TECHNICAL MEMORANDUM

ANL-CT-76-25

VIBRATION OF A GROUP OF CIRCULAR CYLINDERS IN A CONFINED FLUID

by

Ho Chung and Shoei-sheng Chen

Components Technology Division

ARGONNE NATIONAL LABORATORY

U of C-AUA-USERDA

Base Technology

February 1976

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LMFBR Components (UC-79k)

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PREFACE

The work reported herein was performed as part of the base technology activity under the Flow Induced Vibration Program (189a No. CA054-A and CA070-A) sponsored by ERDA/RDD. The overall objective of the activity is to develop new and/or improved analytical methods and guidelines for designing LMFBR components to avoid detrimental flow induced vibration.

Heat exchanger tubes and reactor fuel pins are long, slender, beam-like components typically arranged in bundles and immersed in a flowing fluid. As such, they are susceptible to flow-induced vibration. The excitation mechanism may be associated with vortex-shedding, fluidelastic interaction, or random pressure fluctuations in the turbulent flow. Designing to avoid large amplitude motion, that is, to avoid a resonance condition or instability condition, and the prediction of component response, require knowledge of the dynamic behavior of the components. However, cylinders in a closely spaced bundle do not respond as single cylinders immersed in a fluid, rather, interaction with the fluid causes coupled motion of groups of cylinders. The fundamental natural frequency of the coupled system will be lower than that of a single cylinder immersed in a fluid.

Understanding and modeling fluid/structure interaction in cylinder bundles is a basic requirement in the development of analytical methods and guidelines for designing heat exchanger and reactor fuel assemblies that are free from component vibration problems. As a step toward satisfying this requirement, in this report a method of analysis based on the potential flow theory is developed for analyzing free vibration of a group of cylinders immersed in a fluid contained in a cylinder. The method of analysis presented can determine the added

mass coefficients and natural frequencies of coupled cylinder-fluid systems. To demonstrate the method the coupled natural frequencies for two eccentrically located cylinders with a fluid-filled gap are calculated and discussed. A few other examples are also given to show the added mass coefficients.

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NOMENCLATURE

a_{in}, b_{in}	Arbitrary constants to be determined in Eq. (17)
C_{mn}	Coefficients in Eqs. (4-7, 12, 13); $(m+n-1)!/[m!(n-1)!]$
D_{mn}	Coefficients in Eqs. (9, 10); $n!/[m!(n-m)!]$
e	Eccentricity for two eccentric cylinders
$\vec{e}_i$, $\vec{e}_\ell$	Unit vectors
E	Modulus of elasticity
f	External loading per unit length
f_{pn}	Generalized force in Eq. (42)
$F_{\ell i}$	Hydrodynamic force acting on cylinder ℓ due to the motion of cylinder i
G	Gap size between cylinders
H_i, G_i	Fluid dynamic forces acting on cylinder i in the x-direction and in the y-direction
h_{pn}	Generalized coordinates in Eqs. (41,42)
$\bar{h}_{pn}$	Normal modes of cylinders
I	Moment of inertia
k	Total number of inner cylinders under consideration
ℓ	Length of cylinder
m_i	Mass per unit length of cylinder i
p	Fluid pressure
r_i, θ_i	Coordinates associated with cylinder i
R_i	Radius of cylinder i
R_{ij}	Distance between the centers of cylinder i and j
S_i	Surface of cylinder i
S, S_r	Surfaces
t	Time
u_i, v_i	Displacements in the x-direction and in the y-direction of cylinder i

U_i	Displacement of cylinder i
V	Fluid velocity component in radial direction
x,y	Cartesian coordinates
x_j, y_j	Local cartesian coordinates associated with cylinder j
X, Y	Coordinates associated with the outer cylinder 0
z	Axial coordinate
$\alpha_{i\ell}, \beta_{i\ell}, \sigma_{i\ell}, \tau_{i\ell}$	Added mass coefficients
$\alpha_{in\ell}, \beta_{in\ell}, \sigma_{in\ell}, \tau_{in\ell}$	Constants determined from Eqs. (18-20)
γ_{pq}	Element of added mass matrix
$\delta_{i\ell}$	Kronecker's delta
ϵ	Nondimensionalized eccentricity; $e/(R_0 - R_1)$
λ_{01}	Coefficient in Eq. (50)
$\vec{\nu}$	Unit vector normal to cylinder surface
ρ	Mass density of fluid
Φ_i	Velocity potential associated with the motion of cylinder i assuming all other cylinders are stationary
Φ	Total velocity potential
ϕ_i	Normalized velocity potential of Φ_i
$\psi_n(z)$	Orthonormal function of cylinders in vacuo
ψ_{ij}	Angle between the x-axis and the vector from the center of cylinder i to that of cylinder j
$\bar{\omega}_{pn}$	nth natural frequency of cylinder p in vacuo
ω_{pn}	nth uncoupled natural frequency of cylinder p in fluid
Ω_{pn}	Coupled natural frequency of the p mode associated with the axial mode number n

Subscripts

$i,j,\ell = 0, 1, 2, \ldots k$

$p,q = 0, 1, 2, \ldots 2k$

$m = 0, 1, 2, \ldots \infty$

$n = 1, 2, 3, \ldots \infty$

Superscripts

i = Variables written in terms of coordinates associated with cylinder i

VIBRATION OF A GROUP OF CIRCULAR CYLINDERS IN A CONFINED FLUID

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ABSTRACT

This report presents an analytical method for evaluating the hydrodynamic masses of a group of circular cylinders immersed in a fluid contained in a cylinder. The analysis is based on the two dimensional potential flow theory. The fluid coupling effect among cylinders is taken into account; self-added masses and mutual-added masses for both inner and outer cylinders are evaluated. Based on the proposed method, the free vibration of two eccentric cylinders with a fluid-filled gap is analyzed as an example.

I. INTRODUCTION

The dynamic behavior of a group of cylinders submerged in a fluid has been of great interest to many engineers in recent years. In a group of cylinders, because of fluid coupling effects, the motion of any cylinder will excite the others and all cylinders will respond as a group rather than as an individual one. Therefore, a group of cylinders will have coupled natural modes in which all cylinders vibrate at the same frequency with definite phase relations among them. Coupled motions of multiple cylinders in infinite fluid are studied by several authors [1-12].* Studies of two cylinders located concentrically with a fluid-filled gap have also been made [13-14]. However, to the best of the authors' knowledge, coupled vibrations of eccentric cylinders and a group of cylinders in a confined fluid have not been reported. The objective of this report is to present a general method of analysis for coupled vibrations of multiple cylinders in a confined fluid. The results of this study have important application in the design evaluation of nuclear reactor internals and heat exchangers.

The fluid is assumed to be incompressible, inviscid and irrotational; thus, the potential flow theory is applied. The cylinders are assumed to be infinitely long and their axes are parallel to one another; i.e., the two dimensional problem is solved. The mathematical technique employed in this report is the method of coordinate transformations [11, 12]. The fluid velocity potential is expressed in terms of a series with unknown coefficients. The velocity potential of one coordinate system is then expressed in terms of the similar potential function of a second coordinate system using coordinate transformations. Unknown expansion coefficients of the series formulation are determined by matrix inversion of a truncated set of infinite equations obtained by imposing the prescribed boundary conditions. These coefficients are then used to calculate fluid pressure and hydrodynamic forces acting on

*Numbers in brackets designate References at end of report.

each cylinder. Once the hydrodynamic forces associated with the motions of each cylinder are determined, coupled vibrations of the cylinders can be studied in a straight forward manner.

After a general method of solution is presented, a few numerical examples are given and discussed in order to demonstrate the method. Numerical results for the added mass coefficients are given for three cases: (1) two eccentric cylinders with a fluid-filled gap, (2) two cylinders in a fluid contained in an outer cylinder, and (3) seven cylinders in a fluid-containing cylinder. The rate of convergence for the added mass coefficients is studied with respect to the number of terms used in calculations and compared for two typical cases with small and large gap sizes between inner cylinders. The effects on the added mass coefficients due to the variation of eccentricity for case (1), and due to the variation of radius of the containing cylinder for case (3) are studied. Natural frequencies and mode shapes for coupled vibrations of two eccentric cylinders with a fluid-filled gap are calculated and discussed.

II. ADDED MASS MATRIX

A. Formulation and Solution

Consider the motion of a group of k circular cylinders in a cylindrical container (Fig. 1). The container is filled with incompressible, inviscid and irrotational fluid. The axes of the cylinders are assumed to be perpendicular to the x-y plane. Let R_0 be the radius of the container and (X,Y) be global coordinates with the origin at the center of the container. Let R_j be the radius of cylinder j and (x_j,y_j) be the local coordinates associated with inner cylinder j.

The velocity potential associated with the motions of these cylinders can be written in terms of their own local coordinates. Assuming all inner cylinders are stationary, the velocity potential associated with the motion of the outer cylinder 0 is

$$\Phi_0 = \sum_{n=1}^{\infty} \frac{r_0^n}{R_0^{n-1}} (a_{0n} \cos n\theta_0 + b_{0n} \sin n\theta_0) \quad . \tag{1}$$

The velocity potential for inner cylinder j assuming all other cylinders are stationary is

$$\Phi_j = \sum_{n=1}^{\infty} \frac{R_j^{n+1}}{r_j^n} (a_{jn} \cos n\theta_j + b_{jn} \sin n\theta_j) \quad , \quad (j = 1, 2, 3, \ldots k) \quad . \tag{2}$$

In equations (1) and (2), (r_0,θ_0) and (r_j,θ_j) are cylindrical coordinates referred to the outer cylinder 0 and inner cylinder j respectively, and a_{0n}, b_{0n}, a_{jn} and b_{jn} are arbitrary constants to be determined. The total velocity potential is the sum of all the partial potentials generated by every cylinder, i.e.,

$$\Phi = \Phi_0 + \sum_{j=1}^{k} \Phi_j \quad . \tag{3}$$

In order to impose the boundary conditions at a particular cylinder, the total velocity potential associated with all cylinders must be expressed

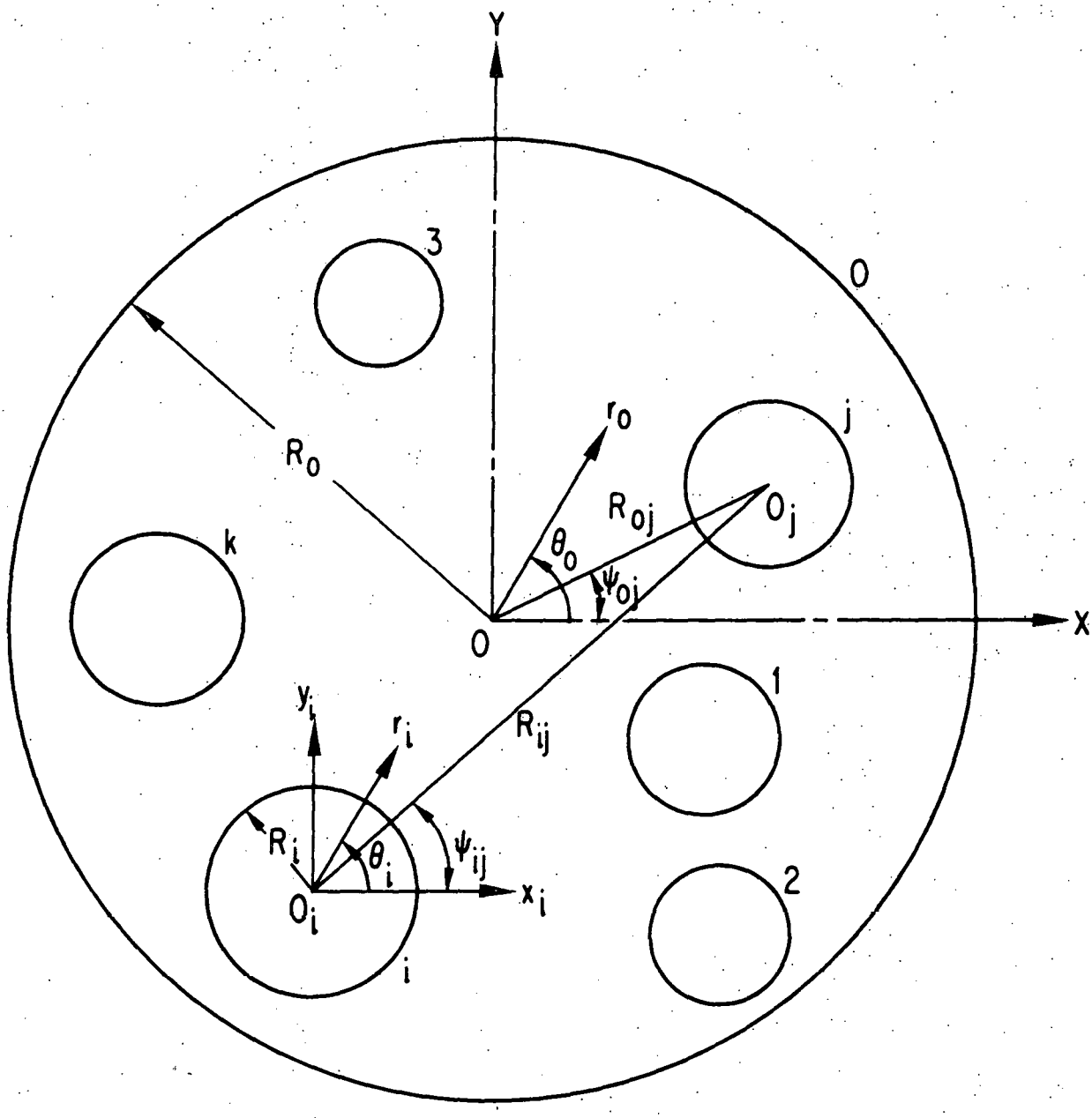

FIGURE 1. Schematic and Coordinate System for a Group of Circular Cylinders in a Fluid-Containing Cylinder

in terms of the coordinates of that cylinder. Therefore, it is necessary to acquire three transformation relationships for coordinates of the outer and inner cylinders; these transformations are accomplished by using Taylor series expansion.

Expansion of the function $[r_0\exp(i\theta_0) - R_{0j}\exp(i\psi_{0j})]^{-n}$ in terms of Taylor series leads to the expressions of the coordinates of inner cylinder j written in terms of (r_0,θ_0) associated with the outer cylinder 0. The above function can be expanded in powers of either $r_0\exp(i\theta_0)$ or $R_{0j}\exp(i\psi_{0j})$, i.e.,

$$\frac{\cos n\theta_j}{r_j^n} = \sum_{m=0}^{\infty} C_{mn} \frac{R_{0j}^m}{r_0^{m+n}} \cos[m\psi_{0j} - (m+n)\theta_0] \quad ,$$

$$\frac{\sin n\theta_j}{r_j^n} = -\sum_{m=0}^{\infty} C_{mn} \frac{R_{0j}^m}{r_0^{m+n}} \sin[m\psi_{0j} - (m+n)\theta_0] \quad , \tag{4}$$

$$(\text{for } n = 1, 2, 3, \ldots,\infty)$$

or

$$\frac{\cos n\theta_j}{r_j^n} = (-1)^n \sum_{m=0}^{\infty} C_{mn} \frac{r_0^m}{R_{0j}^{m+n}} \cos[m\theta_0 - (m+n)\psi_{0j}] \quad ,$$

$$\frac{\sin n\theta_j}{r_j^n} = (-1)^{n+1} \sum_{m=0}^{\infty} C_{mn} \frac{r_0^m}{R_{0j}^{m+n}} \sin[m\theta_0 - (m+n)\psi_{0j}], \tag{5}$$

$$(\text{for } n = 1, 2, 3, \ldots,\infty)$$

where

$$C_{mn} = \frac{(m+n-1)!}{m!(n-1)!}$$

and R_{0j} is the distance between the centers of the outer cylinder 0 and inner cylinder j, and ψ_{0j} is the angle between the x-axis and the vector from the center of the outer cylinder 0 to that of inner cylinder j. Any velocity potential Φ_j in Eq. (2) can be written in terms of the coordinates (r_0,θ_0) associated with the outer cylinder 0 using the above relationships, i.e.,

$$\Phi_j^0 = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} C_{mn} \frac{R_j^{n+1} R_{0j}^m}{r_0^{m+n}} \{a_{jn} \cos[m\psi_{0j} - (m+n)\theta_0]$$

$$- b_{jn} \sin(m\psi_{0j} - (m+n)\theta_0]\} \ , \ (j = 1, 2, \ldots, k) \qquad (6)$$

or

$$\Phi_j^0 = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (-1)^n C_{mn} \frac{R_j^{n+1} r_0^m}{R_{0j}^{m+n}} \{a_{jn} \cos[m\theta_0 - (m+n)\psi_{0j}]$$

$$- b_{jn} \sin[m\theta_0 - (m+n)\psi_{0j}]\} \ , \ (j = 1, 2, \ldots, k) \qquad (7)$$

where the superscript 0 denotes a variable written in terms of the coordinates associated with the outer cylinder 0. If the origin of an inner cylinder j coincides with that of the outer cylinder, then, it becomes

$$\Phi_j^0 = \sum_{n=1}^{\infty} \frac{R_j^{n+1}}{r_0^n} (a_{jn} \cos n\theta_0 + b_{jn} \sin n\theta_0) \ . \qquad (8)$$

Expansion of the function $[r_i \exp(i\theta_i) + R_{0i} \exp(i\psi_{0i})]^n$ in terms of Taylor series leads to the following relationships of the coordinates (r_0, θ_0) of the outer cylinder written in terms of (r_i, θ_i) associated with inner cylinder i, i.e.,

$$r_0^n \cos n\theta_0 = \sum_{m=0}^{n} D_{mn} \frac{R_{0i}^n r_i^m}{R_{0i}^m} \cos[(n-m)\psi_{0i} + m\theta_i] \ ,$$

and (9)

$$r_0^n \sin n\theta_0 = \sum_{m=0}^{n} D_{mn} \frac{R_{0i}^n r_i^m}{R_{0i}^m} \sin[(n-m)\psi_{0i} + m\theta_i] \ ,$$

$$(\text{for } n = 0, 1, 2, \ldots, \infty)$$

where

$$D_{mn} = \frac{n!}{m!(n-m)!} \ .$$

The velocity potential Φ_0 in Eq. (1) written in terms of the local coordinates associated with inner cylinder i can be now written

$$\Phi_0^i = \sum_{n=1}^{\infty} \sum_{m=0}^{n} D_{mn} \frac{r_i^m R_{0i}^{n-m}}{R_0^{n-1}} \{a_{0n}\cos[(n-m)\psi_{0i} + m\theta_i]$$

$$+ b_{0n}\sin[(n-m)\psi_{0i} + m\theta_i]\} \quad , (i = 1, 2, \ldots, k). \quad (10)$$

If the origin of the outer cylinder is the same as that of inner cylinder i, it becomes

$$\Phi_0^i = \sum_{n=1}^{\infty} \frac{r_i^n}{R_0^{n-1}} (a_{0n} \cos n\theta_i + b_{0n} \sin n\theta_i) \quad . \quad (11)$$

Finally the velocity potential Φ_j in Eq. (2) can be expressed in terms of the local coordinates of cylinder i using the following relationships [12].

$$\frac{\cos n\theta_j}{r_j^n} = (-1)^n \sum_{m=0}^{\infty} C_{mn} \frac{r_i^m}{R_{ij}^{m+n}} \cos[m\theta_i - (m+n)\psi_{ij}] \quad ,$$

and (12)

$$\frac{\sin n\theta_j}{r_j^n} = (-1)^{n+1} \sum_{m=0}^{\infty} C_{mn} \frac{r_i^m}{R_{ij}^{m+n}} \sin[m\theta_i - (m+n)\psi_{ij}] \quad ,$$

(for $n = 1, 2, 3, \ldots, \infty$)

where C_{mn} is defines previously, and R_{ij} and ψ_{ij} are defined in Fig. 1. Thus, using Eqs. (2) and (12) gives

$$\Phi_j^i = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (-1)^n C_{mn} \frac{R_j^{n+1} r_i^m}{R_{ij}^{m+n}} \{a_{jn}\cos[m\theta_i - (m+n)\psi_{ij}]$$

$$- b_{jn}\sin[m\theta_i - (m+n)\psi_{ij}]\} \quad , (i,j = 1, 2, \ldots, k). \quad (13)$$

It is necessary to consider two cases: (1) one of the inner cylinders is concentric with the outer cylinder; and (2) none of the inner cylinders are concentric with the outer cylinder. The former is the general case and treated in this report. Thus, the system consists of k+1 cylinders: cylinder 0 is the outer cylinder, cylinder 1 is the inner cylinder which is concentric with the outer cylinder, and the cylinders

$2 \sim k$ are other inner cylinders. Note that case 2 may be treated as a special case of case 1. The total velocity potential (3) can be written as

$$\Phi^i = \Phi_i + \sum_{j=0}^{k}{}^{*} \Phi_j^i , \quad (i = 0,1,2,\ldots,k) \tag{14}$$

where Σ^* denotes the summation for j from 0 to k except j = i.

The displacement components of cylinder i in the x and y directions are u_i and v_i, respectively. The fluid velocity component in radial direction is V and can be written in terms of the coordinates of cylinder i ;

$$V^i = \frac{\partial \Phi^i}{\partial r_i} , \quad (i = 0,1,2,\ldots,k). \tag{15}$$

At the interface of the cylinders and fluid, the following conditions must be satisfied;

$$V^i\Big|_{r_i = R_i} = \frac{\partial u_i}{\partial t} \cos\theta_i + \frac{\partial v_i}{\partial t} \sin\theta_i , \quad (i = 0,1,2,\ldots,k) . \tag{16}$$

Let the coefficients a_{in} and b_{in} be defined as follows:

$$\begin{aligned} a_{in} &= \sum_{\ell=0}^{k} \left(\alpha_{in\ell} \frac{\partial u_\ell}{\partial t} + \sigma_{in\ell} \frac{\partial v_\ell}{\partial t} \right) , \\ b_{in} &= \sum_{\ell=0}^{k} \left(\tau_{in\ell} \frac{\partial u_\ell}{\partial t} + \beta_{in\ell} \frac{\partial v_\ell}{\partial t} \right) , \\ &(i = 0, 1, 2, \ldots, k) , \end{aligned} \tag{17}$$

where, $\alpha_{in\ell}$, $\beta_{in\ell}$, $\sigma_{in\ell}$ and $\tau_{in\ell}$ are solutions of the following system of algebraic equations. For i = 0, substituting equations (1), (2) and (6) into (14-16) yields

$$\begin{aligned} &n\alpha_{On\ell} - n\left(\frac{R_1}{R_0}\right)^{n+1} \alpha_{1n\ell} - \sum_{j=2}^{k} \sum_{m=0}^{n-1} \frac{R_j^{n-m+1} R_{0j}^m}{R_0^{n+1}} \frac{n!}{m!(n-m-1)!} \\ &\{\alpha_{j(n-m)\ell} \cos m\psi_{0j} - \tau_{j(n-m)\ell} \sin m\psi_{0j}\} = \delta_{n1}\delta_{0\ell} , \end{aligned} \tag{18}$$

$$n\sigma_{0n\ell} - n\left(\frac{R_1}{R_0}\right)^{n+1}\sigma_{1n\ell} - \sum_{j=2}^{k}\sum_{m=0}^{n-1}\frac{R_j^{n-m+1}R_{0j}^{m}}{R_0^{n+1}}\frac{n!}{m!(n-m-1)!}$$

$$\{\sigma_{j(n-m)\ell}\cos m\psi_{0j} - \beta_{j(n-m)\ell}\sin m\psi_{0j}\} = 0 \quad ,$$

$$n\tau_{0n\ell} - n\left(\frac{R_1}{R_0}\right)^{n+1}\tau_{1n\ell} - \sum_{j=2}^{k}\sum_{m=0}^{n-1}\frac{R_j^{n-m+1}R_{0j}^{m}}{R_0^{n+1}}\frac{n!}{m!(n-m-1)!} \qquad \text{(Contd.)}$$

$$\{\alpha_{j(n-m)\ell}\sin m\psi_{0j} + \tau_{j(n-m)\ell}\cos m\psi_{0j}\} = 0 \quad ,$$

$$n\beta_{0n\ell} - n\left(\frac{R_1}{R_0}\right)^{n+1}\beta_{1n\ell} - \sum_{j=2}^{k}\sum_{m=0}^{n-1}\frac{R_j^{n-m+1}R_{0j}^{m}}{R_0^{n+1}}\frac{n!}{m!(n-m-1)!}$$

$$\{\sigma_{j(n-m)\ell}\sin m\psi_{0j} + \beta_{j(n-m)\ell}\cos m\psi_{0j}\} = \delta_{n1}\delta_{0\ell} \quad , \qquad (18)$$

where

$\ell = 0, 1, 2, \ldots, k$

$n = 1, 2, 3, \ldots, \infty$

$m = 0, 1, 2, \ldots, (n-1)$.

For i = 1, substituting equations (1), (2) and (7) into (14)-(16) yields

$$n\left(\frac{R_1}{R_0}\right)^{n-1}\alpha_{0n\ell} - n\alpha_{1n\ell} + \sum_{j-2}^{k*}\sum_{m=1}^{\infty}\frac{R_1^{n-1}R_j^{m+1}}{R_{0j}^{m+n}}\frac{(-1)^m(m+n-1)!}{(m-1)!(n-1)!}$$

$$\{\alpha_{jm\ell}\cos(m+n)\psi_{0j} + \tau_{jm\ell}\sin(m+n)\psi_{0j}\} = \delta_{n1}\delta_{1\ell} \quad ,$$

$$n\left(\frac{R_1}{R_0}\right)^{n-1}\sigma_{0n\ell} - n\sigma_{1n\ell} + \sum_{j=2}^{k*}\sum_{m=1}^{\infty}\frac{R_1^{n-1}R_j^{m+1}}{R_{0j}^{m+n}}\frac{(-1)^m(m+n-1)!}{(m-1)!(n-1)!}$$

$$\{\sigma_{jm\ell}\cos(m+n)\psi_{0j} + \beta_{jm\ell}\sin(m+n)\psi_{0j}\} = 0 \quad ,$$

$$n\left(\frac{R_1}{R_0}\right)^{n-1}\tau_{0n\ell} - n\tau_{1n\ell} + \sum_{j=2}^{k*}\sum_{m=1}^{\infty}\frac{R_1^{n-1}R_j^{m+1}}{R_{0j}^{m+n}}\frac{(-1)^m(m+n-1)!}{(m-1)!(n-1)!} \qquad (19)$$

$$\{\alpha_{jm\ell}\sin(m+n)\psi_{0j} - \tau_{jm\ell}\cos(m+n)\psi_{0j}\} = 0 \quad ,$$

$$n\left(\frac{R_1}{R_0}\right)^{n-1}\beta_{On\ell} - n\beta_{1n\ell} + \sum_{j=2}^{k}{}^{*}\sum_{m=1}^{\infty}\frac{R_1^{n-1}R_j^{m+1}}{R_{Oj}^{m+n}}\frac{(-1)^m(m+n-1)!}{(m-1)!(n-1)!} \qquad (19)$$

(Contd.)

$$\{\sigma_{jm\ell}\sin(m+n)\psi_{Oj} - \beta_{jm\ell}\cos(m+n)\psi_{Oj}\} = \delta_{n1}\delta_{1\ell} \quad ,$$

where

$\ell = 0, 1, 2, \ldots, k$

$m,n = 1, 2, 3, \ldots, \infty$.

For $i \geq 2$, substituting equations (2), (10) and (13) into (14)-(16) yields

$$-n\alpha_{in\ell} + \sum_{m=1}^{\infty}\frac{R_{Oi}^{m-n}R_i^{n-1}}{R_O^{m-1}}\frac{m!}{(n-1)!(m-n)!}\{\alpha_{Om\ell}\cos(m-n)\psi_{Oi}$$

$$+\tau_{Om\ell}\sin(m-n)\psi_{Oi}\} + \sum_{j=1}^{k}{}^{*}\sum_{m=1}^{\infty}\frac{R_i^{n-1}R_j^{m+1}}{R_{ij}^{m+n}}\frac{(-1)^m(m+n-1)!}{(m-1)!(n-1)!}$$

$$\{\alpha_{jm\ell}\cos(m+n)\psi_{ij} + \tau_{jm\ell}\sin(m+n)\psi_{ij}\} = \delta_{n1}\delta_{i\ell} \quad ,$$

$$-n\sigma_{in\ell} + \sum_{m=1}^{\infty}\frac{R_{Oi}^{m-n}R_i^{n-1}}{R_O^{m-1}}\frac{m!}{(n-1)!(m-n)!}\{\sigma_{Om\ell}\cos(m-n)\psi_{Oi}$$

$$+\beta_{Om\ell}\sin(m-n)\psi_{Oi}\} + \sum_{j=1}^{k}{}^{*}\sum_{m=1}^{\infty}\frac{R_i^{n-1}R_j^{m+1}}{R_{ij}^{m+n}}\frac{(-1)^m(m+n-1)!}{(m-1)!(n-1)!} \qquad (20)$$

$$\{\sigma_{jm\ell}\cos(m+n)\psi_{ij} + \beta_{jm\ell}\sin(m+n)\psi_{ij}\} = 0 \quad ,$$

$$-n\tau_{in\ell} + \sum_{m=1}^{\infty}\frac{R_{Oi}^{m-n}R_i^{n-1}}{R_O^{m-1}}\frac{m!}{(n-1)!(m-n)!}\{-\alpha_{Om\ell}\sin(m-n)\psi_{Oi}$$

$$+\tau_{Om\ell}\cos(m-n)\psi_{Oi}\} + \sum_{j=1}^{k}{}^{*}\sum_{m=1}^{\infty}\frac{R_i^{n-1}R_j^{m+1}}{R_{ij}^{m+n}}\frac{(-1)^m(m+n-1)!}{(m-1)!(n-1)!}$$

$$\{\alpha_{jm\ell}\sin(m+n)\psi_{ij} - \tau_{jm\ell}\cos(m+n)\psi_{ij}\} = 0 \quad ,$$

$$-n\beta_{in\ell} + \sum_{m=1}^{\infty} \frac{R_{Oi}^{m-n} R_i^{n-1}}{R_O^{m-1}} \frac{m!}{(n-1)!(m-n)!} \{-\sigma_{Om\ell} \sin(m-n)\psi_{Oi} \qquad (20)$$
(Contd.)

$$+ \beta_{Om\ell} \cos(m-n)\psi_{Oi}\} + \sum_{j=1}^{k*} \sum_{m=1}^{\infty} \frac{R_i^{n-1} R_j^{m+1}}{R_{ij}^{m+n}} \frac{(-1)^m (m+n-1)!}{(m-1)!(n-1)!}$$

$$\{\sigma_{jm\ell} \sin(m+n)\psi_{ij} - \beta_{jm\ell} \cos(m+n)\psi_{ij}\} = \delta_{n1}\delta_{i\ell} \quad ,$$

where

$i = 2, 3, 4, \ldots, k$

$\ell = 0, 1, 2, \ldots, k$

$m, n = 1, 2, 3, \ldots, \infty$.

For the special case in which there is no concentric inner cylinder with the outer cylinder, equation (19) is not needed. Equations (18)-(20) form the systems of infinite equations for infinite numbers of the undetermined coefficients: $\alpha_{in\ell}$, $\beta_{in\ell}$, $\delta_{in\ell}$, and $\tau_{in\ell}$. Taking a finite number of terms in the expressions of velocity potentials and coordinate transformations leads to the systems of finite equations. It will be shown that fast convergence is achieved and accurate solutions can be obtained with a few terms.

The fluid forces acting on the cylinders can be calculated from fluid pressure p;

$$p = -\rho \frac{\partial \Phi}{\partial t} \quad , \qquad (21)$$

where ρ is mass denisty of fluid. The two components in the x and y directions of fluid force acting on cylinder i are H_i and G_i;

$$H_i = -\int_0^{2\pi} p^i \Big|_{r_i=R_i} \cdot R_i \cos\theta_i d\theta_i \quad ,$$

and (22)

$$G_i = -\int_0^{2\pi} p^i \Big|_{r_i=R_i} \cdot R_i \sin\theta_i d\theta_i \quad .$$

$(i = 0, 1, 2, \ldots, k)$

Using equations (14), and (17)-(22) yields

$$H_i = -\rho\pi \sum_{\ell=0}^{k} \left(\frac{R_i + R_\ell}{2}\right)^2 \left(\alpha_{i\ell} \frac{\partial^2 u_\ell}{\partial t^2} + \sigma_{i\ell} \frac{\partial^2 v_\ell}{\partial t^2}\right) ,$$

and

$$G_i = -\rho\pi \sum_{\ell=0}^{k} \left(\frac{R_i + R_\ell}{2}\right)^2 \left(\tau_{i\ell} \frac{\partial^2 u_\ell}{\partial t^2} + \beta_{i\ell} \frac{\partial^2 v_\ell}{\partial t^2}\right) . \qquad (23)$$

$$(i = 0, 1, 2, \ldots, k)$$

Here the so-called added mass coefficients $\alpha_{i\ell}$, $\beta_{i\ell}$, $\sigma_{i\ell}$ and $\tau_{i\ell}$ are obtained as follows:

(1) For the outer cylinder (i = 0),

$$\alpha_{0\ell} = \frac{4R_0^2}{(R_0+R_\ell)^2} \sum_{i=0}^{k} \left(\frac{R_i}{R_0}\right)^2 \alpha_{i1\ell} ,$$

$$\beta_{0\ell} = \frac{4R_0^2}{(R_0+R_\ell)^2} \sum_{i=0}^{k} \left(\frac{R_i}{R_0}\right)^2 \beta_{i1\ell} ,$$

$$\sigma_{0\ell} = \frac{4R_0^2}{(R_0+R_\ell)^2} \sum_{i=0}^{k} \left(\frac{R_i}{R_0}\right)^2 \sigma_{i1\ell} , \qquad (24)$$

$$\tau_{0\ell} = \frac{4R_0^2}{(R_0+R_\ell)^2} \sum_{i=0}^{k} \left(\frac{R_i}{R_0}\right)^2 \tau_{i1\ell} ,$$

$$(\ell = 0, 1, 2, \ldots, k) .$$

(2) For the inner cylinder (i = 1) concentric with the outer cylinder,

$$\alpha_{1\ell} = -\frac{4R_1^2}{(R_1+R_\ell)^2} \left[\alpha_{01\ell} + \alpha_{11\ell} + \sum_{n=1}^{\infty} \sum_{j=2}^{k} (-1)^n n \left(\frac{R_j}{R_{0j}}\right)^{n+1} \right.$$
$$\left. \{\alpha_{jn\ell} \cos (n+1)\psi_{0j} + \tau_{jn\ell} \sin (n+1)\psi_{0j}\}\right] ,$$

$$\beta_{1\ell} = -\frac{4R_1^2}{(R_1+R_\ell)^2} \left[\beta_{01\ell} + \beta_{11\ell} + \sum_{n=1}^{\infty} \sum_{j=2}^{k} (-1)^n n \left(\frac{R_j}{R_{0j}}\right)^{n+1} \right.$$
$$\left. \{\sigma_{jn\ell} \sin (n+1)\psi_{0j} - \beta_{jn\ell} \cos (n+1)\psi_{0j}\}\right] , \qquad (25)$$

$$\sigma_{1\ell} = -\frac{4R_1^2}{(R_1+R_\ell)^2}\left[\sigma_{01\ell} + \sigma_{11\ell} + \sum_{n=1}^{\infty}\sum_{j=2}^{k}(-1)^n n\left(\frac{R_j}{R_{0j}}\right)^{n+1}\right.$$

$$\left.\{\sigma_{jn\ell}\cos(n+1)\psi_{0j} + \beta_{jn\ell}\sin(n+1)\psi_{0j}\}\right], \qquad (25)\ \text{(Contd.)}$$

$$\tau_{1\ell} = -\frac{4R_1^2}{(R_1+R_\ell)^2}\left[\tau_{01\ell} + \tau_{11\ell} + \sum_{n=1}^{\infty}\sum_{j=2}^{k}(-1)^n n\left(\frac{R_j}{R_{0j}}\right)^{n+1}\right.$$

$$\left.\{\alpha_{jn\ell}\sin(n+1)\psi_{0j} - \tau_{jn\ell}\cos(n+1)\psi_{0j}\}\right],$$

$$(\ell = 0, 1, 2, \ldots, k)\ .$$

(3) For inner cylinders, i = 2, 3, 4, ...,k,

$$\alpha_{i\ell} = \frac{4R_i^2}{(R_i+R_\ell)^2}\left[-\alpha_{il\ell} - \sum_{n=1}^{\infty} n\left(\frac{R_{0i}}{R_0}\right)^{n-1}\{\alpha_{0n\ell}\cos(n-1)\psi_{0i}\right.$$

$$\left. + \tau_{0n\ell}\sin(n-1)\psi_{0i}\} - \sum_{n=1}^{\infty}\sum_{j=1}^{k*}(-1)^n n\left(\frac{R_j}{R_{ij}}\right)^{n+1}\right.$$

$$\left.\{\alpha_{jn\ell}\cos(n+1)\psi_{ij} + \tau_{jn\ell}\sin(n+1)\psi_{ij}\}\right],$$

$$\beta_{i\ell} = \frac{4R_1^2}{(R_i+R_\ell)^2}\left[-\beta_{il\ell} - \sum_{n=1}^{\infty} n\left(\frac{R_{0i}}{R_0}\right)^{n-1}\{-\sigma_{0n\ell}\sin(n-1)\psi_{0i}\right.$$

$$\left. + \beta_{0n\ell}\cos(n-1)\psi_{0i}\} - \sum_{n=1}^{\infty}\sum_{j=1}^{k*}(-1)^n n\left(\frac{R_j}{R_{ij}}\right)^{n+1}\right.$$

$$\left.\{\sigma_{jn\ell}\sin(n+1)\psi_{ij} - \beta_{jn\ell}\cos(n+1)\psi_{ij}\}\right], \qquad (26)$$

$$\sigma_{i\ell} = \frac{4R_i^2}{(R_i+R_\ell)^2}\left[-\sigma_{il\ell} - \sum_{n=1}^{\infty} n\left(\frac{R_{0i}}{R_0}\right)^{n-1}\{\sigma_{0n\ell}\cos(n-1)\psi_{0i}\right.$$

$$\left. + \beta_{0n\ell}\sin(n-1)\psi_{0i}\} - \sum_{n=1}^{\infty}\sum_{j=1}^{k*}(-1)^n n\left(\frac{R_j}{R_{ij}}\right)^{n+1}\right.$$

$$\left.\{\sigma_{jn\ell}\cos(n+1)\psi_{ij} + \beta_{jn\ell}\sin(n+1)\psi_{ij}\}\right],$$

$$\tau_{i\ell} = \frac{4R_i^2}{(R_i+R_\ell)^2}\left[-\tau_{il\ell} - \sum_{n=1}^{\infty} n\left(\frac{R_{0i}}{R_0}\right)^{n-1}\{-\alpha_{0n\ell}\sin(n-1)\psi_{0i}\right.$$

$$+ \tau_{0n\ell} \cos (n-1)\psi_{0i}\} - \sum_{n=1}^{\infty} \sum_{j=1}^{k*} (-1)^n n \left(\frac{R_j}{R_{ij}}\right)^{n+1} \{\alpha_{jn\ell} \sin (n+1)\psi_{ij} - \tau_{jn\ell} \cos (n+1)\psi_{ij}\} \Bigg] \quad (26)$$

(Contd.)

$$(\ell = 0, 1, 2, 3, \ldots, k)$$

α_{ii}, β_{ii}, σ_{ii} and τ_{ii} are self-added mass coefficients, which are proportional to the hydrodynamic force acting on cylinder i due to its own acceleration, while the others are mutual-added mass coefficients, which are proportional to the hydrodynamic force acting on a cylinder due to the acceleration of another cylinder.

B. Reciprocal Relations

Useful general reciprocal relationships regarding the hydrodynamic forces due to the movement of cylinders can be developed based on the theory of potential [15].

Let us consider the hydrodynamic force acting on cylinder ℓ in the direction of $\vec{e}_\ell$ due to the movement of cylinder i in the direction of $\vec{e}_i$. Assuming all other cylinders are stationary except cylinder i, the hydrodynamic force can be written as follows,

$$F_{\ell i} = -\gamma_{\ell i} \frac{\partial^2 U_i}{\partial t^2} \quad , \tag{27}$$

where

$$\gamma_{\ell i} = -\iint_{S_\ell} \rho \phi_i \frac{\partial \phi_\ell}{\partial \nu} \, dS_\ell \quad . \tag{28}$$

Here, ϕ_i and ϕ_ℓ are normalized velocity potentials associated with cylinder i and ℓ, (ϕ_i is the velocity potential for cylinder i having a unit acceleration) and U_i is the displacement of cylinder i in the direction of the vector $\vec{e}_i$. ν is a unit vector normal to the cylinder surface toward the fluid and S_ℓ is the surface of cylinder ℓ.

Similarly, consider the case that all cylinders are stationary except cylinder ℓ moving in the direction of $\vec{e}_\ell$. The hydrodynamic force exerting on cylinder i in the direction of $\vec{e}_i$ is

$$F_{i\ell} = -\gamma_{i\ell} \frac{\partial^2 U_\ell}{\partial t^2} \quad , \tag{29}$$

where

$$\gamma_{i\ell} = -\iint_{S_i} \rho \phi_\ell \frac{\partial \phi_i}{\partial \nu} \, dS_i \quad . \tag{30}$$

As the velocity potentials ϕ_i and ϕ_ℓ are harmonic functions, Green's theorem prevails [15], i.e.,

$$\iint_S \left(\phi_i \frac{\partial \phi_\ell}{\partial \nu} - \phi_\ell \frac{\partial \phi_i}{\partial \nu} \right) dS = 0 \quad . \tag{31}$$

This equation holds for any closed surface S within a region in which ϕ_i and ϕ_ℓ are harmonic functions. We may choose $S = S_0$ which is the inner surface of the outer cylinder, then it follows that

$$\iint_{S_\ell} \phi_i \frac{\partial \phi_\ell}{\partial \nu} \, dS_\ell - \iint_{S_i} \phi_\ell \frac{\partial \phi_i}{\partial \nu} \, dS_i$$

$$= \iint_{S_0} \left(\phi_i \frac{\partial \phi_\ell}{\partial \nu} - \phi_\ell \frac{\partial \phi_i}{\partial \nu} \right) dS_0 = 0 \quad . \tag{32}$$

If ϕ_ℓ is associated with the outer cylinder, letting $S = S_0$ in equation (30) yields

$$\iint_{S_0} \phi_i \frac{\partial \phi_0}{\partial \nu} dS_0 - \iint_{S_i} \phi_0 \frac{\partial \phi_i}{\partial \nu} dS_i = 0 \quad . \tag{33}$$

Therefore from equations (28), (30), (32) and (33)

$$\gamma_{i\ell} = \gamma_{\ell i} \qquad (i,\ell = 0, 1, 2, \ldots k) \quad . \tag{34}$$

This relationship delivers a physical meaning that the hydrodynamic force acting on cylinder ℓ in the $\vec{e}_\ell$ direction due to a unit acceleration of cylinder i in the $\vec{e}_i$ direction is equal to the hydrodynamic force acting on cylinder i in the $\vec{e}_i$ direction due to a unit acceleration of cylinder ℓ in the $\vec{e}_\ell$ direction. This general reciprocal relationship is valid between any pair of cylinders including inner and outer cylinders.

The reciprocal relations of the added mass coefficients defined in section II-A are generated from equation (34) as special cases. When $\vec{e}_i$ and $\vec{e}_\ell$ are equal, and if they are in x-direction

$$\alpha_{i\ell} = \alpha_{\ell i} \quad , \tag{35}$$

if they are in y-direction

$$\beta_{i\ell} = \beta_{\ell i} \quad . \tag{36}$$

When $\vec{e}_i$ is in x-direction and $\vec{e}_\ell$ is in y-direction, or vice versa

$$\sigma_{i\ell} = \tau_{\ell i} \quad . \tag{37}$$

The added mass coefficients $\alpha_{i\ell}$, $\beta_{i\ell}$, $\sigma_{i\ell}$ and $\tau_{i\ell}$ can be combined to form an added mass matrix γ_{pq}, i.e.,

$$[\gamma_{pq}] = \left[\begin{array}{c|c} \rho\pi \left(\frac{R_i + R_\ell}{2}\right)^2 \alpha_{i\ell} & \rho\pi \left(\frac{R_i + R_\ell}{2}\right)^2 \sigma_{i\ell} \\ \hline \rho\pi \left(\frac{R_i + R_\ell}{2}\right)^2 \tau_{i\ell} & \rho\pi \left(\frac{R_i + R_\ell}{2}\right)^2 \beta_{i\ell} \end{array}\right] \tag{38}$$

$$(p,q = 0, 1, 2, \ldots, 2k + 1).$$

Since γ_{pq} is symmetric, for a group of k cylinders contained in an outer cylinder, there are $(k + 1)(2k + 3)$ independent added mass components. We

can also find a group of 2(k+1) axes as the principal axes of added mass coefficient such that

$$\gamma_{pq} = 0 \quad \text{for } p \neq q \quad . \tag{39}$$

The principal values of added mass matrix may be called effective added masses for the system [12].

C. Numerical Results

The preceding analytical method to evaluate the added mass matrix was programmed (AMASS) on a digital computer and listed in the Appendix of this report. This program can be used to calculate all elements of the added mass matrix for a group of cylinders submerged in a fluid-containing cylinder. The cylinders may have different radii and may be arranged in arbitrary pattern; however, all cylinders must be parallel to one another.

Added mass coefficients are written in terms of infinite series; only a limited number of terms is needed to give results with sufficient accuracy. In order to check the rate of convergence of the solution, the added mass coefficients for two cylinders submerged in a fluid-containing cylinder are shown in Fig. 2 as functions of the number of terms n taken in calculations. The rate of convergence is much faster in the case of widely-spaced inner cylinders than in the closely-spaced case. In general, n = 10 will give sufficiently accurate solutions in practical situations. In the following calculations, the number of terms are taken to be 10.

Table 1 and Fig. 3 show the added mass coefficients as functions of the eccentricity of two eccentric cylinders with a fluid-filled gap. As the eccentricity increases the magnitude of all added mass coefficients increase. It is interesting to note that the self-added mass coefficients of the inner cylinder are strongly influenced by eccentricity and the mutual-added mass coefficients have also considerable dependence on eccentricity. However, the variation of the self-added mass coefficients

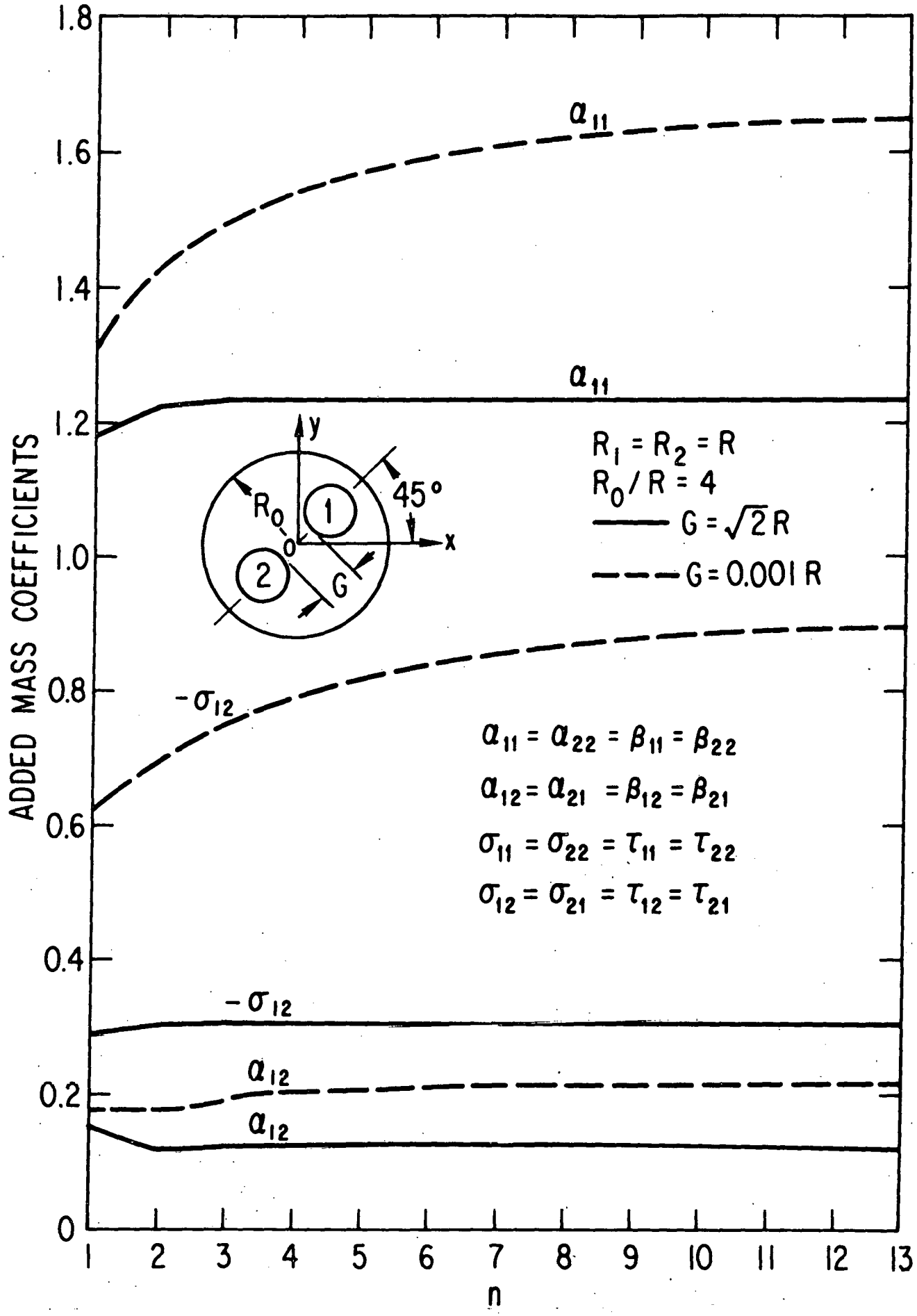

FIGURE 2. Variations of Added Mass Coefficients with the Number of Terms n Used in Calculations

Table 1. Added Mass Coefficients for Two Eccentric Cylinders with a Fluid-Filled Gap

Eccentricity (ε)	α_{00}	α_{11}	α_{12}
0	1.667	1.667	-1.185
0.2	1.672	1.686	-1.194
0.4	1.688	1.751	-1.223
0.6	1.722	1.889	-1.284
0.8	1.799	2.196	-1.420
0.9	1.884	2.537	-1.572
0.95	1.966	2.865	-1.718
0.97	2.020	3.080	-1.813
0.99	2.104	3.414	-1.962
0.999	2.170	3.681	-2.081

where $R_o/R_1 = 2$, $\varepsilon = e/(R_o - R_1)$

$\alpha_{ij} = \beta_{ij}$, $\sigma_{ij} = \tau_{ij} = 0$, $(i,j = 0,1)$

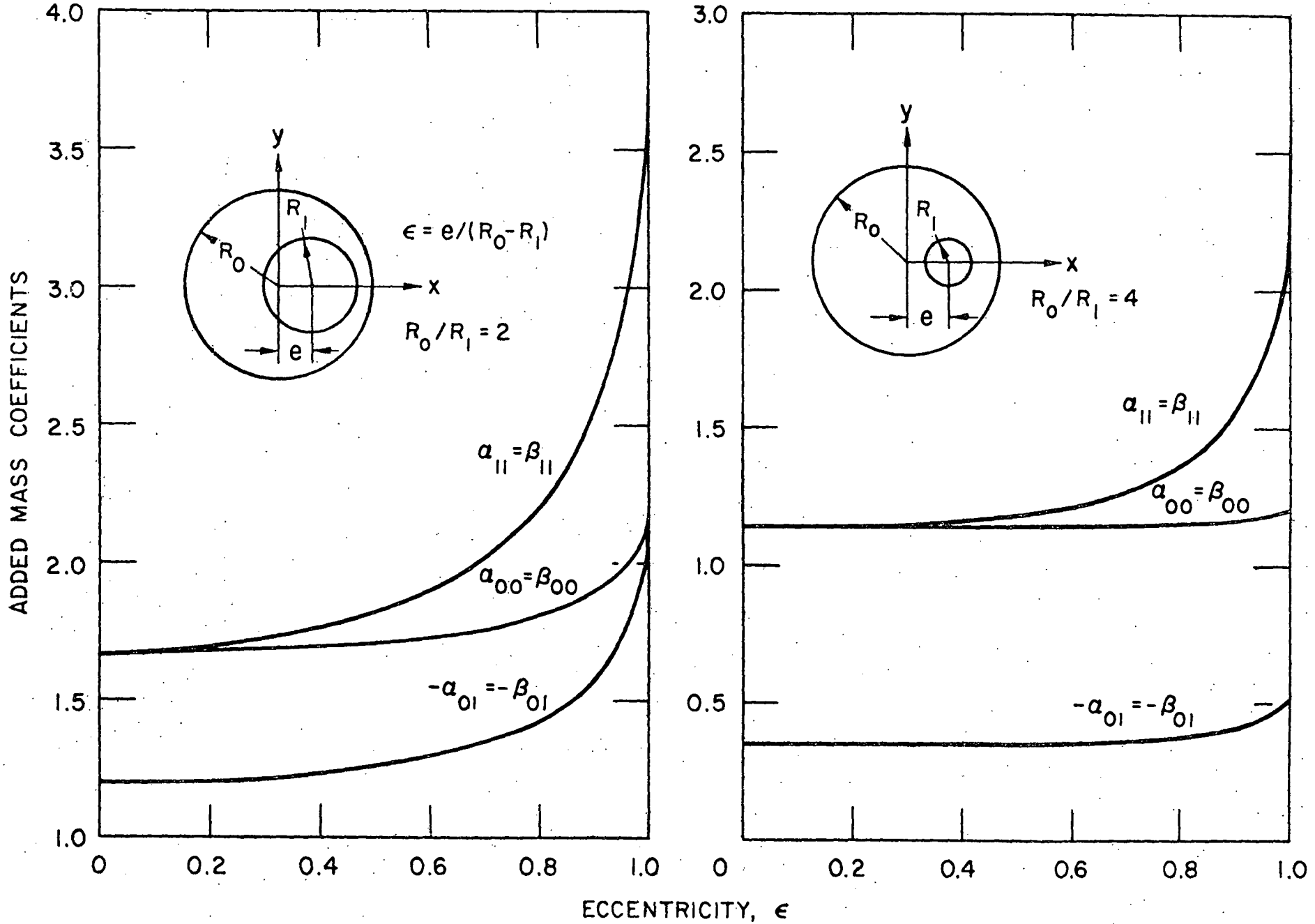

FIGURE 3. Added Mass Coefficients for Two Eccentric Cylinders with a Fluid-Filled Gap

of outer cylinder with the eccentricity is relatively small, and is negligible in the case of $R_0/R_1 = 4$.

The added mass coefficients for a seven-cylinder bundle in a fluid-containing cylinder are shown in Table 2 and Fig. 4. The arrangement of the inner seven cylinders is fixed, while the radius of the outer cylinder is varied in order to investigate the influence of the gap size on the added mass coefficients. The self-added mass coefficients associated with the innermost cylinder 1, the inner cylinder 4 and the outer cylinder 0 are presented. As the gap size increases all of them decrease; specifically α_{oo} approaches 1, whereas α_{11} and α_{44} approach the asympotic values for the case of seven cylinders in an infinite fluid medium. It is interesting to note that the sensitivities of α_{11} and α_{44} to variations of the gap size are very similar to each other even though the cylinder 1 is surrounded by more fixed cylinders. Among the self-added mass coefficients considered, α_{00} is most sensitive to gap size. The mutual added mass coefficients, α_{01}, α_{04} and α_{14} are also presented. As the gap size increases the magnitudes of α_{01} and α_{04} decrease and finally approach zero, yet the magnitude of α_{14} increases and approaches the asymptote for the case of infinite fluid. These results reveal that as the gap size increases the coupling between the inner cylinders and the outer cylinder becomes smaller, and the coupling between inner cylinders 1 and 4 becomes larger. This study also shows that the solutions based on the assumption of infinite fluid medium are not sufficiently accurate for the case of G/R, say less than 5.

Table 2. Variation of Added Mass Coefficients for a Seven Cylinder Bundle in a Fluid-Containing Cylinder

G/R	α_{00}	α_{11}	α_{44}	α_{01}	α_{04}	α_{14}
0.1	7.084	5.129	4.070	-2.211	-1.545	-0.944
0.2	5.571	4.826	3.712	-1.699	-1.200	-1.114
0.3	4.708	4.653	3.510	-1.402	-0.997	-1.214
0.4	4.127	4.538	3.378	-1.200	-0.857	-1.284
0.5	3.702	4.454	3.284	-1.051	-0.753	-1.335
0.6	3.375	4.389	3.214	-0.935	-0.672	-1.375
0.7	3.115	4.339	3.159	-0.843	-0.607	-1.408
0.8	2.902	4.297	3.116	-0.766	-0.553	-1.435
0.9	2.725	4.263	3.081	-0.702	-0.507	-1.458
1.0	2.575	4.234	3.052	-0.647	-0.468	-1.477
1.5	2.075	4.139	2.962	-0.461	-0.334	-1.542
2.0	1.795	4.086	2.916	-0.353	-0.256	-1.580
3.0	1.496	4.030	2.871	-0.233	-0.169	-1.620
4.0	1.344	4.002	2.851	-0.168	-0.122	-1.641
5.0	1.254	3.985	2.839	-0.128	-0.093	-1.653
10.0	1.090	3.954	2.820	-0.049	-0.036	-1.675
20.0	1.028	3.943	2.814	-0.016	-0.012	-1.683
30.0	1.014	3.940	2.812	-0.008	-0.006	-1.685
50.0	1.002	3.938	2.811	-0.001	-0.001	-1.686
100.0	1.001	3.937	2.811	-0.0007	-0.0009	-1.687
∞	1.000	3.936	2.810	0	0	-1.687

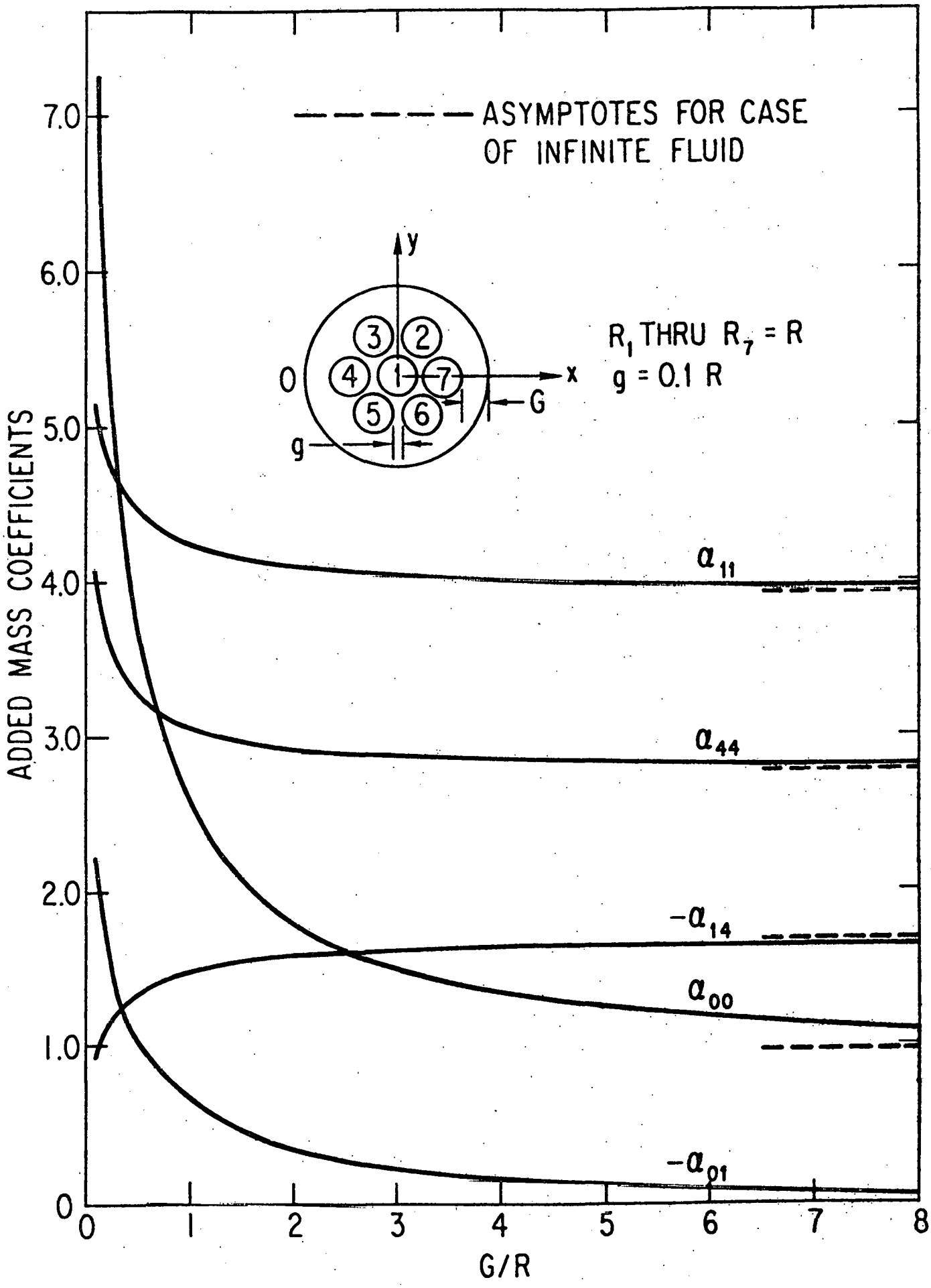

FIGURE 4. Added Mass Coefficients for Two Eccentric Cylinders with a Fluid-Filled Gap

III. VIBRATION OF A GROUP OF CYLINDERS

As discussed in earlier chapters, added mass matrix can accurately be obtained for a group of circular cylinders submerged in a confined liquid; the cylinders may have different radii and can be arranged in an arbitrary manner as long as they are parallel to one another. The added mass matrix can be used to study the free and forced vibration of cylinders. In this section, a frequency equation is derived for a group of cylinders with the same boundary conditions and the same length, and numerical results are presented for free vibration of two eccentric cylinders with a fluid-filled gap.

A. Frequency Equation

Motions of cylinders consist of displacements along the x-axis and y-axis. Assuming the fluid is stationary and neglecting the effects of fluid static pressure, damping, drag force and gravity effect the equations of motions for a group of k cylinders in a fluid-containing cylinder are as follows:

$$E_p I_p \frac{\partial^4 u_p}{\partial z^4} + m_p \frac{\partial^2 u_p}{\partial t^2} + \sum_{q=0}^{2k+1} \gamma_{pq} \frac{\partial^2 u_q}{\partial t^2} = f_p \quad , \tag{40}$$

$$(p,q = 0, 1, 2, \ldots, 2k + 1)$$

where the subscripts p and q denote variables associated with cylinders p and q in the x-z plane, while p+k+1 and q+k+1 in the y-z plane. $E_p I_p$ is flexural rigidity, m_p is mass per unit length, f_p is external excitation force, and γ_{pq} is the element of added mass matrix defined in equation (38).

When all cylinders are of the same length and have the same type of boundary conditions, the displacement of any cylinder can be written as

$$u_p(z,t) = \sum_{n=1}^{\infty} h_{pn}(t)\psi_n(z) \quad , \tag{41}$$

where $\psi_n(z)$ is the nth orthonormal function of the cylinders in vacuo.

Using equations (40) and (41) yields

$$m_p \ddot{h}_{pn} + \sum_{q=0}^{2k} \gamma_{pq} \ddot{h}_{qn} + m_p \bar{\omega}_{pn}^2 h_{pn} = f_{pn} \quad , \tag{42}$$

$$(p,q = 0, 1, 2, ..,2k+1), \quad (n = 1, 2, 3, ...,\infty)$$

where

$$f_{pn} = \frac{1}{\ell} \int_o^\ell f_p \psi_n dz \quad , \qquad \bar{\omega}_{pn}^2 = \frac{E_p I_p}{m_p} \frac{\psi_n''''}{\psi_n} \quad .$$

Here $\bar{\omega}_{pn}$ is the nth natural frequency of the cylinder p in vacuo and f_{pn} is the generalized external force. Note that equations (42) can be applied to all values of n. For each n, there are coupled equations the number of which is twice the number of cylinders. However, there is no coupling among the equations for different n. This is true for a group of cylinders with the same type of boundary conditions and of the same length. For a given n, equations (42) may be written in matrix form,

$$[M]\{\ddot{h}\} + [K]\{h\} = \{F\} \quad , \tag{43}$$

where M is symmetric square matrix and K is diagonal matrix. The elements of these matrices are as follows:

$$\begin{aligned} m_{pq} &= m_p \delta_{pq} + \gamma_{pq} \quad , \\ k_{pq} &= m_p \bar{\omega}_{pn}^2 \delta_{pq} \quad , \end{aligned} \tag{44}$$

$$(p,q = 0, 1, 2, \ldots,2k + 1) \quad .$$

For free vibration, neglect external forcing terms, and let

$$\{h\} = \{\bar{h}\}\exp(i\Omega t) \quad . \tag{45}$$

The coupled natural frequency Ω and the coupled normal modes $\{\bar{h}\}$ of the system are computed from the homogeneous equations

$$[[K] - \Omega^2[M]]\{\bar{h}\} = 0 \quad . \tag{46}$$

B. Sample Problem: Free Vibration of Two Eccentric Cylinders with a Fluid-Filled Gap

As an example, consider the motion of two eccentric cylinders with a fluid-filled gap as illustrated in Fig. 5. The axes of the cylinders are located on the x-z plane and parallel to the z-axis. In this case, it is found that

$$\begin{aligned} &\alpha_{00} = \beta_{00} \ , \\ &\alpha_{11} = \beta_{11} \ , \\ &\alpha_{01} = \alpha_{10} = \beta_{01} = \beta_{10} \ , \\ &\sigma_{j\ell} = \tau_{j\ell} = 0 \ , \quad (j,\ell = 0, 1) \ . \end{aligned} \tag{47}$$

Hence, the added matrix can be written

$$[\gamma_{pq}] = \rho\pi \begin{bmatrix} R_0^2\alpha_{00} & \left(\frac{R_0 + R_1}{2}\right)^2 \alpha_{01} & 0 & 0 \\ \left(\frac{R_0 + R_1}{2}\right)^2 \alpha_{01} & R_1^2\alpha_{11} & 0 & 0 \\ 0 & 0 & R_0^2\alpha_{00} & \left(\frac{R_0 + R_1}{2}\right)^2 \alpha_{01} \\ 0 & 0 & \left(\frac{R_0 + R_1}{2}\right)^2 \alpha_{01} & R_1^2\alpha_{11} \end{bmatrix} . \tag{48}$$

Examining this added mass matrix we may conclude that the x-z plane and the y-z plane motions can be considered separately. For the x-z plane motion equation (46) becomes

$$\begin{bmatrix} m_o\bar{\omega}_{on}^2 - \Omega^2(m_o+\gamma_{oo}) & -\Omega^2\gamma_{01} \\ -\Omega^2\gamma_{01} & m_1\bar{\omega}_{1n}^2 - \Omega^2(m_1+\gamma_{11}) \end{bmatrix} \begin{Bmatrix} \bar{h}_{0n} \\ \bar{h}_{1n} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} . \tag{49}$$

The frequency equation for the coupled modes yields

$$(1 - \lambda_{01})\Omega^4 - (\omega_{0n}^2 + \omega_{1n}^2)\Omega^2 + \omega_{0n}^2\omega_{1n}^2 = 0 \ , \tag{50}$$

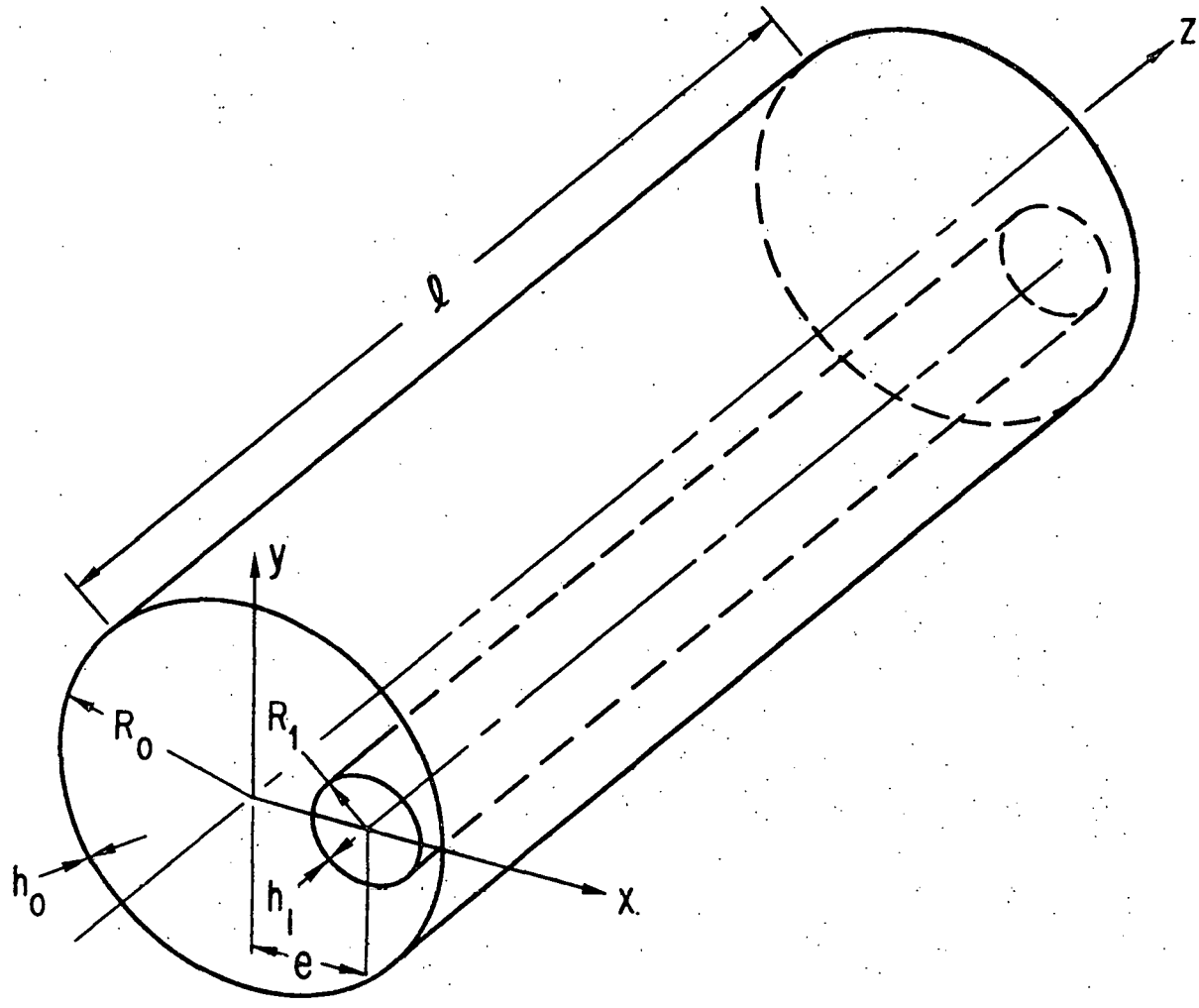

FIGURE 5. Two Eccentric Cylinders with a Fluid-Filled Gap

where

$$\lambda_{01} = \frac{\gamma_{01}\gamma_{10}}{(m_0+\gamma_{00})(m_1+\gamma_{11})} ,$$

$$\omega_{pn} = \bar{\omega}_{pn}\sqrt{\frac{m_p}{m_p+\gamma_{pp}}} , \quad (p = 0, 1) .$$

Here ω_{0n} and ω_{1n} are the uncoupled natural frequencies for each cylinder. The coupled natural frequencies Ω_{1n} and Ω_{2n} are found from equation (50),

$$\Omega_{1n} = \left\{(\omega_{1n}^2+\omega_{2n}^2) - [(\omega_{1n}^2-\omega_{2n}^2)^2 + 4\lambda_{01}\omega_{1n}^2\omega_{2n}^2]^{1/2}\right\}^{1/2} \Big/ [2(1-\lambda_{01})]^{1/2},$$

and (51)

$$\Omega_{2n} = \left\{(\omega_{1n}^2+\omega_{2n}^2) + [(\omega_{1n}^2-\omega_{2n}^2)^2 + 4\lambda_{01}\omega_{1n}^2\omega_{2n}^2]^{1/2}\right\}^{1/2} \Big/ [2(1-\lambda_{01})]^{1/2} .$$

For each n there are two coupled normal modes. The amplitude ratio $\bar{h}_{1n}/\bar{h}_{0n}$ is given by

$$\frac{\bar{h}_{1n}}{\bar{h}_{0n}} = \frac{\omega_{0n}^2 - \Omega^2 (m_0 + \gamma_{00})}{\Omega^2 \quad \gamma_{01}} . \tag{52}$$

Natural frequencies for the y-z plane motion can be found in an analogous manner; the results are exactly the same as the x-z plane motion.

Numerical results presented here are based on two eccentric stainless steel tubes with a water-filled gap. Both tubes have the same length of 1.27 m (50 in.) and wall thickness of 0.15875 cm (0.0625 in.). The radii of outer and inner tube are 5.08 cm (2 in.) and 2.54 cm (1 in.) respectively. The distance between the centers of two cylinders is allowed to vary in order to investigate the effects of eccentricity on natural frequencies.

Table 3 and Fig. 6 summarize the results. For both cases of x-z plane and y-z plane vibrations, the lower frequency is associated with the mode in which two cylinders move out-of-phase, while the higher frequency is associated with in-phase mode. Fig. 6 shows the lowest two natural frequencies (n=1) for uncoupled as well as coupled modes. It can be also seen from Figure 6 and Table 3 that for each n the frequencies for the first coupled modes

Table 3. Natural Frequencies (Hz) for Two Eccentric Cylinders with a Fluid-Filled Gap

n	ε	Uncoupled Vibration		Coupled Vibration	
		Cylinder 0	Cylinder 1	First Mode	Second Mode
1	0	83.660	52.279	48.571	108.54
	0.1	83.637	52.232	48.529	108.54
	0.2	83.566	52.087	48.400	108.53
	0.3	83.442	51.836	48.177	108.51
	0.4	83.254	51.461	47.843	108.47
	0.5	82.985	50.934	47.372	108.43
	0.6	82.604	50.205	46.719	108.37
	0.7	82.053	49.188	45.808	108.29
	0.8	81.211	47.714	44.481	108.17
	0.9	79.747	45.344	42.338	108.00
	0.95	78.428	43.392	40.564	107.88
	0.975	77.394	41.967	39.263	107.79
	0.995	76.169	40.384	37.815	107.69
2	0	334.64	209.12	194.28	434.18
	0.1	334.55	208.93	194.12	434.16
	0.2	334.27	208.35	193.60	434.11
	0.3	333.77	207.35	192.71	434.02
	0.4	333.02	205.85	191.37	433.89
	0.5	331.94	203.74	189.49	433.71
	0.6	330.41	200.82	186.88	433.47
	0.7	328.21	196.75	183.23	433.14
	0.8	324.85	190.86	177.93	432.69
	0.9	318.99	181.37	169.35	432.02
	0.95	313.71	173.57	162.25	431.50
	0.975	309.57	167.87	157.05	431.15
	0.995	304.68	161.53	151.26	430.78
3	0	752.94	470.51	437.14	976.89
	0.1	752.74	470.09	436.76	976.86
	0.2	752.10	468.79	435.60	976.74
	0.3	750.98	466.53	433.59	976.55
	0.4	749.29	463.15	430.58	976.25

Table 3. Natural Frequencies (Hz) for Two Eccentric Cylinders with a Fluid-Filled Gap (Contd.)

n	ε	Uncoupled Vibration		Coupled Vibration	
		Cylinder 0	Cylinder 1	First Mode	Second Mode
3	0.5	746.87	458.40	426.34	975.85
	0.6	743.43	451.84	420.47	975.31
	0.7	738.47	442.70	412.27	974.58
	0.8	730.90	429.43	400.33	973.56
	0.9	717.73	408.09	381.04	972.04
	0.95	705.85	390.53	365.07	970.88
	0.975	696.54	377.70	353.37	970.08
	0.995	685.52	363.45	340.33	969.25

R_0 = 5.08 cm (2 in.)

R_1 = 2.54 cm (1 in.)

$h_0 = h_1$ = 0.15875 cm (0.0625 in.)

$\ell_0 = \ell_1$ = 127 cm (50 in.)

ρ = 8.015×10^3 Kg/m^3 (7.5×10^{-4} lb-sec^2/in^4)

E = 2.017×10^{11} Pa (3.0×10^7 psi)

ρ_f = 9.982×10^2 Kg/m^3 (9.34×10^{-5} lb-sec^2/in^4)

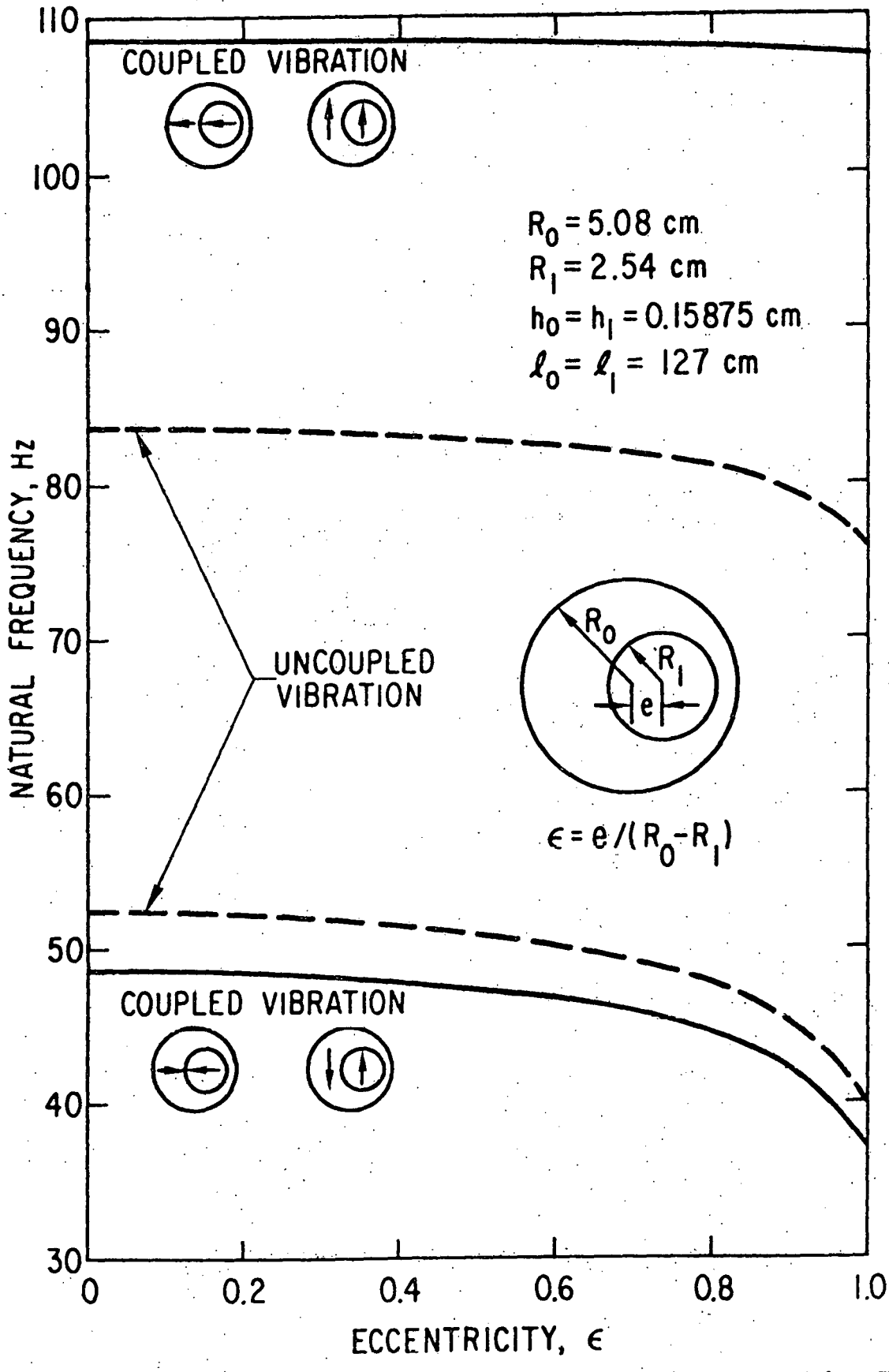

FIGURE 6. Natural Frequencies for Two Eccentric Cylinders with a Fluid-Filled Gap as Functions of Eccentricity

(out-of-phase modes) are lower than either of the uncoupled natural frequencies, while the frequencies for the second coupled modes (in-phase modes) are higher than either of the uncoupled natural frequencies.

It is also interesting to note the effects of eccentricity on coupled and uncoupled modes. For coupled frequencies the effects are pronounced for the out-of-phase modes, but these effects are much smaller and considered to be negligible for the in-phase modes. However, for uncoupled vibrations the effects of eccentricity remain comparable for both cylinders. Fig. 6 also reveals that as eccentricity becomes larger, its influence on frequencies becomes larger.

IV. CONCLUSIONS

In this report, a general method of analysis is presented for coupled motions of a group of cylinders in a fluid-containing cylinder; the coupling effects are accounted for using self- and mutual-added mass coefficients of cylinders. The method can be applied to a group of cylinders with different radii. These cylinders may be arranged arbitrarily as long as they are parallel to one another. With this method of analysis, free and forced vibrations of a group of cylinders in a fluid-containing cylinder can readily be analyzed.

The analysis is based on the two dimensional flow theory. In many practical situations, the displacement of the cylinders are much smaller than the radius. Therefore, the potential flow theory is applicable. Furthermore, many system components, such as nuclear fuel bundles and heat exchanger tubes are long enough such that the hydrodynamic forces based on the two dimensional theory are sufficiently accurate.

Numerical results are given for a few selected problems. From the study of seven cylinders in a fluid-containing cylinder, it is seen that the influence of the outer cylinder on the motions of a group of inner cylinders is similar to that of two concentric cylinders. When the outer cylinder is relatively close to the inner cylinders, the coupling effects become pronounced and have to be considered. For the problem of two eccentric cylinders with a fluid-filled gap, the self-added mass coefficient of the inner cylinder is sensitive to the eccentricity of cylinders. The effects of eccentricity are also found to be distinct for the out-of-phase modes of coupled vibrations while they are almost negligible for the in-phase modes.

ACKNOWLEDGMENT

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REFERENCES

1. Livesey, J. L., and Dye, C. F., "Vortex Exicted Vibration of a Heat Exchanger Tube Row," Journal of Mechanical Engineering Science, 4(4), 1962, pp. 349-352.

2. Laird, A. D. K., and Warren, R. P., "Groups of Vertical Cylinders Oscillating in Water," Journal of Engineering Mechanics Division, Proceedings ASCE, Vol. 89(EM 1), February, 1963, pp. 25-35.

3. Kito, F., "On Vibration of Two Circular Cylinders, which are Immersed in a Water Region of Infinite Extent," Journal of Society Naval Architecture in Japan, Vol. 122, 1967, pp. 1-9.

4. Mazur, V. Y., "Motion of Two Circular Cylinders in an Ideal Fluid," Izvestiya Akademii Nauk SSSR, Mekhanika Zhidkosti i Gaza, No. 6, 1970, pp. 80-84.

5. Connros, H. J., Jr., "Fluidelastic Vibration of Tube Arrays Excited by Cross Flow," Flow Induced Vibration in Heat Exchangers, ASME publication, 1970, pp. 42-56.

6. Shimogo, T., Inui, T., "Coupled Vibration of Many Elastic Circular Bars in Water," Proceedings 21st Japan National Congress for Applied Mechanics, 1971, pp. 495-505.

7. Dalton, C., and Helfinstine, R. A., "Potential Flow Past a Group of Circular Cylinders," Journal of Basic Engineering, Trans. ASME, Vol. 93, December, 1971, pp. 636-642.

8. Shimogo, T., Niino, T., and Setogawa, S., "Coupled Vibration of Elastic Circular Bars in Viscous Fluid," ASME Paper 75-DET-76, 1975.

9. Moretti, P. M., and Lowery, R. L., "Hydrodynamic Inertia Coefficients for a Tube Surrounded by Rigid Tubes," ASME Paper 75-PVP-47, 1975.

10. Chen, S. S., "Dynamic Response of Two Parallel Circular Cylinders in a Liquid," Journal of Pressure Vessel Technology, Trans. ASME, Vol. 97(2), 1975, pp. 77-83.

11. Chen, S. S., "Vibration of a Row of Circular Cylinders in a Liquid," Journal of Engineering for Industry, Trans. ASME, Vol. 97, 4, 1975, pp. 1212-1218.

12. Chen, S. S., "Vibration of Fuel Bundles," Nuclear Engineering and Design; Vol. 35, 1975, pp. 399-422.

13. Fritz, R. J., "The Effect of Liquids on the Dynamic Motions of Immersed Solids," ASME Paper No. 71-Vibr-100, 1971.

REFERENCES (Cont.d)

14. Chen, S. S., "Dynamics of a Rod-Shell System Conveying Fluid," Nuclear Engineering and Design, Vol. 30, 1974, pp. 223-233.

15. MacMillan, W. D., The Theory of Potential, Dover, New York, 1958.

APPENDIX: Computer Program (AMASS)

This appendix gives a list of the computer program, AMASS, which was used to calculate the added mass coefficients for this report. This program can be used for a group of cylinders immersed either in a fluid-containing cylinder or in an infinite fluid medium. The computer program is written in Fortran IV for IBM/370 computer system. The program is not necessarily written in an optimal manner in terms of computing time.

```
C    ************************************************************
C    *                                                          *
C    *              *** SUBROUTINE  AMASS ***                   *
C    *                                                          *
C    *   ADDED MASS COEFFICIENTS OF A GROUP OF CIRCULAR         *
C    *           CYLINDERS IN A CONFINED FLUID                  *
C    *                                                          *
C    ************************************************************
C
C    THIS PROGRAM CALCULATES ADDED MASS COEFFICIENTS OF A GROUP OF CIRCULAR
C    CYLINDERS IMMERSED IN A FLUID CONTAINED IN A CYLINDER OR IN AN INFINITE
C    FLUID MEDIUM
C
C      INPUT VARIABLES
C      ***************
C        NNN: NUMBER OF TERMS USED (NNN=10 WILL GIVE RESULTS WITH SUFFICIENT
C             ACCURACY IN GENERAL)
C        KKI: TOTAL NUMBER OF CYLINDERS INCLUDING THE OUTER CYLINDER
C        ICHK: IF ICHK=1, GENERAL CASE WITH OUTER, CONCENTRIC & NON-CONCENTRI
C                         INNER CYLINDERS, IK=0
C              IF ICHK=2, OUTER & NON-CONCENTRIC INNER CYLINDERS, IK=1
C              IF ICHK=3, NO OUTER & CONCENTRIC INNER CYLINDERS: INFINITE
C                         FLUID, IK=2
C        RA(I): RADIUS OF CYLINDER I
C        XA(I),YA(I): COORDINATES OF CYLINDER I
C          ** INPUT SEQUENCE OF CYLINDER GEOMETRIES **
C               OUTER CYLINDER, CONCENTRIC INNER CYLINDER & OTHER CYLINDERS
C
C      OUTPUT VARIABLES
C      ****************
C        ALP(I,J),BET(I,J),SIG(I,J),TAU(I,J): ADDED MASS COEFFICIENTS
C
C
C  **  DIMENSION SPECIFICATIONS **
C        (KKI)                 RA,XA,YA
C        (KKK)                 R,X,Y
C        (KKI,KKI)             RAIJ,PHYA,ALP,BET,SIG,TAU
C        (KKK,KKK)             RIJ,PHY
C        (KKI,NNN,KKI)         AL,BE,SI,TA,H
C        (KKK,NNN,KKK,NNN)     AA,BB,CC,DD
C        (KN*2,KN*2)           E
C        (KN*2,KKK)            F
C           KKK=KKI+IK
C           KN=KKK*NNN
C
C
C
      SUBROUTINE AMASS(NNN,KKI,ICHK,RA,XA,YA,ALP,BET,SIG,TAU)
      DIMENSION RA(8),XA(8),YA(8),R(8),X(8),Y(8)
      DIMENSION RAIJ(8,8),PHYA(8,8),RIJ(8,8),PHY(8,8)
      DIMENSION AA(8,10,8,10),BB(8,10,8,10)
      DIMENSION CC(8,10,8,10),DD(8,10,8,10)
      DIMENSION H(8,10,8),E(160,160),F(160,8)
      DIMENSION AL(8,10,8),BE(8,10,8),SI(8,10,8),TA(8,10,8)
      DIMENSION ALP(8,8),BET(8,8),SIG(8,8),TAU(8,8)
      GO TO (4001,4002,4003),ICHK
4001  IK=0
      IKK=2
```

```
      GO TO 4004
4002  IK=1
      IKK=1
      GO TO 4004
4003  IK=2
      IKK=0
4004  KKK=KKI+IK
C ** CLEAR WORK SPACES **
      DO 4005 I=1,KKK
      R(I) =0.0
      X(I) =0.0
      Y(I) =0.0
      DO 4005 J=1,KKK
      RIJ(I,J)=0.0
      PHY(I,J)=0.0
      DO 4005 N=1,NNN
      DO 4005 M=1,NNN
      AA(I,N,J,M)=0.0
      BB(I,N,J,M)=0.0
      CC(I,N,J,M)=0.0
4005  DD(I,N,J,M)=0.0
C ** GO TO INDEX KKK FROM KKI **
      GO TO (4031,40331,4033),ICHK
4031  DO 4032 I=1,KKK
      X(I)=XA(I)
      Y(I)=YA(I)
      R(I)=RA(I)
      DO 4032 J=1,KKK
      RIJ(I,J)=RAIJ(I,J)
4032  PHY(I,J)=PHYA(I,J)
      GO TO 4040
40331 X(1)=XA(1)
      Y(1)=YA(1)
      R(1)=RA(1)
      RIJ(1,1)=RAIJ(1,1)
      PHY(1,1)=PHYA(1,1)
4033  DO 4036 I=3,KKK
      IIK=I-IK
      X(I)=XA(IIK)
      Y(I)=YA(IIK)
      R(I)=RA(IIK)
      GO TO (4034,4035),IK
4034  IM1=I-1
      RIJ(I,1)=RAIJ(IM1,1)
      RIJ(1,I)=RAIJ(1,IM1)
      PHY(I,1)=PHYA(IM1,1)
      PHY(1,I)=PHYA(1,IM1)
4035  DO 4036 J=3,KKK
      JJK=J-IK
      RIJ(I,J)=RAIJ(IIK,JJK)
4036  PHY(I,J)=PHYA(IIK,JJK)
4040  CONTINUE
C ** FORM AA, BB, CC, DD MATRICES **
      GO TO (4050,4050,5010),ICHK
4050  DO 5000 I=1,IKK
```

```
      DO 5000 N=1,NNN
      GO TO (8881,8882),I
8881  MMM=N
      GO TO 8883
8882  MMM=NNN
8883  CONTINUE
      DO 5000 J=1,KKK
      IF(ICHK.EQ.1) GO TO 4051
      IF(J.EQ.2) GO TO 5000
4051  CONTINUE
      DO 5000 M=1,MMM
      GO TO (4052,7770),I
4052  CONTINUE
      IF (J-2) 4053,4053,4100
4053  IF (N-M) 4061,4054,4061
4054  IF (I-J) 4058,4055,4058
4055  AA(I,N,J,M)=N
      BB(I,N,J,M)=N
      CC(I,N,J,M)=0.0
      DD(I,N,J,M)=0.0
      GO TO 5000
4058  AA(I,N,J,M)=-N*(R(2)/R(1))**(N+1)
      BB(I,N,J,M)=-N*(R(2)/R(1))**(N+1)
      CC(I,N,J,M)=0.0
      DD(I,N,J,M)=0.0
      GO TO 5000
4061  AA(I,N,J,M)=0.0
      BB(I,N,J,M)=0.0
      CC(I,N,J,M)=0.0
      DD(I,N,J,M)=0.0
      GO TO 5000
4100  MM1=M-1
      NMM=N-M
      ARG11=(M-1)*PHY(1,J)
      AGG1=R(J)*R(J)/(R(I)*RIJ(1,J))
      DO 4101 MC=1,M
4101  AGG1=AGG1*RIJ(1,J)/R(J)
      CNA1=1.0
      DO 4102 MC=1,N
      CNA1=CNA1*MC
4102  AGG1=AGG1*R(J)/R(I)
      CMA1=1.0
      IF (M-1)4103,4105,4103
4103  DO 4104 MC=1,MM1
4104  CMA1=CMA1*MC
4105  CNMMA=1.0
      IF (NMM) 4106,4108,4106
4106  DO 4107 MC=1,NMM
4107  CNMMA=CNMMA*MC
4108  AA(I,N,J,NMM+1)=-AGG1*CNA1*COS(ARG11)/(CMA1*CNMMA)
      BB(I,N,J,NMM+1)=AA(I,N,J,NMM+1)
      CC(I,N,J,NMM+1)=AGG1*CNA1*SIN(ARG11)/(CMA1*CNMMA)
      DD(I,N,J,NMM+1)=-CC(I,N,J,NMM+1)
      GO TO 5000
7770  CONTINUE
```

```
      IF(J.GT.2) GO TO 7774
      IF(N.NE.M) GO TO 7772
      IF(I.EQ.J) GO TO 7771
      IF(N.EQ.1) GO TO 7773
      AA(I,N,J,M)=N*(R(2)/R(1))**(N-1)
      BB(I,N,J,M)=N*(R(2)/R(1))**(N-1)
      CC(I,N,J,M)=0.0
      DD(I,N,J,M)=0.0
      GO TO 5000
7771  AA(I,N,J,M)=-N
      BB(I,N,J,M)=-N
      CC(I,N,J,M)=0.0
      DD(I,N,J,M)=0.0
      GO TO 5000
7772  AA(I,N,J,M)=0.0
      BB(I,N,J,M)=0.0
      CC(I,N,J,M)=0.0
      DD(I,N,J,M)=0.0
      GO TO 5000
7773  AA(I,N,J,M)=1.0
      BB(I,N,J,M)=1.0
      CC(I,N,J,M)=0.0
      DD(I,N,J,M)=0.0
      GO TO 5000
7774  IN1=N-1
      IM1=M-1
      INM1=N+M-1
      ARG1=(N+M)*PHY(I,J)
      AGG=R(I)/RIJ(I,J)*R(J)/R(I)
      DO 15051 MC=1,M
15051 AGG=AGG*(-1.)*R(J)/R(I)
      CNA=1.0
      IF (IN1) 15052,15054,15052
15052 DO 15053 MC=1,IN1
15053 CNA=CNA*MC
15054 CMA=1.0
      IF (IM1) 15055,15057,15055
15055 DO 15056 MC=1,IM1
15056 CMA=CMA*MC
15057 CNMA=1.0
      DO 15058 MC=1,INM1
15058 CNMA=CNMA*MC/(RIJ(I,J)/R(I))
      AA(I,N,J,M)=AGG*CNMA/CNA/CMA*COS(ARG1)
      BB(I,N,J,M)=-AA(I,N,J,M)
      CC(I,N,J,M)=AGG*CNMA/CNA/CMA*SIN(ARG1)
      DD(I,N,J,M)=CC(I,N,J,M)
5000  CONTINUE
      IF (KKK-2) 206,206,5010
5010  CONTINUE
      DO 205 I=3,KKK
      DO 205 N=1,NNN
      DO 205 J=1,KKK
      GO TO (5011,5012,5013),ICHK
5012  IF (J-2) 5011,205,5011
5013  IF (J-3) 205,205,5011
```

```
5011  CONTINUE
      DO 205 M=1,MNN
      IF (I-J) 5017,5014,5017
5014  IF (N-M) 5015,5015,5016
5015  AA(I,N,J,M)=-N
      BB(I,N,J,M)=-N
      CC(I,N,J,M)=0.0
      DD(I,N,J,M)=0.0
      GO TO 205
5016  AA(I,N,J,M)=0.0
      BB(I,N,J,M)=0.0
      CC(I,N,J,M)=0.0
      DD(I,N,J,M)=0.0
      GO TO 205
5017  IN1=N-1
      IM1=M-1
      INM1=N+M-1
      MMN=M-N
      IF (J-1) 5050,5018,5050
5018  IF (M-N) 5019,5020,5020
5019  AA(I,N,J,M)=0.0
      BB(I,N,J,M)=0.0
      CC(I,N,J,M)=0.0
      DD(I,N,J,M)=0.0
      GO TO 205
5020  ARG12=(M-N)*PHY(1,I)
      AGG2=R(1)/RIJ(1,I)
      CMA2=1.0
      DO 5021 MC=1,M
      AGG2=AGG2*RIJ(1,I)/R(1)
5021  CMA2=CMA2*MC
      CNA2=1.0
      IF (N-1) 5022,5024,5022
5022  DO 5023 MC=1,IN1
      AGG2=AGG2*R(I)/RIJ(1,I)
5023  CNA2=CNA2*MC
5024  CMMNA=1.0
      IF (MMN) 5025,5027,5025
5025  DO 5026 MC=1,MMN
5026  CMMNA=CMMNA*MC
5027  AA(I,N,J,M)=AGG2*CMA2*COS(ARG12)/(CNA2*CMMNA)
      BB(I,N,J,M)=AA(I,N,J,M)
      CC(I,N,J,M)=AGG2*CMA2*SIN(ARG12)/(CNA2*CMMNA)
      DD(I,N,J,M)=-CC(I,N,J,M)
      GO TO 205
5050  ARG1=(N+M)*PHY(I,J)
      AGG=R(I)/RIJ(I,J)*R(J)/R(I)
      DO 5051 MC=1,M
5051  AGG=AGG*(-1.)*R(J)/R(I)
      CNA=1.0
      IF (IN1) 5052,5054,5052
5052  DO 5053 MC=1,IN1
5053  CNA=CNA*MC
5054  CMA=1.0
      IF (IM1) 5055,5057,5055
```

```
5055   DO 5056 MC=1,IM1
5056   CMA=CMA*MC
5057   CNMA=1.0
       DO 5058 MC=1,INM1
5058   CNMA=CNMA*MC/(RIJ(I,J)/R(I))
       AA(I,N,J,M)=AGG*CNMA/CNA/CMA*COS(ARG1)
       BB(I,N,J,M)=-AA(I,N,J,M)
       CC(I,N,J,M)=AGG*CNMA/CNA/CMA*SIN(ARG1)
       DD(I,N,J,M)=CC(I,N,J,M)
205    CONTINUE
206    CONTINUE
       GO TO (5200,5100,5100),ICHK
5100   DO 50 N=1,NNN
       DO 50 M=1,NNN
       DO 50 I=3,KKK
       II1=I-1
       IIK=I-IK
       GO TO (11,12),IK
11     AA(II1,N,1,M)=AA(I,N,1,M)
       BB(II1,N,1,M)=BB(I,N,1,M)
       CC(II1,N,1,M)=CC(I,N,1,M)
       DD(II1,N,1,M)=DD(I,N,1,M)
12     DO 50 J=3,KKK
       GO TO (21,23),IK
21     IF (I-4) 22,23,23
22     JJ1=J-1
       AA(1,N,JJ1,M)=AA(1,N,J,M)
       BB(1,N,JJ1,M)=BB(1,N,J,M)
       CC(1,N,JJ1,M)=CC(1,N,J,M)
       DD(1,N,JJ1,M)=DD(1,N,J,M)
23     JJK=J-IK
       AA(IIK,N,JJK,M)=AA(I,N,J,M)
       BB(IIK,N,JJK,M)=BB(I,N,J,M)
       CC(IIK,N,JJK,M)=CC(I,N,J,M)
       DD(IIK,N,JJK,M)=DD(I,N,J,M)
50     CONTINUE
5200   KN=KKI*NNN
C ** PREPARE TO SOLVE SYSTEM OF EQUATIONS FOR X-AXIS **
       DO 230 I=1,KKI
       DO 230 N=1,NNN
       IN=(I-1)*NNN+N
       II=IN+KN
       DO 230 J=1,KKI
       DO 230 M=1,NNN
       JM=(J-1)*NNN+M
       JJ=JM+KN
       E(IN,JM)=AA(I,N,J,M)
       E(II,JJ)=BB(I,N,J,M)
       E(IN,JJ)=CC(I,N,J,M)
       E(II,JM)=DD(I,N,J,M)
230    CONTINUE
       DO 220 I=1,KKI
       DO 220 N=1,NNN
       DO 220 L=1,KKI
       IF (N-1) 62,60,62
```

```
60      IF (I-L) 62,61,62
61      H(I,N,L)=1.0
        GO TO 220
62      H(I,N,L)=0.0
220     CONTINUE
        DO 240 I=1,KKI
        DO 240 N=1,NNN
        IN=(I-1)*NNN+N
        II=IN+KN
        DO 240 L=1,KKI
        F(IN,L)=H(I,N,L)
240     F(II,L)=0.0
        KNKN=2*KN
C ** SOLVE SYSTEM OF EQUATIONS **
        CALL CROUT(E,KNKN,F,KKI,DETERM,160)
        DO 250 I=1,KKI
        DO 250 N=1,NNN
        IN=(I-1)*NNN+N
        II=IN+KN
        DO 250 L=1,KKI
        AL(I,N,L)=F(IN,L)
        TA(I,N,L)=F(II,L)
250     CONTINUE
C ** PREPARE TO SOLVE SYSTEM OF EQUATIONS FOR Y-AXIS **
        DO 270 I=1,KKI
        DO 270 N=1,NNN
        IN=(I-1)*NNN+N
        II=IN+KN
        DO 270 L=1,KKI
        F(IN,L)=0.0
270     F(II,L)=H(I,N,L)
        DO 231 I=1,KKI
        DO 231 N=1,NNN
        IN=(I-1)*NNN+N
        II=IN+KN
        DO 231 J=1,KKI
        DO 231 M=1,NNN
        JM=(J-1)*NNN+M
        JJ=JM+KN
        E(IN,JM)=AA(I,N,J,M)
        E(II,JJ)=BB(I,N,J,M)
        E(IN,JJ)=CC(I,N,J,M)
231     E(II,JM)=DD(I,N,J,M)
C ** SOLVE SYSTEM OF EQUATIONS **
        CALL CROUT(E,KNKN,F,KKI,DETERM,160)
        DO 280 I=1,KKI
        DO 280 N=1,NNN
        IN=(I-1)*NNN+N
        II=IN+KN
        DO 280 L=1,KKI
        SI(I,N,L)=F(IN,L)
        BE(I,N,L)=F(II,L)
280     CONTINUE
C ** CALCULATE ADDED MASS COEFFICIENTS **
        GO TO (6001,6002,6003),ICHK
```

```
6001  I=1
      R21=RA(2)*RA(2)/(RA(1)*RA(1))
      DO 600 L=1,KKI
      ALP(1,L)=AL(1,1,L)+AL(2,1,L)*R21
      SIG(1,L)=SI(1,1,L)+SI(2,1,L)*R21
      TAU(1,L)=TA(1,1,L)+TA(2,1,L)*R21
      BET(1,L)=BE(1,1,L)+BE(2,1,L)*R21
      IF (KKI.LE.2) GO TO 600
      DO 6000 J=3,KKI
      RJI2=RA(J)*RA(J)/(RA(I)*RA(I))
      ALP(1,L)=ALP(1,L)+AL(J,1,L)*RJI2
      SIG(1,L)=SIG(1,L)+SI(J,1,L)*RJI2
      TAU(1,L)=TAU(1,L)+TA(J,1,L)*RJI2
6000  BET(1,L)=BET(1,L)+BE(J,1,L)*RJI2
600   CONTINUE
      I=2
      DO 7004 L=1,KKI
      ALP(2,L)=AL(1,1,L)+AL(2,1,L)
      SIG(2,L)=SI(1,1,L)+SI(2,1,L)
      TAU(2,L)=TA(1,1,L)+TA(2,1,L)
      BET(2,L)=BE(1,1,L)+BE(2,1,L)
      IF (KKI.LE.2) GO TO 7004
      DO 7003 J=3,KKI
      AGG=0.0
      BGG=0.0
      CGG=0.0
      DGG=0.0
      RJIJ=RA(J)/RAIJ(2,J)
      DO 7002 N=1,NNN
      ARG=(N+1)*PHYA(2,J)
      AG1=N*RJIJ*(COS(ARG)*AL(J,N,L)+TA(J,N,L)*SIN(ARG))
      BG1=N*RJIJ*(COS(ARG)*SI(J,N,L)+BE(J,N,L)*SIN(ARG))
      CG1=N*RJIJ*(SIN(ARG)*AL(J,N,L)-TA(J,N,L)*COS(ARG))
      DG1=N*RJIJ*(SIN(ARG)*SI(J,N,L)-BE(J,N,L)*COS(ARG))
      DO 7001 MC=1,N
      AG1=-AG1*RJIJ
      BG1=-BG1*RJIJ
      CG1=-CG1*RJIJ
7001  DG1=-DG1*RJIJ
      AGG=AGG+AG1
      BGG=BGG+BG1
      CGG=CGG+CG1
7002  DGG=DGG+DG1
      ALP(2,L)=ALP(2,L)+AGG
      SIG(2,L)=SIG(2,L)+BGG
      TAU(2,L)=TAU(2,L)+CGG
7003  BET(2,L)=BET(2,L)+DGG
7004  CONTINUE
      DO 605 L=1,KKI
      ALP(2,L)=-ALP(2,L)
      SIG(2,L)=-SIG(2,L)
      TAU(2,L)=-TAU(2,L)
605   BET(2,L)=-BET(2,L)
      GO TO 6003
6002  DO 700 L=1,KKI
```

```
      ALP(1,L)=AL(1,1,L)
      SIG(1,L)=SI(1,1,L)
      TAU(1,L)=TA(1,1,L)
      BET(1,L)=BE(1,1,L)
      DO 700 J=2,KKI
      RJ12=RA(J)*RA(J)/(RA(1)*RA(1))
      ALP(1,L)=ALP(1,L)+AL(J,1,L)*RJ12
      SIG(1,L)=SIG(1,L)+SI(J,1,L)*RJ12
      TAU(1,L)=TAU(1,L)+TA(J,1,L)*RJ12
700   BET(1,L)=BET(1,L)+BE(J,1,L)*RJ12
6003  ISS=IKK+1
      IF(KKI-ISS)899,6004,6004
6004  CONTINUE
      DO 800 I=ISS,KKI
      DO 800 L=1,KKI
      ALP(I,L)=AL(I,1,L)
      SIG(I,L)=SI(I,1,L)
      TAU(I,L)=TA(I,1,L)
      BET(I,L)=BE(I,1,L)
      DO 790 J=1,KKI
      AGG=0.0
      BGG=0.0
      CGG=0.0
      DGG=0.0
      IF (I-J) 801,770,801
801   GO TO (802,802,805),ICHK
802   IF (J-1) 805,803,805
803   DO 750 N=1,NNN
      ARG1=(N-1)*PHYA(1,I)
      R1IJ=RA(1)/RAIJ(1,I)
      RIJ1=RAIJ(1,I)/RA(1)
      AG11=N*R1IJ*(COS(ARG1)*AL(1,N,L)+TA(1,N,L)*SIN(ARG1))
      BG11=N*R1IJ*(COS(ARG1)*SI(1,N,L)+BE(1,N,L)*SIN(ARG1))
      CG11=N*R1IJ*(-SIN(ARG1)*AL(1,N,L)+TA(1,N,L)*COS(ARG1))
      DG11=N*R1IJ*(-SIN(ARG1)*SI(1,N,L)+BE(1,N,L)*COS(ARG1))
      DO 804 MC=1,N
      AG11=AG11*RIJ1
      BG11=BG11*RIJ1
      CG11=CG11*RIJ1
804   DG11=DG11*RIJ1
      AGG=AGG+AG11
      BGG=BGG+BG11
      CGG=CGG+CG11
750   DGG=DGG+DG11
      GO TO 770
805   RJIJ=RA(J)/RAIJ(I,J)
      DO 760 N=1,NNN
      ARG=(N+1)*PHYA(I,J)
      AG1=N*RJIJ*(COS(ARG)*AL(J,N,L)+TA(J,N,L)*SIN(ARG))
      BG1=N*RJIJ*(COS(ARG)*SI(J,N,L)+BE(J,N,L)*SIN(ARG))
      CG1=N*RJIJ*(SIN(ARG)*AL(J,N,L)-TA(J,N,L)*COS(ARG))
      DG1=N*RJIJ*(SIN(ARG)*SI(J,N,L)-BE(J,N,L)*COS(ARG))
      DO 806 MC=1,N
      AG1=-AG1*RJIJ
      BG1=-BG1*RJIJ
```

```
      CG1=-CG1*RJIJ
806   DG1=-DG1*RJIJ
      AGG=AGG+AG1
      BGG=BGG+BG1
      CGG=CGG+CG1
760   DGG=DGG+DG1
770   CONTINUE
      ALP(I,L)=ALP(I,L)+AGG
      SIG(I,L)=SIG(I,L)+BGG
      TAU(I,L)=TAU(I,L)+CGG
790   BET(I,L)=BET(I,L)+DGG
800   CONTINUE
      DO 299 I=ISS,KKI
      DO 299 L=1,KKI
      ALP(I,L)=-ALP(I,L)
      SIG(I,L)=-SIG(I,L)
      TAU(I,L)=-TAU(I,L)
299   BET(I,L)=-BET(I,L)
899   CONTINUE
      DO 8800 I=1,KKI
      DO 8800 L=1,KKI
      ALP(I,L)=ALP(I,L)*RA(I)*RA(I)
      SIG(I,L)=SIG(I,L)*RA(I)*RA(I)
      TAU(I,L)=TAU(I,L)*RA(I)*RA(I)
8800  BET(I,L)=BET(I,L)*RA(I)*RA(I)
      DO 810 I=1,KKI
      DO 810 L=1,KKI
      RIL=4./((RA(I)+RA(L))**2)
      ALP(I,L)=ALP(I,L)*RIL
      SIG(I,L)=SIG(I,L)*RIL
      TAU(I,L)=TAU(I,L)*RIL
810   BET(I,L)=BET(I,L)*RIL
      RETURN
      END
```

```
C    ****************************************
C    *                                      *
C    *      *** SUBROUTINE  CROUT ***       *
C    *                                      *
C    ****************************************
C
C    THIS PROGRAM SOLVES MATRIX EQUATION AX=B BY THE CROUT METHOD
C
C      INPUT VARIABLES
C      ***************
C        A:  MATRIX
C        N:  ORDER OF A
C        B:  COLUMN MATRIX
C        M:  NUMBER OF COLUMN VECTOR B
C        DETERM:  RETURN CODE, IF ZERO A IS SINGULAR
C        NMAX:  ROW DIMENSION FOR A & B
C
C      OUTPUT
C      ******
C        B:  SOLUTION X STORED
C
      SUBROUTINE CROUT(A,N,B,M,DETERM,NMAX)
      DIMENSION A(NMAX,N),B(NMAX,M)
      DIMENSION V(160)
      DETERM=1.0
      DO 1000 K=1,N
      KP1=K+1
      KM1=K-1
      TEMP=0.0
      DO 20 I=K,N
      DO 2  L=1,K
2     V(L)=A(I,L)
      A(I,K)=A(I,K)-DOTP(V,A(1,K),KM1)
      IF(ABS(A(I,K)).LT.TEMP) GO TO 20
3     TEMP=ABS(A(I,K))
      IMAX=I
20    CONTINUE
      AMAX=A(IMAX,K)
      IF (DETERM.EQ.0.0) RETURN
      IF (IMAX.EQ.K) GO TO 600
      DETERM=-DETERM
      DO 50 J=1,N
      TEMP=A(K,J)
      A(K,J)=A(IMAX,J)
      A(IMAX,J)=TEMP
50    CONTINUE
      IF(M.LE.0) GO TO 600
      DO 400 J=1,M
      TEMP=B(K,J)
      B(K,J)=B(IMAX,J)
      B(IMAX,J)=TEMP
  0   CONTINUE
600   DO 666 L=1,K
666   V(L)=A(K,L)
      IF(K.EQ.N) GO TO 850
```

```
      DO 700 I=KP1,N
700   A(I,K)=A(I,K)/AMAX
      DO 800 J=KP1,N
800   A(K,J)=A(K,J)-DOTP(V,A(1,J),KM1)
850   IF(M.LE.0) GO TO 1000
      DO 900 J=1,M
900   B(K,J)=B(K,J)-DOTP(V,B(1,J),KM1)
1000  CONTINUE
      IF(M.LE.0) RETURN
      DO 8000 I=1,N
      K=N+1-I
      DO 6666 L=K,N
6666  V(L)=A(K,L)
      DO 7000 J=1,M
7000  B(K,J)=(B(K,J)-DOTP(V(K+1),B(K+1,J),I-1))/A(K,K)
8000  CONTINUE
      RETURN
      END
C     ****************************************
C     *                                      *
C     *      *** FUNCTION  DOTP ***          *
C     *                                      *
C     ****************************************
C
C     MATRIX MULTIPLICATION SUBPROGRAM
C
      FUNCTION DOTP(A,B,N)
      REAL*4 A,B,DOTP
      DIMENSION A(1),B(1)
      DOTP=0.0
      IF (N.EQ.0) RETURN
      DO 100 I=1,N
100   DOTP=DOTP+A(I)*B(I)
      RETURN
      END
```