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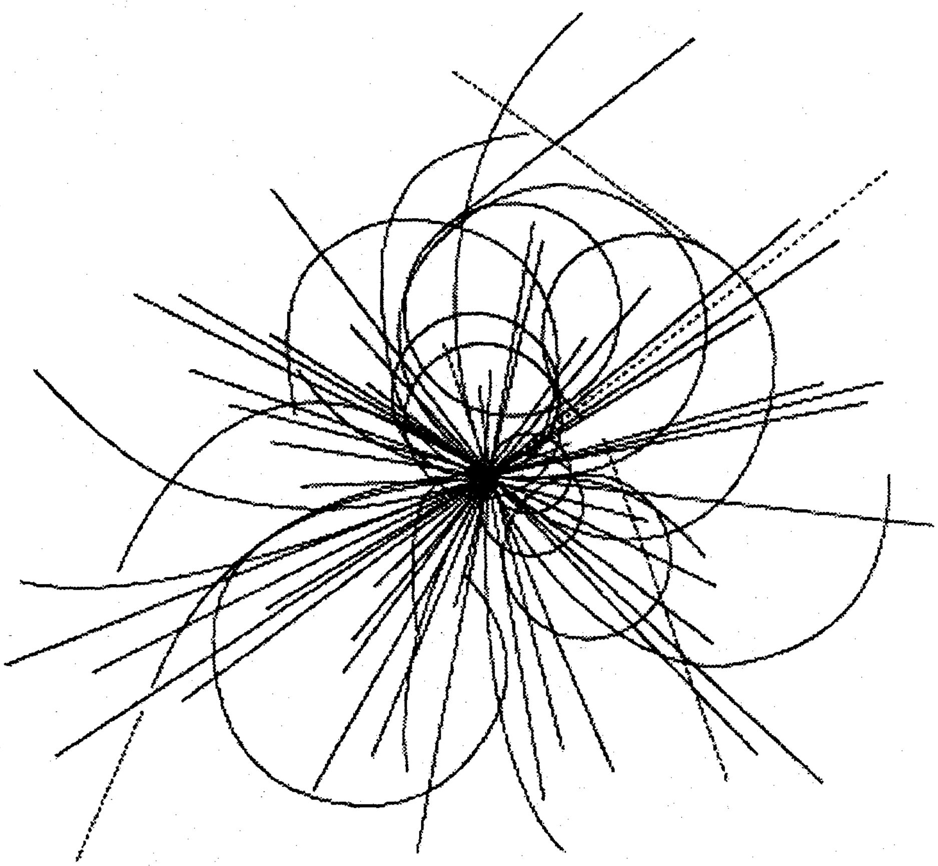

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Constant of Motion for a One-Dimensional and nth-Order Autonomous System and Its Relation to the Lagrangian and Hamiltonian*

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Abstract

A constant of motion is defined for a one-dimensional and nth-differential-order autonomous system. A generalization of the Legendre transformation is given that allows one to obtain a relation among the constant of motion, the Lagrangian, and the Hamiltonian. The approach is used to obtain the constant of motion associated with the nonrelativistic third-differential-order Abraham-Lorentz radiation damping equation.

I. INTRODUCTION

Finding the constant of motion of a dynamical system is one of the best approaches to understanding some of the system's local and global characteristics [1]. In addition, the constants of motion are of considerable importance in physics [2,3], since they are closely connected with the concept of "energy" and invariants of motion [4]. The constant of motion approach has been used to study the relationship among this constant, the Lagrangian, and the Hamiltonian for one-dimensional and second-differential-order dynamical systems [5,6]. In this paper, the study of the constant of motion and its relation with the Lagrangian and the Hamiltonian is extended to one-dimensional and nth-differential-order autonomous systems. The approach is applied to the nonrelativistic third-differential-order Abraham-Lorentz radiation damping problem.

II. CONSTANT OF MOTION

A one-dimensional autonomous system of nth-order of differentiation is described by the equation

$$\frac{d^n x}{dt^n} = F(x, x^{(1)}, \ldots, x^{(n-1)}) \ , \tag{1}$$

where t represents the time, x represents the position, $x^{(i)}$, $i = 1, \ldots, n-1$ is the ith-differentiation with respect to time ($x^{(i)} = d^i x/dt^i$), and the function F does not depend explicitly on time. Using the definitions

$$\xi_i = x^{(i)}, \ i = 1, \ldots, n \tag{2a}$$

and

$$\xi_o = x \quad . \tag{2b}$$

Eq. (1) can be written as the following dynamical system:

$$\frac{d\xi_{i-1}}{dt} = \xi_i \ , \ i = 1, \ldots, n-1 \tag{3a}$$

and

$$\frac{d\xi_n}{dt} = F(x, \xi_1, \ldots, \xi_{n-1}) \ . \tag{3b}$$

A constant of motion of this system is a function,

$$K = K(x, \xi_1, \ldots, \xi_{n-1}) \ , \tag{4}$$

such that its total differentiation with respect to time is null ($dK/dt = 0$), i.e., the function K must be a solution of the following partial differential equation:

$$\sum_{i=1}^{n-1} \xi_i \frac{\partial K}{\partial \xi_{i-1}} + F(x, \xi_1, \ldots, \xi_{n-1}) \frac{\partial K}{\partial \xi_{n-1}} = 0 \ . \tag{5}$$

This equation can be solved by the characteristics method [7], where the equations for the characteristics are given by

$$\frac{dx}{d\xi_1} = \frac{d\xi_1}{d\xi_2} = \ldots = \frac{d\xi_{n-1}}{F(x, \xi_1, \ldots, \xi_{n-1})} = \frac{dK}{0} \ . \tag{6}$$

Finding $n-1$ characteristics curves, $C_i = C_i(x, \xi_1, \ldots, \xi_{n-1}), i = 1, \ldots, n$, the general solution of Eq. (6) is given by any arbitrary function of the characteristics,

$$K = K(C_1, \ldots, C_n) \ . \tag{7}$$

III. RELATION WITH THE LAGRANGIAN

Define the operator Ω^μ as

$$\Omega^\mu = \sum_{i=1}^{\mu} (-1)^{\mu-i} \xi_i \frac{d^{\mu-i}}{dt^{\mu-i}} , \tag{8}$$

and the generalized Legendre transformation as

$$\sum_{\mu=1}^{n-1} \Omega^\mu \frac{\partial L}{\partial \xi_\mu} - L = G , \tag{9}$$

where L is the Lagrangian associated with Eq. (1) and $G(x, \xi_1, \ldots, \xi_{n-1})$ is any arbitrary function. Applying the total time differentiation operator,

$$\frac{d}{dt} = \sum_{\nu=1}^{n-1} x^{(\nu+1)} \frac{\partial}{\partial x^{(\nu)}} , \tag{10}$$

to Eq. (9) and making some rearrangements, the result is

$$\xi_1 \left\{ \sum_{\mu=1}^{n-1} (-1)^{\mu+1} \frac{d^\mu}{dt^\mu} \frac{\partial L}{\partial \xi_\mu} - \frac{\partial L}{\partial x} \right\} = \frac{dG}{dt} . \tag{11}$$

This expression brings about the relation among the constant of motion, the Legendre transformation, and the Lagrangian. If L is the Lagrangian of the system (1) (meaning that the expression inside the braces of Eq. (11) is null), the function G must be a constant of motion of the system ($G = K$). On the other hand, if G is a constant of motion, then L represents the Lagrangian of the system ($\xi_1 \neq 0$).

IV. RELATION WITH THE HAMILTONIAN

Defining the generalized momenta $\omega_\mu, \mu = 1, \ldots, n-1$, as

$$\omega_\mu(x, \xi_1, \ldots, \xi_{n-1}) = \frac{\partial L}{\partial \xi_\mu} , \mu = 1, \ldots, n-1 , \tag{12}$$

the Legendre transformation (9) becomes

$$H = \sum_{\mu=1}^{n-1} \sum_{i=1}^{\mu} (-1)^{\mu-i} \xi_i \omega_\mu^{(\mu-i)} - L , \tag{13}$$

where $\omega_\mu^{(\mu-i)}$ represents the total $(\mu-i)$th differentiation with respect to the time of the new variable ω_μ, and function H is given by

$$H(x,\omega_1,\ldots,\omega_{n-1}) = K\Big(x,\xi_1(x,\omega_1,\ldots,\omega_{n-1}),\ldots,\xi_{n-1}(x,\omega_1,\ldots,\omega_{n-1})\Big) \ . \tag{14}$$

Of course, the jacobian of the transformation ξ to ω must be different from zero. Making the differential variation in Eq. (13) and rearranging terms, the generalized Hamiltonian equations follow:

$$\frac{\partial H}{\partial x} = \sum_{\mu=1}^{n-1}\sum_{i=1}^{\mu}(-1)^{\mu-i}\left[\xi_i\frac{\partial\omega_\mu^{(\mu-i)}}{\partial x} + \omega_\mu^{(\mu-i)}\frac{\partial\xi_i}{\partial x}\right] - \frac{\partial L}{\partial x} \tag{15a}$$

and

$$\frac{\partial H}{\partial\omega_\lambda} = \sum_{\mu=1}^{n-1}\left\{\sum_{i=1}^{\mu}(-1)^{\mu-i}\left[\omega_\mu^{(\mu-i)}\frac{\partial\xi_i}{\partial\omega_\lambda} + \xi_i\frac{\partial\omega_\mu^{(\mu-i)}}{\partial\omega_\lambda}\right] - \omega_\mu\frac{\partial\xi_\mu}{\partial\omega_\lambda}\right\} , \lambda = 1,\ldots,n-1 \ . \tag{15b}$$

V. SPECIAL CASES

Clearly, Eqs. (9), (11), (13), and (15) are reduced themselves to the well-known expressions for a second-order-differential autonomous system ($n = 2$). For a third-order-differential dynamical system, the Legendre transformation (9) represents a second-order partial differential equation of variable coefficients:

$$\left(a - v\frac{d}{dt}\right)\frac{\partial L}{\partial a} + v\frac{\partial L}{\partial v} - L = K \ , \tag{16a}$$

where $v = \xi_1$ is the velocity, $a = \xi_2$ is the acceleration, and the total time differentiation operator is given by

$$\frac{d}{dt} = v\frac{\partial}{\partial x} + a\frac{\partial}{\partial v} + x^{(3)}(t)\frac{\partial}{\partial a} \ . \tag{16b}$$

The solution of Eq. (16) is not trivial and depends on the dynamics, which is determined by the equation

$$x^{(3)} = F(x,v,a) \ . \tag{17}$$

This fact contrasts with the second-order case of Reference 6, where all the dynamics is kept in the constant of motion K.

VI. ABRAHAM-LORENTZ RADIATION DAMPING

The third-order Abraham-Lorentz equation for the electromagnetic radiation of a charged particle [8] that is moving freely in one-dimensional space can be written as

$$x^{(2)} - \tau x^{(3)} = 0 \ , \tag{18a}$$

where the parameter τ characterizes the dissipation of energy and is given by

$$\tau = \frac{2}{3}\frac{\epsilon^2}{mc^3}, \tag{18b}$$

ϵ being the charge of the particle, m its mass, and c the speed of light. The dynamical system associated with Eq. (18) is expressed by the equations

$$x^{(1)} = v \ , \tag{19a}$$

$$x^{(2)} = a \ , \tag{19b}$$

and

$$x^{(3)} = a/\tau \ , \tag{19c}$$

where v represents the velocity, and a is the acceleration of the charge. The constant of motion satisfies the following partial differential equation:

$$v\frac{\partial K}{\partial x} + a\frac{\partial K}{\partial v} + \frac{a}{\tau}\frac{\partial K}{\partial a} = 0 \ . \tag{20}$$

It is not difficult to find from Eqs. (6) and (19) the following two characteristic curves:

$$C_1 = v - \tau a \tag{21a}$$

and

$$C_2 = \tau\Big[v + (v - \tau a)\log(\tau a)\Big] - x \ . \tag{21b}$$

Therefore, the general solution of Eq. (20) is

$$K(x, v, a) = G(C_1, C_2) \ , \tag{22}$$

where G is an arbitrary function of the characteristic curves. Selecting this function of the form

$$G(C_1, C_2) = \frac{1}{2}mC_1^2 + \tau g(C_2) . \qquad (23)$$

one is allowed to recover the usual expression for the nonrelativistic energy when the damping factor τ goes to zero. (Clearly, the arbitrary function g must remain finite for τ equal to zero.) Thus, the constant of motion for this system can be written as

$$K(x, v, a) = \frac{1}{2}mv^2 - \tau mva + \frac{1}{2}\tau^2 ma^2 + \tau g(\tau v + \tau(v - \tau a)\log(\tau a) - x) . \qquad (24)$$

The function g is arbitrary and may be determined from experimental analysis on the radiating system. Substituting this expression in Eq. (16) and solving the resulting partial differential equation, the Lagrangian should be obtained. However, to do this requires much more elaborate and complex analysis that will not be performed here.

VII. CONCLUSIONS

A constant of motion was defined for one-dimensional and nth-differential-order autonomous systems. By generalizing the Legendre transformation on these systems, a relation is obtained between the Lagrangian and the constant of motion. The relation between the constant of motion and the Hamiltonian also follows from the generalized Legendre transformation. Finally, for the nonrelativistic Abraham-Lorentz radiation damping system, an explicit constant of motion is deduced using the above approach.

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