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SCALAR QUANTUM CHROMODYNAMICS IN TWO DIMENSIONS AND PARTON MODEL†

Sun-Sheng Shei
The Institute for Advanced Study
Princeton, New Jersey 08540

and

Hung-Sheng Tsao
Rockefeller University
New York, New York 10021

ABSTRACT

We study the SU(N) scalar quantum chromodynamics in two space-time dimensions in the large N limit. This is the model of color gauge fields interacting with scalar quarks. We find that the consensual properties of the four dimensional QCD, i.e., the infrared slavery, quark confinement, the charmonium picture etc. are all realized. Moreover, the current in this model mimics nicely the behaviors of current in the four dimensional QCD, in contrast to the original model of 't Hooft.

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I. Introduction

The naive quark[1] model has been very helpful in our understanding of the qualitative features of the hadron spectroscopy. Together with the parton picture[2,3] it has also been useful in describing the asymptotic scaling behaviors[4] in deep-inelastic scatterings, e^+e^- annihilation and other physical processes related to the short-distance or light-cone behaviors of the strong interactions. The dynamical basis for the success remained a mystery until the realization that the quantum chromodynamics (QCD) is asymptotically free.[5,6] In QCD the short-distance behaviors are almost the same as in the free quark model except for some small calculable deviations.[5,6] This explains the almost Bjorken scaling in deep-inelastic experiments. Furthermore, it has been speculated that QCD may give us another bonus. Owing to the infrared instability in QCD, the severe infrared behavior at large distance may provide strong forces to confine quarks.[5,7] The problem of whether QCD indeed gives rise to the quark confinement is still unsolved.

To test the idea of quark confinement in QCD, 't Hooft[8] studies the U(N) QCD in 1 space - 1 time dimensions. This is still a very complicated dynamical system. An important observation was made by 't Hooft that a great simplification can be achieved by considering the limit of large N.[9] He succeeds in showing that the original quarks are removed from the physical Hilbert space. Furthermore, the physical spectrum consists of an infinite number of color singlet bound states of $q\bar{q}$. The model has been further elaborated by Callan, Coote and Gross.[10] The application to the deep inelastic scatterings has also been studied.[11,12,13]

DISCLAIMER

One special feature in the 't Hooft model is the behavior of the currents. In contrast to the QCD in 4 dimensions, the currents in 't Hooft model are softer than as was to be expected from the short-distance (or light-cone) consideration.[14] To be specific, in the short-distance (or light-cone) expansion of the product of currents, $V_\mu(z)V_\mu(0)$, the leading expansions are zero (in the 't Hooft model).

In order to make the comparison with the four dimensional QCD more realistic, it is certainly desirable to have a model in which the leading short-distance (or light-cone) expansion is non-vanishing. We find that the QCD interacting with scalar quarks is such a model. The purpose of this paper is to study the scalar QCD in the large N limit. We find that the spectrum consists of hadrons which are color-singlet bound states of $q\bar{q}$. The quarks are removed from the physical spectrum. In other words, the infrared slavery, quark confinement etc. are realized in our model. The charmonium picture is also found to be valid. Furthermore, the currents mimic nicely the currents in the QCD in 4 dimensions.

The paper is organized in the following way. In section II the scalar QCD is studied in the large N limit. In particular, we obtain the dressed quark propagator and the integral equation for the wave functions describing $q\bar{q}$ bound states. In the section III we examine the behavior of the bound state wave functions. First, we study the non-relativistic limit. Next, we study the end point behavior of wave functions at $x = 0$ and $x = 1$. The semi-classical (or W.K.B.) approximation is used to obtain the

approximate wave functions as well as the spectrum of bound states. The completeness and orthogonality conditions are also discussed. In section IV the full $q\bar{q}$ scattering amplitude is constructed. From this we obtain the hadron-quark-antiquark vertex. The question of unitarity is mentioned briefly. It is indicated that indeed only color-singlet states are needed in the unitarity relation of hadron amplitudes. The matrix elements of scalar densities and vector currents and two point functions are discussed in section V. We study the asymptotic behavior of two point functions and compare with the free quark model. The differences between the behavior in our model and the 't Hooft model are discussed. Section VI deals with the asymptotic behavior of form factors of the scalar densities and vector currents. The deep inelastic scatterings are studied in the section VII. We find that νW scales and the answer for νW agrees with Feynman parton picture exactly. It is also consistent with the light-cone expansion. The Drell-Yan-West[15] and Bloom-Gilman[16] relation is examined. We also study the deep inelastic scattering for scalar densities. The last section consists of summaries and discussions. In the appendix we discuss a numerical method of solving the integral equation. The method is well known in fluid mechanics. As an example, we study the original 't Hooft's equation numerically. The leading behavior in the eigenvalue μ_k^2 is obtained.

II. The Scalar QCD

The model we shall consider consists of scalar quarks interacting via gauge fields in 2 space-time dimensions. Specifically, the Lagrangian is

$$\mathcal{L} = \frac{1}{4} G_{\mu\nu,i}{}^{j} G^{\mu\nu}{}_{,j}{}^{i} + (D_\mu\varphi)^*(D^\mu\varphi) - m_{ao}^2 \ \varphi^{*a} \ \varphi^a \tag{2.1}$$

where

$$G_{\mu\nu,i}{}^{j} = \partial_\mu A_i{}^j{}_\nu - \partial_\nu A_i{}^j{}_\mu + g[A_\mu, A_\nu]_i^j \tag{2.2}$$

$$D_\mu \varphi_i{}^a = \partial_\mu \varphi_i{}^a + g\tilde{A}_{i\mu}{}^j \varphi_j{}^a \tag{2.3}$$

$$\tilde{A}_{i_\mu}{}^j = A^j_{i\mu} - \frac{1}{N}\delta^j_i A^k_{k\mu} = - \tilde{A}^{*i}_{j\mu} \tag{2.4}$$

The color group in this model is SU(N). The scalar fields transform as fundamental representation of color group. They also carry flavor index a. The flavor symmetry may be broken explicitly by assigning $m_{ao}^2 \neq m_{bo}^2$ for $a \neq b$. This theory is superrenormalizable and hence asymptotically free. As in any gauge theory, one has to fix a gauge condition before quantization. The simplest gauge condition that can be imposed here is the light-cone gauge. It has the virtue that (a) no ghost is needed, (b) no self couplings of gauge fields exist. In this gauge,

$$A^+ = A_- = 0 \tag{2.5}$$

and

$$G_{+-} = - \partial_- A_+ \ . \tag{2.6}$$

One can write the Lagrangian effectively as

$$\mathcal{L} = - \frac{1}{2}(\partial_- A_+)(\partial^+ A^-) + \partial_\mu \varphi_i^{a*} \partial^\mu \varphi_i^a - m_{ao}^2 \varphi_i^{a*} \varphi_i^a$$
$$+ g A^j_{+i} [-\varphi_i^{*a} \partial_- \varphi_j^a + (\partial_- \varphi_i^{a*}) \varphi_j^a] \tag{2.7}$$

The Feynman rules can be derived readily and are given in Fig.1. We follow 't Hooft[8] in studying the theory in the large N limit with g^2N held fixed. In this limit only planar diagrams with no quark loops are important.

Let us first study the dressed quark propagator. The lowest non-trivial contribution (Fig.2) is

$$- i \Pi^{(2)} = Ng^2 \int \frac{d^2k}{(2\pi)^2} \frac{-i}{k_-^2} (2p+k)_- \frac{i}{(p+k)^2 - m_o^2 + i\varepsilon} (2p+k)_-$$
$$= - ic + 2p_- Ng^2 \int \frac{d^2k}{(2\pi)^2} \frac{1}{k_-^2} \frac{(2p+k)_-}{(p+k)^2 - m_o^2 + i\varepsilon} \tag{2.8}$$

Here the first term

$$- ic = \frac{Ng^2}{(2\pi)^2} \int \frac{d^2k}{(p+k)^2 - m_o^2 + i\varepsilon}$$

is independent of p and can be absorbed into the mass renormalization, the second integral is infrared divergent due to the singularity at $k_- = 0$. 't Hooft uses a regulation scheme by inserting a factor $\theta(k_-^2 - \lambda^2)$[8] in the integral which is then infrared divergence free. We use the same regulation scheme here. When this is done, we obtain

$$\Pi^{(2)} = c + \frac{Ng^2}{2\pi} 2p_- \left(\frac{\mathrm{sgn}(p_-)}{\lambda} - \frac{1}{p_-}\right) \tag{2.9}$$

To find the full dressed quark propagator in the large N limit, it is necessary to solve the integral equation represented symbolically in Fig. 3. If the dressed propagator is written as

$$\frac{i}{2p_+p_- - m_o^2 + i\epsilon - 2p_-\Pi - c} = \frac{i}{2p_-(p_+ - \Pi) - (m_o^2 + c) + i\epsilon} ,$$

the integral equation is then

$$-i(2p_-\Pi + c) = Ng^2 \int \frac{d^2k}{(2\pi)^2} \frac{-i}{k_-^2} \frac{i(2p+k)_-^2}{2(p+k)_-[(p+k)_+ - \Pi(p+k)] - (m_o^2 + c) + i\epsilon} \tag{2.10}$$

We find that

$$c = i\, Ng^2 \int \frac{d^2k}{(2\pi)^2} \frac{1}{2(p+k)_-[(p+k)_+ - \Pi(p+k)] - (m_o^2 + c) + i\epsilon} \tag{2.11}$$

$$\Pi = i\, Ng^2 \int \frac{d^2k}{(2\pi)^2} \frac{2(p+k)_-}{2(p+k)_-[(p+k)_+ - \Pi(p+k)] - (m_o^2 + c) + i\epsilon} \tag{2.12}$$

Using the regulation scheme of 't Hooft,[8] we obtain

$$\Pi = \frac{Ng^2}{2\pi} \left(\frac{\mathrm{sgn}(p_-)}{\lambda}\right) - \frac{1}{p_-} \tag{2.13}$$

This agrees with the second order calculation in eq. (2.9). The dressed quark propagator D is given by

$$D(p) = \frac{1}{2p_-[p_+ - \frac{Ng^2}{2\pi}(\frac{\mathrm{sgn}(p_-)}{\lambda} - \frac{1}{p_-})] - m^2 + i\epsilon} \tag{2.14}$$

This indicates that the quarks have infinite self-energy as the cutoff $\lambda \to 0$, which removes them from the physical spectrum.

We now turn to the $q\bar{q}$ bound states. The wave functions satisfy the homogeneous Bethe-Salpeter equation depicted in Fig.4, i.e.,

$$\psi(p,r) = Ng^2\, i^2 D(p)\, D(r-p) \times \int \frac{d^2k}{(2\pi)^2} \frac{-i}{k_-^2} \psi(p+k,r)(2p+k)_-(2p-2r+k)_- \tag{2.15}$$

Since the kernel is a function of − components only, the equation can be simplified by introducing

$$\phi(p_-,r) \equiv \int dp_+\, \psi(p_+,p_-;r) \tag{2.16}$$

It follows that

$$\phi(p_-,r) = \int dp_+ \frac{i}{p_+ - \frac{Ng^2}{2\pi}\left(\frac{\mathrm{sgn}(p_-)}{\lambda} - \frac{1}{p_-}\right) + \frac{-m_1^2 + i\epsilon}{2p_-}}$$
$$\times \frac{1}{p_+ - r_+ - \frac{Ng^2}{2\pi}\left(\frac{\mathrm{sgn}(p-r)_-}{\lambda} - \frac{1}{(p-r)_-}\right) + \frac{-m_2^2 + i\epsilon}{2(p-r)_-}}$$
$$\times \int \frac{dk_-}{k_-^2} \phi(p_- + k_-; r) \left(\frac{2p_- + k_-}{2p_-}\right) \left(\frac{2p_- - 2r_- + k_-}{2p_- - 2r_-}\right) \tag{2.17}$$

The integration over p_+ is easily carried out. We end up with

$$\phi(p_-,r) = -\frac{g^2N}{2\pi}\,\frac{\theta(p_-)\;\theta(r_--p_-)}{r_+ - \frac{g^2N}{\pi}\frac{1}{\lambda} - \frac{(m_1^2 - \frac{g^2N}{\pi})}{2p_-} - \frac{(m_2^2 - \frac{g^2N}{\pi})}{2(r-p)_-}}$$

$$\times \int_0^{r_-} \frac{dk'_-}{(k'_--p_-)^2}\,\phi(k'_-,r)\left(\frac{p_-+k'_-}{2p_-}\right)\left(\frac{p_-+k'_- -2r_-}{2(p_--r_-)}\right) \tag{2.18}$$

The integral is infrared divergent. We use the regulating scheme described before. Equation (2.18) becomes

$$\left(r_+ - \frac{g^2N}{\pi}\frac{1}{\lambda} - \frac{m_1^2 - \frac{g^2N}{\pi}}{2p_-} - \frac{m_2^2 - \frac{g^2N}{\pi}}{2(r_--p_-)}\right)\cdot\phi(p_-,r)$$

$$= -\frac{g^2N}{2\pi}\;\theta(p_-)\;\theta(r_--p_-)$$

$$\times\left\{\frac{2}{\lambda}\,\phi(p_-,r) + \int_0^{r_-}\frac{dk'_-}{(k'_--p_-)^2}\,\phi(k'_-,r)\left(\frac{p_-+k'_-}{2p_-}\right)\left(\frac{-2r_-+p_-+k'_-}{2(p_--r_-)}\right)\right\} \tag{2.19}$$

The λ-dependent terms cancel from both sides of the equation. With the change of variables

$$r_+ = \frac{M^2}{2r_-}\,, \quad r_-x = p_-\,,$$

we find the integral equation for $\phi(x)$,

$$M^2\phi(x) = \left(\frac{m_1^2 - \frac{g^2N}{\pi}}{x} + \frac{m_2^2 - \frac{g^2N}{\pi}}{1-x}\right)\phi(x) - \frac{g^2N}{\pi}\int_0^1 \frac{dy}{(x-y)^2}\,\phi(y)\,\frac{(x+y)(2-x-y)}{4x(1-x)} \tag{2.20}$$

Aside from the factor $\frac{(x+y)(2-x-y)}{4x(1-x)}$, this equation resembles the one obtained by 't Hooft for the fermion-antifermion bound states. Similar equation has been obtained by Bardeen and Pearson[17] in different context.

III. The Behaviors of the Bound State Wave Functions

In this section, we examine the behavior of the bound state wave functions described by eq. (2.20) in the last section. In particular, we are interested in the following problems: (a) non-relativistic limit, (b) the end point behavior of the wave functions, (c) a simple approximation to the wave functions and the mass spectrum.

(a) Non-relativistic limit

In this limit we expect that the momentum in the center of mass frame to be very small as compared to the masses of the constituents. For simplicity, we consider here only the case of equal mass $m_1 = m_2$. The momentum fraction x is related to the c.m. momentum by

$$x = \frac{1}{2} + \frac{q}{2m'} \quad \text{with } \left|\frac{q}{m'}\right| << 1 \tag{3.1}$$

where

$$m' = \sqrt{m^2 - \frac{Ng^2}{\pi}} \tag{3.2}$$

The eq. (2.20) can be written as

$$M^2\phi(q) = m'^2\left\{2\left(1+\frac{q}{m'}\right)^{-1} + 2\left(1-\frac{q}{m'}\right)^{-1}\right\}\phi(q)$$

$$-\frac{g^2N}{\pi}\int_{q'=-m'}^{q'=m'}\frac{dq'}{2m'}\;\frac{\phi(q')}{\frac{1}{4m'^2}(q-q')^2}\;\frac{\left(1+\frac{q+q'}{2m'}\right)\left(1-\frac{q+q'}{2m'}\right)}{\left(1-\frac{q^2}{m'^2}\right)} \tag{3.3}$$

We keep only terms up to q^2 in the kinetic terms and neglect q,q' as compared to m' inside the integral. The binding energy $\mathcal{E}$ is related to M by

$$M = 2m' + \mathcal{E}$$

Thus,

$$\mathcal{E}\phi = \frac{q^2}{m'}\phi - \frac{g^2 N}{2\pi}\int_{q'=-m'}^{q'=+m'} dq' \frac{\phi(q')}{(q-q')^2} \tag{3.4}$$

The condition $|q/m'| << 1$ allows us to replace the upper and lower limit of integral by $+\infty$ and $-\infty$ respectively. Therefore the non-relativistic limit of eq. (2.20) is

$$\mathcal{E}\phi = (\frac{q^2}{m'} + \frac{g^2 N}{4\pi}|X|)\phi \tag{3.5}$$

where X is the conjugate variable of q. The solutions of eq. (3.5) are well-known.(18) They can be expressed in terms of Airy function. The energy spectrum is discrete and in the limit of large quantum number

$$\mathcal{E}_n \approx n^{2/3}\left[\frac{3}{16}\frac{g^2 N}{(m^2 - \frac{g^2 N}{\pi})^{1/4}}\right]^{2/3} \tag{3.6}$$

We remark that eq. (3.5) is exactly the same as the equation describing non-relativistic bound state in 't Hooft model. As is well-known, the Schröedinger equation does not distinguish bosons from fermions. The same equation can be used to describe the S-wave charmonium.(19,20) The only diference is in the boundary condition.

(b) <u>End point behaviors of wave functions</u>

In many applications, it is important to know the bahaviors

of the wave functions near the end points x = 0 and x = 1. One can rewrite eq. (2.20) as

$$(\mu^2 - \frac{\alpha_1}{x} - \frac{\alpha_2}{1-x})\, 4x(1-x)\phi(x)$$

$$= -\int_0^1 \frac{dy}{(y-x)^2}\phi(y)[(2-x)x+(2-2x)y-y^2] \tag{3.7}$$

where $M^2 \equiv \frac{Ng^2}{\pi}\mu^2$, $\alpha_i \equiv \frac{\pi m_i^2}{Ng^2} - 1$

The following lemma[1] is useful: If $f \sim y^\beta$ as $y \to 0$, then,

$$\int \frac{f(y)}{(y-x)} dy \sim -\pi\cot(\beta\pi)x^\beta .$$

Suppose $\phi(y) \sim y^{\beta_1}$ near y = 0, eq. (3.7) implies that

$$(\mu^2 - \frac{\alpha_1}{x} - \frac{\alpha_2}{1-x})\, 4x(1-x)\; x^{\beta_1}$$

$$= x(2-x)\pi\beta_1 \cot(\pi\beta_1)x^{\beta_1 - 1} + 2(1-x)\pi(\beta_1+1)\cot(\pi\beta_1)x^{\beta_1-1}$$

$$- \pi(\beta_1+2)\cot(\pi\beta_1)x^{\beta_1+1} + \text{less important terms.}$$

Compare the most important terms from both sides, we find

$$\pi(\beta_1 + \tfrac{1}{2})\cot(\pi\beta_1) = -\alpha_1 \tag{3.8}$$

Similarly, $\phi(y) \sim (1-y)^{\beta_2}$ for $y \to 1$, where

$$\pi(\beta_2 + \tfrac{1}{2})\cot(\pi\beta_2) = -\alpha_2 \tag{3.9}$$

The solution of the transcendental equation is illustrated in Fig. 5. It is easy to see that when $m_i^2 > \frac{g^2 N}{\pi}$, the exponent β_i

increases as m_i^2 increases.

(c) W.K.B. approximation

We would like to have a simple approximation to the wave functions and the mass spectrum. One such approximation which immediately comes to mind is the W.K.B. approximation. It is a good approximation to both wave functions and eigenvalues especially for the states with large quantum number. In this subsection we shall derive the W.K.B. approximation appropriate to the integral equation (2.20). Notice that the kernel of eq. (2.20) is not hermitean. In order to apply W.K.B. method, the kernel has to be made hermitean. This can be achieved by the following change of function

$$\Phi \equiv \sqrt{x(1-x)}\ \phi \tag{3.10}$$

In terms of Φ, we find that

$$\left\{M^2 - \frac{m_1^2 - \frac{g^2N}{\pi}}{x} - \frac{m_2^2 - \frac{g^2N}{\pi}}{1-x}\right\}\Phi(x)$$
$$= -\frac{g^2N}{\pi}\int_0^1 \frac{dy}{(x-y)^2}\ \frac{(x+y)(2-x-y)}{4\sqrt{x(1-x)}\sqrt{y(1-y)}}\ \Phi(y)$$
$$= \int K(x,y)\ \Phi(y)dy \tag{3.11}$$

Here K(x,y) is hermitean. Now we apply the W.K.B. approximation. Let us write $\Phi = \exp(\frac{iS}{h})$ in the classically allowed domain as in any application of W.K.B. approximation.

In the semiclassical limit of $h \to 0$, $\Phi(y)$ is rapidly varying. The only important contribution to the integral in eq. (3.11) comes from the region of integration where $x \sim y$. Therefore we can replace the factor $\frac{(x+y)(2-x-y)}{4\sqrt{x(1-x)}\ \sqrt{y(1-y)}}$ by 1. The rapid variness of Φ also allows us to replace $\int_0^1$ by $\int_{-\infty}^{\infty}$. Hence, one finds that in the W.K.B. approximation

$$\left(M^2 - \frac{m_1^2 - \frac{g^2N}{\pi}}{x} - \frac{m_2^2 - \frac{g^2N}{\pi}}{1-x}\right)\Phi = \frac{g^2N}{\pi}\ \pi\ \left|i\frac{\partial}{\partial x}\ \Phi\right| \tag{3.12}$$

It follows that

$$\Phi = A\exp\left\{\frac{i}{\pi}\left[\mu^2x - \alpha_1 \ln x + \alpha_2\ \ln(1-x)\right]\right\}$$
$$+ B\exp\left\{-\frac{i}{\pi}\left[\mu^2x - \alpha_1 \ln x + \alpha_2\ \ln(1-x)\right]\right\} \tag{3.13}$$

The condition that $\Phi = 0$ at the classical turning point $x = x_\pm$ (i.e., $x_\pm$ are solutions of $\mu^2 = \frac{\alpha_1}{x} + \frac{\alpha_2}{1-x}$ then implies the quantization condition. In the special case of $\alpha_1 = \alpha_2$, we find that

$$\Phi = \sin\left[\frac{\mu^2}{\pi}x - \frac{\alpha}{\pi}\ln(\frac{x}{1-x})\right] \tag{3.14}$$

and

$$\mu^2\sqrt{1-\frac{4\alpha}{\mu^2}} - 2\alpha\ \ln\left(\frac{1+\sqrt{1-\frac{4\alpha}{\mu^2}}}{1-\sqrt{1-\frac{4\alpha}{\mu^2}}}\right) = n\pi^2 \tag{3.15}$$

Equation (3.15) can also be written in the form of Bohr-Sommerfeld condition

$$\oint p dx = 2\pi n$$

where p, the conjugate momentum to x, is given by

$$|p| = \frac{1}{\pi} [\mu^2 - \frac{\alpha_1}{x} - \frac{\alpha_2}{1-x}] .$$

We would like to point out here that in order for W.K.B. (or semiclassical) approximation to make sense, it is essential that $m_i^2 \geq \frac{g^2 N}{\pi}$. This is evident from the fact that the right hand side of eq. (3.15) becomes complex if $\frac{g^2N}{\pi} > m_1^2$ and/or m_2^2 . We can also understand this in the following way. Only for $\alpha_1, \alpha_2 > 0$ will the classical motion described by

$$|p| = \frac{1}{\pi} [\mu^2 - \frac{\alpha_1}{x} - \frac{\alpha_2}{1-x}]$$

be periodic. It is well known that nonperiodic system can never give rise to any bound state. Perhaps this is an indication that the large N limit is meaningful only when $m_i^2 \geq \frac{g^2N}{\pi}$. In some applications further approximation suffices. In this case we can use

$$\phi = \frac{\sqrt{2}}{\sqrt{x(1-x)}} \sin(\frac{\mu^2}{\pi} x) \tag{3.16}$$

or

$$\Phi = \sqrt{2} \sin(\frac{\mu^2}{\pi} x) \tag{3.17}$$

and

$$\mu_n^2 = n\pi^2 \tag{3.18}$$

(d) <u>Completeness and orthogonality conditions</u>

We have discussed in (c) that although the integral equation for ϕ does not have hermitean kernel, the kernel for Φ is hermitean. Therefore the completeness and orthogonality conditions are most easily expressed in terms of Φ_k. We have

$$\sum_k \Phi_k^*(x)\Phi_k(x') = \delta(x-x') \tag{3.19}$$

and

$$\int_0^1 dx \ \Phi_k^*(x) \ \Phi_{k'}(x) = \delta_{kk'} \tag{3.20}$$

Of course, these can be written using $\phi_k \equiv \frac{1}{\sqrt{x(1-x)}} \Phi_k$. They are

$$\sum_k \sqrt{x(1-x)} \sqrt{x'(1-x')} \ \phi_k^*(x) \ \phi_k(x') = \delta(x-x') \tag{3.21}$$

and

$$\int_0^1 dx \ \ x(1-x) \ \phi_k^*(x)\phi_{k'}(x) = \delta_{kk'} \tag{3.22}$$

respectively.

IV. <u>The $q\bar{q}$ Scattering Amplitude</u>

In many appliations, it is necessary to know the full $q\bar{q}$ scattering smplitude. Let us write $S = 1 + T$.
The scattering amplitude T satisfies the integral equation

$$T = \frac{-i}{(p_- - p'_-)^2} g^2(p_- + p'_-)(p'_- + p_- - 2r_-)$$

$$+ \int \frac{d^2k}{(2\pi)^2} \frac{-iN}{(p_- - k_-)^2} g^2 (p_- + k_-)(p_- + k_- - 2r_-) iD(k) iD(k-r)$$

$$\times T(k,p';r) \tag{4.1}$$

Graphically this is shown in Fig. 6.

It is convenient to introduce an auxiliary function

$$\phi(p_-,p'_-;r) \equiv \int dp_+ \, D(p) \, D(p-r) T(p,p';r) \tag{4.2}$$

In terms of $\phi(p_-,p'_-;r)$ one obtains

$$T(p,p';r) = -ig^2 \frac{(p_- + p'_-)(p_- + p'_- - 2r_-)}{(p_- - p'_-)^2}$$

$$+ \frac{ig^2 N}{(2\pi)^2} \int dk_- \frac{(p_- + k_-)(p_- + k_- - 2r_-)}{(p_- - k'_-)^2} \phi(k_-,p'_-;r) \tag{4.3}$$

From the definition of $\phi(p,p';r)$ and the above equation, one finds that

$$\left\{ r_+ - \frac{g^2N}{\pi} \frac{1}{\lambda} - \frac{m_1^2 - \frac{g^2N}{\pi}}{2p_-} - \frac{m_2^2 - \frac{g^2N}{\pi}}{2(r_- - p_-)} \right\} \phi(p_-,p'_-;r)$$

$$= 2\pi g^2 \frac{1}{(p_- - p_-')^2} \left(\frac{p_- + p'_-}{2p_-}\right) \left(\frac{2r_- - p_- - p'_-}{2(r_- - p_-)}\right)$$

$$- \frac{Ng^2}{2\pi} \frac{2}{\lambda} \phi(p_-,p'_-;r)$$

$$- \frac{Ng^2}{2\pi} \int dk_- \frac{\phi(k_-,p'_-;r)}{(p_- - k_-)^2} \left(\frac{k_- + p_-}{2p_-}\right) \left(\frac{2r_- - p_- - k_-}{2(r_- - p_-)}\right)$$

The λ dependent terms cancel from both sides of the equation. With the change of variables

$$M^2 = 2r_+ r_- \quad ,$$

$$p_- = x r_- \quad ,$$

$$p'_- = x' r_- \quad ,$$

$$k_- = y r_- \quad ,$$

We find the integral equation for $\phi(x,x';r)$

$$\left(M^2 - \frac{m_1^2 - \frac{g^2N}{\pi}}{x} - \frac{m_2^2 - \frac{g^2N}{\pi}}{1-x}\right) \phi(x,x';r)$$

$$= \frac{4\pi g^2}{r_-} \frac{1}{(x-x')^2} \frac{(x+x')(2-x-x')}{4x(1-x)}$$

$$- \frac{Ng^2}{\pi} \int dy \frac{\phi(y,x';r)}{(x-y)^2} \frac{(x+y)(2-x-y)}{4x(1-x)} \tag{4.4}$$

We can also express $\phi(x,x';r)$ and $T(x,x';r)$ in terms of the solution of homogeneous B-S equation. The results are

$$\phi(x,x';r) = \frac{\pi c^2}{r_-} \sum_k \frac{1}{M^2 - M_k^2} \int_0^1 dy \frac{(y+x')(2-y-x')\phi_k(x)\phi_k^*(y)}{(y-x)^2} \tag{4.5}$$

$$T = ig^2 \frac{(x+x')(2-x-x')}{(x-x')^2}$$

$$-ig^2 \left(\frac{g^2N}{\pi}\right) \sum_k \frac{1}{M^2 - M_k^2} \left\{ r_- \frac{4}{\lambda} \theta(x(1-x)) \phi_k(x) [x(1-x)] \right.$$

$$\left. + \int_0^1 dy \frac{\phi_k(y)(x+y)(2-x-y)}{2(x-y)^2} \right\} \times \left\{ x \to x', y \to y' \right\}^* \tag{4.6}$$

From this one can read off the $q\bar{q}h$ coupling. It is

$$g\sqrt{\frac{g^2N}{\pi}}\left\{\frac{4}{\lambda}\, r_-\ \left(x(1-x)\right)\phi_k(x)\theta(x(1-x))] + \int_0^1 dy\ \frac{\phi_k(y)(x+y)(2-x-y)}{2(x-y)^2}\right\} \tag{4.7}$$

Knowing the scattering amplitude we can now answer all physical interesting questions at least in principle. We have checked the three and four point hadronic amplitudes. They are all finite in the limit of $\lambda \to 0$. Furthermore, to order $\frac{1}{N}$, the theory is unitary and no colored intermediate states appear. The arguments are similar to those of ref. 10. We will not include them here.

V. Matrix Elements of Scalar Densities and Vector Currents and Two Point Functions

In this section we would like to consider the matrix elements of scalar densities and vector currents as well as the two point functions.

We begin with the matrix elements between vacuum and one hadron states. In the model we studied, the scalar densities and vector currents are given in terms of the boson fields as

$$S^{ab} = \sum_{i=1}^{N} \bar{\varphi}_i^a\ \varphi_i^b \tag{5.1}$$

$$V_\mu^{ab} = (-i)\sum_{i=1}^{N} (\bar{\varphi}_i^a\, \partial_\mu \varphi_i^b - \partial_\mu \bar{\varphi}_i^a \varphi_i^b) \tag{5.2}$$

Here S^{ab} and V_μ^{ab} are color singlets and a, b are flavor indices.

The matrix elements $\langle 0|S^{ab}|h\rangle$, $\langle 0|V^{ab}|h\rangle$[2] can be calculated using the $q\bar{q}\,h$ vertex we obtained in Sec. IV. Diagrammatically this is represented in Fig. 8. The results are

$$\langle 0|S^{ab}|h\rangle = -\frac{i}{2}\sqrt{\frac{N}{\pi}}\int_0^1 dx\ \phi_k(x) \tag{5.3}$$

$$\langle 0|V_-^{ab}\ |h\rangle = \frac{i}{2}\sqrt{\frac{N}{\pi}}\, q_- \int_0^1 dx\ (1-2x)\ \phi_k(x), \tag{5.4}$$

Other interesting objects to study are the two point functions of scalar densities and vector currents. For the scalar densities, we have

$$M(q^2) = \int dx\ e^{iqx} \langle 0|T\ S^{ab^+}(x)\ S^{ab}(0)|0\rangle \tag{5.5}$$

In contrast to the fermion case, the scalar density S has dimension 0 instead of 1. In the leading $\frac{1}{N}$ approximation, one finds

$$M(q^2) = i\sum_n \frac{|\langle n|S^{ab}|0\rangle|^2}{q^2 - M_n^2} \tag{5.6}$$

which approaches

$$i\,\frac{1}{q^2}\sum_n |\langle n|S^{ab}|0\rangle|^2 \tag{5.7}$$

provided that $\sum_n \langle n|S^{ab}|0\rangle|^2$ is finite. We can evaluate this sum by eq. (5.3)

$$\sum_n |\langle n|S^{ab}|0\rangle|^2 = \frac{N}{4\pi}\sum_n \int_0^1 dx\phi_n(x)\int_0^1 dx'\phi_n^*(x') \tag{5.8}$$

Now, $\{\sqrt{x(1-x)}\ \phi_n\}$ forms a complete orthonormal set, i.e.,

$$\sum_n \sqrt{x(1-x)}\ \phi_n(x)\ \sqrt{x'(1-x')}\ \phi_n^*(x') = \delta(x-x') \tag{5.9}$$

The sum becomes

$$\sum_n |\langle n|S^{ab}|0\rangle|^2 = \frac{N}{4\pi}\int_0^1 dx \int_0^1 dx' \frac{\delta(x-x')}{\sqrt{x(1-x)x'(1-x')}}$$

$$= \frac{N}{4\pi}\int_0^1 \frac{dx}{x(1-x)}$$

This is a logarithmically divergent integral. A more careful treatment yields

$$M(q^2) \sim \frac{Ni}{2\pi q^2}\ \ell n(-q^2) \text{ as } -q^2 \to \infty \tag{5.10}$$

For comparison, one can evaluate M in the free boson theory. The relevant graph is in Fig. 9. It is easy to obtain

$$M(g=0) = \frac{Ni}{4\pi}\frac{1}{q^2}\int_0^1 du \frac{1}{u(1-u)+\frac{m^2}{-q^2}} \tag{5.11}$$

One sees readily that

$M \to M(g=0)$ in the limit of $-q^2 \to \infty$. This agrees with the expected behavior of an asymptotically free field theory. We would like to point out that although the original quarks disappear from the physical spectrum due to infinite self energy, their presence in the original lagrangian is reflected here in the large

q^2 limit. The physical bound states collectively give rise to the right short distance behavior. We turn to the two point function of vector current. The simplest component to study is M_{--}, where

$$M_{\mu\nu} = \int d^2x\ e^{iqx} \langle 0|TV_\mu(x)V_\nu(0)|0\rangle . \tag{5.12}$$

In the leading 1/N approximation, one finds

$$M_{--} = i \sum \frac{|\langle n|V_-|0\rangle^2}{q_-^2 - M_n^2}$$

$$\sim \frac{i}{q^2}\ q_-^2\ \frac{N}{4\pi}\sum_n \left|\int_0^1 dx(1-2x)\phi_n(x)\right|^2$$

$$= \frac{i}{q^2}\ q_-^2\ \frac{N}{4\pi}\int_0^1 dx \frac{(1-2x)^2}{x(1-x)}$$

In the last step, the completeness relation is used. The integral is again logarithmically divergent. The correct treatment yields

$$M_{--} \sim \frac{i}{q^2}\ q_-^2\ \frac{N}{2\pi}\ \ell n(-q^2) + \ldots \tag{5.13}$$

For comparison, one finds in the free quark theory,

$$M_{--}(g=0) = \frac{N}{4\pi^2}\int d^2\ell \frac{(2\ell+q)_-(2\ell+q)_-}{(2\ell_+\ell_- - m^2+i\varepsilon)[2(\ell+q)_+(\ell+q)_- - m^2+i\varepsilon]}$$

$$= \frac{iN}{4\pi}\frac{q_-^2}{q^2}\int_0^1 dx \frac{(1-2x)^2}{x(1-x)-\frac{m^2}{q^2}} \tag{5.14}$$

One sees again that $M_{--} \to M_{--}\ (g=0)$ in the limit of $-q^2 \to \infty$ as was to be expected of an asymptotically free field theory.

Before we part with this section, we would like to point out the differences between two point functions in fermion and boson theories. We list again the asymptotic behaviors of M, M_{--} in both theories:

$$M(\text{boson theory}) \sim \frac{iN}{2\pi q^2} \ln(-q^2)+\ldots$$

$$M_{--}(\text{boson theory}) \sim \frac{iN}{2\pi q^2} q_-^2 \ln(-q^2)+\ldots$$

and

$$M(\text{fermion theory}) \sim \frac{N}{2\pi i} \ln(-q^2)+\ldots$$

$$M_{--}(\text{fermion theory}) \sim \frac{iN}{\pi} \frac{q_-^2}{q^2} +\ldots$$

The difference of the asymptotic behaviors of M is due to the difference in the dimensions of scalar densities. In the boson theory dim S = 0 while in the fermion dim S = 1. Although the vector current V has the same dimension in both theories, their asymptotic behaviors still differ by a factor of $\ln(-q^2)$. This is due to the fact that in the fermion theory, if m = 0, then

$$\langle 0|V|h\rangle \equiv 0 ,$$

except for zero mass hadron state(10) . Or in other words, if m = 0, there will be no absorptive part of $M_{\mu\nu}$ for $s\neq 0$.

VI. Form Factors

In this section we discuss the form factors of scalar densities and vector currents between hadron states. In particular we are interested in their asymptotic behavior as $q^2 \to \infty$.

To leading 1/N expansion, three diagrams may contribute to the form factor $\langle p_1|V_\mu|p_2\rangle$. They are depicted in Fig.(10). It is easy to see, by counting power in λ, that diagram b will not contribute in the limit of $\lambda \to 0$. First let us concentrate on diagram (a). It follows that

$$\begin{aligned}\langle p_1|V_-|p_2\rangle &= \frac{1}{2\pi} F_- \\ &= \frac{1}{2\pi}\int \frac{d^2k}{(2\pi)^2}\, iD(k)\, iD(k-p_1)\, iD(k-p_2)\,(2k-p_1-p_2)_- \\ &\times \left(\frac{4g}{\lambda}\right)^2 \frac{g^2N}{\pi} \quad (p_1)_-\, u(1-u)\phi_1(u)\,(p_2)_-\, u'(1-u')\phi_2(u')\end{aligned} \tag{6.1}$$

where

$$u \equiv \frac{k_-}{p_{1-}} , \quad u' \equiv \frac{k_-}{p_{2-}} = \frac{p_{1-}}{p_{2-}} u .$$

A straightforward calculation, by first integrating over k_+ and expressing k_- in terms of u, yields the simple result.

$$\begin{aligned}F_- &= (p_2)_- \int du\, u'\phi_2(u')\left[1+(1-2u)\frac{p_{1-}}{p_{2-}}\right] \phi_1^*(u) \\ &- (p_1)_- \int du\, u \left[1+(1-2u)\frac{p_{1-}}{p_{2-}}\right] \phi_1^*(u)\phi_2\left(\frac{p_{1-}}{p_{2-}} u\right)\end{aligned} \tag{6.2}$$

Now we can study the asymptotic behavior as $-q^2 \to \infty$. We have shown that $\phi_2(x) = c_2 x^\beta$ as $x \to 0$. With this one finds

$$F_- \sim (p_{1-}+p_{2-})\, F$$

and

$$F \sim \left(\frac{p_{1-}}{p_{2-}}\right) \int_0^1 du\ \ u\ \phi_1^*(u) \left(\frac{p_{1-}}{p_{2-}}\, u\right)^\beta c_2$$

$$= \left(\frac{M_1^2}{Q^2}\right)^{1+\beta} c_2 \int_0^1 du\ u^{1+\beta}\ \phi_1^*(u) \qquad (6.3)$$

where the kinematic relation $\frac{p_{1-}}{p_{2-}} \sim \frac{M_1^2}{Q^2}$ has been used.

The calculation of diagrams (c) is more involved. We will not include it here. We find that it gives the same Q^2 dependence as the contribution from diagram (a).

The form factor of the scalar current $\langle p_1|S|p_2\rangle = \frac{1}{2\pi} F_s$ can be studied in the similar fashion. One finds the contribution from the diagram (a) has the asymptotic behavior

$$F_s \sim \int_0^1 du\ u'\ \phi_1^*(u)\ \phi_2\left(\frac{p_{1-}}{p_{2-}}\, u\right)$$

$$= \left(\frac{p_{1-}}{p_{2-}}\right) \int du\ u\ \phi_1^*(u)\ \phi_2\left(\frac{p_{1-}}{p_{2-}}\, u\right)$$

$$\sim \left(\frac{M_1^2}{Q^2}\right)^{1+\beta} c_2 \int_0^1 du\ u^{1+\beta}\ \phi_1^*(u)$$

The contribution from diagram (c) yields similar Q^2 behavior.

VII. Deep Inelastic Scattering

The deep inelastic lepton-hadron scattering is related to the matrix element of vector currents in the Bjorken limit. The structure function $W^{[2]}$ is defined as

$$W_{\mu\nu} = \left(p_\mu - \frac{q_\mu (q\cdot p)}{q^2}\right)\left(p_\nu - \frac{q_\nu (q\cdot p)}{q^2}\right) \frac{W}{M_2^2}$$

$$= \langle p| \int d^2x\ e^{iqx}\ \ V_\mu(x)\ V_\nu(0)\,|p\rangle \ . \qquad (7.1)$$

In particular,

$$W_{--} = \langle p| \int d^2x\ e^{iqx}\ V_-(x)\ V_-(0)\,|p\rangle$$

$$= (2\pi)^2 \sum_n |\langle p|V_-|n\rangle|^2\ \delta((p+q)^2 - M_n^2)$$

$$= \sum_n |F_-|^2\ \delta((p+q)^2 - M_n^2) \qquad (7.2)$$

In the leading $\frac{1}{N}$ expansion only singlet hadron states has to be included in the summation. The form factor has been studied in the last section. In the Bjorken limit, it is obvioius that only diagram (a) contribute to F_-

$$F_- = p_{2-}(1-x) \int du\ u[1+(1-2u)(1-x)]\,\phi_1^*(u)\ \ \phi_2(u(1-x)) \qquad (7.3)$$

Here the kinematic relation $p_{1-} = (1-x)p_{2-}$ (in Bjorkens limit) has been used. In order to evaluate W_{--} in the leading order in $\frac{1}{N}$, we have to evaluate eq. (7.3). Since in the Bjorken limit $k \to \infty$, therefore the asymptotic expression for $\phi_1^*(u)$ can be used

$$\phi_1^*(u) \sim \sqrt{\frac{2}{u(1-u)}} \sin(k\pi u) \quad \text{for large } k.$$

Notice that in this limit, eq. (7.3) is nothing but the Fourier coefficient of

$$u[1+(1-2u)x] \sqrt{\frac{2}{u(1-u)}} \qquad \phi_2(u(1-x)) \; .$$

The asymptotic behavior of Fourier coefficient are well known.[22] One finds that

$$F_- = p_{2-}(1-x) \frac{1}{\sqrt{k}} (-1)^{k+1} \{x\phi_2(1-x)e_a + \ldots \;\} \tag{7.4}$$

in the Bjorken limit.

In eq. (7.2) we should really replace sum over delta function by the density of states

$$\rho = \frac{1}{g^2 \pi N} \; ,$$

Therefore,

$$W_{--} = |F_-|^2 \rho$$

$$= (p_{2-}(1-x))^2 \frac{1}{g^2 \pi N k} \{x^2\phi_2^2(1-x)e_a^2 + x^2\phi_2^2(x)e_b^2\} \tag{7.5}$$

$$= \frac{(p_{2-}(1-x))^2}{M_k^2} \{x^2 e_a^2 \phi_2^2(1-x) + x^2 e_b^2 \phi_2^2(x)\}$$

$$= \frac{1}{4} W(p_-)^2 \frac{1}{M_2^2}$$

The interference terms in $|F_-|^2$ cancel due to rapid variness of $(-1)^k$.[12] With the kinematic relation

$$q^2 + 2M_2\nu = 2M_2\nu(1-x) = M_k^2 = \frac{g^2 N}{\pi} k\pi^2$$

We obtain for W,

$$\frac{\nu W}{M_2^2} = \frac{1}{M_2} 2(1-x)\, x^2 \{e_a^2\, \phi_2^2(1-x) + e_b^2 \phi_2^2(x)\} \tag{7.6}$$

In contrast to the fermion theory, νW scales in the boson theory. This can be understood using the light cone expansion. We have near the light cone

$$V_-(z)\; V_-(0) \sim -\frac{1}{4\pi^2} \frac{z_- z_-}{(z^2)^2} [\varphi^*(z)\, \varphi(0) + \varphi(z)\varphi^*(0)\;] \tag{7.7}$$

Here $[\varphi^*(z)\varphi(0) + \varphi(z)\;\varphi^*(0)\;]$ has dimension 0. We would like to point out that in the fermion theory the only operator with dimension 0 is the unity operator.

It is interesting to compare with the calculation based on Feynman's parton picture.[23] The calculation is depicted in Fig. 11. We follow reference 23 closely. For W_{--} we obtain

$$\frac{1}{4}(p_{2-})^2 \frac{W}{M_2^2}$$

$$= \sum_a (2\pi)^2 e_a^2 \int \frac{(2xp_{2-} + q_-)^2}{(2\pi)^2} \; \delta(2M_2\nu(x-\xi)) \frac{f^a(\xi)}{\xi} d\xi$$

$$= \sum_a \frac{x^2 p_-^2}{2M_2\nu} \frac{f^a(x)}{x} e_a^2 \tag{7.8}$$

Thus,

$$\nu W = M_2 \sum_a 2x\; f^a(x)\; e_a^2 \tag{7.9}$$

Here $f^a(x)$ is the probability of finding parton a with momentum fraction x. We have discussed before that $\phi^2 = x(1-x)\phi^2$ is the probability density. Therefore,

$$\nu W = M_2 \; 2x^2(1-x) \; \{ \; e_a^2\phi^2(x) \; + \; e_b^2\phi^2(1-x) \; \}$$

This is in perfect agreement with our previous calculation. The structure function for the scalar current can be studied in a similar way. We quote here only the final result

$$\nu W_s = \frac{1}{2} M_2(1-x)(e_a^2\phi^2(x) + e_b^2\phi^2(1-x))$$

This agrees with Feynman's parton picture and is consistent with the light cone expansion.

It is important to check here the consistency of our results on the form factor and the threshold behavior of the structure function W. We have

$$\left.\frac{\nu W}{M_2^2}\right|_{x\to 1} = \frac{1}{M_2} \; 2(1-x) \; \{e_a^2c_a^2 \; (1-x)^{2\beta_a} + e_b^2c_b^2(1-x)^{2\beta_b} \}$$

and from eq. (6.3)

$$F_- = P_{2-} \;\; (1-x) \;\; \left(\frac{M_1^2}{Q^2}\right)^{\beta} c_2 \; \int du \; u^{\beta+1} \; \phi_1^*(u)$$

Notice that we have derived νW starting with the eqs. (7.3) and (7.4). Therefore, to check the validity of Drell-Yan-West[15] as well as Bloom-Grilman relation, all we have to do is to check whether F_- given in eq. (6.3) agrees with eqs. (7.3),(7.4) in

the limit of large k. The agreement follows from the asymptotic evaluation of Fourier coefficient. It is easy to see that

$$\int_0^1 du \; u^{\beta+1} \; \phi_1^*(u) \; \to \; (-1)^{k+1} \; \frac{1}{\sqrt{k}} \; \text{as } k \to \infty \; .$$

This proves the consistency of F_- . The Drell-Yan-West and Bloom-Gilman relations are thus verified in our model.

VII. Summaries and Discussions

We have studied the large N-limit of the SU(N) scalar QCD in two space-time dimensions. In this model the current has the virtue that it mimics nicely the behaviors of the current in the four dimensional QCD. Furthermore the consensual properties of the QCD, i.e., the infrared slavery, the quark confinement, the charmonium picture etc. can all be realized. We have analyzed in this paper the quark-antiquark bound state wave functions, studied the behaviors of the current form factors and e^+e^- annihilation. In the deep inelastic scattering, we have found that νW scales and satisfies the Drell-Yan-West and the Bloom-Gilman relations, and agrees nicely with Feynman parton picture and the light-cone expansion.

It is pertinent to remark here that our results and conclusions are valid in the region of $Ng^2/\pi \leq m_i^2$. We do not expect them to remain true in the other region $\frac{Ng^2}{\pi} > m_1^2$ and/or m_2^2 . A similar situation occurs in the massive Schwinger model. Coleman[24] has shown that the theory behaves quite differently depending on whether $\frac{e^2}{\pi} >> m^2$ or $\frac{e^2}{\pi} << m^2$. However he has not pinned down the critical value of $\frac{e^2}{\pi}$ for the phase transition. The result of our investigation using W.K.B. approximation indicated that perhaps $\frac{Ng^2}{\pi} = m^2$ is the critical value. The W.K.B. approximation indicates that our results are valid only when $\frac{Ng^2}{\pi} \leq m^2$. We have not attempted to study the bahaviors in the other region.(25) Perhaps some kind of bosomization procedure is called for in that region.(26)

It is also worthwhile to point out that Bardeen and Bander(27) have studied the nonlinear O(N) σ-model in 2 dimensions, especially the problem of phase transition. They study the nonlinear theory by taking the $\lambda_0 \to \infty$ limit of the linear theory

$$\mathcal{L} = \frac{1}{2} (D_\mu \vec{\Phi})^2 - \frac{\lambda_0^2}{4} (f_0^2 - \vec{\Phi}^2)^2 - \frac{1}{4} (G_{\mu\nu})^2 .$$

The phase transition they study is the relation between λ_0 and f_0. The difference between the nonlinear σ-model and our model are twofolds. We do not have quantic term $\frac{\lambda_0}{4} (f_0^2 - \vec{\Phi}^2)^2$. Moreover, the group properties in two models are different. The group we discussed is a direct product of color gauge group and flavor group. The group in their work is a flavor gauge group which is spontaneously broken by the vacuum expectation value of Φ.

Another comment concerns the similarities and the lack of them in the scalar QCD and 't Hooft model. It is well known that in two space-time dimensions, bosons and fermions are not fundamentally different.(28) This is related to the fact that spin loses its meaning in 2 dimensions. And therefore there is no spin-

statistics connections. This may account for the fact that both models have many common qualitative features. However, the flavored vector currents behave quite differently in two models mainly due to the softness of vector currents in the fermion theory ('t Hooft model).

Throughout the paper, we have used the singular cutoff (λ-cutoff) procedure of 't Hooft. It is a simple matter to check that all the physical amplitudes and results remain the same if the principal-value prescription of the infrared cutoff is used.

Before we conclude this paper we would like to mention the problems that we have not studied here.

(1) Gauge problem: We restrict our consideration to the light-cone gauge. The situations in the other gauges are much more involved.[3] For example, in the linear gauge n.A = 0, the currents will contain the gauge field A. This makes the expression of A in terms of color current j much more involved. We have not pursued this and other gauge related problems in this paper.

(2) Other physical processes: We have not discussed the other physical processes, for example, the inclusive e^+e^- annihilation,(30) Regge behavior of the scattering amplitudes(31) and the problem of the Pomeron(31) These physical processes have been studied in the 't Hooft model in references 30 and 31. However, we expect that they behave similarly in our model.

(3) Strong coupling limit: We have not studied the strong coupling limit of $Ng^2/\pi >> m^2$ to see if our results remain valid.

(4) Relation to the string model: It has been shown that the 't Hooft model is equivalent to a certain type of string model in the sense that they yield same results for physical amplitudes.(32)

However, we do not know at all whether similar equivalence exists between the scalar QCD and certain other string models.

Added note: While this paper is being typed, we come across the work of M. B. Halpern and P. Senjanovic, Phys. Rev. D 15, 1655 (1977). They discuss the connection between scalar QCD and strings in two dimensions. The equation (2.20) in our paper is also derived.

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APPENDIX

We would like to consider here another method of solution of the integral equation for bound states in fermion theory. The equation under study is

$$\mu^2\phi(p) = \frac{\alpha}{p(1-p)}\,\phi(p) - \int_0^1 \frac{\phi(k)}{(k-p)^2}\,dk$$

$$= \frac{\alpha}{p(1-p)}\,\phi(p) - \int_0^1 \frac{\phi'(k)}{(k-p)}\,dk \qquad \text{A.1}$$

This type of equations appears very frequently in the study of the airfoil.[33] There, one has

$$\frac{\Gamma(t_0)}{B(t_0)} - \frac{1}{\pi}\int_{-a}^{a} \frac{\Gamma'(t)dt}{t-t_0} = f(t) = 4V\alpha(t). \qquad \text{A.2}$$

where Γ is the circulation of the airflow around the profile, B(t) is a function related to the profile, V is the velocity of airflow at infinity and α is the geometrical angel of incidence and 2a is the span of the wing.

The treatment of this type of equation is well documented in the literature. One just casts it into the following form

$$\Gamma(\omega_0) = \frac{a}{\pi}\int_0^{\pi} \frac{\sin\omega}{B(\omega)}\,\ell n\left|\frac{\sin(\frac{\omega-\omega_0}{2})}{\sin(\frac{\omega+\omega_0}{2})}\right|\Gamma(\omega)d\omega$$

$$+ G(\omega_0) + c_0 \qquad \text{A.3}$$

where

$$t_0 = -a \cos \omega_0 \quad ,$$

$$t = -a \cos \omega \quad ,$$

$G(\omega_0)$ is a completely determined function which vanishes when f does, and c_0 is an integration constant.

Equation (1) corresponds to $a = \frac{1}{2}$ and

$$\frac{1}{B(\omega)} = -\frac{1}{\pi}\left(\mu^2 - \frac{4\alpha}{\sin^2\omega}\right)$$

Next the kernel $\ell n \left|\frac{\sin(\frac{\omega-\omega_0}{2})}{\sin(\frac{\omega+\omega_0}{2})}\right|$ is expressed in terms of a complete set of functions $\{\sin n\,\omega\}$,

$$\ell n \left| \frac{\sin(\frac{\omega-\omega_0}{2})}{\sin\frac{\omega+\omega_0}{2}} \right| = -\sum_{h=1}^{\infty} \frac{2}{n} \sin n\omega \sin n\,\omega_0 \qquad \text{A.4}$$

It follows from eqs. (A.3 and A.4) that

$$\Gamma(\omega_0) = c_0 + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{a_n}{n} \sin n\,\omega_0 \qquad \text{A.5}$$

and

$$a_n = \int_0^{\pi} \left(\mu^2 - \frac{4\alpha}{\sin^2\omega}\right) \sin\omega \sin n\omega\, \Gamma(\omega) d\omega \qquad \text{A.6}$$

The boundary conditions $\phi(1) = \phi(0) = 0$ implies that $\Gamma(\omega_0) = 0$ when $\omega_0 = 0$, or, $c_0 = 0$. It is easy to see that a_n satisfies the relation

$$a_n = \sum_{k=1}^{\infty} \frac{1}{k\pi^2} a_k (\mu^2 I_{n.k} - 4\alpha \mathcal{J}_{n,k})$$

$$= \sum_{k=1}^{\infty} \frac{a_k}{k\pi^2} \int_0^{\pi} (\mu^2 - \frac{4\alpha}{\sin^2\omega}) \sin\omega \sin k\omega \sin n\omega\, d\omega \qquad \text{A.7}$$

Here,

$$I_{n,k} = 0 \qquad : n - k \text{ odd}$$

$$= 1 + \frac{1}{4n^2-1} \qquad : n = k$$

$$= \frac{1}{1-(n-k)^2} + \frac{1}{(n+k)^2-1} \qquad : n - k \text{ even}$$

A.8

$$\mathcal{J}_{n.k} = 2 \sum_{j=1+|\frac{n-k}{2}|}^{(n+k)/2} \frac{1}{2j-1} \qquad : n - k \text{ even}$$

$$= 0 \qquad : n - k \text{ odd}$$

A.9

It is a hopeless task to solve the eigenvalue problem of an infinite by infinite matrix. The best one can do is solving the truncated equation. We are interested here in the eigenvalue $\mu^2 >> \alpha$. Therefore we may neglect $4\alpha \mathcal{J}_{n,k}$ as compared to $\mu^2 I_{n,k}$. Thus

$$\frac{1}{\mu^2} a_\ell = \sum_k \frac{1}{k\pi^2} I_{\ell,k}\, a_k \qquad \text{A.10}$$

If we truncated the matrix to an nxn matrix, we have the following relation between eigenvalues μ_i^2 and the diagonal matrix elements.

$$\sum_{i=1}^{n} \frac{1}{\mu_i^2} = \sum_{k=1}^{n} \frac{1}{k\pi^2} I_{k,k} \sim \frac{1}{\pi^2} (1+\frac{1}{2} + \ldots + \frac{1}{n})$$

$$\sim \frac{1}{\pi^2} \ell n\ n \qquad \text{A.11}$$

This diverges as $n \to \infty$. From this one concludes immediately that

$$\frac{1}{\mu_k{}^2} \to \frac{1}{k\pi^2} \quad \text{in the large k limit.}$$

Namely,

$$\mu_k^2 = k\pi^2 + \text{less important term.} \qquad \text{A.12}$$

This agrees with the result of 't Hooft.

The numerical calculation for μ^2 beyond this leading term is much harder. Hansen et al.(34) have done a great deal of work in this direction. We refer the interested readers to their work.

We would like to point out that the equation describing bound states in scalar QCD can be rewritten as

$$(\mu^2 - \frac{\alpha_1}{x} - \frac{\alpha_2}{1-x})\ 4x(1-x)\varphi(x)$$

$$= - \int_0^1 \frac{dy}{(x-y)^2}\ \varphi(y)(x+y)(2-x-y)$$

$$= - \int_0^1 dy\ \frac{\varphi'(y)(x+y)(2-x-y)}{(y-x)} - 2\int_0^1 dy\ \frac{\varphi(y)(1-x-y)}{(y-x)} \qquad \text{A.13}$$

Equations of this type involving φ and its derivatives $\varphi^{(r)}$ occur in the study of aircraft wing of finite span. The general study of these has been done by L. G. Magnaradze.[4]

FOOTNOTES

[1] See e.g. chapter 4 of ref. 20 and chapter 4 of ref. 33

[2] We use the covariant normalization of states $\langle p|p'\rangle = 2E_p\ \delta(p-p')$. In 1 - 1 dimensions the tensors

$$\left(p_\mu - \frac{q_\mu(q\cdot p)}{q^2}\right)\left(p_\nu - \frac{q_\nu(q\cdot p)}{q^2}\right) \text{ and } (q_\mu q_\nu - q_{\mu\nu}q^2)$$

are not linearly independent. Therefore there is only one structure function W in 1 - 1 dimensions.

[3] The gauge problem in the 't Hooft model has been discussed in ref. 29. See also ref. 26.

[4] See chapter 17 of ref. 33.

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FIGURE CAPTIONS

Fig. 1. Feynman rules.

Fig. 2. Second order self energy graph of quark.

Fig. 3. Integral equation for the dressed quark propagator.

Fig. 4. Bethe-Salpeter equation for the bound states.

Fig. 5. Graphic solutions of equations (3.8) and (3.9).

Fig. 6. Bethe-Salpeter equation for the quark-antiquark scattering amplitude.

Fig. 7. The vertex of $q\bar{q}h$ coupling.

Fig. 8. Graph for the matrix elements of $\langle 0|S^{ab}|h\rangle$ and $\langle 0|V^{ab}|h\rangle$, x corresponds to either S^{ab} or V^{ab}.

Fig. 9. Graph for calculating two point functions in the free theory.

Fig. 10. Graphs for the form factor $\langle p_1|V_\mu|p_2\rangle$.

Fig. 11. Graph in the Feynman parton picture.

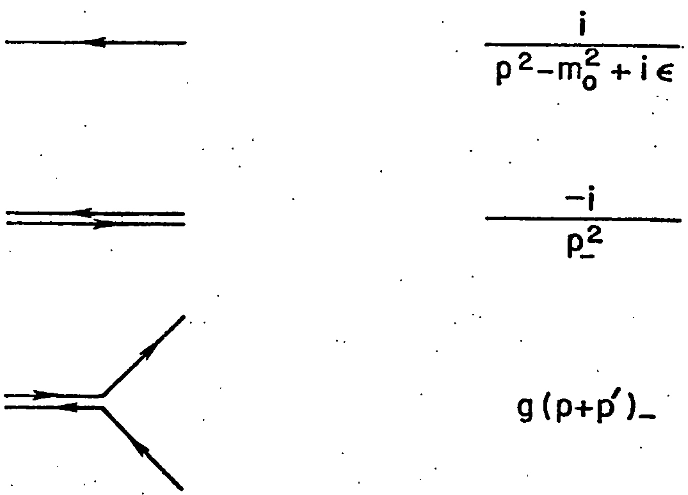
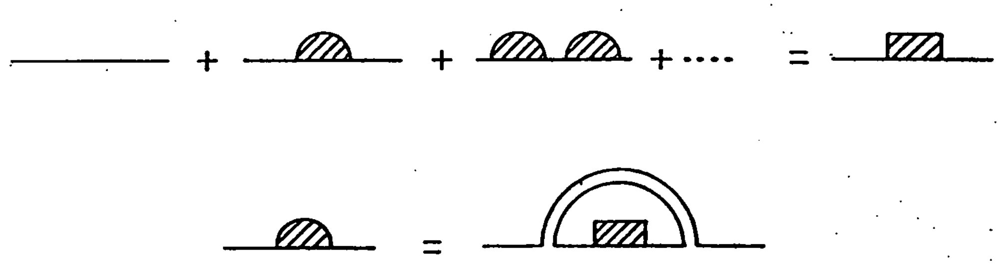


FIG. 1



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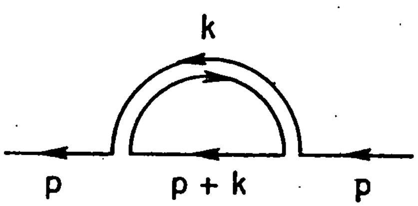

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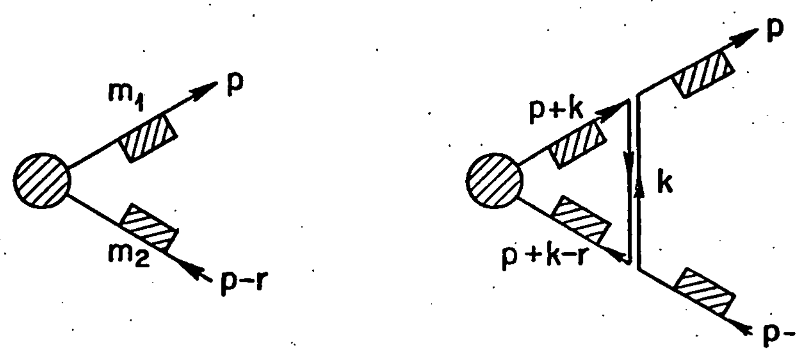

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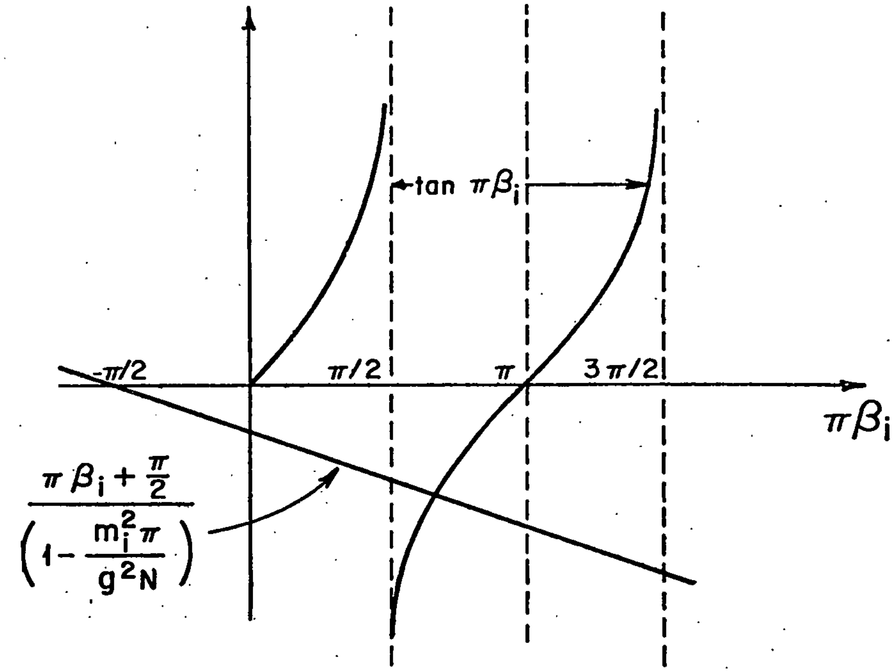

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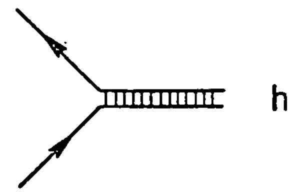

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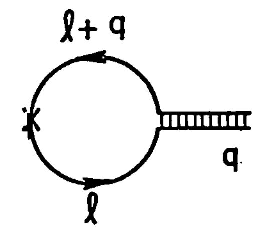

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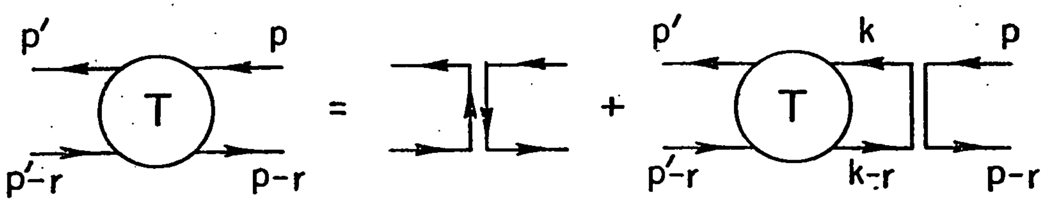

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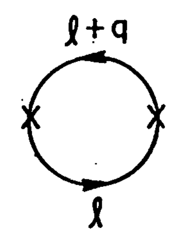

FIG. 9

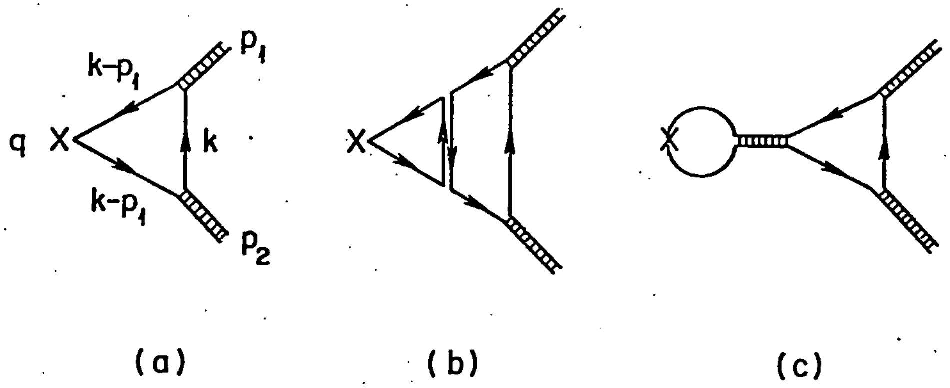

FIG. 10

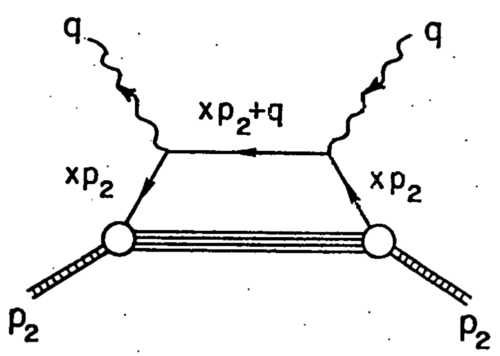

FIG. 11

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