolable in the form of the coefficient functions of class II operators, in $\int d^n q\, D_{\mu\nu}(q) \quad T(\tilde{j}_\mu(q)\, \tilde{j}_\nu(-q))$ it precisely corresponds to the gauge dependence of the $0(e^2)$ terms of the renormalization constants. Therefore, the above quantity minus the renormalization counterterms must be gauge independent.

We further note that imposing renormalization conditions on the independent renormalization parts of this difference determines uniquely all the renormalization constants to $0(e^2)$. We also note that the matrix elements class II operators in the counterterms are generated by an expression of same form as (6.17). This was the form of the matrix elements of the class II operators in the OPE. These facts imply that the matrix elements of the class II operators in the difference between OPE and the counterterms must vanish. OPE may, of course, contain additional operators whose matrix elements vanish identically. Thus, as far as quark-gluon matrix elements (in absence of additional gauge invariant operators) of this difference are concerned, only the class I operators contribute.

The following fact should be noted to understand the above remarks. Given that $m_n \neq m_\Lambda$ ($m_p \neq m'_p$ in the charm model additionally) the only linearly independent Lorentz invariant class II operators whose matrix elements do not vanish (i.e. are of O(1) rather than $O(\mu^2)$) are the class II operators that occur in the wavefunction renormalization counterterms.

VII. GAUGE INDEPENDENCE: HADRONIC MATRIX ELEMENTS

There is a direct way to see how the gauge invariance will work for the matrix elements between physical hadrons. We shall present it here.

It utilizes the fact that one can use a local interpolating field constructed out of quarks (and gluons) with the quantum numbers of physical hadrons to generate the hadronic matrix elements of operator products on the mass shell of the physical hadrons. Thus, let $H_1(x_1)\ldots H_n(x_n)$ be n interpolating local gauge invariant field operators with the quantum numbers of n hadrons. Then the physical matrix elements of an operator $O(y)$ [or any time ordered product of operators] will be generated by considering the fully renormalized Green's function at the poles at the physical masses of the n hadrons viz.

$$\lim_{q_i^2 \to m_i^2} \prod_i (q_i^2 - m_i^2) \int \prod dx_i e^{iq_i x_i} \langle 0 | T\, O(y)\, H_1(x_1)\ldots H_n(x_n) | 0 \rangle^R \quad (7.1)$$

will be proportional to the physical matrix element of $O(y)$ between the n hadrons, the proportionality constant is independent of external momenta.

We shall always use gauge invariant local operators as the interpolating fields. We shall have to make the following assumption which is highly reasonable.

The physical matrix element of a renormalized interpolating gauge invariant field operator $\{H(x) + C.T.\}$ between physical vacuum and single hadron state

$$\langle 0 | H(x) | h(q) \rangle^R$$

is (i) a finite (nonvanishing quantity at $q^2 = m^2$

(ii) a gauge-independent quantity at $q^2 = m^2$.

With these assumptions we shall prove the following two results:

(i) The physical matrix elements of a class II or a class III, IV operator between n hadrons vanish identically at $q \neq 0$ (except of a set an exceptional momenta of the external hadrons).

(ii) The physical matrix elements of <u>Lorentz invariant</u> class II operators at $q = 0$ between n hadrons are porportional to the corresponding S-matrix element for the n-hadrons [i.e. zero if the S-matrix element is zero] ; the proportionality constant being independent of external momenta. [If the matrix element under consideration is between two hadrons "the corresponding S-matrix element" is to be interpreted as the inverse propagator.] For class III and IV they vanish identically.

Given result (i), the discussion of gauge-independence goes through as in the previous section, for the case in which there are non-hadronic external lines. Given the result (ii), the gauge independence of the other case (in which there is a single exchange of a weak or electromagnetic gauge boson) goes through since a contribution strictly proportional to the strong interaction S-matrix cannot be isolated. This is precisely the analogue of the external line wavefunction renormalization in the case of the quark gluon matrix elements.

We shall not be specific as to what sort of gauge invariant interpolating field operators we shall be using. Some of the arguments given below can be simplified slightly with a specific and simple choice of the

interpolating field operators, but we shall keep the discussion general.

Let us first consider the unrenormalized but dimensionally regularized Green's functions of any linear combination of a class II and III operator. Let the operator be

$$O_x = \frac{\delta \tilde{S}}{\delta A_i} F_{ix}[A, c, \bar{c}] + \mathcal{G}_0 F_x[A, c, \bar{c}] + \frac{\delta S}{\delta c_\alpha} x_\alpha[A, c, \bar{c}] \qquad (7.2)$$

where $F_{ix} = \frac{\delta F_x}{\delta(\partial\bar{c})_i}$ when i refers to a gauge field. Here the fact that the operator injects a momentum q is taken into account by simply taking its source

$$N_x \sim e^{iq\cdot x_n} \quad . \qquad (7.3)$$

We then consider the generating functional

$$\bar{W} = \int [dAdcd\bar{c}]\, H^{(1)}_{x_1} \ldots H^{(n)}_{x_n} \exp i\,\{\mathcal{L}_{eff} + N_x O_x\} \quad . \qquad (7.4)$$

The unrenormalized Green's function under consideration is $\frac{1}{\bar{W}} \frac{\delta \bar{W}}{\delta N_x}\Big|_{N=0}$.

$$\bar{W} = H_1[-i\frac{\delta}{\delta j}] \ldots H_n[-i\frac{\delta}{\delta j}] \int [dAdcd\bar{c}] \exp i\,\{\mathcal{L}_{eff} + N_x O_x + j_i A_i\}\Big|_{j=0} . \qquad (7.5)$$

Now we shall use Eq. (5.) of ref. viz

$$\frac{\delta}{\delta N_x} \int [dAdcd\bar{c}] \exp i\{\mathcal{L}_{eff} + N_x O_x + j_i A_i\}\,|_{N=0} = -\int [dAdcd\bar{c}]\, j_i (F_{ix} + D^\alpha_i c_\alpha F_x) \exp i\,\mathcal{L}_{eff}$$

and the gauge invariance of $H^{(i)}$ to conclude that

$$\frac{1}{\bar{W}} \frac{\delta \bar{W}}{\delta N_y}\Big|_{N=0} = \frac{1}{\bar{W}} \int [dAdcd\bar{c}]\, \{\sum_{\gamma=1}^{n} (H^{(1)}_{x_1} \ldots H^{(\gamma)'}_{x_r y} \ldots H^{(n)}_{x_n}) \exp i\,\mathcal{L}_{eff}$$

where

$$H^{(\gamma)'}_{x_r y} = \frac{\delta H^{(\gamma)}_{x_r}}{\delta A_i} F_{iy} \quad . \qquad (7.6)$$

In Sec. III we proved the above result for the <u>renormalized</u> Green's

functions

$$\langle H^{(1)}_{x_1} \ldots H^{(n)}_{x_n} O_y \rangle^R = \sum_{r=1}^{n} \langle H^{(1)}_{x_1} \ldots H^{(\gamma)'}_{x_r y} \ldots H^{(n)}_{x_n} \rangle^R \quad . \qquad (7.7)$$

It is obtained from Eq. (3.20) by differentiating it with respect to sources $N^{(i)}$. Note that we have proven the relation (7.7) in each order in the perturbation series. We are now going to <u>assume</u> that the same result holds nonperturbatively before we apply it to extract information on the hadronic matrix elements.

Let us first consider the case $q \neq 0$ for an arbitrary (not necessarily Lorentz invariant) class II or III operator. Let $H^{(\gamma)}_{x_r}$ be the interpolating field for a hadron of mass m_r. Now the local functional $H^{(r)}_{x,y}$ may also be an interpolating field for some hadron (it could be the same depending on O_y) of mass m'_r. Now consider the expression of Eq. (7.7). We will have to consider that part of it that has poles $\frac{1}{p_i^2 - m_i^2}$ for each i. However, it is easy to see that the rth term in the sum can only have the pole structure.

$$\prod_{i \neq r} \frac{1}{p_i^2 - m_i^2} \times \frac{1}{(p_r + q)^2 - m'^2_r} + \text{terms with fewer poles} \quad . \qquad (7.8)$$

Thus for $q \neq 0$, the physical matrix element under consideration will vanish identically.

Now consider the $q = 0$ case. In this case, the terms in the sum can have the right pole structure only if $m_r = m'_r$. Now if we make the assumption (valid in nature) that hadrons with the same electric charge and which can be connected by second order weak interactions are non degenerate in mass, the only case when the above

hadronic matrix element will be nonvsnishing is when $q = 0$ and O_y does not carry any quantum numbers. [It may still carry Lorentz indices.]

Now we shall show that a purely hadronic matrix element of such a Lorentz invariant class II operator at $q = 0$ are proportional to the strong interaction S-matrix element (proportionality constant being independent of hadronic external momenta) and as such cannot be distinguished from it. This is, of course, an analogue of additional wave function renormalization in the case one is considering the quark gluon matrix elements. Let us emphasize however that an analogous result will not be true for a hadronic matrix element of such an operator in presence of another gauge invariant operator (such as a current) injecting a variable 4-momentum. (We shall give examples of this case later.) In other words, such quantities can be observed in principle.

Let $H^{(r)}$ be the interpolating field for the hadron $h^{(r)}$ and let it include its wavefunction. [Thus for a vector particle $H^{(r)}_{x_r} \sim \int \epsilon_\mu(p)\, d^3\vec{p}\; e^{ip\cdot x_r}\, \tilde{H}^{(r)}_\mu(p)$; $p\cdot\epsilon = 0$, for a spin 2 particle $H^r_{x_r} \sim \int \epsilon_{\mu\nu}(p)\, e^{ipx_r}\, \tilde{H}_{\mu\nu}(p)\, d^3\vec{p}$ $p_\mu \epsilon_{\mu\nu} = 0 = \epsilon_{\mu\nu} p_\nu$ for a spin $\frac{1}{2}$ particle $H^r_{x_r} \sim \int [\bar{u}(p)]_\gamma\; e^{ip\cdot x_r} \tilde{H}_\gamma(p)$ etc.] Now from Lorentz covariance

$$\text{spin zero:}\quad \langle 0|\tilde{H}(p)|h(p)\rangle\big|_{p^2=m^2} = f(p^2)\big|_{p^2=m^2} = \text{const}$$

$$\text{spin } \tfrac{1}{2}:\quad \bar{u}_\gamma(p)\langle 0|\tilde{H}_\gamma(p)|h_\delta(p)\rangle\big|_{p^2=m^2} = \bar{u}_\gamma(p)[a(p^2)+b(p^2)\not{p}]_{\gamma\delta}\big|_{p^2=m^2}$$

$$= (\text{const})\, \bar{u}_\delta(p)$$

$$\text{spin 1:}\quad \epsilon_\mu(p)\langle 0|\tilde{H}_\mu(p)|h_\nu(p)\rangle\big|_{p^2=m^2} = \epsilon_\mu(p)(g_{\mu\nu}a(p^2) + b(p^2)p_\mu p_\nu)$$

$$= \epsilon_\nu(p) \times \text{const} \qquad (7.9)$$

etc.

Noting the above results, it is clear that the residue of the right hand side of Eq. (7.7) at the poles of external particles $(\text{mass})^2$, are proportional to the S-matrix elements themwelves. Since one cannot turn off weak and electromagnetic interactions, they cannot be separated from the strong interaction S-matrix elements.

This kind of an argument will not however apply if one is considering the hadronic matrix elements of a class II operator in presence of an additional gauge invariant operator that is not an interpolating field for an external hadron. An example of such a case is the hadronic part[26] of the $O(e^4)$ diagrams for $K \to \pi \ell \bar{\ell}$ (ℓ being a lepton) that proceed via a Z or a γ intermediate state; and one uses the OPE for the two charged weak currents. Here, Z (or γ) can inject a variable (and in principle measurable) external four-momentum k into the hadronic part of the diagram. In this case the relevant matrix element of a class II operator is generated by

$$\langle K(x)\, \Pi(y)\, j_\mu(z)\, O(\omega)\rangle_0^R$$

where K and Π are local gauge invariant interpolating fields for K and π. According to Eq. (7.) this has the structure

$$\langle K'(x,\omega)\, \Pi(y)\, j_\mu(z)\rangle_0^R + \langle K(x)\, \Pi'(y,\omega)\, j_\mu(z)\rangle_0^R + \langle K(x)\, \Pi(y)\, j'_\mu(z,\omega)\rangle_0^R$$

Now, for $q = 0$ (q being the momentum conjugate to ω and $q = 0$ is the case relevant to the above example), the first two terms will not have poles at the hadronic mass shells of K and π, no matter what the quantum numbers of O^{II} are. The third term however has the correct pole

structure at the hadronic mass shells; and thus will in general make a nonvanishing contribution to the $K \to \pi Z(\gamma)$ hadronic matrix element.

Furthermore note that in the general case a similar hadronic matrix element of a class II operator

$$\langle H_1(x)\, H_2(y)\, O(w)\, j_\mu(z)\rangle^R$$

at $q \neq 0$, will not generally vanish owing to the term in it

$$\langle H_1(x)\, H_2(y)\, j'_\mu(z,w)\rangle$$

and at $q = 0$, it will not be proportional to the corresponding strong interaction hadronic Green's function of j_μ generated by

$$\langle H_1(x)\, H_2(y)\, j_\mu(z)\rangle$$

because of the presence of the same term. Thus, under such circumstances, the class II operators in the OPE seem to have observable effects. We have not fully analyzed such a situation however, in view of the absence of known practical applications.

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APPENDIX A

Consider the quantity

$$W[j,k,\ell,N',N] = \int [dAdcd\bar{c}]\, \exp i\{\mathcal{L}_0 - \tfrac{1}{2}\eta_0 (\tfrac{\delta \mathcal{F}}{\delta \bar{c}_\alpha})^2 - \mathcal{G}_0 \mathcal{F} + j_i \frac{\delta \mathcal{F}}{\delta k_i} + O^{GI}(N,A)\} \tag{A.1}$$

where $\mathcal{F}$ is a functional of $A, c, \bar{c}, k, \ell$ and $\underline{N}'^{(i)}$ at $N^{(i)} = N'^{(i)}$, the above expression reduces to the r.h.s. of Eq. (3.8). We treat N and N' as independent variables in (A.1). Now

$$-i\frac{\delta W}{\delta N'^{(i)}} = \int [dAdcd\bar{c}]\; -\eta_0 \frac{\delta \mathcal{F}}{\delta \bar{c}_\alpha} \frac{\delta^2 \mathcal{F}}{\delta N'^{(i)} \delta \bar{c}_\alpha} - \mathcal{G}_0 \frac{\delta \mathcal{F}}{\delta N'^{(i)}} + j_k \frac{\delta^2 \mathcal{F}}{\delta k_k \delta N'^{(i)}}\}$$

$$\exp i\{\ldots\} \quad . \tag{A.2}$$

Now considering the BRS transformations

$$\delta A_i = D_i^\alpha c_\alpha \delta\lambda \quad \delta c_\alpha = -\tfrac{1}{2} g_0 f_{\alpha\beta\gamma} c_\beta c_\gamma \delta\lambda \quad \delta \bar{c}_\alpha = -\frac{\delta \mathcal{F}}{\delta \bar{c}_\alpha}\eta_0\, \delta\lambda \tag{A.3}$$

on the integration variables of

$$\int [dAdcd\bar{c}]\, \frac{\delta \mathcal{F}}{\delta N'^{(i)}}\, \exp i\{\ldots\} \tag{A.4}$$

and equating the change to zero one obtains

$$O = \int [dAdcd\bar{c}]\; -\eta_0 \frac{\delta \mathcal{F}}{\delta \bar{c}_\alpha} \frac{\delta^2 \mathcal{F}}{\delta \bar{c}_\alpha \delta N'^{(i)}} - {}_0 \frac{\delta \mathcal{F}}{\delta N'^{(i)}} +$$

$$+ i j_k \Big[\mathcal{G}_0 \frac{\delta \mathcal{F}}{\delta {}_k} - \frac{\delta^2 \mathcal{F}}{\delta \bar{c}_\alpha \delta {}_k} \frac{\delta \mathcal{F}}{\delta \bar{c}_\alpha}\eta_0 \Big] \frac{\delta \mathcal{F}}{\delta \bar{c}_\alpha}\eta_0 \Big] \frac{\delta \mathcal{F}}{\delta n'^{(i)}}\; \exp i\{\ldots\} \; . \tag{A.5}$$

Here, one has used the fact that the exponent except for the source term is invariant under (A.3). Thus we obtain an equation of the form

$$\frac{\delta W}{\delta N'^{(i)}} = j_k \int [dAdcd\bar{c}]\, X'_k[A, c, \bar{c}, N']\, \exp i\{\ldots\} \quad . \tag{A.6}$$

The above set of equations can be expanded in powers of N' up to a

desired power and then integrated from $N'^{(i)} = 0$ to $N^{(i)}$. The result will be an expression of the form

$$W[j, k, \ell, N' = N, N] = \text{r.h.s. of Eq. (3.8)}$$

$$= W[j, k, \ell, N' = 0, N]$$

$$+ j_k \int [dA dc d\bar{c}]\, X_k[A, c, \bar{c}, N] \exp\{\ldots\}\,|_{N'=N} \qquad (A.7)$$

where $W[j, k, \ell, N' = 0, N]$ contains only gauge invariant ($O^{GI}[N, A]$) N-dependent terms in its exponential.

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FOOTNOTES

1. We shall refer to ref. 3 as I.

2. As emphasized in I, we are using the terms "matrix elements," "On shell renormalization," "gauge independence of matrix elements" in a certain limiting sense. These phrases will be used merely as a shorthand.

3. Here, by hadronic matrix elements we mean the real physical matrix elements between physical hadrons. These of course are assumed to exist.

4. These assumptions are listed in Sec. VII.

5. S-matrix elements are singular at these exceptional momenta. The most singular part of the S-matrix at exceptional momenta is indeed gauge independent. However, a less singular (or non singular) part does depend on gauge.

6. These are again the real physical S-matrix elements.

7. When one is considering a matrix element of an operator with only two external hadrons the "corresponding S-matrix element" is to be interpreted as the inverse propagator.

8. We here mean that the conclusions as to which operators appear in the expansion and about the gauge independence of their Wilson coefficients will remain valid in the exact theory.

9. I would like to thank Professor D. Gross for a brief discussion of this point.

10. Most of our discussion will hold for other compact gauge groups as well.

11. We have suspended the summation-integration convention and written gauge fields and fermion fields separately.

12. e_α are weak and electromagnetic coupling constants. $c_{\alpha_1} = c_{\alpha_2}$ are all equal for when T_{α_1} and T_{α_2} denote the generators of the same simple factor group of G_W.

13. It may not be obvious as to what the condition $J_k F_{k,j}\, \Phi_j [J]|_{J=J(\text{phy})} = 0$ means for F_k. Examples of such an F_k are (i) F_k containing at least two fields, (ii) $F_k \sim \partial_k^\lambda \tilde{F}_\lambda$, (iii) $F_{k,j} \sim \partial_j^\beta F'^\beta_k[\partial]$, (iv) F_k linear in field but $F_{k,j}(\partial)$ vanishes on the mass shell of A_j etc.

14. An operator may be case in the form of a class II as well as a class III operator. In this case it will be called a class III operator.

15. Class IV operators may not be all linearly independent of the class II and III operators. Consequently the set S_0 is not a set of linearly independent operators.

16. For the definition of $\mathcal{G}_0$ see Eq. (2.9) of I.

17. On account of (a), the F_k function of these may be chosen to be linear in A.

18. For the WT identity we are referring to, please see Eqs. (4.3)-(4.5) of Ref. 8.

19. Here one has to solve an equation of the form

$$\mathcal{G}'[\Phi, \omega, \bar{\omega}, k, L, N^{(i)}]\, T[\Phi, \omega, \bar{\omega}, k, L, N^{(i)}] = 0 \ .$$

We proceed as done in Dixon-Taylor lemma (see Sec. 4 of ref. 8) by expanding $\mathcal{G}'$ and T in powers of $N^{(i)}$. Thus the role of β

there is taken by many parameters $N^{(i)}$. One uses the result of 8 that given that

$$\mathcal{G}'^{(0)}[\Phi, \omega, \bar{\omega}, K, L]\ T'[\Phi, \omega, \bar{\omega}, K, L] = 0\ \mathcal{G}'^{(0)^2} = 0$$

for a local functional F' net ghost number zero, F' can be expressed as

$$T' = \text{a local gauge invariant functional of } \Phi + \mathcal{G}'^{(0)} F[\Phi, \omega, \bar{\omega}, K, L]$$

where F is local. Then the Eq. $\mathcal{G}' T = 0$ can be solved as in the Dixon Taylor lemma to yield that

$$T = \text{a local gauge invariant functional of } \Phi \text{ and } N^{(i)} + \mathcal{G}' F[\Phi, \omega, \bar{\omega}, K, L, N^{(i)}]$$

where F is local in all its arguments.

20. This is <u>not</u> the canonical transformation that Dixon has discussed in his paper. His generating functional was a function of the unprimed commuting variables (A, ℓ) and the primed anticommuting variables (κ', c'). Our generating functional is a function of the unprimed sources (κ, ℓ) and the primed fields (A', c'). Under such a transformation <u>also</u> the equation $\frac{\delta \tilde{S}}{\delta A_i} \frac{\delta \tilde{S}}{\delta \kappa_i} + \frac{\delta \tilde{S}}{\delta c_\alpha} \frac{\delta \tilde{S}}{\delta \ell_\alpha} = 0$ is preserved. The latter transformation is more useful for the purpose of our discussion.

21. In the sense defined in I.

22. The domain for p^2 is so restricted because the two point function will contain logarithmic singularity $\sim(\ell n\, p^2)^2$.

23. Note that the Bose symmetry requires that $\Gamma^{(2)\mu\nu}_{\lambda\sigma}(p, q, -q, -p) = \Gamma^{(2)\mu\nu}_{\sigma\lambda}(-p, q, -q, p)$.

24. Bose symmetry places constraint on $S^{(n)}$ viz.

$$S^{(n)}_{\mu\nu,\lambda\sigma}(p) = S^{(n)}_{\mu\nu,\sigma\lambda}(-p) \quad .$$

25. The $\pm$ sign is to be so chosen that $S^{(n)}$ is Bose-symmetric.

26. We are not necessarily suggesting that this is a good way of an actual practical estimate; this is meant to be an example of a situation under consideration.