Conf-9410254--6

UCRL-JC-118788
PREPRINT

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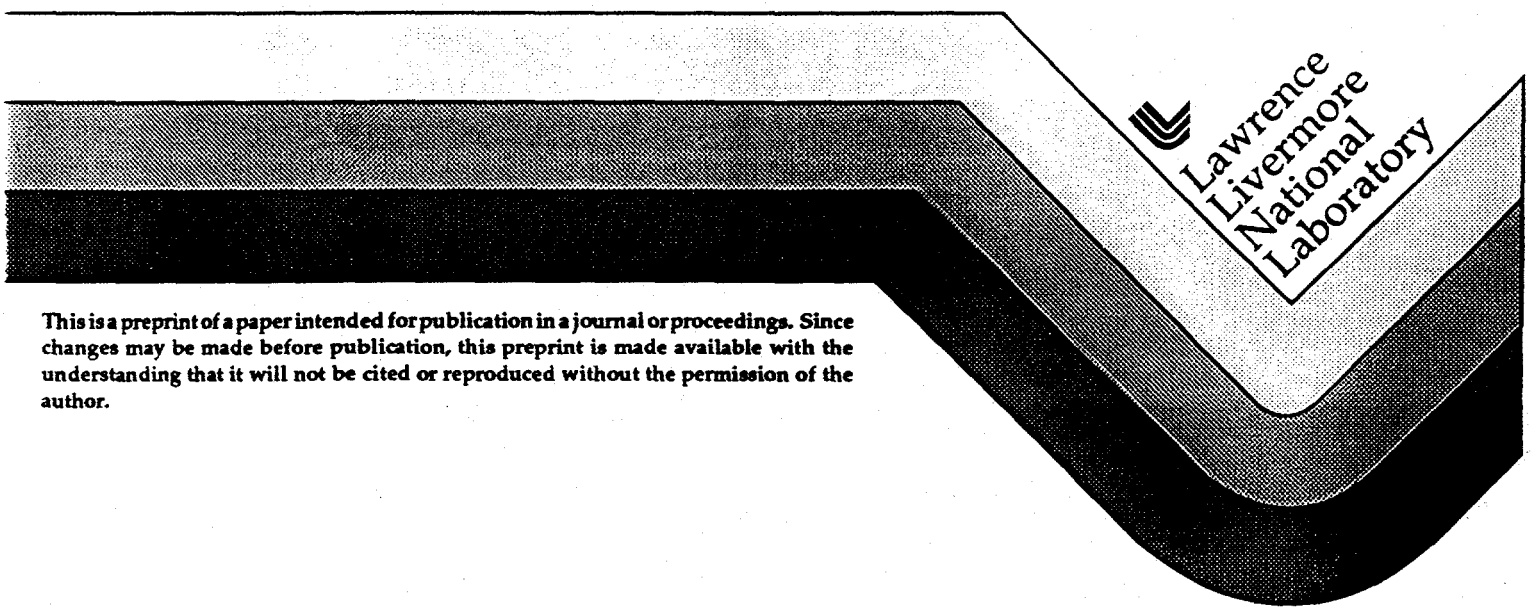
This paper was prepared for submittal to the Eighth Nuclear Explosives Code Developers' Conference (NECDC)
Las Vegas, Nevada
October 25-28, 1994

October 17, 1994



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Consistent Finite-Volume Discretization of Hydrodynamic Conservation Laws for Unstructured Grids

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We consider the conservation properties of a staggered-grid Lagrange formulation of the hydrodynamics equations (SGH). Hydrodynamics algorithms are often formulated in a relatively ad hoc manner in which independent discretizations are proposed for mass, momentum, energy, and so forth. We show that, once discretizations for mass and momentum are stated, the remaining discretizations are very nearly uniquely determined, so there is very little latitude for variation. As has been known for some time, the kinetic energy discretization must follow directly from the momentum equation; and the internal energy must follow directly from the energy currents affecting the kinetic energy. A fundamental requirement (termed isentropicity) for numerical hydrodynamics algorithms is the ability to remain on an isentrope in the absence of heating or viscous forces and in the limit of small timesteps. We show that the requirements of energy conservation and isentropicity lead to the replacement of the usual volume calculation with a conservation integral. They further forbid the use of higher order functional representations for either velocity or stress within zones or control volumes, forcing the use of a constant stress element and a constant velocity control volume. This, in turn, causes the point and zone coordinates to formally disappear from the Cartesian formulation. The form of the work equations and the requirement for dissipation by viscous forces strongly limits the possible algebraic forms for artificial viscosity. The momentum equation and a center-of-mass definition lead directly to an angular momentum conservation law that is satisfied by the system. With a few straightforward substitutions, the Cartesian formulation can be converted to a multidimensional curvilinear one. The formulation in 2D axisymmetric geometry preserves rotational symmetry.

Introduction

This paper discusses a Lagrangian hydrodynamics formulation that is both multidimensional and suitable for arbitrarily connected polygonal or polyhedral zones. In this section, we discuss why such a method is of interest. The *free-Lagrange* method (Crowley, 1970) is one of several principally Lagrange methods suitable for large deformations. The method is characterized by a set of *mesh optimization* operations that provide the ability to reconnect, relax, refine, and reconstruct the calculational mesh as necessary.

The specific algorithms used for the mesh optimization determines the connectivity of the mesh as well as the geometrical complexity of the cells; in 3D, for example, a Delaunay scheme produces a tetrahedral grid while a Voronoi (1908) method yields a polyhedral one. Each of these mesh types has been shown to perform well for some class of problems. The operations (and consequently the mesh types) tend to be mutually exclusive, so that choosing one makes it difficult to take advantage of the others. We wish the choice of mesh optimization to be made at execution time by the code user who may even wish to employ different methods in different parts or at different times in a problem.

Further, difference schemes are often tailored to a particular type of mesh, excluding the possibility of user selection at the time of execution. Because of the foregoing, we have chosen an alternative approach that focuses on developing differencing techniques suitable for arbitrary polygons in 2D and polyhedra in 3D. In this way, the numerical differencing is unaffected by choice of mesh type or optimization algorithm. The only spatial discretization paradigm that seems suited to this situation is the *finite volume* method that replaces differential operators with surface integrals. It is appropriate for multidimensional formulations on unstructured grids formed from polygonal (2D) or polyhedral (3D) cells. In particular we excluded from consideration (a) the *finite element* method that is limited to a relatively small set of polyhedra, and (b) *finite difference* schemes such as that of Schulz (1961) because they are restricted to grids that are logically rectangular. We have recently developed data structures and discretization templates that accommodate arbitrary polygonal or polyhedral zoning and are well suited to the construction of finite volume integrals (Burton, 1992).

Notation and Conventions

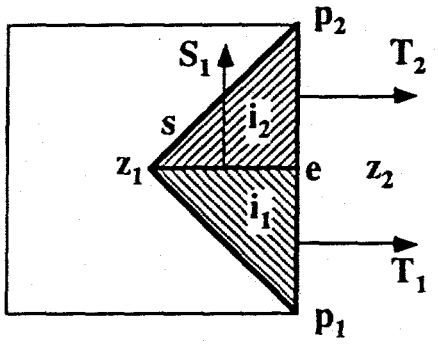

Figure 1. Side template for a rectangular zone.

For ease of visualization, we will present our derivations in the context of the 2D case, although the results are valid in 1D and 3D. Further, although most figures will picture quadrilateral zones, the results are valid for arbitrary polygons in 2D and polyhedra in 3D.

For 2D geometry, the zones are divided into triangular areas that will be called *sides* as shown (darkened) in Figure 1 for the special case of a rectangular zone. The *sides* are significant because they are the templates that provide the connectivity between *points*, *zones*, and so forth. The templates are generalizations of those described by Cooper (1985). Each side is divided into two triangular areas called *corners* and labeled *i*.

The analogous 3D *side* is shown in Figure 2. To aid visualization, we have pictured a simple brick zone, but again the scheme works for any polyhedron. Although it may appear that we have introduced very fine detail into the differencing templates, such geometrical detail is necessary because the polyhedral faces are generally non-planar. Each side is further divided into two tetrahedral *corner* volumes labeled *i*.

Figures 1 and 2 also show the surface area vectors $\mathbf{S}_i$ and $\mathbf{T}_i$ that are fundamental in defining finite volume surface integrals.

Indices are used to point to memory locations in which data, such as physical coordinates, might be stored. The *p* index refers to points that define the mesh. To form numerical integrals, other auxiliary points are also needed corresponding to zone (*z*), edge (*e*), and face (*f*) centers. For each corner index *i* there is an implied set of indices $\{p, z, e, f, s\}$, and for each side index *s* there is an implied set of primary indices $\{p_1, p_2, z, f\}$ and an auxiliary set $\{i_1, i_2, z_2, e\}$.

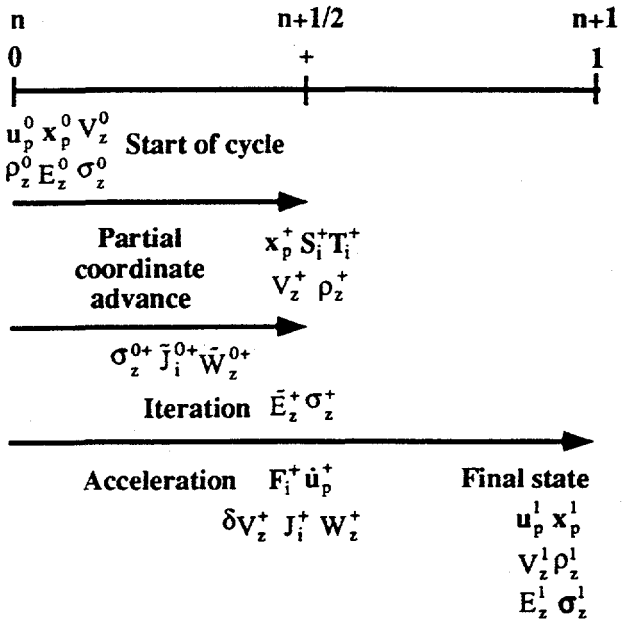

Figure 3. Time centering of variables.

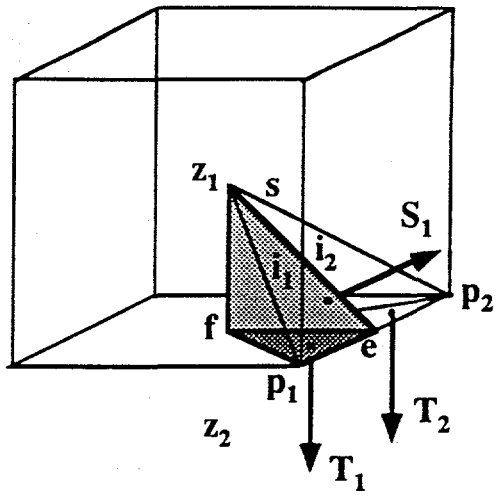

Figure 2. Side template for a hexahedral zone. S_1 is normal to plane *efz1* and T_1 is normal to plane *fep1*.

Conservation Laws in Cartesian Geometry

We consider the conservation properties of a staggered-grid Lagrange formulation of the hydrodynamics equations (SGH) which is an extension of a 1D scheme, denoted VNR, due to von Neumann and Richtmyer (1950). The method is second-order accurate in the discretized momentum equation on uniform grids. The term *staggered* refers to spatial centering in which position, velocity, and kinetic energy are centered at nodes, while density, pressure, and internal energy are within cells. Available space permits only discussion of a multidimensional Cartesian formulation. Formulations in curvilinear geometry are described by Burton (1994).

Unlike the original VNR scheme which was also temporally staggered, our method uses a temporal centering, termed *even-time*, in which most final-state variables are centered at the full timestep. The simultaneous advancement of both position and internal energy requires a logically implicit calculation that is approximated using a predictor-corrector procedure to calculate a half-timestep acceleration used for advancing velocity.

The time-centering of the variables is shown in Figure 3 for purposes of establishing notation. The exact definitions of the variables will be discussed later. We use the superscript notation {-, 0, +, 1} to indicate time centering of {n-1/2, n, n+1/2, and n+1} respectively. Subscripts refer to spatial centering. Capitalized quantities are extensive while lower cased quantities are intensive or specific.

Mass Conservation

Since we consider only a Lagrangian formulation, we postulate that the mass in each corner volume is constant.

$$M_i = \text{constant} \tag{1}$$

Zone and point mass are also constant in the Cartesian formulation and are given by

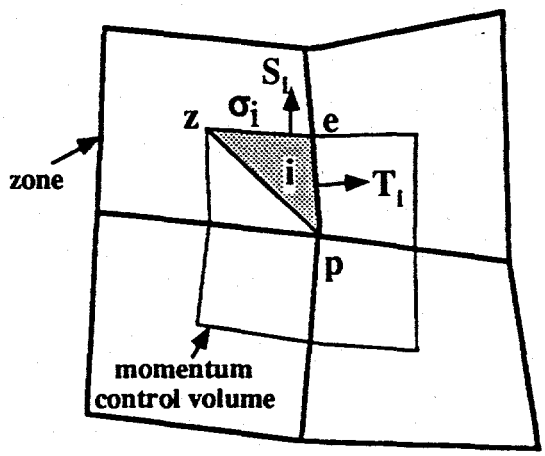

Figure 4. Zone and momentum control volumes, showing surface vectors.

$$M_z = \sum_i^z M_i \qquad M_p = \sum_i^p M_i \tag{2}$$

where the notation $\sum_i^p$ and $\sum_i^z$ refer to sums over corners *i* surrounding respectively a point *p* or a zone *z*. In practice, the zone mass is first computed from the specified initial density and volume, and the resulting mass is partitioned to M_i based upon constant mass fractions φ_i

$$M_i = \varphi_i M_z \tag{3}$$

The fractions φ_i are calculated from the initial configuration and can be defined in several ways. Our experience to date has been that best numerical results are obtained from a surface area weighting

$$\varphi_i = \frac{\left|\mathbf{T}_i \bullet \hat{\mathbf{x}}_{pz}\right|}{\sum_i^z \left|\mathbf{T}_i \bullet \hat{\mathbf{x}}_{pz}\right|} \tag{4}$$

where $\hat{\mathbf{x}}_{pz}$ is a unit vector from the zone center to the respective point and $\mathbf{T}_i$ is a zone surface vector.

Forces

As shown in Figure 4, each point *p* is surrounded by surfaces that define the *momentum control volume*. Forces result from reversible and irreversible stress fields σ and π that exist within the zone or side. The stress on each element of surface S_i of the momentum control volume results in forces R_i due to reversible fields and Q_i due to the irreversible fields. We will show in a later section that the reversible stress fields must be centered at the zone, but will not at this point restrict the notation to reflect this. Sign conventions are such that the pressure is positive in compression while stress is positive in tension.

Since stress at + depends upon energy while energy is incremented from 0 to + by the Pδv work, the two equations must be solved simultaneously. This can be done in many ways, and we simply note that we use a predictor-corrector iteration.

Then let us assume that the stresses are known, and we can calculate forces F_i time centered at + on the surfaces S_i

$$\mathbf{F}_i^+ = \mathbf{R}_i^+ + \mathbf{Q}_i^+ \tag{5}$$

$$\mathbf{R}_i^+ = \mathbf{S}_i^+ \bullet \sigma_i^+ \qquad \mathbf{Q}_i^+ = \mathbf{S}_i^+ \bullet \pi_i^+ \tag{6}$$

so that the total force on point *p* is

$$\mathbf{F}_p^+ = \sum_i^p \mathbf{F}_i^+ \tag{7}$$

Conservation requires that the forces sum to zero on the boundary of the momentum control volume (Newton's third law); that is,

$$\mathbf{F}_{i_1}^+ = -\mathbf{F}_{i_2}^+ \tag{8}$$

Since the opposing surface vectors are of opposite sign, the finite volume method automatically accomplishes this providing $\sigma_{i_1}^+ = \sigma_{i_2}^+$ at the surface.

Acceleration and Velocity

The control volume plays the role of an accounting device in which the conserved momentum of the system can always be accumulated. The force at + imparts an acceleration to the momentum control volume, presumably to the center of mass. Later, we will show that we must take this acceleration to be constant throughout the volume, although we do not explicitly make this assumption at this point. The acceleration is calculated from

$$M_p \dot{\mathbf{u}}_p^+ = \mathbf{F}_p^+ \tag{9}$$

which is actually a finite volume surface integral.

Velocity is integrated using a central difference scheme

$$\mathbf{u}_p^1 = \mathbf{u}_p^0 + \dot{\mathbf{u}}_p^+ \delta t^+ \tag{10}$$

The value at + is not arbitrary, but is chosen for consistency with kinetic energy equations to be described below.

$$\mathbf{u}_p^+ \equiv \tfrac{1}{2}\left(\mathbf{u}_p^0 + \mathbf{u}_p^1\right) \tag{11}$$

Kinetic Energy

Trulio and Trigger (1961a, 1961b) observed that the VNR method was not energy-conserving and proposed conservative methods for the one-dimensional equations. Their 1D formulation retained the spatial staggering of VNR but relinquished the temporal staggering. Burton (1990a) derived a temporally staggered form for unstructured multidimensional grids. The following derivation is for an even-time scheme. By considering the momentum equations in the half intervals [0, +] and [+, 1] and dotting them into the respective velocities at 0

and 1, the following evolution equation for kinetic energy can be derived

$$\begin{aligned} K_p^1 - K_p^0 &= \left(K_p^+ - K_p^0\right) + \left(K_p^1 - K_p^+\right) \\ &= \tfrac{\delta t^+}{2} \mathbf{F}_p^+ \bullet \left(\mathbf{u}_p^0 + \mathbf{u}_p^1\right) \\ &= \sum_i^p J_i^+ \end{aligned} \tag{12}$$

where we have identified the following form preserving definitions of kinetic energy

$$K_p^0 \equiv \tfrac{M_p}{2} \mathbf{u}_p^0 \bullet \mathbf{u}_p^0 \qquad K_p^+ \equiv \tfrac{M_p}{2} \mathbf{u}_p^1 \bullet \mathbf{u}_p^0 \tag{13}$$

The latter corresponds to a definition originally made by Trulio and Trigger. We also identified energy currents J between the zone and the point arising from forces on the surface S_i

$$J_i^+ \equiv \delta t^+ \mathbf{F}_i^+ \bullet \mathbf{u}_p^+ \tag{14}$$

Work

The discrete SGH system has only two energy reservoirs, internal in the zone and kinetic at the node. The zone is viewed as a system without explicitly modeled kinetic energy but with a velocity boundary condition. Similarly, the node is a system having only kinetic energy. Thus, the work done by a zone on the surrounding nodes is simply the sum of the exiting energy currents about the zone

$$W_z^+ = \sum_i^z J_i^+ = \delta t^+ \sum_i^z \mathbf{F}_i^+ \bullet \mathbf{u}_p^+ \tag{15}$$

This work expression conserves energy exactly but does not reduce to PδV where δV is a difference of volumes that are functions of coordinates $V_Z=V(x_p)$. Most SGH hydrodynamics formulations simply assert that the work is equal to PδV. Such formulations (which we will denote as PDV), do not exactly conserve energy. Implications of this will be discussed below.

Internal Energy

Energy is exchanged through the zone boundary via heat transfer δH and work W. The zonal internal energy is then formally obtained from the *first law of thermodynamics*

$$\delta E_z^+ = \delta H_z^+ - W_z^+ \tag{16}$$

Relationships which are taken for granted in analytical thermodynamics are not automatically satisfied in numerical work. In particular, if the work and heat terms correctly represent the energy transfer with the surroundings, Equation (16) is also a statement of energy conservation . This is true of our work expression. However, a PDV algorithm which simply substitutes PδV in (16) might formally claim to satisfy an equation like the first law, but the result is no longer a statement of energy conservation.

Entropy

The work can be resolved into parts W_r resulting from reversible forces **R** and W_q due to irreversible viscous or plastic forces **Q**. The energy balance for the zone can then be written

$$\begin{aligned} \delta E_z^+ &= T_z^+ \, \delta S_z^+ - W_r \\ &= \left(\delta H_z^+ - W_q\right) - W_r \end{aligned} \tag{17}$$

The *second law of thermodynamics*

$$T_z^+ \, \delta S_z^+ \equiv \delta H_z^+ - W_q \geq \delta H_z^+, \tag{18}$$

where S is the *entropy* and T is the *temperature*, can be formally satisfied if we require that all viscous or plasticity models be dissipative, *i.e.*,

$$W_q = \delta t^+ \sum_i^z Q_i^+ \bullet \mathbf{u}_p^+ \leq 0 \tag{19}$$

To clearly establish terminology, a *reversible* process has $W_q = 0$, an *adiabatic* process $\delta H = 0$, and an *isentropic* or reversible adiabatic process has both.

Isentropic processes. A fundamental requirement for hydrodynamics algorithms is the ability to remain numerically on an isentrope of the equation of state (EOS) in the absence of heating or viscous forces and in the limit of small timesteps. We will refer to this as the *isentropicity condition.*

Our formulation formally conserves energy and seems to satisfy the second law, but needs further refinement to satisfy the isentropicity condition. That is, under isentropic conditions, the energy change for the zone involves only reversible fields and is exactly given by

$$\begin{aligned} \delta E_z^+ &\to -\delta t^+ \sum_i^z \mathbf{R}_i^+ \bullet \mathbf{u}_p^+ \\ &= -W_r \end{aligned} \tag{20}$$

so that

$$T_z^+ \, \delta S_z^+ = \delta E_z^+ + W_r \to 0 \tag{21}$$

Further, if we were to explicitly construct an *integrated entropy* from this, it would properly not change under these conditions. Although it may appear that (21) is sufficient to satisfy the isentropicity condition, this is deceptive. Although this integrated entropy does provide the theoretical basis for constraining the form of the viscous stresses, it is not used explicitly except perhaps as a calculational diagnostic.

Again we are faced with relationships that are consistent analytically, but not necessarily numerically. We have not yet guaranteed consistency between our integrated entropy and that defined by the EOS. For points sufficiently near each other, the EOS was explicitly constructed to satisfy $\delta E = -P\delta V$ along an isentrope; and we must be certain the integrated entropy also satisfies this.

In our formulation, δE is fixed by the energy conserving formulation, and we have not as yet defined δV. If we make no attempt to guarantee exact consistency of the two, our numerical model will generally yield $\delta E \neq -P\delta V$ even though no viscous forces are present. Failure of a differencing scheme to satisfy such a consistency relationship will appear as unintended entropy errors (deviations from the isentrope). The solution, of course, is to define a consistent δV which is done in the next section.

Since, in the traditional PDV formulation, δE is simply set to $-P\delta V$, any expression for δV is consistent with the EOS. Consequently, PDV formulations automatically satisfy the isentropicity condition independent of their inability to exactly conserve energy. It should be noted that there will always be accuracy issues associated with large steps because the $P\delta V$ calculation is implicit and not algebraically exact. We are not concerned with this type of error which is controllable, but rather with a more serious potential error associated with the form of δE.

Volume

Constraint on the stress field. *It follows from the preceding discussion that the form of the strain or volume calculation cannot be arbitrarily chosen, but is in fact dictated by the initial discretization chosen for the momentum equation.* That is, it must be defined such that the following is true for a fluid

$$W_z^+ = -\delta t^+ \sum_i^z P_i^+ S_i^+ \bullet \mathbf{u}_p^+ \rightarrow P_z^+ \delta V_z^+ \tag{22}$$

The first step in establishing the desired relationship is to factor the zonal pressure P_Z from the sum. This can only be done if $P_i = P_Z$. We are then constrained to use only a so called *constant stress element*, resulting in

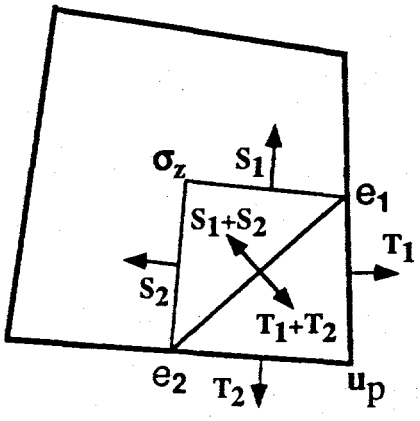

Figure 5. Path independence of the volume integral.

$$W_z^+ = P_z^+ \left\{ -\delta t^+ \sum_i^z S_i^+ \bullet \mathbf{u}_p^+ \right\} \tag{23}$$

In particular, extensions to SGH such as side-centered stresses used in the TTS method (Browne and Wallick, 1971) must be excuded. Note that we are discussing constraints only on reversible fields, not viscous fields. For solids, it is necessary to factor the reversible work into a product of a stress tensor and a deformation tensor (Burton, 1994). In the next section, we establish that the above corresponds to a reasonable discretization for δV; *i.e.*,

$$\delta V_z^+ = -\delta t^+ \sum_i^z S_i^+ \bullet \mathbf{u}_p^+ \tag{24}$$

Volume integral. Obviously, we cannot simply postulate that $V=V(x)$. Instead, we propose an evolution equation for volume in conservation form and will verify that it is equivalent to Equation (24)

$$\delta V_z^+ = \delta t^+ V_z^+ (\nabla \bullet \mathbf{u}) = \delta t^+ \oint_z d\mathbf{T} \bullet \mathbf{u} \rightarrow \delta t^+ \sum_i^z T_i^+ \bullet \mathbf{u}_p^+ \tag{25}$$

Note that volume is rigorously conserved even though it may not be exactly what might be calculated directly from the coordinates. This is easily shown to reduce to the form required by the reversibility constraint. We rely upon the fact that the velocity $\mathbf{u}_p$ is constant within the momentum control volume and that $S_1+S_2 = -(T_1+T_2)$ as shown in Figure 5. This is simply a statement of the path independence of the integral between e_1 and e_2 providing σ_Z and $\mathbf{u}_p$ are constant along the path. The same result obtains in 3D. Then we have shown that the volume change can be written as in Equation (24)

$$\delta V_z^+ = -\delta t^+ \sum_i^z S_i^+ \bullet \mathbf{u}_p^+ \tag{26}$$

As was the case with stress, use of a higher order spatial dependence for $\mathbf{u}_p$ would destroy the path independence of the integral. In particular, velocity or acceleration interpolation cannot be used, such as has been suggested to correct for the center-of-mass of the momentum control volume (Margolin and Nichols, 1983). Our point is that such extensions to this energy-conserving form of SGH can be unintentionally dissipative, not that they fail to improve spatial accuracy or should not be employed.

Defining V_z^{00} to be the initial zone volume, the zone volume and specific volume used in EOS calculations is given by

$$V_z = V_z^{00} + \sum_n \delta V_z^+ \qquad v_z = \frac{V_z}{M_z} \tag{27}$$

Coordinates

After substituting a conservative integral for the explicit volume expression, examination of the difference equations reveals an extremely significant fact. The momentum contained in the control volume about a point remains important as the primary momentum accounting device, but *the specific point coordinates* $\mathbf{x}_p$ *do not play a role in the differencing*. What actually matters are the coordinates of the edge centers $\mathbf{x}_e$ (and face centers $\mathbf{x}_f$ in 3D) that are advanced using a momentum conserving average of adjacent point velocities given by

$$\mathbf{u}_e^+ = \frac{1}{M_e}\sum_i^e M_i \mathbf{u}_p^+ \tag{28}$$

with similar expressions for *f* and *e* at times 0 and 1. In order to mechanically form the surface S_i vectors, control points $\mathbf{x}_z$ and $\mathbf{x}_p$ are needed in addition to $\mathbf{x}_e$ and $\mathbf{x}_f$, but mathematically drop out of the integrals.

The coordinates for the auxiliary points can be directly integrated. However, since we use constant mass weights, a more economical alternative procedure for calculating the auxiliary points is possible. If the point velocity is formally integrated

$$\begin{aligned} \mathbf{x}_p^+ &= \mathbf{x}_p^0 + \tfrac{\delta t^+}{2}\mathbf{u}_p^0 \\ \mathbf{x}_p^1 &= \mathbf{x}_p^+ + \tfrac{\delta t^+}{2}\mathbf{u}_p^1 = \mathbf{x}_p^0 + \delta t^+ \mathbf{u}_p^+ \end{aligned} \tag{29}$$

then the auxiliary coordinates can be found using the same weighted averages as in the velocity equations

$$\mathbf{x}_e^+ = \frac{1}{M_e}\sum_i^e M_i \mathbf{x}_p^+ \tag{30}$$

with similar expressions for $\mathbf{x}_f$ and $\mathbf{x}_z$. We emphasize that (30) is simply a calculational shortcut and the differencing remains logically independent of the point coordinates.

Hourglass instability and coordinates. It is well known that SGH suffers from spurious modes on the scale of the mesh size because of degrees of freedom unconstrained by the difference equations. One such mode is the *hourglass* shown in Figure 6a. By definition, an hourglass mode is any mode of deformation that does not change the zone volume or strain and therefore produces no response from the constitutive model. There exist many *ad hoc* artificial viscosity schemes successful in reducing hourglass distortion without affecting physical *shear* modes. The smoothing viscosities discussed in a later section are effective against instabilities such as hourglassing.

However, unless we introduce special artificial viscosities that are themselves sensitive to hourglassing, the only manifestation of the spurious mode is through the point coordinates $\mathbf{x}_p$. But, as we have shown, the specific point coordinates do not play a role in the differencing either. It follows that the major consequence of hourglass modes is simply that the grid may appear distorted, not that the quality of the solution has been compromised. We are justified therefore in independently moving the points, without adjusting velocity, anywhere aesthetics demands. There is not even a requirement that edges formed from pairs of points be straight lines. For example, we could advance the point using the center-of-mass velocity of the surrounding zones

$$\begin{aligned} \mathbf{U}_p &= \frac{1}{M_p}\sum_i^p M_i \mathbf{u}_z \\ \mathbf{X}_p^+ &= \mathbf{X}_p^0 + \tfrac{\delta t^+}{2}\mathbf{U}_p^0 \\ \mathbf{X}_p^1 &= \mathbf{X}_p^+ + \tfrac{\delta t^+}{2}\mathbf{U}_p^1 = \mathbf{X}_p^0 + \delta t^+ \mathbf{U}_p^+ \end{aligned} \tag{31}$$

As a practical matter, these results do not eliminate from concern other instabilities such as *chevron* modes (Figure 6b) that occur in fluids because the constitutive model does not respond to shear deformation. Further, although pure hourglass modes do not themselves degrade the solution, they are seldom pure. That is, velocity patterns that appear in one zone as hourglassing typically show up in an adjacent one as a chevron or other mode.

Angular Momentum

In Burton (1990b, 1994), we showed that in Cartesian geometry, a zone-centered conservation law for angular momentum can be derived since the zone-center is also the center-of-mass (CM). No similar law can be proved at the point because it is not a CM. Although conservation of angular momentum on the global scale has never been in question, this work showed conservation on a scale somewhat larger than a zone. The lack of such conservation on the scale of momentum control volumes gives rise to numerical instabilities such as chevron and hourglass modes.

Artificial Viscosity

Artificial viscosity serves two principal functions: first to attain the correct shock dissipation, and second to smooth numerical noise. The requirement that the viscous work be dissipative $W_q \le 0$ greatly constrains the permissible form of artificial viscosity equations. Two spatial centerings of the viscosity tensor are common. We consider here a side-centered form in which the viscous force is produced on the surfaces S. A more common zone-centered form is discussed in Burton (1994).

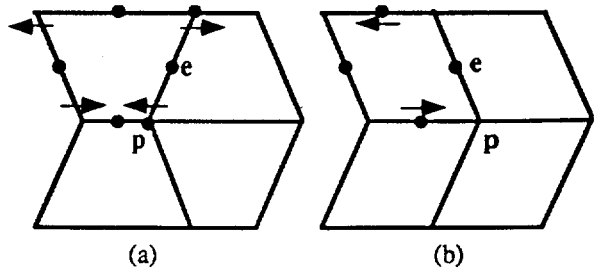

Figure 6. Typical hourglass (a) and chevron (b) instabilities.

For a side centered viscous stress, the viscous work reduces to

$$\begin{aligned} W_q &= \delta t^+ \sum_i^z Q_i^+ \bullet \mathbf{u}_p^+ \\ &= \delta t^+ \sum_i^z \boldsymbol{\pi}_s^+ : \mathbf{S}_i^+ \mathbf{u}_p^+ \\ &= \delta t^+ \sum_s^z \boldsymbol{\pi}_s^+ : \mathbf{S}_s^+ \mathbf{u}_{12} \end{aligned} \tag{32}$$

where

$$\begin{aligned} \mathbf{u}_{12} &\equiv \mathbf{u}_{p_1}^+ - \mathbf{u}_{p_2}^+ \\ \mathbf{S}_s^+ &\equiv \mathbf{S}_{i_1}^+ = -\mathbf{S}_{i_2}^+ . \end{aligned} \tag{33}$$

We can guarantee dissipation by choosing a viscous stress of the form

$$\begin{aligned} \boldsymbol{\pi}_s^+ &= -\rho_z f \, \hat{\mathbf{S}}_s^+ |\hat{\mathbf{u}}_{12} \bullet \hat{\mathbf{c}}| \, \mathbf{u}_{12} \\ Q_s^+ &= \mathbf{S}_s^+ \bullet \boldsymbol{\pi}_s^+ \\ &= -\rho_z f \, \mathbf{S}_s^+ \bullet \hat{\mathbf{S}}_s^+ |\hat{\mathbf{u}}_{12} \bullet \hat{\mathbf{c}}| \, \mathbf{u}_{12} \\ &= -\rho_z f \, |\mathbf{S}_s| |\hat{\mathbf{u}}_{12} \bullet \hat{\mathbf{c}}| \, \mathbf{u}_{12} \end{aligned} \tag{34}$$

where

$$f = (q_2 |\mathbf{u}_{12}| + q_1 c) \Theta(\mathbf{u}_{12}) \tag{35}$$

$\hat{\mathbf{c}}$ is the direction of propagation, c is the soundspeed, and ρ_z is the zone density. The function Θ is unity in compression and zero in expansion. The quantities q_1 and q_2 are multipliers for the linear and quadratic viscosities. In the above form, the restoring force on each side is proportional to the velocity difference $\mathbf{u}_{12}$ between the two points. The *ad hoc* factor $|\hat{\mathbf{u}}_{12} \bullet \hat{\mathbf{c}}|$ has been introduced to eliminate non-shock components of $\mathbf{u}_{12}$ such as those which might arise from convergent flow.

A variation on a form due to Barton (1988)

$$\begin{aligned} \boldsymbol{\pi}_s^+ &= -\rho_z f \, |\hat{\mathbf{u}}_{12} \bullet \hat{\mathbf{c}}| \, \hat{\mathbf{u}}_{12} \mathbf{u}_{12} \\ Q_s^+ &= \mathbf{S}_s^+ \bullet \boldsymbol{\pi}_s^+ \\ &\rightarrow -\rho_z f \, |\mathbf{S}_s \bullet \hat{\mathbf{u}}_{12}| |\hat{\mathbf{u}}_{12} \bullet \hat{\mathbf{c}}| \, \mathbf{u}_{12} \end{aligned} \tag{36}$$

also produces a restoring force proportional to the velocity difference $\mathbf{u}_{12}$ and is rigorously dissipative. The absolute value of the dot product $\mathbf{S}_s \bullet \hat{\mathbf{u}}_{12}$ has been taken to avoid pathological attractive forces that could otherwise occur.

Yet another variation comes from calculating a side gradient $\nabla_s \mathbf{u}$ from the motion of the side points {*p1, p2, z, f*}

$$\begin{aligned} \boldsymbol{\pi}_s^+ &\propto -\rho_z \, f \, |\mathbf{x}_{12}| \, \nabla_s \mathbf{u}^T \\ Q_s^+ &= \mathbf{S}_s^+ \bullet \boldsymbol{\pi}_s^+ \end{aligned} \tag{37}$$

which produces a resulting force that is not necessarily parallel to $\mathbf{u}_{12}$, so that dissipation is not guaranteed but has not been a problem in practice. This form (and the next) can be made rigorously dissipative by retaining only the component of the gradient parallel to $\mathbf{u}_{12}$

$$\nabla_s \mathbf{u} \rightarrow \left[\nabla_s \mathbf{u} : \hat{\mathbf{u}}_{12} \hat{\mathbf{u}}_{12} \right] \hat{\mathbf{u}}_{12} \hat{\mathbf{u}}_{12} \tag{38}$$

We have not fully investigated the consequences of this modification.

A variation of the previous form is a side-centered viscosity used only for purposes of smoothing

$$\begin{aligned} \boldsymbol{\pi}_s^+ &\propto -\rho_z \, c \, |\mathbf{x}_{12}| \left(\nabla_s \mathbf{u} - \overline{\nabla_s \mathbf{u}} \right)^T \\ Q_s^+ &= \mathbf{S}_s^+ \bullet \boldsymbol{\pi}_s^+ \end{aligned} \tag{39}$$

Again dissipation is not guaranteed. After decomposing the gradient into volumetric, deviatoric, and rotational components, this form becomes a multidimensional generalization of the spurious-vorticity-damping method (Burton, 1990b) and is effective against both chevron and hourglass instability.

Conclusions

Hydrodynamics algorithms are often formulated in a relatively *ad hoc* manner in which independent discretizations are proposed for mass, momentum, energy, and so forth. We have shown that, once discretizations for mass and momentum are stated, the remaining discretizations can be determined in a consistent manner, so there is little latitude for variation. The resulting analysis provided some known results and several previously unreported surprises.

- As has been known (and largely ignored for some years) the kinetic energy discretization must follow directly from the momentum equation; and the internal energy must follow directly from the energy currents affecting the kinetic energy.
- Traditional PDV formulations of SGH do not exactly conserve energy, although they do satisfy the isentropicity condition.
- In our energy-conserving form of SGH, the requirement of isentropicity unexpectedly forces the replacement of the usual volume calculation with a conservation integral.
- Isentropicity further restricts the use of higher order functional representations for either velocity or stress within zones or control volumes, forcing the use of a constant stress element and a constant velocity control volume.
- In turn, the constant stress model causes the zone center coordinates to formally disappear from the formulation. Likewise the constant velocity model causes the point coordinates to formally disappear from the Cartesian formulation, removing the direct need for hourglass corrections
- The form of the work equations and the requirement for dissipation by viscous forces strongly limits the possible algebraic forms for artificial viscosity. We have proposed a distinction based upon the shock dissipation and numerical smoothing functions of artificial viscosity and have presented forms for

both, as well as two spatial centerings of the viscosity tensor.

- The momentum equation and a center-of-mass definition lead directly to an angular momentum conservation law that is satisfied by the system. This work shows conservation on a scale somewhat larger than a zone. The lack of such conservation within the momentum control volumes gives rise to numerical instabilities such as chevron and hourglass modes.
- In Burton (1994), it was shown that, by a few straightforward substitutions, the Cartesian formulation can be converted to a multidimensional curvilinear one. The resulting equations for momentum and quantities derived from it are not in strict conservation form and some conservation error occurs near the axis.
- The formulation in 2D axisymmetric geometry was also shown to preserve rotational symmetry.

Acknowledgments

The author wishes to acknowledge A. Harrison, D. Miller, D. Lappa, and T. Palmer for their collaboration and helpful criticisms of the manuscript. He further gratefully acknowledges D. Nielsen Jr., D. Nowak, the Defense & Nuclear Technologies Directorate, and the LDRD committee for their continuing support of the work. This work was performed under the auspices of the U.S. Department of Energy, by Lawrence Livermore National Laboratory under Contract #W-7405-Eng-48.

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