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ANALYSIS OF DISCRETE-TIME QUEUES FOR RANDOM OUTPUT PROCESSES AND NETWORK SOLUTIONS*

Andrew S. Noetzel
Applied Mathematics Department
Brookhaven National Laboratory
Upton, NY 11973

ABSTRACT

The relationships between the local balance and random output properties of queues, and the product form solution for networks of queues, are established by analyzing finite discrete-time queues, with Bernoulli processes for the random output condition. The results can be extended to continuous-time queues with Poisson output processes. It is shown that the random output condition is necessary but not sufficient for the network product form solution. Though not all the necessary steps have been proven, intuitive arguments and sample cases are shown to indicate an equivalence between a generalized form of local balance and the network product form solution.

1. Introduction

The analysis of queueing networks has been demonstrated in the pioneering works of Jackson [6] and Gordon and Newell [5], and others [1,4,9]. All the known solutions for the equilibrium state probabilities of arbitrary networks of queues are of the product form type; the state probability of the network is expressed as the product of the state probabilities of each of the individual queues of the network in isolation with a Poisson input of the same mean rate as the queue has in the network. The class of queues known to yield the product form solution is limited to queues with exponential servers, queues with an infinite number of servers, the processor-sharing discipline, and the preemptive last-come-first-served discipline.[1,10] The queues of this class all share the properties of having a Poisson output process when the input is a Poisson process [8,9], and of local balance [2,4,14].

A queue that has a Poisson output process when the input is Poisson apparently behaves in a network of similar queues as it does in isolation (with a Poisson input); the product form solution for the network is then simply a consequence of the independence of the queue behaviors. This explanation is satisfactory for an open network without loops, since in that case, the input to each queue actually is a Poisson process, and the queues are truly independent. But it fails when the network has loops, since the input process of a queue is then not independent of its output and, therefore, not independent of past inputs. Hence, the relationship between the Poisson output condition and the product form for general networks remains incomplete.

In this paper, we summarize current research that establishes the relationships between queues that yield product form network solutions, local balance, and the Poisson output condition. Not all the results reported have been conclusively proven; the incomplete proofs will be indicated, and an intuitive explanation and sample case will be shown.

The technique used in this analysis effort is to consider finite-state discrete time queues rather than continuous-time queues, and Bernoulli processes instead of Poisson processes. However, the results shown here can be shown to hold for continuous Markov processes and Poisson inputs by the well-known limiting process in which the time increment approaches zero. Throughout, input and output processes are described as 'random', indicating Bernoulli processes for the discrete case, Poisson processes for the continuous case.

The queues considered also have finite capacity; when the queue is full, arrivals pass through to become part of the output process. But, as long as the mean service rate of the queue is greater than the mean arrival rate, the probability of an overflow approaches zero as the capacity of the queue is increased. Hence, by letting the capacity of the queue increase beyond all bounds, the results can be shown to be valid for queues with an infinite number of states.

2. Definitions and Notation

Consider the finite-capacity queue with Bernoulli input process. Let $S(k)$, called level k, $0 \le k \le m$, be the set of states for which the queue holds k customers; m is the capacity of the queue. For each possible transition there is a conditional probability: $\lambda_{j|i}$ is the conditional probability of an input transition to S_j from S_i, given S_i.

The probability of an arrival at any discrete time is λ, so all states have a total probability λ of an input transition. Thus, if $S_i \varepsilon S(k)$, $0 \le k < m$,

$$\sum_{S_j \varepsilon S(k+1)} \lambda_{j|i} = \lambda .$$

At level m, there are transitions from each state back to itself, with probability λ. These represent overflows; both input and output with no change of state.

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The conditional probability of a departure transition from state $S_i \varepsilon S(k)$ (given S_i) to state $S_j \varepsilon S(k-1)$ is $\mu_{j|i}$. Let

$$\mu_i = \sum_{S_j \varepsilon S(k-1)} \mu_{j|i}.$$

The conditional probability of a transition from S_i to some S_j at the same level is $\eta_{j|i}$. Let $\eta_i = \sum_{\substack{S_j \varepsilon S(k) \\ j \neq i}} \eta_{j|i}$.

If the queue is in S_i, and neither an input, output, or transition at the same level occurs, the queue will remain in S_i until the next discrete interval. This conditional event has probability $1 - \lambda - \mu_i - \eta_i$.

Let A be the n x n matrix of transition probabilities for an n-state queue, and let P be the n-element vector of mean state probabilities. Then AP = P.

The matrix A is a block tri-diagonal matrix. Along the main diagonal are the probabilities $a_{ii} = 1 - \lambda - \mu_i - \eta_i$. Let $S(k) = \{S_{k_1}, S_{k_2}, \ldots S_{k_u}\}$ be the set of states at level k. Let $S(j) = \{S_{j_1}, \ldots S_{j_v}\}$. With j=k-1,

$$\text{let } U_k = \begin{bmatrix} \mu_{j_1|k_1} & \mu_{j_1|k_2} & \cdots & \mu_{j_1|k_u} \\ \mu_{j_2|k_1} & \cdot & & \cdot \\ \vdots & \cdot & & \cdot \\ \mu_{j_v|k_1} & \mu_{j_v|k_2} & \cdots & \mu_{j_v|k_u} \end{bmatrix}$$

be the v x u submatrix of A, giving the (output) transition rates from S(k) to S(k-1). Each of the U_k, $1 \leq k \leq m$, will be above the main diagonal of A.

Let T be the entire matrix of output transitions: this will consist of all submatrices U_k, $1 \leq k \leq m$, and the submatrix λI for level m. For example,

$$T = \begin{bmatrix} 0 & [U_1] & 0 & 0 \\ 0 & 0 & [U_2] & 0 \\ 0 & 0 & 0 & [U_3] \\ 0 & 0 & 0 & [\lambda I] \end{bmatrix}$$

And let V = A-T be the matrix of transitions without output.

Let $\Sigma = \underbrace{\overbrace{11\ldots1}}$ be a row vector of ones. The dimension of Σ is assumed to conform to the row dimension of the matrix it premultiples. Thus, Σ may be considered an operator that converts a rectangular matrix into a row vector by summing the columns. It converts a column vector into a scalar.

Thus, for a queue with states S_o to S_n, in which states S_j to S_n are at the highest level, $\Sigma T = \underbrace{\overbrace{\mu_1\ \mu_2 \ldots \mu_{j-1} \mu_j{+}\lambda\ \ldots \mu_n{+}\lambda}}$. And, for any vector P of state probabilities, $\Sigma P = 1$.

Any queue such that

$$TP = \lambda P \qquad \text{(1a)}$$

for all λ, will be said to satisfy local balance. The local balance condition requires that the absolute probability (or rate) of leaving each state due to an arrival, is equal to the absolute rate of entering the state due to a departure.

The local balance condition (1a) restated for $S_i \varepsilon S(k)$, k<m, is

$$\sum_{j \varepsilon S(k+1)} \mu_{i|j} P_j = \lambda P_i. \qquad \text{(1b)}$$

Let $P_1(\lambda,n) = Pr[S(n)]$ be the marginal probability that queue Q_1 contains n customers, when the input to the queue is a random process with mean rate λ. Let W be a finite open or closed network of queues $\{Q_1, \ldots, Q_w\}$. The structure of W is determined by the matrix B, of which the elements $\{b_{j|i}\}$, $1 \leq i,j \leq w$, are the probabilities that a customer goes to queue j when leaving queue i. (For open networks, one queue number may be used to designate both the source and sink). Let L be any vector of relative flow through the network; L satisfies the relation BL=L. If the network is open then L is determined absolutely by the rate of the source; otherwise L is determined to within a constant.

Let $P_W(n_1, n_2, \ldots, n_w)$ be the equilibrium probability of n_i customers at queue i, $1 \leq i \leq w$, of network W. Then network W has a product form solution if, for some L,

$$P_W(n_1, n_2 \ldots n_w) = \frac{1}{c} \prod_{i=1}^{w} P_i(\lambda_i, n_i) \qquad (2)$$

The normalization constant $c = \Sigma \prod_{i=1}^{w} P_i(\lambda_i, n_i)$, where the summation is over all possible sets of states $(n_1 \ldots n_w)$. The summation is infinite if this network is open and has infinite storage capacity.

3. Results: Product Form, Local Balance, and Random Output

A queue has random output if, when the probability of an input in any discrete interval is λ, the probability of an output in any discrete interval after any sequence of output and non-output intervals, is λ. This is stated in terms of the queue parameters in the following theorem.

Theorem 1: A queue has random output if and only if, for all strings $R_{a,t}$ consisting of a occurrences of A and t occurrences of T,

$$\Sigma T R_{a,t} P = \lambda^{t+1} \qquad (3)$$

The proof of this is found in [12]. Intuitively, it states that the probability of any output sequence that contains t identified outputs is λ^t, given that the intervals that are not identified as producing an output are not observed; they may or may not have an output. Equivalently, any string of t output intervals and n non-output intervals will have probability $\lambda^t(1-\lambda)^n$.

Theorem 2: Local balance implies random output.

The proof follows from AP=P and TP=λP. Then $R_{a,t}P = \lambda^t P$, satisfying Theorem 1.

Theorem 3: Random output does not imply local balance.

This is demonstrated by the simple counter-example of Figure 1. Assume that (1) holds for the queue Q of Figure 1a, which then has random output. But the queue Q' of Figure 1b may be derived from Q by splitting S_a into S_{a1} and S_{a2}, with the rates as shown. Then $P_a = P_{a1} + P_{a2}$, since the probability of a transition into either S_{a1} or S_{a2} is the same as that into S_a, and the combined probability of transition out of S_{a1} and S_{a2} is the same as that of S_a. If Q is in local balance then $\lambda P_a = \mu_{a|2} P_2$. But no such relation is then possible for either S_{a1} or S_{a2} of the queue Q' of Figure 1b. (Although, of course $(P_{a1}+P_{a2}) = \mu_{a|2}P_2$).

The future behavior of Q' in no way depends on which of S_{a1} or S_{a2} has been entered, or on the probabilities of transition between them, because of the equivalence of the transition probabilities from each of these to the other states of Q'. Therefore, Q' has the same output probability as Q, and therefore has random output.

Definition: A set of states S is an equivalent set if a) S is contained in a single level of the queue and b) the transition probabilities from each state of S to any state, or other equivalent set, outside of S is the same as that of every other state of S.

In other words, S is an equivalent set if it could have been derived by means of state splitting, as in the example of Figure 1. And any equivalent set may be merged into a single state without changing the output behavior of the queue.

But equivalent sets of states may be derived by splitting more than one state of the queue simultaneously. Then the equivalent set may not be identifiable until the other equivalent sets are identified. An algorithm for identifying equivalent sets is given in [12].

Definition: A queue will be said to have generalized local balance if the queue obtained by merging all equivalent state sets has local balance.

Theorem 4: A queue with generalized local balance has random output.

This theorem follows from the output equivalence of a queue with equivalent state sets, and a queue with all equivalent state sets merged to single states.

Theorem 5: Random output does not imply generalized local balance.

This is demonstrated by the queue of Figure 2. (The transition probabilities in this example have absolute values greater than one in order that the derived probability expressions will have a simple form. However, if all of the probabilities representing a change of state are divided by a large number, and the probabilities of remaining in a state correspondingly adjusted, as in the limiting processes to a continuous-time case, the example still holds.)

Since μ_a, μ_b and μ_c are all different, there are no equivalent state sets. The equilibrium state probabilities are:

$$P = \begin{bmatrix} P_0 \\ P_a \\ P_b \\ P_c \\ P_2 \end{bmatrix} = \begin{bmatrix} 3\lambda^2 + 18\lambda + 24 \\ 2\lambda^3 + 11\lambda^2 + 12\lambda \\ \lambda^3 + 5\lambda^2 + 4\lambda \\ \lambda^3 + 8\lambda^2 + 16\lambda \\ \lambda^4 + 6\lambda^3 + 8\lambda^2 \end{bmatrix} \Big/ D \qquad (4)$$

where $D = D_1 D_2$, $D_1 = \lambda^2 + 6\lambda + 8$,

and $D_2 = \lambda^2 + 4\lambda + 3.$ (5)

From this, it can be seen that $TP \neq \lambda P$, hence the queue does not have generalized local balance. In [12], it is shown that the conditions of Theorem 1 hold, hence, the queue has random output.

The marginal state probabilities for this queue are:

$$P(\lambda) = \begin{bmatrix} P(\lambda,0) \\ P(\lambda,1) \\ P(\lambda,2) \end{bmatrix} = \begin{bmatrix} 3 \\ 4\lambda \\ \lambda^2 \end{bmatrix} D_2. \qquad (6)$$

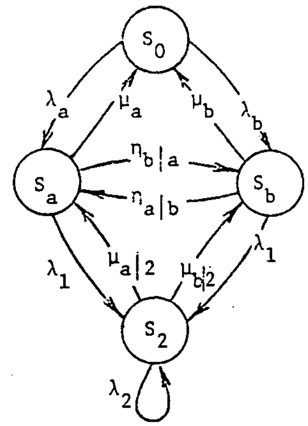

Figure 1a

<u>Theorem 6</u>: Random output is not a sufficient condition for a product form network solution.

The queue of Figure 2 is again used.[12] It is necessary to show only that there is one network containing the queue, that does not have a product form solution. The network can be as simple as the queue itself, with a single feedback loop, as shown in Figure 3. (An even simpler network, the closed network obtained in the q=0 case, would not serve as a counterexample. In that case, there would be a constant N customers in the network and the queue would always be in a state of level N. Through normalization, this would meet the product form requirements.) With $q=\frac{1}{2}$, the mean state probabilities for the queue in the network are:

$$P_N = \begin{bmatrix} P_{N0} \\ P_{Na} \\ P_{Nb} \\ P_{Nc} \\ P_{N2} \end{bmatrix} = \begin{bmatrix} 3\lambda^2 + 60\lambda + 192 \\ 6\lambda^2 + 54\lambda^2 + 96\lambda \\ \lambda^3 + 18\lambda^2 + 32\lambda \\ \lambda^3 + 24\lambda^2 + 128\lambda \\ \lambda^4 + 12\lambda^3 + 32\lambda^2 \end{bmatrix} \Big/ D_3 \qquad (7)$$

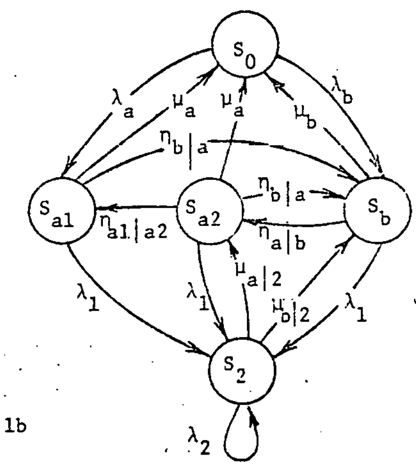

Figure 1b

Where $D_3 = \lambda^4 + 20\lambda^3 + 131\lambda^2 + 316\lambda + 192$. The marginal state probabilities are:

$$P_W = \begin{bmatrix} P_W(0) \\ P_W(1) \\ P_W(2) \end{bmatrix} = \begin{bmatrix} 3\lambda^2 + 60\lambda + 192 \\ 8\lambda^3 + 96\lambda^2 + 256\lambda \\ \lambda^4 + 12\lambda^3 + 32\lambda^2 \end{bmatrix} \Big/ D_3 \qquad (8)$$

The marginal state probabilities are not equal to the marginal probabilities of the queue in isolation, with random input with rate λ, as given in (6).

<u>Definition</u>: A queue will be said to have state-dependent input local balance (s.d.i.l.b.) if TP=LP, where L is a vector of state-dependent input rates, $\{\lambda_n | n=0,1,2\ldots\}$ for each state for which the queue contains n customers.

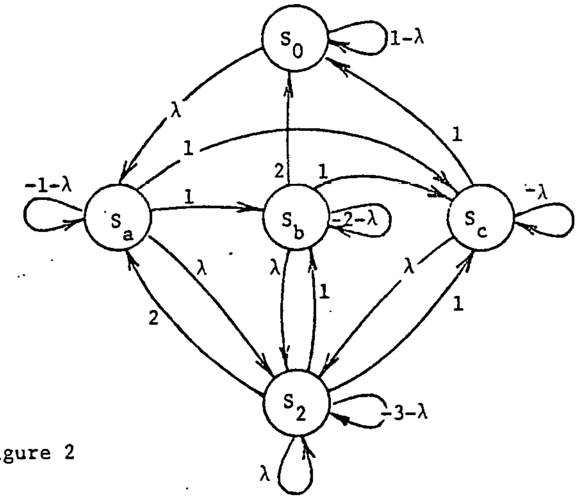

Figure 2

Obviously, if $\lambda_n = \lambda$ for all n, then the s.d.i.l.b. is local balance. Hence, s.d.i.l.b. implies local balance.

Theorem 7: Local balance implies state-dependent input local balance.

This theorem has not been proven in general. Intuitively, if local balance holds, the state probabilities of a queue are not modified by either an output or the nonoccurrence of an output. In particular, there is no shift of probability between states at a particular level n. Therefore, if λ_n is increased, the probabilities of all states at level n+1 are proportionately increased, and the probabilities of all states at level n correspondingly decreased, but the absolute rate out of a level n state due to an arrival, will still equal the absolute rate into that state due to a departure.

The theorem has been shown to hold in examples of up to six states. (Beyond that, the explosion of the combinations causes serious difficulties.) The simplest case, that of the queue of Figure 1a, will be shown.

$$P=\begin{bmatrix}P_0\\P_a\\P_b\\P_2\end{bmatrix}=\left.\begin{bmatrix}(\eta_a\mu_b+\eta_b\mu_a+\mu_a\mu_b)\mu_2+\lambda_1(\mu_a\mu_{a|2}+\mu_b\mu_{b|2})\\ \lambda_a\mu_b\mu_2+\lambda_0\eta_b\mu_2+\lambda_0\lambda_1\mu_{a|2}\\ \lambda_b\mu_a\mu_2+\lambda_0\eta_a\mu_2+\lambda_0\lambda_1\mu_{b|2}\\ \lambda_0\lambda_1(\lambda_1+\eta_a+\eta_b)+\lambda_1(\lambda_a\mu_b+\lambda_b\mu_a)\end{bmatrix}\right/ D \qquad (9)$$

where

$$D=(\eta_a\mu_b+\eta_b\mu_a+\mu_a\mu_b)\mu_2+\lambda_1(\mu_a\mu_{a|2}+\mu_b\mu_{b|2})$$
$$+(\lambda_a\mu_b+\lambda_b\mu_a+\lambda_0(\eta_a+\eta_b))(\lambda_1+\mu_2)+\lambda_0\lambda_1(\lambda_1+\mu_2). \qquad (10)$$

Also, $\lambda_a+\lambda_b=\lambda_0$.

The state-dependent input local balance condition is expressed by

$$TP=\begin{bmatrix}0 & \mu_a & \mu_b & 0\\ 0 & 0 & 0 & \mu_{a|2}\\ 0 & 0 & 0 & \mu_{b|2}\\ 0 & 0 & 0 & \lambda_2\end{bmatrix}\begin{bmatrix}P_0\\P_a\\P_b\\P_2\end{bmatrix}=\underbrace{\lambda_0\lambda_1\lambda_1\lambda_2}\begin{bmatrix}P_0\\P_a\\P_b\\P_2\end{bmatrix} \qquad (11)$$

Of these equations, the only independent condition not equivalent to a balance equation is

$$\mu_{a|2}P_2 = \lambda_1 P_a, \qquad (12)$$

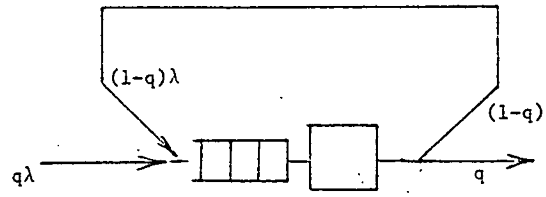

Figure 3

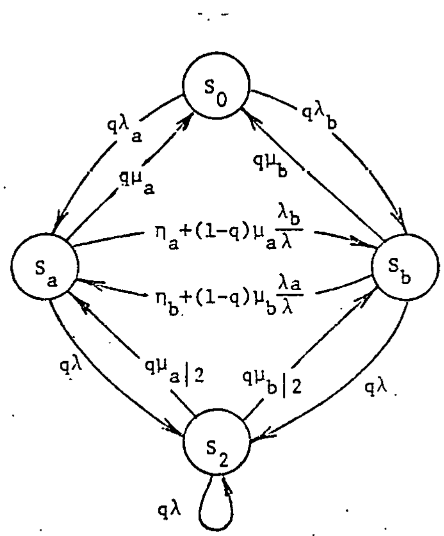

Figure 4: Queue of Figure 1a in Network of Figure 3

which can be reduced to the condition

$$\lambda_a\mu_b\mu_{b|2}+\lambda_0\eta_b\mu_{b|2} = \lambda_0\eta_a\mu_{a|2}+\lambda_b\mu_a\mu_{a|2}. \qquad (13)$$

If local balance holds, that is, (12) is satisfied for all $\lambda_0 = \lambda_1 = \lambda_2$, then s.d.i.l.b., since (12) holds for all λ_0, λ_1, λ_2.

Theorem 8: Local balance implies a product form network solution.

Consider two locally balanced queues in a closed two queue network containing N customers. Since each queue is s.d.i.l.b. (by Theorem 7), the state probabilities are independent of the occurrence or nonoccurrence of an output. Let $\mu_{1,n} = \Sigma\mu_i P_i$, where the summation is over all states of queue one for which it contains n customers, and let $\mu_{2,N-n}$ be similarly defined for queue two. Then $\mu_{1,n}$ and $\mu_{2,N-n}$ will be the input rates to queues two and one, respectively.

The marginal state probabilities of each of the queues in isolation, with input λ, are

$$P_{1,n} = \frac{\lambda^n}{\prod_{i=1}^{n} \mu_{1,i}} P_{1,0} \text{ and } P_{2,N-n} = \frac{\lambda^{N-n}}{\prod_{i=1}^{N-n} \mu_{2,i}} P_{2,0} \quad (14)$$

The network is simply represented by the Markov chain with states $(n,N-n)$, $0 \le n \le N$, and rates $\mu_{1,n}$ and $\mu_{2,N-n}$. The solution is

$$P_W(n,N-n) = \frac{\prod_{j=N-n+1}^{N} \mu_{2,j}}{\prod_{i=1}^{n} \mu_{1,i}} P_W(0,N) = \frac{1}{c} \frac{\lambda^N}{\prod_{i=1}^{n} \mu_{1,i} \prod_{j=1}^{N-n} \mu_{2,j}}, \quad (15)$$

which is seen to satisfy the product form requirement (2).

This simple example may be extended to the general network case, using induction and the reasoning of Norton's Theorem for queueing networks. See, for example, [3] and [10].

<u>Theorem 9</u>: The product form network solution implies that each queue in the network has generalized local balance.

This Theorem has also not been demonstrated for all queues, since, for a large number of states, there are many possible state equivalences, and so many possibilities for generalized local balance. However, it has been established for queues with a small number of states.

If a queue has a product form network solution, its marginal state probabilities must have the product form in any network; in particular, it must hold for the simple network of Figure 3. As an example, consider the simple queue of Figure 1a, whose state probabilities are given by (9), with $\lambda_0 = \lambda_1 = \lambda$, when the queue is isolated with random input of mean rate λ. If the input to the network is a random process of mean rate $q\lambda$, the mean throughput of the queue is λ. Since all states of the queue in isolation are possible for the queue in the network, the normalization factor for the product form will be one.

The state diagram for the queue in the network is given in Figure 4. The probability of a departure from the queue, when in S_a, is μ_a, but the probability of the departure from the network is $q\mu_a$. If the customer remains in the network, he returns to the queue in S_a with probability λ_a/λ and to S_b with probability λ_b/λ. Therefore, the total probability of a transition from S_a to S_b is $\eta_a + (1-q)\mu_a \frac{\lambda_b}{\lambda}$. Substituting the rates from Figure 4 for those of Figure 1 in (9) and (10), the following network equilibrium probabilities are obtained:

$$\begin{bmatrix} P_{N0} \\ P_{Na} \\ P_{Nb} \\ P_{N2} \end{bmatrix} = \begin{bmatrix} (\eta_a\mu_b + \eta_b\mu_a + \mu_a\mu_b)\mu_2 + q\lambda(\mu_a\mu_{a|2} + \mu_b\mu_{b|2}) \\ \lambda_a\mu_b\mu_2 + \lambda\eta_b\mu_2 + q\lambda^2\mu_{a|2} \\ \lambda_b\mu_a\mu_2 + \lambda\eta_a\mu_2 + q\lambda^2\mu_{b|2} \\ \lambda(\lambda_a\mu_b + \lambda_b\mu_a) + \lambda^2(\eta_a + \eta_b) + q\lambda^3 \end{bmatrix} \Big/ D_N, \quad (16)$$

with $D_N = (\eta_a\mu_b + \eta_b\mu_a + \mu_a\mu_b)\mu_2 + q\lambda(\mu_a\mu_{a|2} + \mu_b\mu_{b|2})$

$$+ (\lambda_a\mu_b + \lambda_b\mu_a + \lambda(\eta_a + \eta_b))(\lambda + \mu_2) + q\lambda^2(\lambda + \mu_2). \quad (17)$$

If the queue has a product form network solution, then $P_0 = P_{N0}$, $P_a + P_b = P_{Na} + P_{Nb}$, and $P_2 = P_{N2}$. The first relation yields

$$P_0 = [(\eta_a\mu_b + \eta_b\mu_a + \mu_a\mu_b)\mu_2 + \lambda(\mu_a\mu_{a|2} + \mu_b\mu_{b|2})]/D$$

$$= P_{N0} = [(\eta_a\mu_b + \eta_b\mu_a + \mu_a\mu_b)\mu_2 + q\lambda(\mu_a\mu_{a|2} + \mu_b\mu_{b|2})]/D_N. \quad (18)$$

After cancellations, this expression yields

$$(\mu_a\mu_{a|2} + \mu_b\mu_{b|2})(\lambda_a\mu_b + \lambda_b\mu_a + \lambda(\eta_a + \eta_b))$$

$$= \lambda(\eta_a\mu_b + \eta_b\mu_a + \mu_a\mu_b)\mu_2.$$

Multiplying through, this can be placed in the form

$$\lambda_b\mu_a\mu_{a|2}(\mu_a - \mu_b) + \lambda_a\mu_b\mu_{b|2}(\mu_b - \mu_a)$$

$$+ \lambda(\eta_a\mu_{a|2}(\mu_a - \mu_b) + \eta_b\mu_{b|2}(\mu_b - \mu_a)) = 0,$$

which results in either $\mu_a = \mu_b$, or

$$(\lambda_b\mu_a + \lambda\eta_a)\mu_{a|2} = (\lambda_a\mu_b + \lambda\eta_b)\mu_{b|2}. \quad (19)$$

But for the queue, $\mu_a = \mu_b$ is all that is required for the equivalence of S_a and S_b. And (19) is the condition (13) required for local balance. Hence, either S_a and S_b are equivalent, or the queue has local balance. Product form implies generalized local balance.

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