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# Ground States of Semi-Linear Diffusion Equations *

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**Abstract**

This article is concerned with the uniqueness of nontrivial nonnegative solutions of semilinear diffusion equations of the type $\Delta u + f(u) = 0$ in radially symmetric domains in $\mathbf{R}^N$ $(N \geq 2)$, where $f$ is defined and continuous on $[0, \infty)$ and Lipschitz continuous on $(0, \infty)$, with $f(0) = 0$.



## 1 Statement of Results

Let $f$ be defined on $[0, \infty)$, such that $f$ is continuous on $[0, \infty)$ and Lipschitz continuous on $(0, \infty)$, with $f(0) = 0$. We consider the boundary value problem

$$\Delta u + f(u) = 0,\ x \in \mathbf{R}^N\,(N \geq 2);\ \lim_{|x| \to \infty} u(x) = 0. \tag{1}$$

A *ground state solution* of (1) is a nontrivial solution that does not change sign. (We assume that it is nonnegative everywhere.) We prove the following result.

**Theorem 1** *Let*

$$F(u) = \int_0^u f(v)dv,\ u \geq 0, \tag{2}$$

*and*

$$\beta = \inf\{u > 0 : F(u) > 0\}. \tag{3}$$

*If $\beta > 0$ and $u \mapsto f(u)/(u - \beta)$ is monotone nonincreasing on $(\beta, \infty)$, then (1) admits at most one ground state solution.*

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This result generalizes and extends earlier results of McLeod and Serrin [1], Peletier and Serrin [2], and Peletier and Serrin [3].

In Section 2 we present a brief outline of the proof of the theorem; details can be found in our article [4]. A further generalization for quasilinear diffusion equations of the type $\nabla \cdot (a(|\nabla u|)\nabla u + f(u) = 0$ can be found in our article [6].

The proof of the theorem generalizes to the case of a bounded radially symmetric domain in the following sense. If $f$ satisfies the conditions of the theorem, then the boundary value problem

$$\Delta u + f(u) = 0,\ x \in B_R;\ u(x) = 0,\ \mathbf{n} \cdot \nabla u(x) = 0,\ x \in \partial B_R, \tag{4}$$

admits at most one ground state solution. Here, $B_R$ is the ball of finite radius $R$ centered at the origin in $\mathbf{R}^N$; $\partial B_R$ is its boundary, and $\mathbf{n}$ is the outward unit normal at the boundary. Notice that the zero normal gradient condition at the boundary must be specified in this case. (It is satisfied automatically in the unbounded case.)

# 2 Proof of Theorem 1

## 2.1 Preliminaries

Any ground state solution of (1) is radially symmetric (see [5]), so $u$ depends only on $r = |x|$. If $'$ denotes differentiation with respect to $r$, then $u$ satisfies

$$u'' + \frac{N-1}{r} u' + f(u) = 0,\ r > 0; \tag{5}$$

$$u'(0) = 0;\ \lim_{r \to \infty} u(r) = 0. \tag{6}$$

Two identities play a crucial role in the following analysis. They are obtained by multiplying (5) by $u'$ and $r^{2(N-1)}u'$, respectively, and integrating over $(r_1, r_2)$,

$$\left[\frac{1}{2}(u'(r))^2 + F(u(r))\right]_{r_1}^{r_2} = -(N-1) \int_{r_1}^{r_2} \frac{1}{s}(u'(s))^2\, ds, \tag{7}$$

$$\left[r^{2(N-1)} \left(\frac{1}{2}(u'(r))^2 + F(u(r))\right)\right]_{r_1}^{r_2} = 2(N-1) \int_{r_1}^{r_2} s^{2N-3} F((u(s))\, ds. \tag{8}$$

One can show, using (7), that $\lim_{r\to\infty} u'(r)$ exists and

$$\lim_{r \to \infty} u'(r) = 0; \tag{9}$$

and similarly, using (8), that $\lim_{r\to\infty} r^{2(N-1)}(\frac{1}{2}(u'(r))^2 + F(u(r)))$ exists and

$$\lim_{r \to \infty} r^{2(N-1)} \left(\frac{1}{2}(u'(r))^2 + F(u(r))\right) = K, \tag{10}$$

where $K = 0$ if $N = 2$; if $N > 2$, then $\lim_{r\to\infty} r^{N-2}u(r) = (N-2)^{-1}\sqrt{2K}$. Letting $r_2 \to \infty$ in (7), we obtain the identity

$$\frac{1}{2}(u'(r))^2 + F(u(r)) = (N-1)\int_r^\infty \frac{1}{s}(u'(s))^2\, ds,\ r \geq 0, \tag{11}$$

and similarly, from (8),

$$r^{2(N-1)}\left(\frac{1}{2}(u'(r))^2 + F(u(r))\right) = K - 2(N-1)\int_r^\infty s^{2N-3}F((u(s))\, ds,\ r \geq 0. \tag{12}$$

We observe that, whereas (9) follows from the definition of a ground state solution, the analogous condition $u'(R) = 0$ must be *imposed* if the domain under consideration is the ball $B_R$; cf. (4).

**Lemma 1** *If $u$ is a ground state solution of (1), then $u(0) > \beta$.*

**Proof.** Taking $r = 0$ in (11), we find that $F(u(0)) > 0$; hence, $u(0) > \beta$. □

**Lemma 2** *Any ground state solution of (1) is monotonically decreasing on its support.*

**Proof.** Let $u$ be a ground state solution of (1) and let $R$ be the lowest upper bound (possibly $\infty$) of the support of $u$. Let $a = \inf\{r \in [0, R) : u'(s) < 0 \text{ for all } s \in (r, R)\}$. We have $u(a) > 0$ and $u'(a) = 0$. Suppose $a > 0$ and $u$ has a local maximum at $a$. Then there exists a point $b \in [0, a)$, such that $u'(b) = 0$ and $u' > 0$ on $(b, a)$. Because $\lim_{r\to R} u(r) = 0$, there must be a point $c \in (a, R)$ where $u(c) = u(b)$.

Taking $r_1 = b$ and $r_2 = c$ in (7), we arrive at a contradiction. We must therefore conclude that either $a = 0$ or, if $a > 0$, then $u''(a) = 0$. The latter configuration is impossible, because $f$ is Lipschitz at $u(a)$, so $u(r) \equiv u(a)$ is the (unique) solution of (5) that starts at $u(a)$ with zero slope. It must therefore be the case that $a = 0$. □

## 2.2 Distinct Solutions Do Not Intersect

We assume that $u_1$ and $u_2$ are two ground state solutions of (1) and show that, if the graphs of $u_1$ and $u_2$ intersect, then $u_1$ and $u_2$ are identical.

**Lemma 3** *If $u_1(r) = u_2(r) > \beta$ for some $r \geq 0$, then $u_1$ and $u_2$ are identical.*

**Proof.** The lemma follows from the sublinearity of $f$. Suppose that $u_1(a) = u_2(a) = \tau$ for some $a \geq 0$, where $\tau > \beta$, and that $u_1 > u_2$ on $[0, a)$. The equality $u_1'(a) = u_2'(a)$ is ruled out, because $f$ is Lipschitz at $u_1(a)$, so it must be the case that $u_1'(a) < u_2'(a)$. We have $(u_2 - \beta)f(u_1) - (u_1 - \beta)f(u_2) \leq 0$ on $[0, a]$. Since $u_1$ and $u_2$ satisfy (5), it follows that

$$(u_2 - \beta)(u_1'' + ((N-1)/r)u_1') \geq (u_1 - \beta)(u_2'' + ((N-1)/r)u_2'), \tag{13}$$

and therefore

$$((u_2 - \beta)r^{N-1}u_1')' \geq ((u_1 - \beta)r^{N-1}u_2')', \tag{14}$$

on $[0, a]$. Upon integration over $[0, a]$, we find that $u_1'(a) \geq u_2'(a)$, a contradiction. □

**Lemma 4** *If $0 < u_1(r) = u_2(r) \leq \beta$ for some $r \geq 0$, then $u_1$ and $u_2$ are identical.*

**Proof.** We prove the lemma in two steps. In the first step we rule out the possibility that the graphs of $u_1$ and $u_2$ have more than one point in common, once they are at or below the horizontal line $u = \beta$. In the second step we show that they cannot even have a single point in common.

Suppose that there are two distinct points $a$ and $b$ ($a < b$) such that $\beta \geq u_1(a) = u_2(a) > u_1(b) = u_2(b) > 0$. Without loss of generality, we may assume that $u_1 > u_2$ on $(a, b)$. By continuity, there exists a pair of points $(c, d)$, with $a < c < d < b$, such that $u_1(d) = u_2(c)$ and $u_1'(d) = u_1'(c)$. Applying (8) to $u_1$ on $[a, d]$ and to $u_2$ on $[a, c]$ and subtracting the two expressions, we arrive at the identity

$$(d^{2(N-1)} - c^{2(N-1)})\left(\frac{1}{2}(u_1'(d))^2 + F(u_1(d))\right) - \frac{1}{2}a^{2(N-1)}[(u_1'(a))^2 - (u_2'(a))^2]$$

$$= 2(N-1)\int_{u_1(d)}^{u_1(a)} \left(\frac{(r_1(u))^{2N-3}}{|u_1'(r_1(u))|} - \frac{(r_2(u))^{2N-3}}{|u_2'(r_2(u))|}\right) F(u)\, du. \tag{15}$$

Here, $r_1$ and $r_2$ are the inverse functions for $u_1$ and $u_2$, respectively (i.e., $u_j(r_j(u)) = u$ for $0 \leq u \leq u_j(0)$, $j = 1, 2$). The expression in the left member is positive, while the right member is negative, a contradiction. The possibility of two points of intersection is thus ruled out.

Suppose that there is a single point $a > 0$ where $0 < u_1(a) = u_2(a) \leq \beta$. Without loss of generality we may assume that $u_1(r) > u_2(r)$ for $r > a$.

Let

$$K_j = \lim_{r\to\infty} r^{2(N-1)}\left(\frac{1}{2}(u_j'(r))^2 + F(u(r))\right),\ j = 1, 2. \tag{16}$$

Applying (8) to $u_1$ and $u_2$ on $[a, r]$ and subtracting the resulting equations, we obtain

$$\frac{1}{2}a^{2(N-1)}[(u_1'(a))^2 - (u_2'(a))^2]$$

$$= K_1 - K_2 - 2(N-1) \int_{u(r)}^{u(a)} \left( \frac{(r_1(u))^{2N-3}}{|u_1'(r_1(u))|} - \frac{(r_2(u))^{2N-3}}{|u_2'(r_1(u))|} \right) F(u)\, du. \quad (17)$$

The expression in the left member is negative. Under the integral sign, the expression inside the parentheses is positive, while $F(u)$ is zero or negative, so the integral is certainly negative. Hence, if $K_1 \geq K_2$ (which is certainly true if $N = 2$), the expression in the right member is positive, and we have a contradiction.

It remains to investigate those cases where $N > 2$ and $K_1 < K_2$. Take $\epsilon < \frac{1}{8}(K_2 - K_1)$ and choose $r$ sufficiently large that

$$r^{2(N-1)} \left( \frac{1}{2}(u_1'(r))^2 + F(u_1(r)) \right) < K_1 + \epsilon, \quad (18)$$

$$r^{2(N-1)} \left( \frac{1}{2}(u_2'(r))^2 + F(u_2(r)) \right) > K_2 - \epsilon. \quad (19)$$

From (18) we obtain

$$u_1(r) < \frac{\sqrt{2(K_1 + \epsilon)} + \epsilon}{(N-2)r^{N-2}}. \quad (20)$$

By reducing $\epsilon$ if necessary, we can certainly achieve that $\sqrt{2(K_1+\epsilon)} + \epsilon < \sqrt{2(K_2 - \epsilon)}$. Thus,

$$u_1(r) < \frac{\sqrt{2(K_2 - \epsilon)}}{(N-2)r^{N-2}}. \quad (21)$$

On the other hand, it follows from (19) that

$$u_2(r) > \frac{\sqrt{2(K_2 - \epsilon)}}{(N-2)r^{N-2}}. \quad (22)$$

These results imply that $u_1(r) < u_2(r)$ for $r$ sufficiently large. But this conclusion contradicts the earlier assumption that $u_1(r) > u_2(r)$ for all $r > a$. Thus, the possibility that the graphs of $u_1$ and $u_2$ intersect is ruled out. □

On the basis of Lemmas 3 and 4 we conclude that distinct ground state solutions of (1) do not intersect.

## 2.3 Distinct Solutions Must Intersect

According to Lemma 2, any ground state solution of (1) is (strictly) decreasing on its support. Thus, if $r \mapsto u(r)$ is a ground state solution, the inverse $u \mapsto r(u)$ is well defined on $[0, u(0)]$ by the identity $u(r(u)) = u$. Let $v$ be defined by the expression

$$v(u) = \frac{1}{2}(u'(r(u)))^2, \; 0 \leq u \leq u(0). \quad (23)$$

Thus,

$$u'(r) = -\sqrt{2v(u(r))},\ r \geq 0. \tag{24}$$

We now use the pair $(u, v)$ as the coordinates for a phase plane analysis.

From (23) we obtain $dv/du = u''(r(u))$. As $u$ satisfies (5), it follows that

$$\frac{dv}{du} = \frac{N-1}{r(u)}\sqrt{2v} - f(u),\ 0 < u < u(0). \tag{25}$$

Furthermore,

$$v(0) = 0,\ v(u(0)) = 0. \tag{26}$$

We prove the following lemma.

**Lemma 5** *If $u_1$ and $u_2$ are two distinct ground state solutions of (1), then $u_1(r) = u_2(r)$ for at least one value $r > 0$.*

**Proof**. Let $u_1$ and $u_2$ denote two *distinct* ground state solutions of (1). The graphs of $u_1$ and $u_2$ do not intersect; without loss of generality we assume that $u_1(r) > u_2(r)$ for all $r > 0$. Denoting the inverse functions for $u_1$ and $u_2$ by $r_1$ and $r_2$, we then have $r_1(u) > r_2(u)$ for all $u \in (0, u_2(0))$.

We now analyze the trajectories of the two solutions in the $(u, v)$-phase plane, distinguishing them by their respective indices.

Because $r_1(u) > r_2(u)$ near 0, $v_1$ and $v_2$ satisfy

$$\frac{dv_1}{du} \leq \frac{N-1}{r_2(u)}\sqrt{2v_1} - f(u),\ u > 0;\ v_1(0) = 0; \tag{27}$$

and

$$\frac{dv_2}{du} = \frac{N-1}{r_2(u)}\sqrt{2v_2} - f(u),\ u > 0;\ v_2(0) = 0. \tag{28}$$

Notice that the right hand sides of the differential equations are not Lipschitz. Hence, it is only possible to compare the *maximal* solutions of these initial value problems, unless we can somehow guarantee that there are no other solutions. The condition $\beta > 0$ serves this purpose.

We refer to our article [7], where we investigated initial value problems of the type

$$x' = p(t)x^\alpha + q(t),\ t > 0;\ x(0) = 0, \tag{29}$$

where $0 < \alpha < 1$ and $p$ and $q$ are integrable near 0. We showed that (29) has at most one nontrivial nonnegative solution if (i) $p$ and the first integral $Q$ of $q$ are nonnegative near 0; and (ii) for every $t > 0$, there is a point $\tau \in (0, t)$, where $Q(\tau) > 0$.

In the case of (27) and (28), where $\alpha = \frac{1}{2}$, $p(t) = (N-1)\sqrt{2}/r(t)$, and $q(t) = -f(t)$, the condition (i) is satisfied, and (ii) is satisfied if $\beta > 0$, unless $f$ vanishes identically near 0. If $f$ vanishes identically near 0, a trivial modification suffices to establish uniqueness, again provided that $\beta > 0$.

The direct comparison yields the inequality

$$v_1(u) \leq v_2(u),\ u \in [0, u_2(0)]. \tag{30}$$

If $u_1(0) > u_2(0)$, then $v_1(u_2(0)) > 0$, while $v_2(u_2(0)) = 0$. This would clearly contradict (30), so at this point we must conclude that $u_1(0) = u_2(0)$.

The inequality (30) implies that $|u_1'(r_1(u))| \leq |u_2'(r_2(u))|$. Furthermore, $r_1(u) \geq r_2(u)$, so

$$\frac{|u_1'(r_1(u))|}{r_1(u)} \leq \frac{|u_2'(r_2(u))|}{r_2(u)},\ u \in [0, u_2(0)]. \tag{31}$$

Next, we apply (11) to $u_1$ and $u_2$ at $r = 0$ and subtract the resulting equations. We find

$$\int_0^\infty \frac{1}{r}(u_1'(r))^2\, dr = \int_0^\infty \frac{1}{r}(u_2'(r))^2\, dr, \tag{32}$$

or, after a transformation of variables,

$$\int_0^{u_1(0)} \frac{|u_1'(r_1(u))|}{r_1(u)} du = \int_0^{u_2(0)} \frac{|u_2'(r_2(u))|}{r_2(u)} du. \tag{33}$$

We recall that $u_1(0) = u_2(0)$ and conclude that the inequality (31) is compatible with the identity (33) if and only if $u_1'(r_1(u)) = u_2'(r_2(u))$ for all $u \in [0, u_1(0)]$. This equality, in turn, implies that $u_1$ and $u_2$ coincide everywhere. But here we have arrived at a contradiction, since we had assumed that $u_1$ and $u_2$ were distinct. Hence, if $u_1$ and $u_2$ are distinct, their graphs must intersect at some point $r > 0$. □

The Monotone Separation Lemma of Peletier and Serrin [2, Lemma 9] is an immediate consequence of Lemma 5. We formulate it as a corollary.

**Corollary 1** *If $u_1$ and $u_2$ are two distinct ground state solutions of (1) and $u_1(r) = u_2(r) = \tau$ for some $r > 0$, then $u \mapsto r_1(u) - r_2(u)$ is monotone nonincreasing on $[0, \tau]$.*

## 2.4 Completion of the Proof of Theorem 1

In Section 2.2 we found that if the graphs of two ground state solutions of (1) intersect at some point, then they coincide everywhere. On the other hand, according to Lemma 5, the graphs of two distinct ground state solutions must intersect at some point. Clearly, we have a contradicition, unless (1) admits no more than one ground state solution, as asserted.

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