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## APPROACH TO UNCERTAINTY IN RISK ANALYSIS

by

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## APPROACH TO UNCERTAINTY IN RISK ANALYSIS\*

### ABSTRACT

In the Fall of 1985 EPA's Office of Radiation Programs (ORP) initiated a project to develop a formal approach to dealing with uncertainties encountered when estimating and evaluating risks to human health and the environment. Based on a literature review of modeling uncertainty, interviews with ORP technical and management staff, and input from experts on uncertainty analysis, a comprehensive approach was developed. This approach recognizes by design the constraints on budget, time, manpower, expertise, and availability of information often encountered in "real world" modeling. It is based on the observation that in practice risk modeling is usually done to support a decision process. As such, the approach focuses on how to frame a given risk modeling problem, how to use that framing to select an appropriate mixture of uncertainty analyses techniques, and how to integrate the techniques into an uncertainty assessment that effectively communicates important information and insight to decision-makers.

The approach is presented in this report. Practical guidance on characterizing and analyzing uncertainties about model form and quantities and on effectively communicating uncertainty analysis results is included. Examples from actual applications are presented.

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## 1. INTRODUCTION

The Environmental Protection Agency's Office of Radiation Programs (ORP) is responsible for regulating on a national level the risks associated with technological sources of ionizing radiation in the environment. A critical activity at the ORP as part of developing regulatory policy is analyzing and evaluating risk. Those involved in the analysis of risk are often confronted with a formidable obstacle to producing reliable risk estimates -- uncertainties about the data, parameters, phenomena, models and methods involved.

The ORP believes that the analysis of uncertainty should be an integral part of any risk analysis. Accordingly, in the fall of 1985 the ORP initiated a project to develop a formal approach to uncertainty in risk analysis. In order to establish a basis for the approach, three activities were undertaken:

1. A literature review of studies related to uncertainty in risk analysis was prepared [Rish, 1988]. The following areas of study were included in this review:

- philosophical discussions of uncertainty and its relationship to risk,
- frameworks for the treatment of uncertainty in risk analysis,
- methodologies for uncertainty analysis,
- software available to facilitate uncertainty analysis, and
- applications of uncertainty analysis methodologies.

2. A glossary of risk and uncertainty related terminology was prepared. (see Glossary)

3. Structured discussions on uncertainty and risk analysis were held with ORP staff members having a diversity of backgrounds and responsibilities. This was done to obtain inputs needed to make the approach to uncertainty realistic with respect to ORP activities, needs, and mode of operation. It was also the first step toward achieving internal ORP consensus on an acceptable approach.

This report presents a draft ORP approach to uncertainty in risk analysis based on results from the three activities above and the insight gained from experience. The purpose of this report is to begin the development of a consistent, organized, and well-reasoned approach to uncertainty that ORP can apply to any of its risk assessment problems. The goals of the approach are:

1. to make the reasoning and judgments made about how to handle uncertainties encountered in a risk analysis explicit, so that they can be determined to be reasonable, and
2. to identify where uncertainties matter by assessing the sensitivity of risk management decisions to uncertainties, assessing the level of confidence in decision outcomes and identifying steps that can be taken to reduce or eliminate uncertainties.

With little exception, discussion of the details of available techniques for analyzing uncertainty has not been included in this report. Such techniques are summarized in an organized manner in the companion literature review.\* Instead this approach focuses on how to frame a given risk problem, how to select an appropriate mixture of uncertainty analysis techniques, and how to integrate the techniques into an uncertainty assessment that effectively communicates important information and insight to decision-makers.

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\* See Review of Studies Related to Uncertainty in Risk Analysis [Rish, 1988].

## 2. APPROACH TO UNCERTAINTY IN RISK ANALYSIS

The explicit consideration of uncertainties and their implications is an important part of risk analysis activities for the following reasons:

- The EPA has a responsibility to provide--through its regulations, guidelines, practices, and rulings--a reasonable level of assurance that protection of human health and the environment are maintained. In order to have confidence that this goal is achieved, the implications of uncertainties on regulatory decisions must be carefully assessed.
- There can be considerable costs associated with a decision based upon analysis with a high level of inherent uncertainty. These potential costs come from adopting a course of action which results in unexpected negative consequences, misplaced or practically irreversible commitments of resources, or policies which are difficult to alter at a later date when new information becomes available. Analysis of uncertainties can help to identify a risk management strategy which is most flexible to uncertain or changing conditions, and can provide a higher degree of confidence that risk management goals will be achieved.
- Environmental risk analysis results can be highly sensitive to uncertainties in inputs or model formulations. Once the sources of uncertainties in the assessment are identified, their relative contribution to the overall uncertainty in risk estimates can be examined. This is useful information for planning measurement and modeling activities.
- When there is disagreement among sources of information a good decision requires knowing the extent to which the disagreement would affect risk analysis results. An example would be disagreement among health experts about dose-response relationships.

- "There is considerable empirical evidence to suggest that due to a variety of heuristics employed in human thought processes cognitive biases may result in "best estimates" that are not actually very good. Even if all that is needed is a "best estimate" answer, the quality of that answer may be improved by an analysis that requires people to incorporate and deal with the full uncertainty." [Morgan et al., 1982]
- Many technological risk management problems involve complex mixtures of technical fact and value judgments. Explicit characterization of uncertainties can help to distinguish disagreements over technical uncertainties from those which are due to divergent values.
- The act itself of examining uncertainties in a quantitative manner results in a broader understanding of the processes being modelled, and the sources and nature of the controversial issues involved. It forces a careful review and characterization of the present state of knowledge, and it provides a structure for updating the risk assessment as information and understanding evolve.

These reasons underlie the approach to uncertainty in risk analysis described in this section. The overall approach includes an institutional approach applied at the program level and a technical approach applied at the analysis and evaluation level. The institutional approach consists of a set of policies and procedures adopted by the ORP to ensure adequate consideration of uncertainty in risk analyses. The technical approach consists of guidance for (a) framing a risk problem with respect to some specific policy, risk analysis, and uncertainty considerations, (b) developing an appropriate uncertainty assessment strategy for the risk problem, and (c) evaluating and effectively communicating the results of the uncertainty assessment.

## 2.1 INSTITUTIONAL APPROACH

The policies and practices that together constitute an institutional approach to handling uncertainties in risk analysis include:

- Initiating a task to develop a program-wide approach to uncertainty in risk analysis, of which this report and its accompanying literature review are a part. The goal is a consistent, organized, and well-reasoned approach that reflects an awareness of the state-of-the-art in uncertainty treatment and is compatible with the ORP's mode of operation.
- Establishing lines of communication between ORP technical staff and expert practitioners of uncertainty analysis in order to keep abreast of the state-of-the-art and have a source of consultation. In addition, continuing collaboration exists between ORP experts and other leading experts on the scientific basis for radiation-related processes and effects.
- Training of the staff in current uncertainty analysis techniques and software. The EPA computer code MOUSE [Klee, 1985] and the Carnegie-Mellon University code DEMOS [Henrion and Morgan, 1985] are being evaluated for possible use.
- Encouraging through guidelines and criteria, the selection of facility designs and sites that can be reliably characterized and evaluated.
- Developing in-house analytical methodologies for uncertainty analysis. A discrete probability distribution methodology for analyzing input parameter uncertainties in geosphere transport modeling for low-level waste disposal sites has been developed [Hung, 1985]. In addition, a probabilistic version has been

developed of the river release pathways model used to derive the radionuclide release limits specified in 40CFR191, subpart B, [Rish et al., 1985]. This version employs Latin Hypercube Sampling (LHS) to propagate parameter uncertainties through model calculations.

- Obtaining independent peer review of risk analyses done in support of rulemakings.
- Establishing research, field and test programs aimed at reducing uncertainties about the processes and parameters associated with assessing the risks of radiation in the environment.

## 2.2 TECHNICAL APPROACH

In addition to the institutional steps described above to deal with uncertainty in its risk analysis activities, the ORP has developed technical guidance for the treatment of uncertainty.

The technical guidance, presented in the remainder of this report, addresses the following three basic elements of the proposed technical approach to uncertainty:

1. framing the risk problem from policy, risk analysis, and uncertainty perspectives,
2. developing an uncertainty assessment strategy, and
3. communicating the results of the uncertainty assessment.

### 2.2.1 FRAMING THE PROBLEM

In order to design an appropriate strategy for assessing the uncertainties in a given risk problem, it is first necessary to frame the problem with respect to some specific policy, risk analysis, and uncertainty considerations. This is because the choice of appropriate uncertainty characterizations and analysis techniques depends upon these considerations. A framework for organizing these considerations is presented below.

#### 2.2.1.1 Policy Considerations

The following policy considerations important to designing an appropriate uncertainty assessment strategy should be addressed.

- (1) The type of decision that the risk analysis will support should be characterized. There are at least four basic types of risk management decisions that risk analyses can be used to support.
  - a. site and facility design selection or approval,
  - b. compliance and variance determinations for licensing,
  - c. "act versus study" decisions about whether or not to implement risk control actions or wait until further data collection and analysis reduce uncertainties about decision outcomes (In other words, when is "enough" information known to justify taking action or not taking action?), and
  - d. "level of control" decisions about the proper levels for standards, criteria, thresholds, and compensation.

For the last type of decision, determining appropriate levels of control, the control strategy alternatives that are under consideration should be identified. These include:

- establishing design specifications, siting criteria or licensing conditions for the technology being considered for regulations.
- setting limits on radionuclide source inventory or release rate,
- setting radionuclide concentration limits in various media,
- setting limits on exposure or dose,
- setting limits or goals for acceptable level of risk,
- creating incentives to control risks, and
- specifying compensation mechanisms.

(2) Decision criteria to be used should be identified. These can be generally categorized as either "rights-based" or "utility-based" criteria.\* Rights-based criteria include:

- zero risk,
- a specified bound on risk (i.e., de minimus, consistent level, acceptable or reasonable level, risk/safety goal),
- protecting the most-sensitive individual (this can also be a response to uncertainty),
- Best Available Control Technology (BACT) or the like, and
- approval, compensation, and other legal determinations.

---

\* The concepts in this paragraph were developed from conversations with M. Granger Morgan with his kind permission.

Utility-based criteria include:

- cost-effectiveness,
- cost-benefit,
- value-impact,
- As Low As Reasonably Achievable (ALARA), and
- other such economic preference tradeoffs.

(3) Policy strategies being considered with respect to time and space factors should be identified. Alternative strategies include:

- a one-time solution (e.g., limit on cumulative releases during next 1000 years) versus an adaptive "look-ahead" solution (e.g., control imminent hazard now and determine long-term control later when better information is available, or adopt a time-phased policy),
- a generic solution versus a site-specific solution, and
- population versus individual protection.

(4) The key value parameters and decision variables should be identified.\* Value parameters measure the preferences of the decision-makers. Key value parameters include:

- the appropriate investment rate to reduce health risk (e.g., "value of life"),
- the discount rate for combining benefits and costs accruing at different times, and
- the level of confidence desired by the policy-maker in the estimated outcomes of alternative decisions.

---

\* Based on Henrion and Morgan [1984]

Examples of this last value parameter, the confidence level criterion, include:

- based on "best-estimates,"
- based on conservative estimates, of which worst-case is an extreme example, and
- based on a subjectively-determined reasonable level of assurance.

Key decision variables should be identified. These are quantities over whose values the decision-maker exercises direct control. An example of a decision variable is the permitted maximum emission rate from the technology being evaluated. In some cases, the decision variable is specified as input to the risk analysis, and the sensitivity of outcomes to alternative levels of the variable is analyzed. In other cases, it is desired to use the analysis to determine an "optimal" level for a decision variable as an output.

An important measure of the significance of an uncertainty is the effect it can have on the key decision variables involved. Accordingly, in framing the risk problem it is useful to identify "breakpoints" where changes to risk analysis results would lead to an alternative decision. The criteria that the decision-maker will use to determine such breakpoints should be identified, to the extent possible.

#### 2.2.1.2 Risk Analysis Considerations

The risk analysis considerations that are important to designing an appropriate uncertainty assessment strategy can be organized around the generic risk analysis framework depicted in Figure 2-1. Each of the processes in the boxes and outputs on the arrows on the framework must be understood and analyzed as part of an integrated risk analysis. These processes and outputs include:

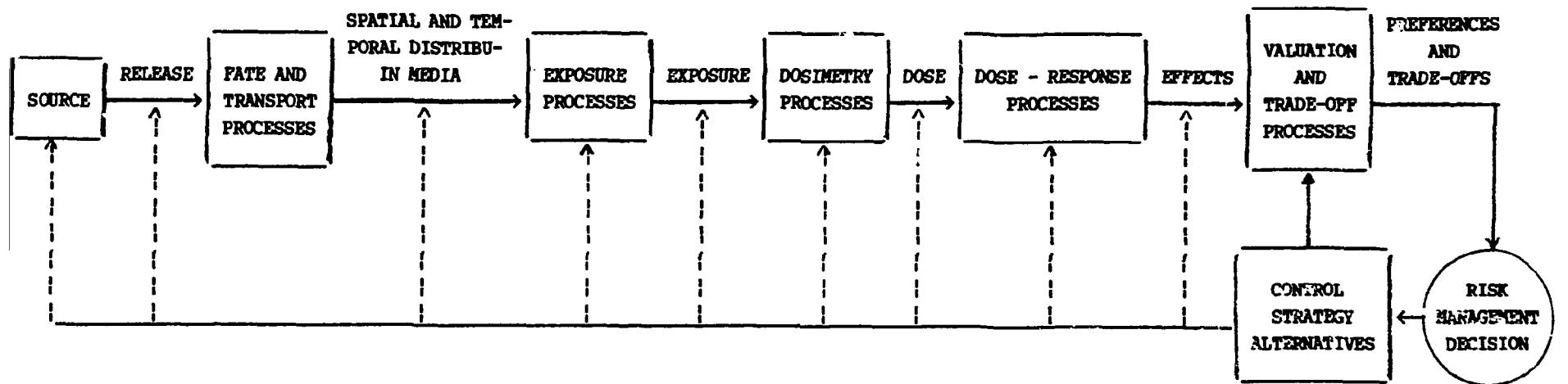


Fig. 2-1. Risk analysis framework.

- Source: the technology, activity, or conditions resulting in a release of radioactivity to the environment.
- Release: the types, amounts, timing, and probabilities of releases.
- Fate and transport: physical and chemical transport, transformation and loss processes occurring to releases in the geosphere, hydrosphere and atmosphere.
- Spatial and temporal distribution in media: the concentration as a function of time and space in soil, air and water.
- Exposure processes: population characteristics, migration patterns (time and motion), biosphere pathways, and micro environments (e.g., indoor levels).
- Exposure: the amounts and timing of radioactivity ingested, inhaled, absorbed, and directly exposed to by persons (or animals, plants, or objects).
- Dosimetry: processes involved in going from exposure to organ-specific and equivalent whole-body dose (rads to rems).
- Dose
- Dose-response: biological (or other) effects of radiation.
- Control strategy alternatives: the performance, effectiveness, and costs of alternative control strategies under consideration.
- Valuations and tradeoffs: the processes of weighing and/or placing an economic value on risks, comparing impacts, costs, and benefits of alternative control strategies, and determining the optimal decisions indicated by the analysis based on preference structures.

Using the generic risk analysis framework described above, the following risk analysis considerations should be addressed when developing an uncertainty assessment strategy for a particular problem:

(1) The models, data, and judgments that will be used to assess each of the processes and outputs in the risk analysis framework depicted in Figure 2-1, and how they will be combined to form an integrated risk analysis, should be outlined.

(2) The types of models to be used in the risk analysis should be identified. These include:

- conceptual models
- natural analogue, microcosm, or prototype models
- mathematical or logical expressions, and
- computer codes.

(3) The critical dimension of the models should be established. These include (from Henrion and Morgan [1984]):

- Predictive versus optimizing: Is the model simply intended to describe or predict a situation, or is it intended to find an optimal decision? In the latter case, an explicit objective function is required to rank possible outcomes.
- Analytic versus implicit: Can the vector of outputs,  $y$ , be computed directly as a function of the input values,  $x$

$$y = f(p, x)$$

or is the model specified implicitly,

$$f(x, y) = 0?$$

In the latter case, if the function is non-linear, an iterative solution algorithm may be required.

- Static versus dynamic: Does it model changes over time?
- Level of spatial and temporal aggregation: Does it model variations over space and time? If so, in how many dimensions, and what is the level of aggregation?
- Deterministic versus stochastic: Does it represent phenomena as deterministic or probabilistic? Note that this is distinct from whether uncertainty is represented. For example, a fault tree for a nuclear reactor may compute probabilities of failures, but not necessarily the uncertainties in those probabilities.
- Size: How many input values, state variables, equations, and outputs does it contain? Is it small (tens), moderate (hundreds), or large (thousands)?

(4) The types of quantities should be established for key model parameters. Types of quantities include those listed and described in Table 2-1.

(5) The types of data to be used in the risk analysis should be identified. Types of data include:

- direct empirical data (i.e., laboratory and field data),
- indirect empirical data (i.e., observations from analogues, microcosms, prototypes, surrogate measures),
- calculated or inferred data, and
- constants or specified parameters.

Table 2-1. Types of quantities used in risk analysis models\*

Quantity type	Examples	Description	Recommended uncertainty characterization	Rationale
Empirical parameters	Thermal efficiency, oxidation rate, price, toxicity.	Input parameters that measure aspects of processes being modeled.	Treat parametrically, establish ranges, or develop probabilistic measures. Depends on a number of factors.	There exists a "correct value" which is not precisely known and must be estimated.
Defined constants	Atomic weight of 0, Joules per kwh.	Quantities that are exact and certain by definition.	Treat as certain.	The value is fixed by definition and is not empirical.
Value parameters	Investment rate to prevent mortality, discount rate, risk aversion.	Parameters used to model preferences or utilities of the decision-makers or those that they represent.	Establish a set of alternative parametric levels over value systems of interest.	If one is uncertain about what one's values are, the impact of alternative value assumptions should be systematically explored.
Decision variables	Air quality standard (for EPA), plant size and type (for utility).	Quantities over whose values the decision-maker exercises direct control.	Establish a set of alternative parametric levels of interest to the decision-maker.	The decision-maker controls the value of this variable. As with value parameter, if he is uncertain he should systematically explore the implications of alternative choices.
Outcome variables	Estimated excess deaths per year, expected net present value.	Output variables computed by the models used.	Describe qualitatively, present parametrically, present ranges, or present probabilistic measures. Depends on a number of factors.	Depends on: the type of decision that the risk analysis supports, the confidence level criteria used, and how input uncertainties are treated.

\*Adapted from a more comprehensive table prepared by M. Granger Morgan and Max Henrion of Carnegie-Mellon University, with their permission.

(6) The types of judgments to be used in the risk analysis should be identified. Types of judgments include:

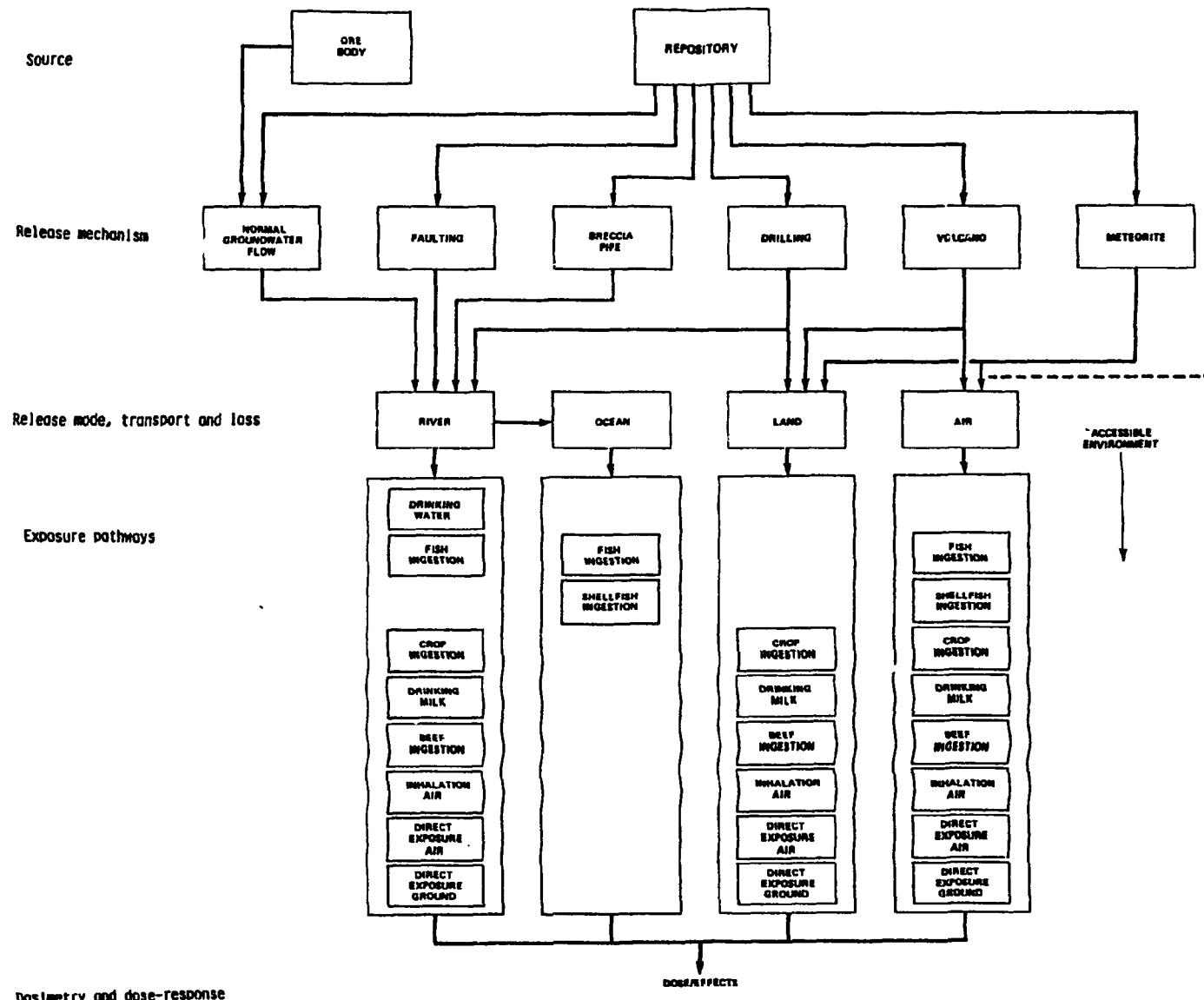
- assumptions (e.g., that a process is insignificant, that future conditions will be similar to past conditions, that processes and events are independent)
- choice of valid or appropriate model (including approximation methods).
- inferences
- "weight of evidence" judgments, and
- opinions on uncertain parameter values, ranges or probability distributions.

The framework shown in Figure 2-1 is generic to all risk analysis problems; however, it is useful in approaching a specific problem to use the framework to create a more detailed version, herein called a risk analysis flow diagram. An example of such a diagram is shown in Figure 2-2 for the problem of estimating the population mortality effects from possible releases of radionuclides from a high-level radioactive waste repository. As can be seen in the figure, the risk analysis flow diagram shows in a modular fashion each of the processes which must be analyzed and the interrelationships between the process inputs and outputs. It is useful to relate the risk analysis flow diagram for the specific example shown in Figure 2-2 to the generic risk analysis framework in Figure 2-1, as follows:

---

GENERIC FRAMEWORK (Figure 2-1)	HIGH-LEVEL WASTE RISK ANALYSIS FLOW DIAGRAM (Figure 2-2)
TECHNOLOGICAL ACTIVITY	High-level waste geologic repository
RELEASE	Releases can occur by several mechanisms (normal groundwater, faulting, breccia piping, drilling, volcano, meteorite) to four release modes (river, ocean, land, air). One curie is assumed to be released. Flow paths are shown.
TRANSPORT, TRANSFORMATION, AND LOSS PROCESSES	Transport processes depend on release mode. No transformation is assumed. Loss is assumed to be from sedimentation in river or ocean. Half-lives are too long to be a significant loss mechanism.
EXPOSURE PROCESSES	Each mode has some subset of nine possible exposure pathways. For example, the ocean mode has two associated pathways: fish and shellfish ingestion.
EXPOSURE	The exposure from all pathways for the four modes are summed to yield total population dose.
EFFECTS PROCESSES	The population dose is multiplied by a risk coefficient (linear/no threshold) with units of fatalities per dose.
EFFECTS	Effects are fatalities per curie released.

---



**Fig. 2-2. Risk analysis flow diagram for the problem of estimating population mortality risk from possible releases of radionuclides from a high-level radioactive waste geologic repository. Based upon the framework shown in Fig. 2-1.**  
 Source: Rish et al., 1983.

### 2.2.1.3 Uncertainty Considerations

After a risk problem has been framed from a policy and risk analysis perspective by the considerations above, it should be framed with respect to the sources and nature of the uncertainties associated with assessing the risk. The generic risk analysis framework in Figure 2-1 can be used to structure the uncertainty considerations necessary to complete the framing of the risk problem, as follows:

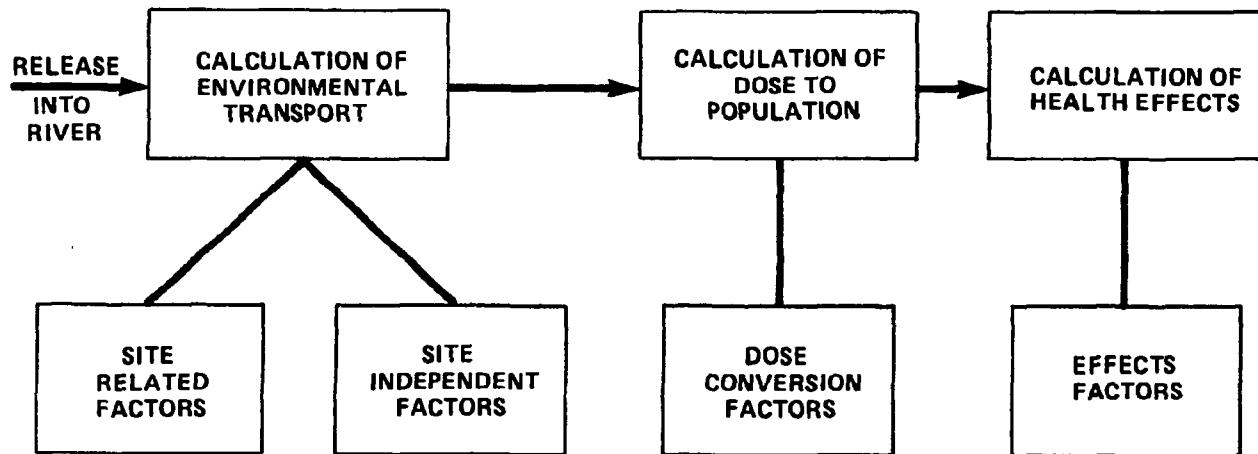
(1) The extent and quality of information and understanding available to analyze the processes in each box and to estimate the outputs on each arrow of Figure 2-1 should be summarized.

(2) The sources of uncertainty in these models and data should be summarized. Sources of uncertainty include:

- uncertainty about model form or validity,
- uncertainty introduced by assumptions and approximations made in model implementation,
- inherent randomness,
- random error in direct measurements,
- incomplete or inconsistent data,
- variability not included in the analysis due to level of aggregation used,
- uncertainty about inferences, extrapolations, and analogies used, and
- basic disagreements about theory, phenomenology, conceptual models, or interpretations of available scientific evidence.

(3) Preliminary bounds or ranges on uncertainties should be estimated. Conditions and assumptions leading to credible upper and lower bounds should be summarized (e.g., different conceptual models associated with upper versus lower bound).

Detailed risk analysis flow diagrams can also be used to identify important quantity uncertainties in the analysis and how they propagate through the problem. Let us examine, for example, the river mode exposure pathways portion of Figure 2-2. Just as Figure 2-2 is a more detailed version of the generic framework in Figure 2-1, Figure 2-3 is a detail of the river mode portion of Figure 2-2 showing key uncertain quantities which must be addressed in estimating population mortality effects from a radionuclide release to a river.



- $B_{NV}$  SOIL TO PLANT CONCENTRATION FACTOR
- $\lambda_{SN}$  SOIL REMOVAL RATE CONSTANT
- $CF_{NP}$  BIOACCUMULATION FACTOR FOR FISH
- $RF$  SOIL TO AIR RESUSPENSION FACTOR
- $SED$  RIVER SEDIMENT-ATION REMOVAL FACTOR
- $F_r$  FRACTION OF RIVER FLOW USED FOR IRRIGATION
- TREAT FRACTION REMOVED BY WATER TREATMENT

- $F_{MN}$  TRANSFER RATE FROM GRASS TO MILK
- $F_{FN}$  TRANSFER RATE FROM GRASS TO BEEF
- $\lambda_{EN}$  VEGETATION WEATHERING COEFFICIENT

- $D_{NP}$  DOSE CONVERSION FACTORS FOR INGESTION, INHALATION, EXTERNAL AIR AND EXTERNAL GROUND

- $HE$  MORTALITY RISK COEFFICIENT

Fig. 2-3. Detail of river mode portion of Fig. 2-2 showing key uncertain factors in risk analysis for this portion of the problem. Source: Rish et al., 1983.

### 2.2.2 DEVELOPING AN UNCERTAINTY ASSESSMENT STRATEGY

Once the policy, risk analysis, and uncertainty consideration are addressed, the risk problem is framed in a manner that facilitates developing a logical and appropriate strategy for uncertainty assessment. Uncertainty assessment involves two basic activities: characterizing uncertainties and analyzing uncertainties. Numerous different approaches and techniques are available to accomplish each of these activities. The framework developed in the previous section provides a basis for selecting an appropriate combination of techniques that will result in the insights needed about uncertainties to support a particular risk management decision.

Selecting appropriate ways to characterize and analyze uncertainties should be done in parallel since these activities depend on each other. For example, limitations on available time, resources, and information affecting the extent to which uncertainties can be characterized will also limit the choice of analytical strategies. Conversely, the choice of an appropriate analytical strategy for the decision being supported carries with it requirements on the type of uncertainty characterizations to be used. In fact, uncertainty assessment involves a series of tiered decisions about the levels of uncertainty characterization and analysis needed. The assessment process begins with simpler measures of uncertainty (i.e., ranges) and simpler analytical techniques (i.e., sensitivity analysis) and progresses, to the extent needed to support the decision, to more complex measures and techniques.

The development and implementation of an appropriate uncertainty assessment strategy can be viewed as a decision process by the risk analyst. Decisions are made on ways to characterize uncertainties, ways to analyze uncertainties, and whether to proceed to increasingly refined (and complex) levels of uncertainty assessment for particular uncertainties involved.

#### 2.2.2.1 Characterizing Uncertainty

Generally speaking, uncertainty about a quantity, model, or other aspect of a risk analysis can be characterized in any of the following ways:

- describe the uncertainty in qualitative terms,
- specify a set of alternative "scenarios" or models to be analyzed,
- specify a range of values of uncertain quantities,
- use data analysis techniques to develop a frequency distribution, standard error or confidence interval for uncertain quantities, and/or
- use expert judgments to develop subjective probabilistic measures for uncertain quantities.

Selecting the appropriate way to characterize an uncertainty associated with a particular risk analysis problem depends on the type of decision that the risk analysis supports, confidence level criteria to be used, type of model, type of quantity, extent and quality of information and understanding available, and the method used to propagate uncertainty in the risk analysis. All but the last of these considerations are addressed by the guidance for framing the risk problem described in the previous section.

Recommended ways to characterize uncertainties about models and model quantities based on these considerations are presented below; however, before deciding on appropriate characterization it is necessary for the analyst to decide on an appropriate level of aggregation to be used in risk analysis models. The level of aggregation that "works" for the analyst depends on, among other things, the type of information available to him on processes being studied and the "comfortableness" of the analyst with that information. For example, rem per curie estimates

are derived from models of dose conversion processes. An analyst who performs a risk analysis for radionuclide exposures is undoubtedly aware that these more detailed models exist and can include them as sub-models in his pathways model in place of using dose conversion factors. He might, however, choose the use of dose conversion factors as an appropriate level of aggregation because he feels more able to make good judgments about dose conversion factors than about the inputs to the more detailed models from which dose factors are derived. Of course, he will review the more detailed models in making his judgments about appropriate dose factors. Standard environmental risk models often involve simple mathematical equations which are relatively uncontroversial because they consolidate detailed complex dependencies inside of several aggregate model parameters. These parameters, consequently, have very large inherent uncertainties because they become surrogates for modeling complex processes across a variable population. Thus, there is a tradeoff available to analysts between structural detail and degree of parameter uncertainty.

#### 2.2.2.1.1 Characterizing Model Uncertainty

Guidance for characterizing model uncertainty is presented below for the different types of models that enter into a risk analysis. Uncertainty about the appropriateness of the models used in a risk analysis is an important potential source of systematic error in the analysis.

Conceptual models -- Uncertainty about conceptual models for events and processes involved in the risk problem should be characterized by qualitatively describing the nature of the uncertainty and identifying alternative plausible conceptual models. Expert judgment should be used to assess the relative likelihoods, in qualitative terms, of the validity of alternative conceptual models identified.

Natural analogue, microcosm, and prototype models -- The extent to which these models are or are not representative of the actual risk processes being evaluated should be described. A qualitative, and in

some cases quantitative, assessment of the extent to which these models might over- or under-estimate the outcomes of actual risk processes involved should be developed.

Mathematical or logical expressions -- A set of alternative plausible solution techniques, analytic methods, and mathematical or logical functions (e.g., order of the exponent in a power law function, linearity, eularian or langrangian, static or dynamic, compartmental, finite element, etc.) should be identified. The validity of the assumptions and approximations associated with each plausible alternative should be described.

Computer codes -- The approximations used in the codes to represent mathematical and logical expressions should be described. The reliability of the codes and the extent to which they have been verified and validated should be described. Verification is the process of showing that the code produces correct solutions of the encoded mathematical model within defined limits for each parameter used. Validation is the process of showing that the encoded mathematical model produces a valid solution to the physical problem associated with the particular application.

#### 2.2.2.1.2 Characterizing Uncertainty About Quantities

Guidance is presented in this section on how to characterize uncertainty about the different types of quantity in a risk analysis. This guidance is organized according to the quantities identified in Table 2-1, and is summarized in the last two columns of the table.

Empirical parameters -- At a minimum, a range of values for each uncertain parameter (lower-bound, "best-estimate," upper-bound) should be established. The range should be justified by available data and/or expert judgments, and this justification should be documented.

The way to characterize uncertainties about empirical parameters depends on, among other things, the confidence level criteria to be used by the policy-maker in the decision(s) that the risk analysis supports as follows:

(a) Based on "best-estimates" -- There is considerable evidence in the literature of a variety of heuristics employed by experts in processing information that can result in significant biases in single-valued "best-estimates" for empirical parameters. It is theorized that the quality of "best-estimates" can be improved by explicit consideration by the experts of the full range of uncertainty about empirical parameters and the conditions associated with different values within the range, especially the upper and lower bounds. This practice is recommended where the results of sensitivity analyses indicate that the risk analysis results or choice of decision alternatives are significantly affected by variations within the parameter range. The information, assumptions, and conditions associated with the "best-estimate" should be documented. In addition, the meaning of "best-estimate" should be specified and consistently applied (e.g., Is it the mean, mode, or median of the range?)

(b) Based on conservative estimates -- The same guidance provided above for the "best-estimate" confidence level criterion applies to characterizing empirical parameter uncertainty when basing decisions on conservative estimates of risk. Conservative estimates can also be improved by consideration of the full range of uncertainty about a parameter. The meaning of "conservative" should be specified and consistently applied. A special case of a conservative estimate is the "worst-case" or "upper-bound" estimate. The extent to which the worst-case estimate differs from the best estimate should be indicated, and the conditions and assumptions associated with each estimate should be provided (i.e., the reasons for the difference).

(c) Based on a reasonable level of confidence -- Risk management decisions can be based on a subjectively determined confidence level criterion corresponding to a "reasonable level of confidence" in the

risks associated with decision alternatives. This reasonable level is usually a relatively high degree of confidence; however, the marginal cost of being more certain of a decision outcome is taken into account. For example, adopting a lower release limit will increase the degree of confidence that dose criteria will be met, but an 85 per cent confidence level might be "reasonable" if lowering the limit to achieve higher confidence means a quantum leap in control technology costs or the use of a more efficient but less reliable technology.

In order to determine what level of protection provides a reasonable level of confidence, the decision-maker needs to have an assessment of the relative levels of confidence associated with basing actions on different risk estimates across the range of uncertainty in risk analysis results. He then can factor confidence levels into his decision. This is especially important since parameter uncertainties in environmental models usually have skewed probability density distributions with relatively low likelihoods associated with a significant portion of the upper or lower half of the output uncertainty range. Thus, there might be negligible increases in confidence level associated with decisions based on these higher risk estimates. Using single-valued "conservative" or upper-bound estimates for uncertain risk analysis parameters, especially when their uncertainty tends to be log-normally distributed, can result in risk estimates that are orders of magnitude above estimates having what one would consider a reasonable level of associated confidence. As North notes, "a plausible upper-bound or worst-case projection may not be helpful when there is a potential for large impacts but a high likelihood that the large impacts will not occur" [North and Balson, 1985].

Uncertainty about the parameters of risk analyses employing a "reasonable level of assurance" criteria should be characterized using probability distributions. The method used to establish the probability distribution depends on the extent and quality of data available on the parameter. If the results of sensitivity analyses (see Section 2.2.2.3) indicate that the risk analysis results or choice of decision alternatives are significantly affected by variations within the

parameter range, then a probabilistic measure should be developed to represent uncertainty about the parameter so that likelihoods of alternative outcomes can be assessed. If enough data are available to develop a statistically meaningful representation of the uncertainty, then standard statistical methods can be used to establish a probabilistic measure. If available data are inadequate, then expert judgments should be used to encode a subjective probabilistic measure of uncertainty about the parameter. The particular type of probabilistic measure depends on the method used to propagate uncertainties through the risk analysis, as discussed below. The probabilistic measure used should adequately characterize significant features of the distribution of probability across the quantity uncertainty range. Expert judgments about probabilities should be obtained in a consistent, well-documented manner reflecting current professional practices. A proposed draft procedure for elicitation of expert judgment is presented in Appendix A.

The way to characterize uncertainties about empirical parameters used will also depend on the technique selected to propagate uncertainties through the risk analysis which, in turn, depends on a number of considerations (see Section 2.2.2.2.2). If uncertainties are to be propagated by scenario or parametric analysis, then a range of values for the parameter will be used and the conditions associated with the range will be identified. If a method of moments is to be used, then the range will be further specified by associating confidence levels with the lower and upper bounds of the range. If the discrete probability distribution (DPD) method is used, then probabilities will be associated with a number of values within the range of the parameter. If analytical solutions to output uncertainty or a stochastic simulation approach to uncertainty propagation is to be used, then a probability distribution representing uncertainty about the empirical parameter will be developed.

Finally, important guidance regarding the selection of an appropriate probability distribution to represent uncertainty about a quantity is provided by Seiler [1983], as follows:

"In a discussion of errors and of error propagation, the assumption of a probability distribution for a stochastic variable is a decisive step, since it determines all properties of the probabilistic behavior of this quantity. However, the choice is usually made without much further thought and results mostly in the adoption of either a normal or a log-normal distribution. The criteria for this selection are sometimes based on experimental or theoretical evidence, most often, however, on aspects of convenience and ease of use. Since normal and therefore log-normal distributions are the basis of some of the more common statistical tests, and since they also offer attractive mathematical properties, they are by far the most favored choice."

"Normal and log-normal distributions are frequently found in nature. In many cases, however, the evidence for their applicability is not very good. It is sometimes based on a theoretical model, as in the case of radioactive decay where the normal distribution is theoretically indicated for a large number of decays. Whether the distribution of the actual counts registered by the electronic devices is of that type or not, is a question which can only be resolved by experiment."

"As a consequence, it is much safer to perform mathematical operations in the high probability areas than in the tails of the distribution. Means and standard deviations can be determined to a good approximation, whereas calculations of 95% confidence levels or other operations involving the tails are questionable. In the evaluation of experimental data and a possible discussion of confidence limits, this aspect should be borne in mind."

Defined Constants -- These will be treated as certain.

Value Parameters -- Uncertainty about value parameters will be characterized by establishing a set of alternative parametric levels for these parameters. Uncertainty about value parameters is fundamentally different than uncertainty about technical parameters. It is uncertainty about the appropriate level of a measure of one's value system, not uncertainty about a measure that has a "correct" level, which must be represented. By treating value parameter uncertainties

parametrically in the risk analysis, the analyst makes it possible for the decision-maker to examine the implications of alternative value judgments on risk analysis and evaluation results.

Decision Variables -- As with the value parameters, if there is uncertainty about the appropriate level of a decision variable (e.g., the decision-maker is uncertain about which emission rate limit to specify), then alternative parametric levels should be specified. Usually what is desired is to evaluate the effect of alternative levels of decision variables on risk analysis outcomes.

Outcome Variable -- The characterization of uncertainty about the outcome variables of a risk analysis depends on the type of decision that the analysis is supporting (see Section 2.2.1.1 - Policy Considerations), the confidence-level criterion to be used, and limitations on how model input uncertainties can be characterized.

Risk analyses being used to support site and facility design selection decisions should characterize output uncertainty by presenting the range between upper and lower bound estimates. Sites and designs having associated upper-bound risks that are at or below risk goals can be identified, and the relative magnitudes of overall uncertainty about risks from alternatives can be compared.

Characterizing risk analysis outcome uncertainty for compliance determinations depends on the confidence or assurance level criterion specified in the pertinent regulation, or otherwise indicated by the implementing agency. Similarly, characterizing risk analysis outcomes in support of "level of control" decisions depends on the confidence level criterion that applies. The same guidance provided earlier for characterizing empirical parameter uncertainties based on confidence level criterion generally applies to characterizing outcome variable uncertainty. Additional consideration must be given, however, to quantities in the analysis that were treated parametrically (i.e., value parameters and decision variables) and to outcome uncertainty associated with plausible alternative conceptual and/or mathematical models of risk-related processes. Results of analyzing bounding "scenarios"

constructed by forming credible combinations of assumed alternative conceptual and mathematical models (including alternative assumptions) should be presented. Characterizations of output uncertainties should be presented parametrically over the ranges of value parameters and decision variables being considered. Care and creativity must be used to avoid parametric presentations that are confusing because they require the decision-maker to evaluate too many combinations of assumed parameter levels. It is better to present a simplified parametric characterization that illustrates the salient implications on the risk analysis results of assuming different parametric levels.

"Act versus study" decisions address whether or not to implement risk control actions or wait until further studies reduce uncertainties about decision outcomes. For this type of decision uncertainty about risk analysis, outcomes should at first be characterized by providing "best-estimates" and plausible upper-bound estimates of risks. If the analysis indicates that plausible upper-bound risk estimates are relatively low, then this builds confidence in a decision to not regulate. Where plausible upper-bound risk estimates are significant, the best-estimates can be helpful in deciding whether to gather more information before basing decisions on the upper-bound estimates. If changes to the risk analysis outcome magnitude within the range between the best-estimate and upper-bound estimate result in indicated changes to risk management alternatives, then more information about the likelihood of outcomes within this range is needed to support the decision.

In this case the analyst can use "probability trees," as described in the discussion of probabilistic uncertainty analysis in Section 2.2.2.2, to perform a value-of-information analysis. In this type of treatment, alternative uncertain aspects of the risk analysis are represented by branches on a tree diagram, and each branch is assigned a probability. An example is shown in Figure 2-4 in a later section. The choice of whether to take action to control possible undesirable risks that are represented by particular paths through the tree can then be viewed as a decision on whether to buy insurance against the probabilities and outcomes associated with those paths. A good

discussion and example of the application of probability trees and value-of-information analysis to the "act versus study" decision is provided in North and Balson [1985].

#### 2.2.2.2 Analyzing Uncertainty

Along with selecting appropriate ways to characterize uncertainties in a risk analysis it is necessary to select appropriate ways to analyze uncertainties. As explained previously, these two activities should be done in parallel since they depend on each other, and are done iteratively to reach appropriate levels of detail in uncertainty treatment for particular uncertainties involved.

##### 2.2.2.2.1 Analyzing Model Uncertainty

Three approaches to analyzing model uncertainty should be used:

- validation of models,
- verification of models, and
- analysis of credible alternative models.

Guidance on each of these approaches is presented below.

###### (a) Validation of models

Validation is the process of obtaining assurance that a model, usually as embodied in a computer code, is a correct representation of the physical process or system associated with its particular application. Validation of models can be accomplished in three ways:

- calibration and confirmation of models by measurements taken over the range of conditions for which the model is being used,
- comparisons of predicted behavior to the behavior of available analogues of the process being modeled, and
- expert judgments of validity obtained through peer review.

A standard approach to validation is to use empirical measurements to calibrate and confirm model predictions. This should be done over the full range of conditions of the system being modeled in order to adequately address uncertainty about model validity. Also, the temporal and spatial frames of the model should be addressed by measurements taken over similar frames.

Often, only partial validations with measurements are possible. For example, if a model predicts concentrations of a radionuclide at all locations and times downstream in a river, then validation using measurements taken at one location during one season of the year will only be partial. As another example, this approach is of limited use for validating models that predict effects occurring over very long time frames. Only partial validation of model predictions of the early development of these effects is possible. Similarly, validation using comparisons of predicted behavior to the behavior of available analogues can usually only be partial validation. Analogues are usually only available for some of the processes being modeled and/or conditions that are not fully consistent with those being modeled.

Theoretical arguments can be used in some cases as the basis for asserting that partial validations imply overall validity. Such arguments and their bases should be carefully documented.

In most cases, risk analysis models will be validated using a combination of partial validation by comparisons to measurements and analogues, and by expert judgments of validity obtained through peer review. The logic and rationale behind judgments of validity should be clearly documented, and should include a statement by the peer reviewers of the assumptions and physical, spatial, and temporal conditions for which their judgments hold true.

(b) Verification of models<sup>\*</sup>

Verification is the process of obtaining assurance that a computer code correctly performs the operations specified in the mathematical and logical models that it represents. Verification may be accomplished in four ways:

- by comparing code results to hand calculations,
- by comparison with an alternate calculational scheme,
- by comparison with verified computer codes (benchmarking), and
- by performing a detailed independent review of the code.

These four methods may be used singly or in combination to verify all or parts of a computer code.

The most straightforward means for verifying a computer code is to duplicate the code calculations by hand, performing the same calculations that the code performs. This method has the advantage of providing the most direct assurance that the calculational scheme works. Although straightforward, this method becomes excessively cumbersome when the calculations become very complex, such as for a finite element grid, or when a large number of run options need to be checked. Often a simplified problem can be set up to minimize the effort required.

Sometimes an alternate calculational scheme can be constructed to check the results of a computer code. For example, an exact or approximate analytical solution may be available for a problem which the computer code solves by numerical methods. Alternatively, two numerical methods may be used to solve the same problem. Where a different calculational scheme can be constructed, a comparison of the code results with the result of this alternate method can provide assurance that the code is calculating correctly.

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\* The discussion in this section is based on discussions with Dr. John Kircher of Battelle Memorial Institute.

In addition to providing a check that the numerical model is coded properly, analytical solutions assist in checking the ability of the code to simulate a simple problem and provide a means for doing sensitivity analyses of grid size and time-step size. For numerical solutions, the grid size and time-step size used have important effects on round-off errors. Modelers usually adjust the grid size and time-step size in a computer run to get a stable "best" match to the analytical solution.

Site-specific problems generally need more than idealized analytical solution capabilities. Another effective verification activity is comparison of code results to the results of a verified computer code designed to perform the same type of analysis. Such code-to-code comparison is called benchmarking.

Sometimes a code or parts of a code are not involved in straightforward calculations. A graphics package is one example. A simulation model may also fall into this category. In such cases the code may be verified by having one or more independent reviewers walk through the code and assure themselves that it is operating correctly. This type of verification is only suitable when other verification options cannot reasonably be applied.

(c) Analysis of credible alternative models

Consideration of uncertainty about appropriate models for the events and processes involved in a risk analysis can result in the identification of a set of credible alternative conceptual or mathematical models. A systematic search for possible alternative models should be performed, and expert judgments should be used to assess the credibility of each alternative for the system and conditions being modeled. If possible, weighting factors representing the likelihood of each model being the "correct" one should be obtained from the experts.

The sensitivity of risk analysis results to credible alternative models should be bounded. If weighting factors for alternative models are available, results obtained from each alternative can be combined according to the weighting factors as a way to incorporate model uncertainties into overall risk analysis results. Where alternative mathematical functional forms have been identified, in some cases "it is possible to reformulate them as a single form with an extra parameter that can make the model equivalent to each of the (alternative) forms according to the value chosen. For example, it is possible to define a dose-response function with a threshold parameter and dose exponent parameter, which will also reproduce non-threshold models (if the threshold parameter is zero) and linear models (if the exponent is one). Thus, uncertainty about the model form can be converted into uncertainty about parameter values. This often simplifies the analysis, especially if one wants to compare the impact of uncertainty about the model form with other uncertainties" [Henrion and Morgan, 1984].

#### 2.2.2.2.2. Analyzing Uncertainty About Quantities

The approach to analyzing uncertainty about quantities is a "tiered" approach wherein the risk analyst makes decisions along the way about whether to proceed to the next level of detail and complexity in the uncertainty analysis. The approach consists of three basic levels of analysis, to be done progressively until an appropriate level of detail in quantity uncertainty treatment is reached. These levels are:

- Level 1: deterministic sensitivity analysis,
- Level 2: analytical treatment of uncertainty propagation, and
- Level 3: probabilistic uncertainty analysis.

Note that even within these levels, the analyst is required to make judgments about appropriate uncertainty treatment. The logic behind these judgments should always be made explicit.

Two primary considerations in deciding on an appropriate level of detail in quantity uncertainty treatment are (1) the type of decision that the risk analysis supports and (2) the confidence level criterion

involved. Table 2-2 summarizes guidance on the appropriate level of quantity uncertainty treatment for each type of decision that risk analyses support. Appropriate ways to characterize input parameter and outcome variable uncertainties based on these considerations are discussed in Section 2.2.2.1.2, and are summarized in Table 2-1. The uncertainty analysis must be done to a level of detail that, at a minimum, produces these required uncertainty characterizations.

#### Level 1: Deterministic sensitivity analysis

The analysis of uncertainties about the quantities in a risk analysis begins with a deterministic sensitivity analysis. The purposes of this analysis are (1) to assess the potential effect of uncertainties on risk analysis results and (2) to identify important uncertainties that might merit more detailed treatment. The sensitivity analysis should be carefully planned so that it addresses in an integrated manner questions about alternative models and alternative quantity values. In addition, risk analyses often involve combining the results of several sub-models for various processes to get concentration, dose, or risk estimates. The sensitivity analysis must be able to address questions about model output variables of interest and, thus, must allow for sub-model linkages.

The deterministic sensitivity analysis can be done to various levels of detail. Five types of sensitivity analysis are recommended:

1. sensitivity to alternative "scenarios" consisting of credible combinations of alternative models and quantities,
2. sensitivity to credible alternative models,
3. sensitivity to range changes in uncertain quantities,
4. sensitivity to alternative assumptions about possible correlations among model quantities, and
5. response surface methods (in some cases).

Table 2-2. Summary of Guidance on Appropriate Level of Quantity Uncertainty Treatment in Risk Analysis

Type of decision	Guidance*
Site and facility design selection or approval	Develop an uncertainty range for outcome (credible lower and upper bounds); establish a "best-estimate" outcome; provide rationale.
Compliance and variance determinations  "Level of control" decisions about the proper levels for standard criteria, thresholds, and compensation	<p>(1) Parametric treatment of selected value parameters, decision variables and alternative models.</p> <p>(2) Treatment of other uncertain quantities depends on confidence level criterion involved, as follows:</p> <p>(a) decision to be based on "best-estimates" -- deterministic "best-estimates" analysis with careful consideration of credible lower and upper bounds for each quantity to improve quality of "best-estimate."</p> <p>(b) decision to be based on "conservative" estimates -- deterministic "best-estimate" analysis, and credible upper bound based on scenario analysis</p> <p>(c) decision to be based on reasonable level of confidence--propagate probability distributions for uncertain quantities.</p>
"Act versus study" decisions	Deterministic scenario analysis to establish "best-estimate and credible upper bound (first-order analysis); perform value-of-information evaluation on probability trees (higher-order analysis).

\*NOTE: Deterministic sensitivity analysis is recommended for all types of decisions.

These are described below:

(1) Scenario analysis

The sensitivity analysis begins with a macro-level analysis aimed at bounding the potential total effect of combined model and quantity uncertainties on risk analysis results. This is accomplished by constructing "scenarios" consisting of credible combinations of alternative models and model quantity values within their ranges of uncertainty corresponding to worst-case, best-case, and most likely case assumptions. These scenarios should be developed by obtaining consensus judgments on them from a group of experts on the processes, models, and parameters involved in the risk analysis. The rationale behind each scenario should be documented.

The primary question to be addressed by the "scenario" analysis is: Do the differences in the risk analysis results across the scenarios indicate possible changes to risk management decision alternatives? In addition, a credible bound on overall uncertainty about the risk analysis results is obtained. Lave and Epple [1985] have suggested that, for problems that involve many uncertain variables, the point of doing scenario analysis is primarily to provide an opportunity to stretch the analyst's thinking by providing various combinations of possible events and outcomes for consideration.

If the scenario analysis shows that uncertainties could make a difference to the decision being supported, then a more detailed level of analysis should be done to assess the relative contributions of the individual sources of uncertainty. This allows the analyst to focus on important uncertainties for even more detailed uncertainty treatment or planning steps to reduce or eliminate them.

The scenario analysis can be organized around a "scenario tree," such as the one shown in Figure 2-4. Furthermore, probability estimates can be developed for the branches in the scenario tree to extend it for a probabilistic uncertainty analysis (see pages 59 to 63). As an aid to risk management decision-making, the tree can be further extended by

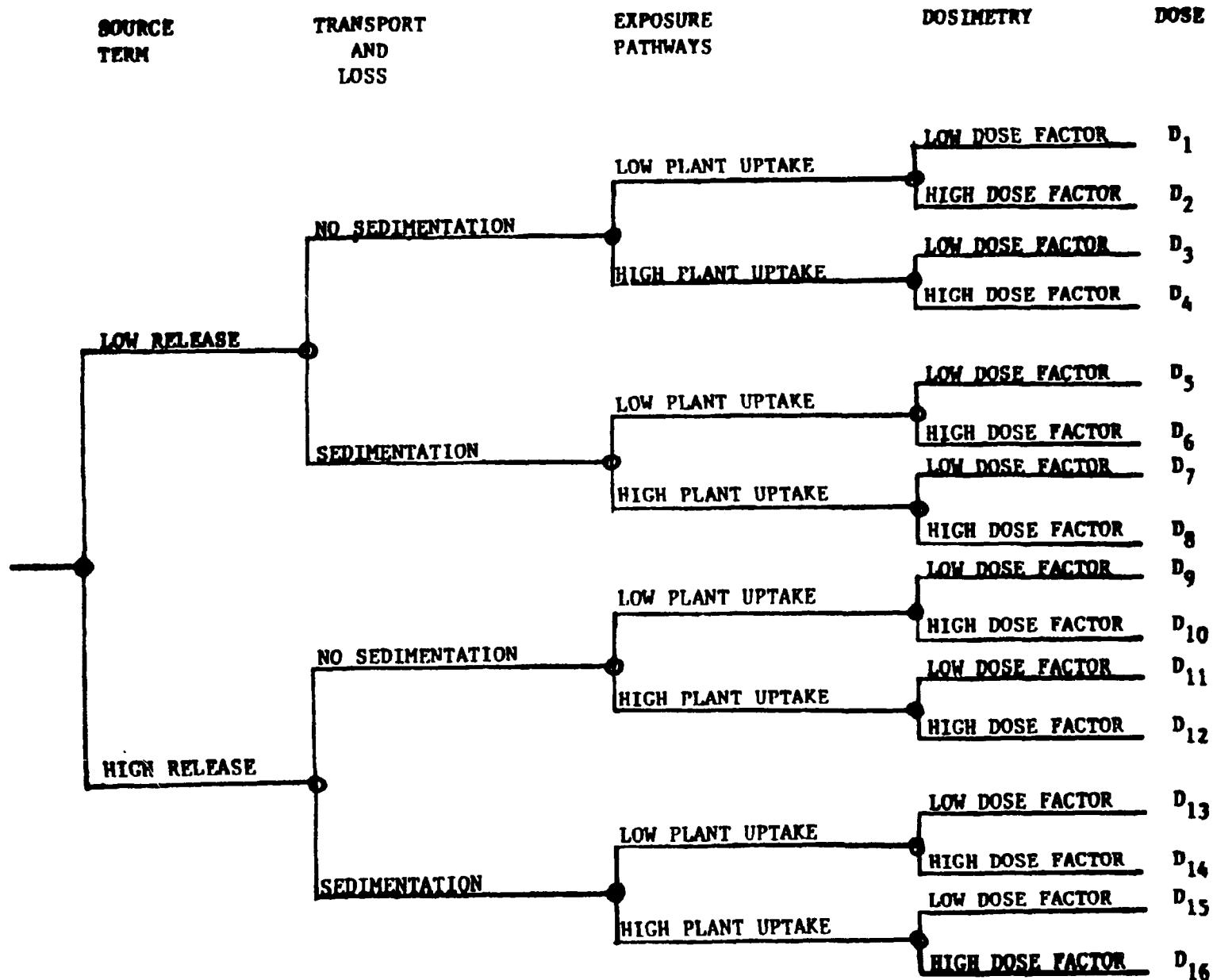


Fig. 2-4. Example of a scenario tree, depicting possible outcomes (dose) from combinations of uncertain processes or parameters.

including decisions on it, as is illustrated in Figure 2-5 (reproduced from North and Balson [1985]). In this tree the current decision is whether or not to adopt immediate additional controls on air emissions of compounds linked to acid deposition. Uncertainty about long-range transport processes in the atmosphere and long-term ecosystem impacts are represented by branches for low and high cases. Future decisions about additional controls, to be based on the uncertain outcome of the current decision, can also be included in the tree, as shown. Probabilities, costs of alternative decisions, and scenario outcomes can also be included.

#### (2) Sensitivity to credible alternative models

The first step of the more detailed level of sensitivity analysis was already performed as part of the analysis of model uncertainty described in Section 2.2.2.2.1. In this analysis, model quantities are fixed at "best-estimate" values and used in runs of credible alternative models. The range of results across alternative models can be compared to the range of overall uncertainty obtained from the scenarios analysis to assess the relative contributions of model uncertainty to overall uncertainty in risk analysis results. This step is a critical, but often overlooked, part of uncertainty assessment. The usefulness of any level of treatment of quantity uncertainties is limited if uncertainty due to credible alternative models has not been addressed. Note that in regulatory applications a common approach to model uncertainty is to use the alternative model that produces the most conservative results. This approach is valid when any one credible alternative model is not clearly and convincingly "correct" by consensus. If this approach is adopted, the rationale for the model chosen and an assessment of the effect of using that model (versus alternative models) on the level of confidence in risk analysis results should be provided.

The remainder of the more detailed level of sensitivity analysis is aimed at identifying and prioritizing those quantity uncertainties which have a significant impact on the output variables of interest. This helps to keep any more detailed uncertainty analysis that might be

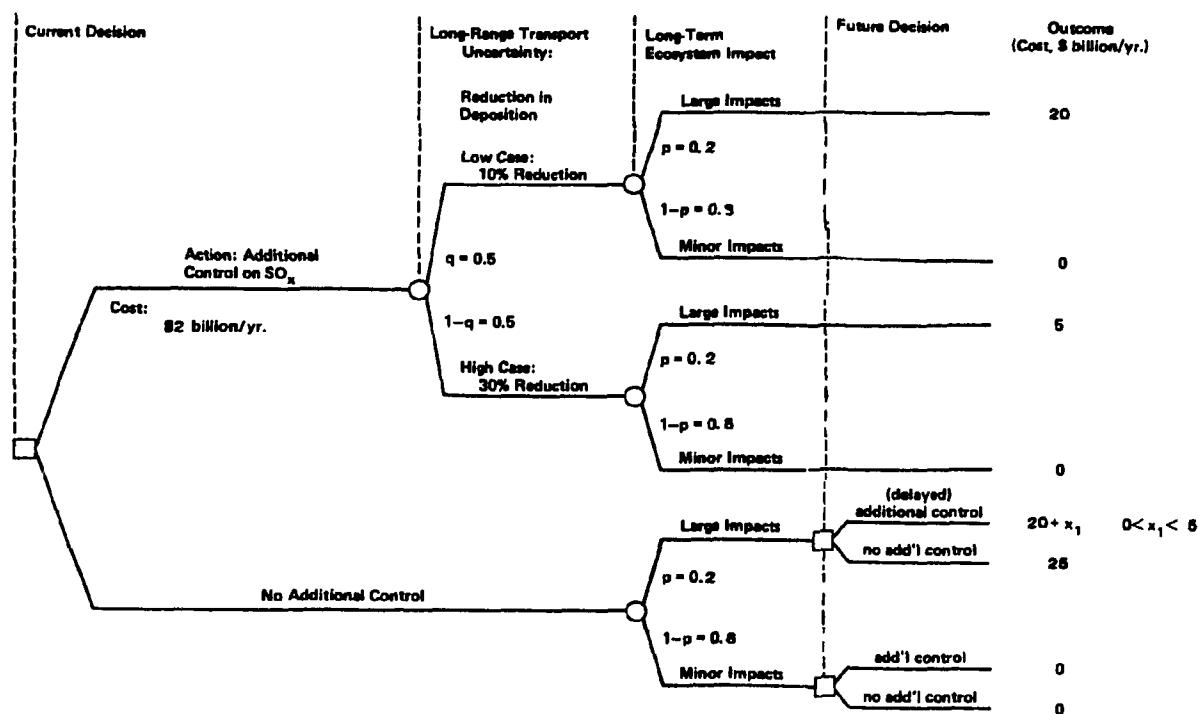


Fig. 2-5. Example of a "scenario tree" for an acid deposition control problem. The tree is extended to include current and future decisions, outcome uncertainties and branch probabilities. Adapted from a figure in North and Balson, 1985. Reproduced with authors' permission.

indicated down to a reasonable level of effort and cost by allowing it to focus on important uncertainties.

(3) Sensitivity to quantity range changes

Since a given model quantity cannot be characterized as any more uncertain than by merely specifying its lower and upper bounds, calculating the change in risk analysis model output for a total range change in the input parameter provides an indication of how important it is to further upgrade the parameter's uncertainty treatment. The greater the change in output produced by the quantity range change, the more important it is to determine how likely it is that the "true" quantity value is at different magnitudes within its bounds. If a total range change in the value of a quantity results in a large change in model output values (and a subsequent change in decision alternatives), it would then be appropriate to make the efforts required to refine the characterization and treatment of uncertainty about the quantity.

The refinement might reveal, for example, that there is high probability assigned to a small interval with the total range and low probability assigned to the majority of the range. Thus, while large differences in risk estimates could result from alternative opinions about the parameter's magnitude within its total range, the probability of such differences would be assessed as low. On the other hand, if a total range change in a model quantity results in only a small change in the model output values (and no subsequent change in decision alternative), then spending further resources to develop a refined uncertainty characterization for this parameter might be considered unnecessary.

The procedure for the quantity range sensitivity analysis is as follows:

- a. fix each uncertain model quantity, one at a time, at its credible lower bound (holding all others at their medians),
- b. compute the output measure,
- c. fix each uncertain quantity, one at a time, at its credible upper bound,
- d. compute the output measure, and
- e. divide the high output by the low output.

An example of the results from such a quantity range sensitivity analysis is shown in Table 2-3.

These results are useful for identifying the quantities which merit closest attention with respect to their uncertainty characterization. Note that these results do not provide any information about the probability of a quantity's value being at any level within the range. As such, these results should be used as a means to focus and prioritize where more attention should be paid to model parameter uncertainties.

#### (4) Sensitivity to correlation assumptions

The sensitivity analyses described above are based on the assumption that the parameters of the model are independent of each other. There is sometimes evidence that to some extent correlation exists between some of the model input quantities. Depending on the extent of correlation and the model structure, correlation effects can either increase or decrease output uncertainty. It is often the case that insufficient information exists to estimate the level of correlation between variables; however, by comparing results assuming no correlation to those assuming fully correlated quantities (where

Table 2-3. Example of Quantity Range Sensitivity Analysis Results  
 [Rish et al., 1983]

Quantity	Factor change in median dose over quantity uncertainty range (Fixing all other quantities at their medians)					
	AM-241	CS-135	NP-237	PU-239	RA-226	TC-999
D <sub>rop</sub> - ingestion	764.0	5.0	982.0	4240.0	1504.0	10,545.0
D <sub>rop</sub> - inhalation	-	-	-	193.0	-	-
D <sub>rop</sub> - external ground	-	-	-	-	1.5	-
D <sub>rop</sub> - external air	-	-	-	-	-	-
B <sub>nv</sub>	-	56.0	8.3	4510.0	8.5	3.0
F <sub>mn</sub> (milk)	-	1.3	-	-	-	1.5
F <sub>fn</sub> (beef)	-	1.4	-	-	-	1.1
Bioaccumulation factor	1.5	7.0	4.9	1.7	1.1	2.8
λ <sub>en</sub>	2.9	1.4	1.4	2.3	1.9	2.9
λ <sub>sn</sub>	-	151.0	25.7	44.0	5.3	3.9
Resuspension factor	-	-	3.0	-	-	-
Irrigation fraction	5.8	6.0	7.0	9.1	9.3	8.2
Sedimentation factor	100.0	10.0	10.0	100.0	-	-
Water treatment factor	1.6	-	1.1	1.4	1.3	1.4

(-) = Negligible

Note: The entries above were derived by (1) fixing each quantity in first column at its lower bound (holding all other quantities at their medians, (2) computing the dose, (3) fixing each at its upper bound, (4) computing the dose, and (5) dividing high dose by low dose.

correlations are suspected) it is possible to assess the sensitivity of model results to such correlations, if they did exist. It is extremely important to treat possible correlation effects in deterministic sensitivity and probabilistic uncertainty analyses. To not do so can seriously flaw an otherwise insightful uncertainty assessment. Several available software systems for uncertainty and sensitivity analysis include capabilities to handle correlations.

A first order sensitivity analysis to possible correlations can be done by first grouping possibly correlated uncertain model input quantities and assuming them to be fully correlated. The change in median model output from a range change in the group of correlated input parameters, varied together, is determined. An example of the results of such an analyses is presented in Table 2-4 for a model estimating doses of the isotopes AM-241 and PU-239 released to a river. The results are expressed as the change in the range of the results when different quantities are treated as correlated. For example, for PU-239, when the inhalation and ingestion dose conversion factors are treated as independent, the dose results vary by a factor of about 4000 when the ingestion dose conversion factor is varied over its range. However, if the inhalation and ingestion dose conversion factors are treated as fully directly correlated and varied together over their ranges, the change in dose results increases to a factor of about 300,000.

The sensitivity analysis to correlation assumptions can be quite useful for providing improved confidence in the analyst's understanding of the structural relationships in the risk model being used. To illustrate this point, it is helpful to examine the example in Table 2-4 in more detail. Correlations were assumed in the example for dosimetry and pathway factors. The dose conversion factors for ingestion and inhalation were directly correlated because the physical and biological processes that affect both absorption into the blood and biological half life should be similar for each route of uptake.  $B_{nv}$ ,  $F_{mn}$ ,  $F_{fn}$ , bioaccumulation factor, and  $\lambda_{sn}$  were assumed to be directly correlated with each other and inversely correlated with sedimentation removal. Direct correlation among  $B_{nv}$ ,  $F_{mn}$ ,  $F_{fn}$ , and bioaccumulation factor was

Table 2-4. Example of Results from Correlated Range Sensitivity Analysis [Rish et al., 1983]

Quantities	Factor change in median dose	
	AM-241	PU-239
D <sub>no</sub> p - ingestion	764.0	4240.0
D <sub>no</sub> p - inhalation	-	193.0
Fully correlated dose conversion factors	778.3	3308,548.0
B <sub>nv</sub>	-	4510.0
F <sub>mn</sub>	-	-
F <sub>fn</sub>	-	-
Bioaccumulation factor	1.5	1.7
λ <sub>sn</sub>	-	44.0
Sedimentation factor	100.0	100.0
Fully correlated B <sub>nv</sub> , F <sub>mn</sub> , F <sub>fn</sub> , and bioaccumulation factor	1.5	4552.0
Fully correlated B <sub>nv</sub> , F <sub>mn</sub> , F <sub>fn</sub> , bioaccumulation factor, and sedimentation factor	N/A	450,000.0
Fully correlated B <sub>nv</sub> , F <sub>mn</sub> , F <sub>fn</sub> , bioaccumulation factor, sedimentation factor, and λ <sub>sn</sub>	148.2	631.6

assumed because biological mobility is the factor which each parameter has in common.  $\lambda_{sn}$ , which is an expression of mobility in soil, was treated as directly correlated with  $B_{nv}$  because the greater a radionuclide's mobility in soil, the greater its availability for uptake by plants. Inverse correlation with sediment removal was assumed because a radionuclide which tends to remain associated with particulates and not be dissolved in water might be less likely available for biological uptake. A higher  $K_d$ , therefore, leads to a lower value for  $B_{nv}$  and bioaccumulation factor. Particle binding also affects the radionuclide removal rate from the soil root zone. Thus, increases in  $K_d$  might also lead to decreases in  $\lambda_{sn}$ . Note that increases in  $B_{nv}$ ,  $F_{fn}$ ,  $F_{mn}$ , and bioaccumulation will increase dose while increases in  $\lambda_{sn}$  will decrease dose. Sediment removal processes are also affected by particle binding. This factor was considered, therefore, to be inversely correlated to the other pathway parameters, because higher  $K_d$  will decrease those parameters but increase sediment removal.

The results for Am-241 demonstrate the obvious point that correlation is only important if the uncertainties about two or more of the correlated parameters are important. Referring to Table 2-4, assuming full correlation between the ingestion and inhalation dose conversion factors, and thus varying them together over their ranges, resulted in approximately the same factor change in the median dose estimate as from varying the ingestion dose factor alone. Also, assuming fully correlated  $B_{nv}$ ,  $F_{mn}$ ,  $F_{fn}$ , bioaccumulation factor, and  $\lambda_{sn}$  parameters resulted in the same range change sensitivity as for the uncorrelated bioaccumulation factor alone.

In contrast to the correlation sensitivity results for Am-241, the results for PU-239 demonstrate that if two or more parameters which might be correlated have significant levels of uncertainty, then correlation assumptions can greatly affect uncertainty analysis results. Referring to the PU-239 results in Table 2-4, both the ingestion and inhalation dose conversion factor uncertainties are important with respect to the model's dose output. Assuming they are fully correlated, and varying them together over their uncertainty ranges, results in a

change in the range of dose output which is much larger than when the parameters are treated as uncorrelated (i.e., 300,000 vs 4000). The effect that the assumed model structure and type of correlation can produce is indicated by the decrease in parameter range sensitivity (from a factor of 4552 to a factor of 631) which occurred when  $\lambda_{sn}$  and the sedimentation factor were added to the correlated group of parameters for PU-239. Adding the inversely correlated sedimentation removal to the  $B_{nv}$ ,  $F_{mn}$ ,  $F_{fn}$ , and bioaccumulation factor grouping increases the change in dose for a range change in these parameters. However, adding  $\lambda_{sn}$  to the grouping more than offsets the sedimentation factor correlation effect. The result is a net decrease in output sensitivity over that produced by a range change in  $B_{nv}$  alone.

In the example, assessing the magnitude and direction of possible model quantity correlations was shown to be important for determining their net effect on the results of the uncertainty analysis; however, due to a lack of sufficient understanding of the processes behind these quantities, any detailed correlation assumptions would be mostly conjecture. Research on the mechanisms behind the model quantities and the factors upon which these mechanisms depend would serve to improve the dose estimates for those radionuclides having important uncertain quantities which are correlated. Thus, the correlation sensitivity analysis can help show where research efforts might effectively reduce risk analysis uncertainty.

Note that it is possible, using a factorial design for the deterministic sensitivity analysis, to accomplish both the quantity range sensitivity and the first-order correlation analysis described above at the same time. The approach is to select two or more values from each uncertain quantity range and combine them into a set of all possible combinations of all quantities at all selected values. According to Rod [1984]: "Such a complete set is called a full factorial design. Three factors at two values ("low" and "high") each would be combined as follows:

Combination #	Factor 1	Factor 2	Factor 3
1	low	low	low
2	low	low	high
3	low	high	low
4	low	high	high
5	high	low	low
6	high	low	high
7	high	high	low
8	high	high	high

"One advantage that full factorial designs have over one-at-a-time sampling is that the interactions among all combinations of factors are estimable from the basic set of runs. From the preceding example the effects of factors 1, 2, 3, 1+2, 1+3, 2+3, and 1+2+3 on outputs can all be estimated."

"The main disadvantage of full factorial designs is that the number of runs required is given by:

(# full factorial samples) =  $(k)^n$   
 where  $k$  = # levels per factor  
 and  $n$  = # factors."

"Two approaches to reducing the required number of samples are the restriction to two levels per input (and so a restriction to the linear assumption) and the use of partial factorial designs."

#### (5) Response surface methods (in some cases)

Response surface methods can be used to screen important model uncertainties and to construct simplified versions of models; however, these methods require a significant effort and therefore should only be used to offset the cost of planned probabilistic uncertainty analyses by simplifying them, or when the models involved are frequently used in other applications for which the response surface results would be valid.

If the model involved in a risk analysis is relatively simple (e.g., dose from a few biosphere pathways), then it can be insightful to develop a "feel" for the model response surface by calculating "elasticities" for uncertain input quantities at various points in their ranges. "Elasticity" is the percent change in model output per percent change in model input quantity value. Elasticity depends upon the specific point about which it is calculated within the uncertainty range of the input quantity. This is because a model might be more or less sensitive to small changes in the quantity depending on the magnitude of the quantity. Elasticity can be thought of as the slope of the model response curve for the given quantity at a specific location on the curve. This concept is illustrated in Figure 2-6. The figure shows increasing elasticity at values of input quantity  $X$  below  $X^*$ , and decreasing elasticity above  $X^*$ . Note that on this figure the quantity range sensitivity is defined as  $Y_2$  divided by  $Y_1$ .

By calculating elasticities across uncertain model quantity ranges, provided the model is simple enough for the analyst to conceptualize, the analyst can improve his or her understanding of the structure of the model and can identify portions of the uncertain quantity ranges where the model output is especially sensitive or insensitive to changes in the quantity. For example, in the hypothetical example in Figure 2-6 the elasticity increases for values of  $X$  below  $X^*$  and decreases for values above  $X^*$ . Thus, the model is more sensitive to small differences in the assumed input parameter value in the lower portion of its uncertainty range than in the upper portion.

As an example of how a quantity range sensitivity analysis combined with elasticity analysis can provide insight as to where uncertainties about model quantities matter and how they interact, consider the results shown in Table 2-5. These results are from analyses performed on the river release models used by the ORP to derive the release limits in 40 CFR 191 [Rish et al., 1983]. The analysis assumes one curie of PU-239 released to a river, and calculates the fatalities from human doses from eight exposure pathways. Table 2-5 (top) shows the allocation of dose by the model to the eight pathways.

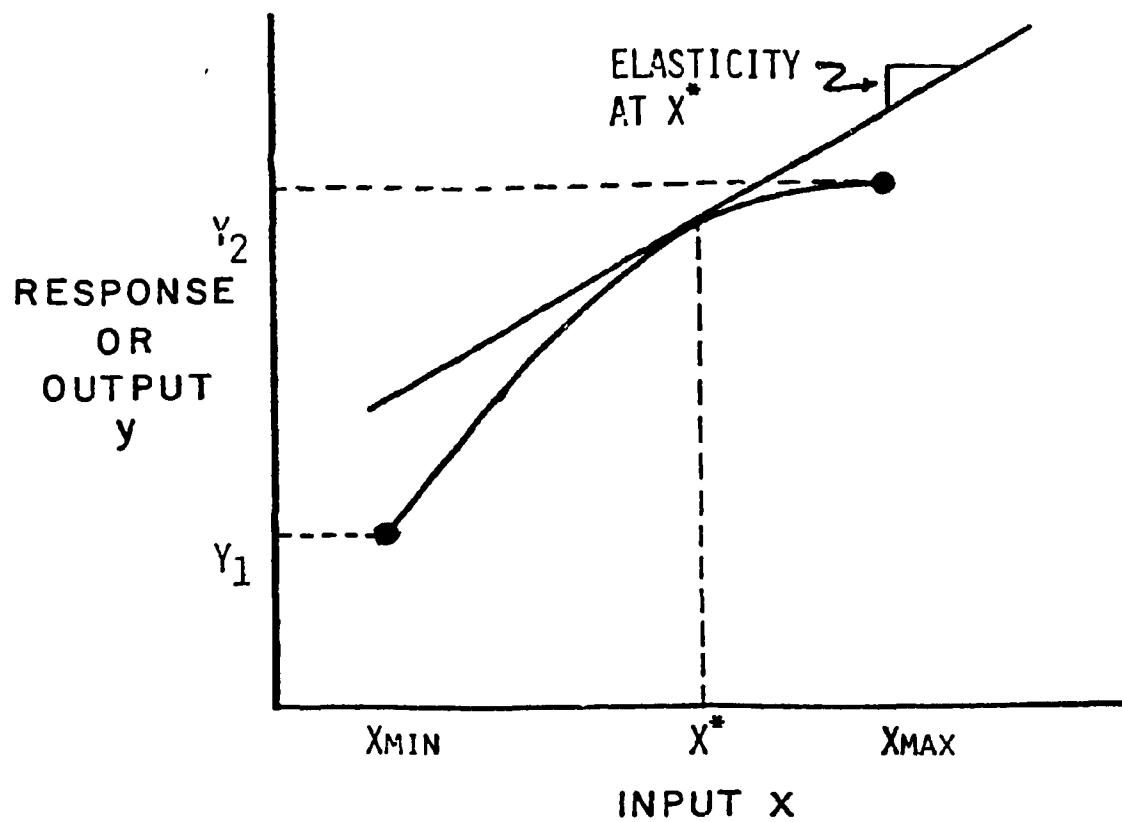


Fig. 2-6. Model response curve for input quantity  $X$  illustrating concept of elasticity (local sensitivity).

Table 2-5 (top). Median dose per curie released to river -- by pathway

River mode pathway	Median dose
Drink Dose	1.7600
Fish Dose	0.0427
Crop Dose	3.4464
Milk Dose	n*
Beef Dose	n
Inhale Dose	1.1088
Ground Dose	0.0073
Air Dose	n
<b>TOTAL DOSE</b>	<b>6.3652</b>

\*n = Negligible

Table 2.5 (bottom). Range Change Sensitivities and Elasticities

Quantity	Factor change in dose for quantity range change	Elasticities		
		% Δdose per % Δquantity at lower bound	median	upper bound
D <sub>rop</sub> - ingestion	424.0	(-)	0.825	0.999
D <sub>rop</sub> - inhalation	193.0	(-)	0.174	0.994
D <sub>rop</sub> - external ground	(-)	(-)	(-)	(-)
D <sub>rop</sub> - external air	(-)	(-)	(-)	(-)
B <sub>nv</sub>	4510.0	(-)	0.114	1.0
F <sub>mn</sub> (milk)	(-)	(-)	(-)	(-)
F <sub>fn</sub> (beef)	(-)	(-)	(-)	(-)
Bioaccumulation factor	1.7	(-)	0.007	0.398
λ <sub>en</sub>	2.3	-0.66	-0.429	-0.221
λ <sub>sn</sub>	44.0	(-)	-0.284	-0.004
Resuspension factor	3.0	(-)	0.174	0.68
Irrigation fraction	9.1	0.34	0.717	0.93
Sedimentation factor	100.0	1.0	1.0	1.0
Water treatment factor	1.4	0.122	0.277	0.385

(-) = Negligible

Source: [Rish et al, 1983]

Table 2-5 (bottom) shows the results of a range sensitivity and elasticity analysis on model quantities that were judged to have possibly significant uncertainty levels. Referring to the table, the dose estimate from the : is significantly sensitive to range changes of several parameters: Dnop-ingestion, Dnop-inhalation,  $B_{nv}$ ,  $\lambda_{sn}$ , and sedimentation. Plutonium is the only radionuclide specified in 40 CFR 191 where the results are sensitive to the inhalation dose conversion factor. This is due in part to the relatively high inhalation dose conversion factor for PU-239. Because of this fact the inhalation pathway plays a significant role in the total dose (see Table 2-5 (top)), which explains the variable elasticities of both ingestion and inhalation dose factors (see Table 2-5 (bottom)). When the ingestion Dnop is at its lower bound, the inhalation exposures dominate and changes in the ingestion dose conversion factor do not affect the dose at this level. When the inhalation Dnop is at its lower bound the total dose is due to the ingestion routes, and changes in inhalation Dnop do not affect the model results. The elasticities also indicate that only difference between the higher values of Dnop inhalation produce significant changes in the total dose.

The range and local sensitivities for the parameter  $B_{nv}$  are noteworthy. It can be seen that the range change for this parameter affects the dose more than any other factor, yet the dose is elastic only to changes near the upper bound of the  $B_{nv}$  distribution. This is exemplified in Table 2-5 (bottom) where the elasticity of the median is only about 10% of that of the upper bound for  $B_{nv}$ . Thus, differences of opinion about the appropriate value for  $B_{nv}$  are more important at the high end of its uncertainty range.

As a final point, Table 2-5 (bottom) reveals that the dose is sensitive to changes in the soil removal rate constant  $\lambda_{sn}$  only between the 0.25 and 0.50 fractiles. This occurs because, at low values for  $\lambda_{sn}$ , radiological decay becomes the controlling mechanism for removal. For high removal rates no activity is retained in soil so other pathways, such as drinking water, fish ingestion, and inhalation, predominate. This is an interesting finding because resolution of  $\lambda_{sn}$  at its extremes is not as important as uncertainty around its mean.

For many risk analysis problems the models involved are complex and have many input quantities. For these models a simplistic elasticity analysis would probably not yield much insight since the complexity of the algorithms involved and the model logical structure, and the large possibilities for interaction effects would make the significance of the results obscure. A more sophisticated level of response surface analysis is required, if warranted. Such an analysis is warranted if:

1. probabilistic analyses of uncertainty are planned (see Table 2-2), and it is desirable to reduce their cost by more refined screening of important uncertain quantities and/or developing a simpler version of the model, or the development of a response surface is a necessary step in the probabilistic analysis (i.e., calculation of first-order derivatives for method-of-moments or Taylor series approximation techniques (see "Level 2" discussion below), or
2. the models involved are planned for frequent use in other applications for which it is expected that the response surface results will be valid.

Response surface methods can begin with a preliminary statistical analysis used to screen the model input quantities to identify those few quantities that have a significant effect on the output. This screening step begins with those quantities identified as being "important" by the previous sensitivity analyses. To these, the analyst applies stage-wise correlation analysis (see [Vaurio, 1982]) followed by step-wise regression analysis (see [Vaurio, 1982] and [Iman, et al., 1980]).

A response surface model is developed using the important model quantities as identified by the screening step. This response surface model is, in effect, a simplified version of the original model. The response surface model can be developed in several different ways, including:

1. surface fitting, and
2. use of differential sensitivity theory (perturbation calculus).

Software is available to facilitate the surface-fitting approach (see [Vaurio, 1981]). The use of differential sensitivity theory to calculate a response surface model involves deriving a set of "adjoint" equations for important uncertain quantities. These are partial differential equations representing changes to model output from perturbations to input quantities.

According to Rod [1984]: "Once the set of forward equations, with initial and boundary conditions, and the set of adjoint equations are established, all system responses to all input changes can be found with just two calculations per desired response."

"One limitation of the method is that computed sensitivities are strictly linear approximations at one point on the system response curve. Sensitivities at points away from that point (far enough for an assumption of linearity to break down) require separate point calculations. The points can be linked by interpolating between the tangential planes generated by the differential sensitivity model."

"The advantage of the differential sensitivity method in calculation time and cost savings is had at the expense of a greatly increased theoretical development effort. The complete sets of forward and adjoint equations must be derived to match the specific computer model under study, effectively requiring creation of a unique sensitivity model for each new physical model."

"In practice, the development of a differential sensitivity model has taken months of effort by experts in the theory of the particular field for which the original physical model was created. This requirement both boosts the cost of implementing the method and discourages its use by anyone other than the original code's developers. Independent review, by regulatory authorities, for example, is more difficult."

"Recent innovations may help to relieve the development cost disadvantage [Oblow, 1983]. A group at the Oak Ridge National Laboratory created a FORTRAN "pre-compiler" GRESS, which generates the necessary differential equations directly from the original source code

and incorporates them into a new source code. This automatic procedure is still being refined, and the breadth of its applicability has not been assessed."

Note that the results of GRESS can, in some cases, be used with analytical solutions (or approximations) for statistical error in the model to obtain estimates of model output uncertainty. This approach is discussed in the next section.

#### Level 2: Analytical treatment of uncertainty propagation

In some cases, when the risk analysis involves the use of models consisting of explicit mathematical expressions, algebraic formulae are available to obtain analytical solutions (or good approximations) for uncertainty in the results of the calculations.

Seiler has developed a set of analytical solutions, or sufficiently close approximations, for the propagation of input parameter uncertainties through "some simple algebraic structures that occur often in risk assessments" [Seiler, 1986]. These structures are:

- linear combinations,
- positive powers of one variable,
- negative powers of one variable,
- non-integer powers of one variable,
- products linear in each normally distributed variable,
- products of powers of log-normally distributed variables,
- non-linear dose-effects relations,
- cumulative incidence functions,
- survival functions, and
- more complex composite forms.

There are important limitations to the applicability of these formulae. They only allow estimation of the mean and standard error of the output variable. They assume independence among the uncertain parameters (quantities), though Seiler has developed formulae for error propagation where large correlated errors are present [Seiler, 1983].

Sums and differences of log-normally distributed quantities cannot be performed. It is necessary to numerically evaluate the partial derivatives with respect to the model output for each uncertain quantity.

Note that this last requirement can be accomplished two ways: (1) by an adjoint sensitivity analysis of the model, or (2) by perturbing the input quantities one at a time while holding all others at their nominal values. For the first approach, the code GRESS [Oblow, 1983] can be used to generate the necessary differential equations by adjoint sensitivity analysis of the risk analysis model. These can then be used in the formulae for analytical treatment of output uncertainty.

A paper [Seiler, 1986] presenting Seiler's formulae and his treatment of the algebraic structures occurring in risk assessments is reproduced in Appendix B.

Depending on the type of decision that the risk analysis supports and the confidence-level criterion involved in the decision, greater specification of the model output uncertainty than is possible by Seiler's formulae (mean and standard error) may be desired. In these cases, an assumption must be made about the shape of the output distribution.

#### Level 3: Probabilistic uncertainty analysis

The deterministic sensitivity analysis identified those model quantity uncertainties that are "important" in that changes to them within their ranges of uncertainty produce changes to risk analysis outcomes, and possibly to the risk management alternative chosen. Depending on the type of decision being supported by the risk analysis and the particular confidence level criteria involved (see Table 2-2), the analyst must decide whether it is appropriate to proceed to the next level of quantity uncertainty treatment -- a probabilistic uncertainty analysis. In general, a probabilistic uncertainty analysis is called for when it is necessary for the decision-maker to know the relative likelihoods of alternative risk analysis results (and thus alternative

risk management decision outcomes) across their full range of uncertainty, and analytical solutions are not practical or sufficient.

(1) Probability trees

"The simplest and probably most common approach to uncertainty propagation is to explore the range of possible outcomes without attempting to quantify their relative likelihoods" [Henrion and Morgan, 1984]. This is the "scenario" approach previously described as part of the deterministic sensitivity analysis. A simple level of probabilistic analysis involves extending the scenario analysis by assigning discrete probabilities to the high, medium, and low values of each important uncertain quantity in each scenario. In order to account for conditional dependencies, the quantities should be ordered conceptually in a "probability tree" such that each quantity is subsequent to any quantities it depends on. An example of a "probability tree" is shown in Figure 2-7. Each node represents a key uncertain factor with branches to each of its possible levels. Each branch is assigned a probability conditional on the outcomes of the previous branches. Each of the endpoints on the right of the tree represents a potential scenario whose probability is the product of the probabilities of the branches leading up to it. Risk analysis outcomes are combined with the scenario's associated probability from the "probability tree." The scenario outcomes are ordered and cumulated to obtain a cumulative probability distribution representing uncertainty about the risk analysis.

This approach to obtaining a probabilistic measure of risk analysis uncertainty has, as advantages, that it is easy to follow and it requires a relatively modest effort for simple models. A major disadvantage is that since the number of separate scenarios to analyze is  $M^N$  for  $N$  uncertain quantities each with  $M$  possible values, the approach is impractical for analyses involving large  $M$  or more than five to ten uncertain quantities ( $N$ ). In these cases, an alternative approach is available, called the method of Discrete Probability Distributions (DPD) [Kaplan, 1981]. According to Henrion and Morgan

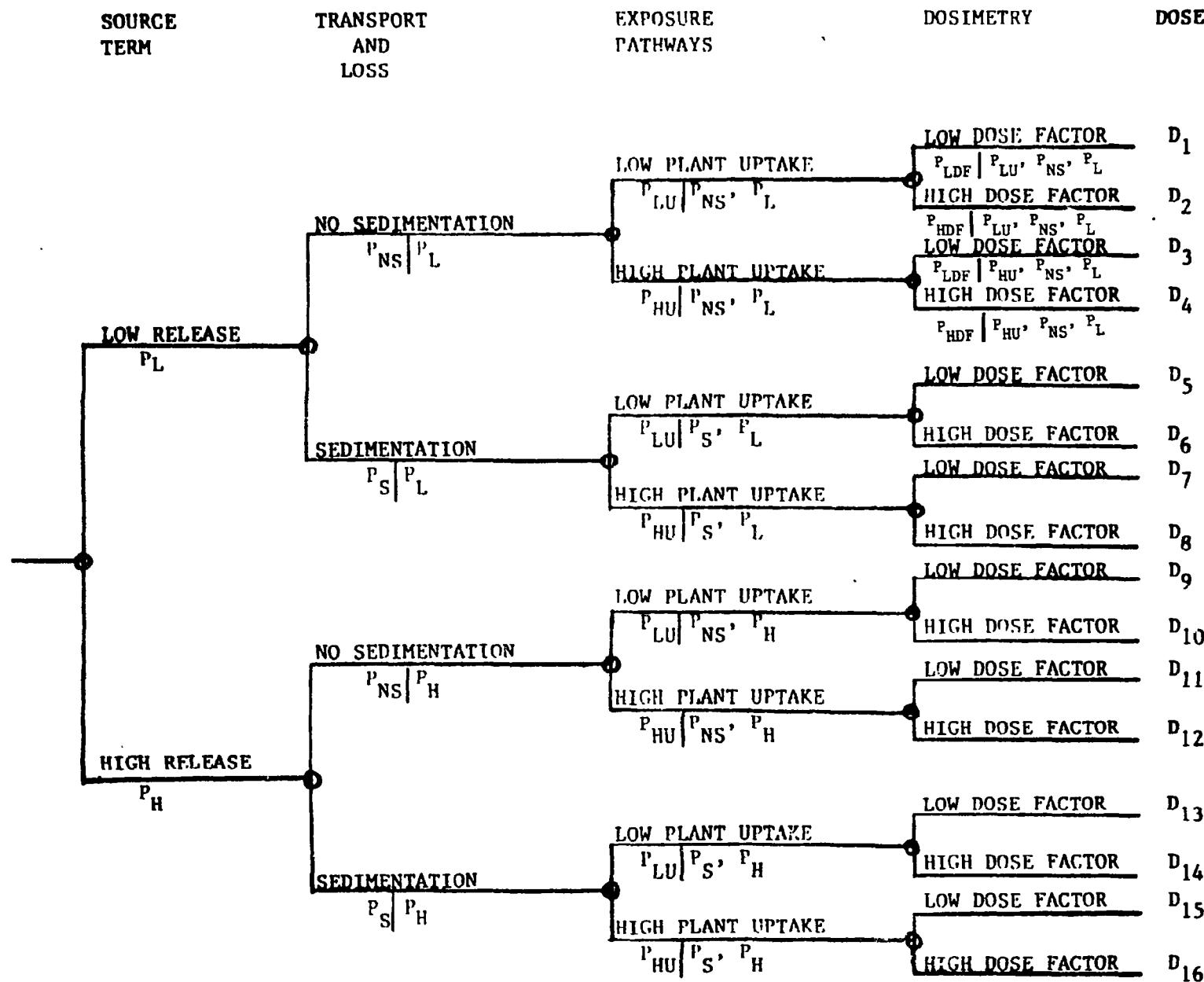


Fig. 2-7. Example of a probability tree, wherein the "scenario" analysis is extended to include discrete probabilities of key uncertain factors.

[1984]: "Suppose every uncertain parameter is discretised to five values. Where two parameters must be multiplied, all 25 possible combinations of the two are computed with their probabilities. These are ordered and the resulting 25 point DPD is "condensed"; that is, it is itself approximated by a 5-point DPD before it takes part in further calculations."

"If factors can always be combined only two at a time, there is never any need for more than 25 calculations at each point, and so the combinatorial explosion is avoided. If the same parameter appears in the calculation in more than one place, then this will not work, since it will create dependent subexpressions to be combined. For example,

$$y = (x_1 + x_2)/x_3 - x_1 \cdot x_2$$

Thus, the method requires the calculations to be reordered to put all repetitions of the same term in the same subexpression. This can require considerable ingenuity, and unfortunately is impossible for many complex computations, which puts a severe limitation on the applicability of the method."

## (2) Stochastic simulation techniques

An alternative to the analytical treatment and probability tree approaches described above is to use stochastic simulation techniques (also known as Monte Carlo techniques) to propagate quantity uncertainties through model calculations.

Probability distributions should be developed for each important model quantity identified by the sensitivity analyses. The guidance for characterizing empirical parameter uncertainty provided in Section 2.2.2.1.2 should be used when developing the probability distributions. Note that some uncertain quantities (e.g., decision variables, value parameters) will be treated parametrically, and some planning should be done on how to best combine parametric and probabilistic analyses.

The concept of stochastic simulation is simple. Values are sampled from the ranges of each uncertain quantity according to the frequencies represented by their probability distributions. At each iteration a set of values are selected and the model is run. After many iterations a histogram of the results provides an estimate of the probability distribution of the model outcome. This probability distribution represents uncertainty about the risk analysis model results due to uncertainty about model input quantities.

Some excellent software is available to facilitate Monte Carlo-type uncertainty propagation. Available software is summarized in the companion literature review;\* however, three systems are worthy of mention here. MOUSE [Klee, 1985] and DEMOS [Henrion, 1979] are interactive computer programs that allow specifying probability distributions for model input quantities and Monte Carlo-type propagation through user-specified models. These systems are highly recommended for risk analysis models consisting of combinations of algebraic expressions, including matrix operations. They can be quite cost-effective for these applications. LHS (Latin Hypercube Sampling) is a computer program for the generation of Latin hypercube and random samples for propagating uncertainties through computer codes [Iman and Shortencarier, 1984]. The program is relatively portable, and can be used as the mechanism to convert a deterministic model into one that propagates input parameters probabilistically. Sampling can be done from standard or user-defined distributions and from empirical data. Correlation among input parameters can be treated. A companion program is available for calculating partial correlation and standard regression coefficients for a data set [Iman et al., 1985].

For a risk analysis model having a large number of uncertain quantities or a relatively complex algorithm, it might be more cost-effective to develop a simplified version of the model on which to perform the Monte Carlo-type analysis. This simplified version can be developed by applying response surface techniques (e.g., adjoint sensitivity analysis) to the model.

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\* See Review of Studies Related to Uncertainty in Risk Analysis [Rish, 1988].

### 2.2.3 EVALUATING AND COMMUNICATING THE RESULTS OF THE UNCERTAINTY ASSESSMENT

Once an uncertainty assessment strategy has been developed and implemented for a particular risk analysis, the significance of the results of the uncertainty assessment must be evaluated. Based on the type of decision that the risk analysis is supporting, the confidence level criterion involved, and the intended "audience," the analyst must devise an effective way to communicate the results of the uncertainty assessment.

Generally speaking, the results of an uncertainty assessment as part of a risk analysis are used to address two types of questions:

1. questions about levels of confidence in the possible outcomes of alternative risk management decisions being considered, and
2. program planning-type questions about:
  - risk control versus uncertainty reduction (act vs. study),
  - effective research, measurement and analytical activities, and
  - institutional responses to uncertainty issues.

A decision-maker should be able to use the uncertainty assessment results to:

1. assess the levels of confidence in risk analysis results,
2. identify the important sources of uncertainty in the risk analysis results,
3. understand the "resiliency" of alternative risk management decisions to uncertainty about their outcomes, and

4. determine possible actions that may be taken to reduce uncertainty in decision outcomes.

In other words, the uncertainty assessment should answer the following corresponding questions:

1. How confident can we be in the risk analysis results?
2. Why are we uncertain?
3. How wrong could we be, how likely is it, and what difference would it make?
4. How might I increase confidence in my decision?

Before describing how to present uncertainty assessment results to address these questions, an important "lesson-learned" from previous applications of uncertainty assessment is worth consideration. Experience has shown that the process of systematically addressing uncertainties in a risk analysis provides those directly involved in the process with important insights for each of the questions listed above. Many of these insights are difficult to reflect in the substantive results of the uncertainty assessment, but they are just the same a "product" of the assessment that contributes to better decision-making. For this reason it is recommended that, in order to increase confidence in decisions based on risk analyses, someone with a role in the risk management decision process be involved in, or at least constantly monitor, the risk analysis and its accompanying uncertainty assessment as they evolve.

How confident can we be in the risk analysis results?

The way to address this question depends on the type of decision that the risk analysis supports and the confidence level criterion involved.

(1) Risk analyses being used to support site and facility design selection decisions should characterize output uncertainty by presenting the range between credible upper and lower bound estimates for each alternative site or design being considered. These ranges should be presented comparatively in a figure also showing their relationship to the particular selection criteria that the risk analysis addresses. For example, Figure 2-8 shows hypothetical results from an uncertainty assessment done in support of a site selection decision where criteria exist for radionuclide concentrations in ground water ( $C^*$ ) and maximum individual dose ( $D^*$ ).

Figure 2-8 (top) shows that while Site A has the potential to not meet the ground water concentration criterion, the best-estimate is well below the criterion. The discussion accompanying these results might further indicate, for example, that low probability scenarios are associated with the portion of the range for Site A that is in excess of the criterion. The credible range of uncertainty about ground water concentration for Site B is entirely within the criterion ( $C^*$ ); however, the best-estimate is significantly higher than that for Site A. In fact, the best-estimate concentration for Site A is less than the credible lower bound concentration for Site B. The results for Site C shows a relatively narrow range of uncertainty around a best-estimate ground water concentration that barely meets the criterion.

Figure 2-8 (bottom) shows that Sites A and B have comparable best-estimates for maximum individual dose; however, the range of uncertainty in the dose estimate for Site B is less than that for Site A. Also, the credible upper bound on the dose estimate uncertainty for Site B is lower than that for Site A, allowing for a greater margin beneath the dose criterion ( $D^*$ ). The dose estimate for Site C is less uncertain than those for Sites A and B, but the best-estimate and range for Site C are significantly higher than those for Sites A and B (and significantly closer to the dose criterion).

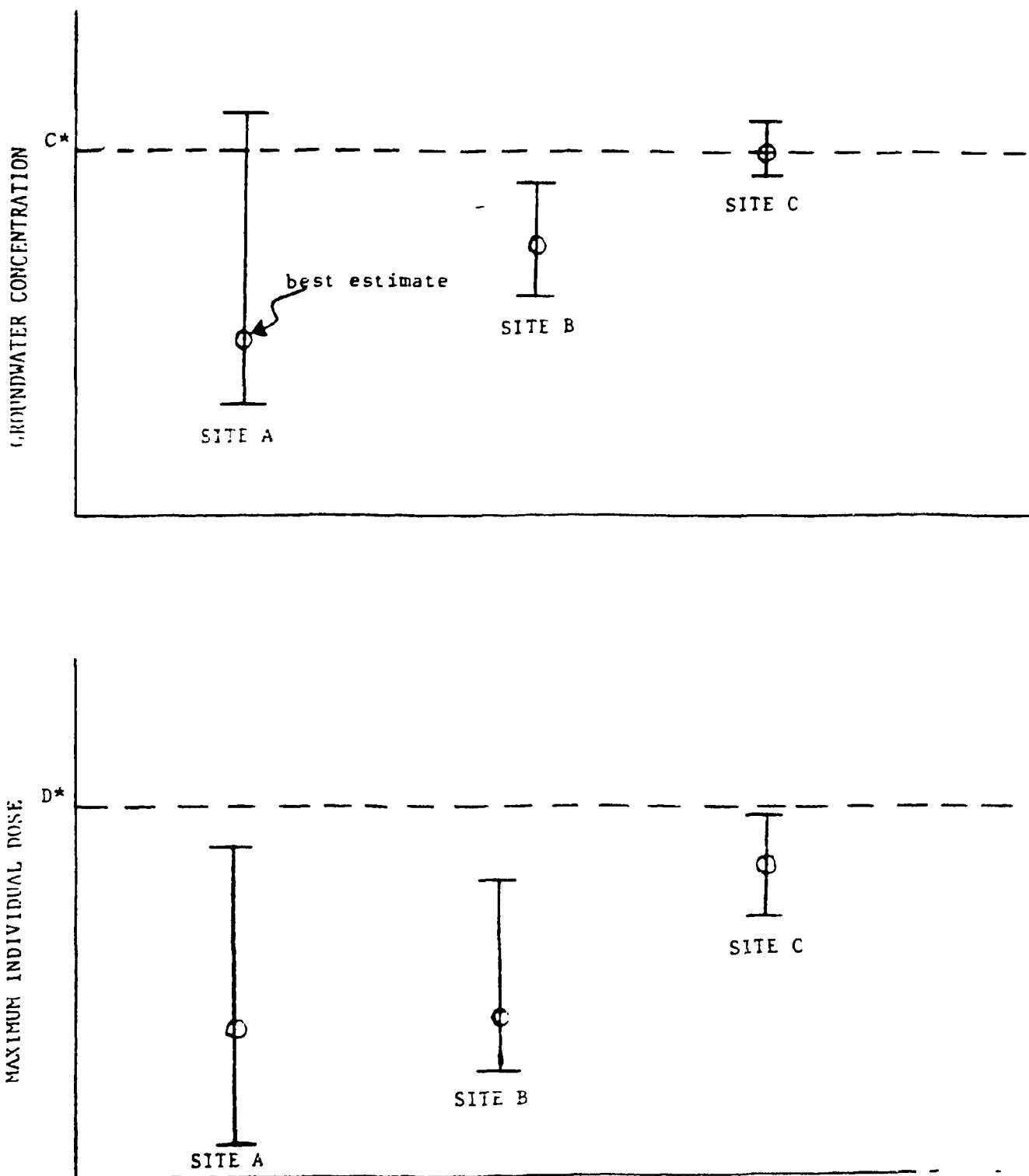


Fig. 2-8. Presentation of hypothetical results of an uncertainty assessment done in support of a site selection decision where criteria exist for ground-water concentration ( $C^*$ ) and maximum individual dose

In addition to presenting graphical results of the uncertainty assessment, like those in Figure 2-8, the conceptual models, assumptions, and conditions associated with the best-estimates and each of the credible bounds should be explained clearly and concisely.

(2) "Act versus study" decisions address whether or not to implement risk control actions or wait until further studies reduce uncertainties about decision outcomes. For this type of decision uncertainty about risk analysis outcomes should be characterized by providing "best-estimates" and plausible upper-bound estimates of risks. If the analysis indicates that plausible upper-bound risk estimates are relatively low, then there is increased confidence in a decision to not regulate. Where plausible upper bound risk estimates are significant, the best-estimates can be helpful in deciding whether to gather more information before basing decisions on the upper bound estimates. If changes to the risk analysis outcome fall within the range between the best-estimate and upper bound estimate, and result in indicated changes to risk management alternative, then more information about the likelihood of outcomes within this range is needed to support the decision. In this case the analyst can use probabilistic uncertainty analysis to perform a value-of-information assessment. An example is shown in Figure 2-9. The choice of whether to take action to control undesirable risks that are represented by particular paths through the tree can then be viewed as a decision on whether to buy insurance against the probabilities and outcomes associated with those paths. A good discussion and example of the application of probability trees and value-of-information analysis to the "act versus study" decision is provided in North and Balson [1985].

Useful guidance regarding "act versus study" decisions is provided in a recent editorial by Morgan as follows:

"Research can never demonstrate that a risk does not exist. It can establish probabilistic bounds on possible risks, and if those bounds are sufficiently low, we should then say "enough". For this to happen two things are needed. First, government agencies need to explicitly consider the question of "stopping rules" before they embark on

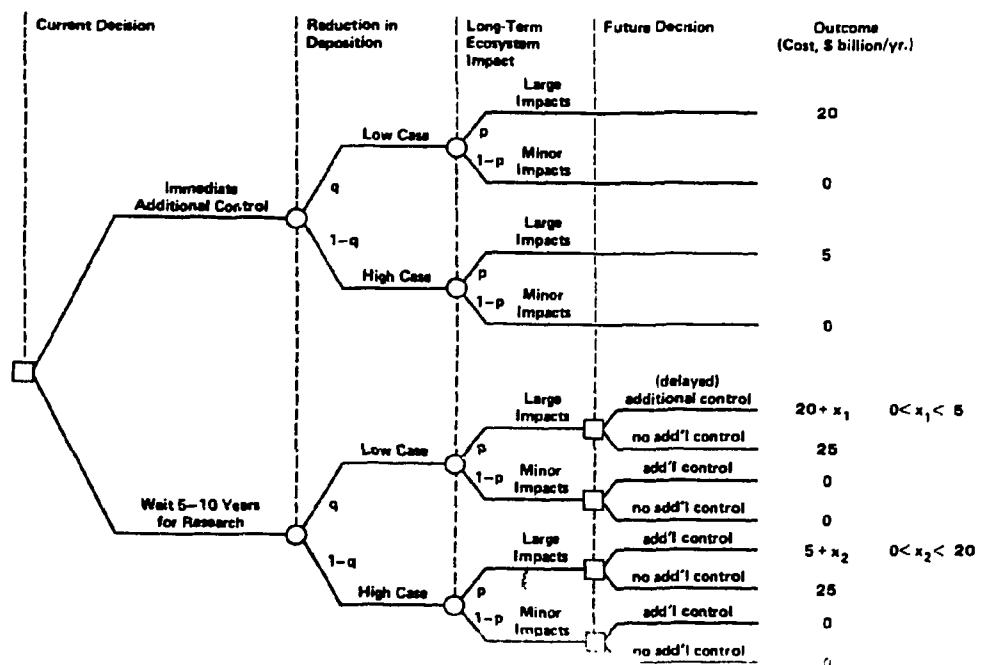


Fig. 2-9. Example of a probability tree for the choice of taking action (immediate additional control) or studying more (wait 5-10 years for research). Trees like this can be useful for performing a value-of-information assessment. Source: North and Balson, 1985. Reproduced with author's permission.

mission-oriented programs of risk research. As the research progresses they need to continue to refine those rules in the light of what has already been investigated and learned; what it is likely to cost to learn more; what the risks might be; and what kinds of findings are still needed before it makes sense to stop. Second, we need to evolve some common understanding between society, risk regulators, and the courts about how to establish acceptable probabilistic upper bounds on possible risks. Without these two developments, well-meaning government investments in risk-motivated applied research may sometimes do more harm than good" [Morgan, 1986].

(3) Characterizing risk analysis outcome uncertainty for compliance determinations depends on the confidence or assurance level criterion specified in the pertinent regulation, or otherwise indicated by the implementing agency. Similarly, characterizing risk analysis outcomes in support of "level of control" decisions depends on the confidence level criterion that applies, as follows:

a. Based on "best-estimates" -- there is considerable evidence in the literature of a variety of heuristics employed by experts in processing information that can result in significant biases in single-valued "best-estimates" for empirical parameters. It is theorized that the quality of "best-estimates" can be improved by explicit consideration by the experts of the full range of uncertainty about empirical parameters and the conditions associated with different values within the range, especially the upper and lower bounds. This practice is recommended where the results of sensitivity analyses indicate that the risk analysis results or choice of decision alternatives are significantly affected by variations within the parameter range. The information, assumptions, and conditions associated with the "best-estimate" should be documented.

b. Based on conservative estimates -- the same guidance provided above for the "best-estimate" confidence level criterion applies to characterizing empirical parameter uncertainty when basing decisions on conservative estimates of risk analysis results. Conservative estimates can also be improved by consideration of the full range of uncertainty

about a parameter. The meaning of "conservative" should be specified and consistently applied. A special case of a conservative estimate is the "worst-case" or "upper-bound" estimate. The extent to which the worst-case estimate differs from the best estimate should be indicated, and the conditions and assumptions associated with each estimate should be provided (i.e., the reason for the difference).

c. Based on a reasonable level of confidence -- risk management decisions can be based on a subjectively-determined confidence level criterion corresponding to a "reasonable level of confidence" in the risks associated with decision alternatives. This reasonable level is usually a relatively high degree of confidence; however, the marginal cost of being more certain of a decision outcome is taken into account. For example, adopting a lower release limit will increase the degree of confidence that dose criteria will be met, but an 85 percent confidence level might be "reasonable" if lowering the limit to achieve 95 percent confidence means a quantum leap in control technology costs or the use of a more efficient but less reliable technology.

Using single-valued "conservative" or upper bound estimates for uncertain risk analysis outcomes, especially when their uncertainty tends to be log-normally distributed, can result in potentially costly decisions based on risk estimates that have negligible likelihood of being "correct" and are orders of magnitude above estimates having what one would consider a reasonable level of associated confidence. As North notes, "a plausible upper bound or worst-case projection may not be helpful when there is a potential for large impacts but a high likelihood that the large impacts will not occur" [North and Balson, 1985].

In order to determine what level of protection provides a reasonable level of confidence, the decision-maker needs to have an assessment of the relative levels of confidence associated with basing actions on different risk estimates across the range of uncertainty in risk analysis results. He then can factor confidence levels into his decision. This is especially important since parameter uncertainties in environmental models usually have skewed probability density

distributions with relatively low likelihoods associated with a significant portion of the upper half of the output uncertainty range. Thus, there are negligible increases in confidence level associated with decisions based on these higher risk estimates.

The relative likelihoods associated with risk analysis outcomes across the range of uncertainty should be assessed for the decision-maker. A sense should be communicated of the marginal change in confidence about achieving risk goals that is associated with using different risk analysis outcomes from the uncertain range as the basis for risk management decisions.

Additional consideration must be given to quantities in the analysis that were treated parametrically (i.e., value parameters and decision variables) and to outcome uncertainty associated with plausible alternative conceptual and/or mathematical models of risk-related processes. Results of analyzing bounding "scenarios" constructed by forming credible combinations of assumed alternative conceptual and mathematical models (including alternative assumptions) should be presented. Characterizations of output uncertainties should be presented parametrically over the ranges of value parameters and decision variables being considered. Care and creativity must be used to avoid parametric presentations that are confusing because they require the decision-maker to evaluate too many combinations of assumed parameter levels. It is better to present a simplified parametric characterization that illustrates the salient implications on the risk analysis results of assuming different parametric levels.

#### Why are we uncertain?

A decision-maker using the results of a risk analysis having significant uncertainty in its outcomes needs to know why the results are uncertain. The risk analyst should use the results of the deterministic sensitivity analysis and uncertainty allocation analysis to develop a summary of the important sources of uncertainty in the risk analysis, including a characterization of their individual relative contribution to the outcome uncertainty. In addition, scenario trees

like that shown in Figure 2-4 can be useful for explaining sources of uncertainty to the decision-maker.

How wrong could we be, how likely is it, and what difference would it make?

It is useful for the risk management decision-maker to have an assessment of the overall range of uncertainty in the risk analysis results supporting her or his decision. The assumptions, models, and conditions associated with the credible upper and lower bounds on the risk analysis outcome should be described.

The likelihoods of scenarios associated with various outcomes within the uncertainty range should be assessed for the decision-maker. These likelihoods can be presented as qualitative or comparative statements (e.g., "unanticipated," "relatively low likelihood," "most-likely"). These qualitative statements can either be translations of quantitative probabilistic results or qualitative judgments reflecting expert consensus.

The effect of changes to the assumed risk analysis outcome (within its range of uncertainty) on achieving risk control goals and criteria should be assessed for the decision-maker. Critical points within the range of uncertainty for risk analysis outcomes, where changes to risk management strategies would be indicated, should be identified. The decision-maker can then combine these with the likelihood assessments to better understand the "risks" of her or his decision.

How might I increase confidence in my decision?

By considering the important sources of uncertainty identified by the uncertainty assessment available, strategies should be identified for reducing or eliminating the uncertainty in risk analysis results, thus increasing confidence in the expected outcome of the risk management decision that the risk analysis supports.

These strategies can be "global," for example, establishing siting criteria that encourages choosing a site where environmental transport processes are better understood, adopting a "look-ahead" risk management policy with progressive decisions about control strategies based on future information, or obtaining additional expert opinions on key uncertain aspects of the risk problem being addressed.

The possible strategies identified to reduce or eliminate uncertainty in the risk analysis results can also be more specific. For example, specific areas where more research, model development, or additional measurements would be most effective in reducing uncertainties can be identified. A balance can then be struck by the decision-maker between the expected reduction in uncertainty from these activities and the cost, in both time and resources, of these information gathering activities. Other relevant factors, such as the expected reliability of the new information and the time frame for obtaining the information, should be summarized for the decision-maker.

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## GLOSSARY

The glossary was prepared by combining definitions presented in the following documents, as referenced by the bracketed numbers at the end of each definition.

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Abatement - The reduction in degree or intensity of pollution. [7]

Absolute risk - An expression of excess risk based on the assumption that the excess risk from radiation exposure adds to the underlying (baseline) risk, by a constant increment dependent on dose; an absolute risk time--response model distributes the radiogenic risk after exposure independently of the underlying natural risk. [2]

Accident - An unwanted energy transfer (an accident) causing property damage and/or human injury. [3]

Accident - That occurrence in a sequence of events which usually produces unintended injury, death, or property damage. [5]

Accuracy - The degree of agreement between a measured value and the true value, usually expressed at +/- percent of full scale. [5]

Artificial variability - Variability induced by procedures used to convert raw data into model inputs; sources include data selection, processing, level of aggregation, ergodicity, and interpretation.

Attributable risk - The rate of the disease in exposed individuals that can be attributed to the exposure. This measure is derived by subtracting the rate (usually incidence or mortality) of the disease among nonexposed persons from the corresponding rate among exposed individuals. [5]

Bayesian Framework (Subjectivist Framework) - A school of thought on the meaning of probability which views probability as an expression of an internal state of knowledge or confidence expressed subjectively. This school of thought is associated with the statistician Bayes, and its inherent logical reasoning is viewed as governed by Bayes' Theorem.

Benefit - The degree to which effects are judged desirable. [5]

Best available control technology - An emission limitation (including a visible emission standard) based on the maximum degree of reduction for each pollutant subject to regulation under the act which would be emitted from any proposed major stationary source or major modification which the Administrator, on a case-by-case basis, taking into account energy, environmental and economic impacts, and other costs, determines is achievable for such source or modification through application of production processes or available methods, systems, and techniques, including fuel cleaning or treatment or innovative fuel combustion techniques for control of such pollutant. [5]

Bias - Any difference between the true value and that actually obtained due to all causes other than sampling variability. [5]

Case-fatality rate - A ratio of the number of deaths due to a disease to the number of cases of that disease in a specified period of time. It expresses the frequency with which affected individuals die of the disease. [7]

Classical Framework (Frequentist Framework) - A school of thought on the meaning of probability which views probability as something external which is a measure of the results of repetitive experiments. From this perspective, probability is a measurable quantity and the outcome of experiments involving repeated trials and observations.

Code - A quantitative procedure to solve a particular mathematical abstract of the physical problem. [4]

Code - A mathematical and logical model that has been translated to computer language.

Common mode failures - Several errors in a technological system occurring simultaneously. [7]

Comparative risk - An expression of the risks associated with two (or more) actions leading to the same goal; may be expressed quantitatively (a ratio of 1.5) or qualitatively (one risk greater than another risk). [5]

Confidence interval - An interval estimate of a statistical parameter, obtained as a particular function of observed values of one or more random variables whose joint distribution depends upon that parameter. The interval-valued function is so defined that, in an infinitely increasing number of independent replications of the experiment yielding the observed values of the random variables, the proportion of times that the interval contains the (unknown) parameter value converges to a number at least as large as some preset value, called the confidence level of the interval. [2]

Confidence interval - A range of values ( $a_1 < a < a_2$ ) determined from a sample of indefinite rules so chosen that, in repeated random samples from the hypothesized population, an arbitrarily fixed proportion ( $1-\epsilon$ ) of that range will include the true value,  $x$ , of an estimated parameter. The limits,  $a_1$  and  $a_2$ , are called confidence limits; the relative frequency ( $1-\epsilon$ ) with which these limits include  $a$  is called the confidence level. As with significance levels, confidence levels are commonly chosen as 0.05 or 0.01, the corresponding confidence coefficients being 0.95, 0.99. Confidence intervals should not be interpreted as implying that the parameter itself has a range of values; it has only one value. On the other hand, the ( $LQ$ ) confidence limits ( $a_1$ ,  $a_2$ ) being derived from a sample either do or do not include the true value,  $a$ , of the parameter. However, in repeated samples, a certain proportion (namely  $1-\epsilon$ ) of these intervals will include  $a$ , provided that the actual population satisfied the initial hypothesis. [5]

Confounding factors - Variables that may introduce differences between cases and controls which do not reflect differences in the variables of primary interest. [5]

Cost-benefit analysis - A formal quantitative procedure comparing costs and benefits of a proposed project or act under a set of preestablished rules. To determine a rank ordering of projects to maximize rate of return when available funds are unlimited, the quotient of benefits divided by costs is the appropriate form; to maximize absolute return given limited resources, benefits-costs is the appropriate form. [5]

Credibility interval - An analogue of confidence interval, in terms of subjective probability. If one's information about the true value of an unknown parameter can be summarized by a probability distribution for that value, a credibility interval of a given probability level for the parameter is an interval such that the subjective probability distribution, integrated over the interval, is not less than the given probability level. [2]

Damage - Damage is the severity of injury or the physical, functional, or monetary loss that could result if control of a hazard is lost. [5]

Danger - Expresses a relative exposure to a hazard. A hazard may be present, but there may be little danger because of the precautions taken. [5]

De minimus risk - From the legal maxim "de minimus non curat lex" or "the law is not concerned with trifles." [5]

Diversity - Pertaining to the variety of species within a given association of organisms. Areas with low diversity are characterized by a few species; often relatively large numbers of individuals represent each species. [7]

Dose - The amount or concentration of undesired matter or energy deposited at the site of effect. [5]

Dose-effect - The relationship between dose (usually an estimate of dose) and the graduation of the effect in a population, that is a biological change measured on a graded scale of severity; although at other times one may only be able to describe a qualitative effect that occurs within some range of exposure levels. [5]

Dose-effect (dose-response) model - A mathematical formulation of the way in which the effect, or response, depends on dose. [2]

Dose-response - A correlation between a quantified exposure (dose) and the proportion of a population that demonstrates a specific effect (response). [5]

Dose-response assessment - The process of characterizing the relation between the dose of an agent administered or received and the incidence of an adverse health effect in exposed populations and estimating the incidence of an adverse as a function of human exposure to the agent. [5]

Effect - A biological change caused by an exposure. [5]

Efficacy - A measure of the probability and intensity of beneficial effects. [5]

Environmental pathway - All routes of transport by which a toxicant can travel from its release site to human populations including air, food chain, and water. [7]

Excess deaths - The excess over statistically expected deaths in a population within a given time interval. Attempts are made to relate excess deaths to specific causes. Note that since every person can (and must) die only once, there can be no excess deaths over all time. [5]

Expected - Assumed to be probable or certain on the basis of existing evidence and in the absence of significant evidence to the contrary. [6]

Expected deaths - The number of deaths statistically expected in a population in a given time interval obtained by summing the product of age-, sex, and race-specific mortality rates from a standard population and person-years in each age, sex, and race category in the study population. [5]

Expected loss - The quantity obtained by multiplying the magnitude of health or environmental effect loss by the probability (or risk) of that loss and adding the products. The expected loss is the average loss over a large number of trials; one must reflect on the appropriateness of its use in cases for which there will be only one, or a few, trials. [5]

... assessment, this process entails postulating a ... based on observable responses and developing a ... model to describe this reality. The model may then be used to extrapolate to response levels which cannot be directly observed. [5]

Failure modes and effects analysis - A tool to systematically analyze all contributing component failure modes and identify the resulting effects on the system. [5]

False negative results - Results which show no effect when one is there. [5]

False positive results - Results which show an effect when one is not there. [5]

Fault tree analysis - A technique by which many events that interact to produce other events can be related using simple logical relationships permitting a methodical building of a structure that represents the system. [5]

Gaussian distribution model - Is expressed by the formula:

$$f(x) = \frac{1}{\sigma_x \sqrt{2\pi}} \exp \frac{(x-\bar{x})^2}{2\sigma_x^2}$$

where  $x$  is the mean,  $\sigma_x$  is the standard deviation. It is also called the normal distribution. For example, a Gaussian air dispersion model is one in which the pollutant is assumed to spread in air according to such a distribution and described by the two parameters  $x$  and  $\sigma_x$  of the normal distribution. [5]

Geometric mean - The geometric mean of a set of positive numbers is the exponential of the arithmetic mean of their logarithms. The geometric mean of a lognormal distribution is the exponential of the mean of the associated normal distribution. [2]

Geometric standard deviation (GSD) - The geometric standard deviation of a lognormal distribution is the exponential of the standard deviation of the associated normal distribution. The geometric standard deviation is not standard for statistical terminology but is more commonly used by physicists. [2]

Hazard - A condition or physical situation with a potential for an undesirable consequence, such as harm to life or limb. [5]

Hazard - A source of risk, peril; the potential for an wanted release of energy to result in personal injury or property damage. [3]

Hazard assessment - An analysis and evaluation of the physical, chemical, and biological properties of the hazard. [5]

Hazard identification - The process of determining whether exposure to an agent can cause an increase in the incidence of a health condition. [5]

Health effect - A deviation in the normal function of the human body. [5]

Health effect assessment - The component of risk assessment which determines the probability of a health effect given a particular level or range of exposure to a hazard. [5]

Health risk - Risk in which an adverse event affects human health. [5]

Hockey stick regression function - A dose-response curve as follows:

$$\begin{aligned} \text{For some } X_0, \\ f(x) &= B_0 & \text{for } X \leq X_0 \\ &= B_0 + B_1 (X - X_0) & \text{for } X > X_0 \end{aligned}$$

This means that for a suitable dose  $X_0$ ,  $f(X)$  remains constant for any  $X$  less than  $X_0$  and increases linearly as  $X$  increases for any  $X$  more than  $X_0$ . The dose  $X_0$  is considered as a physiological threshold value. [7]

Impact - The force of impression of one thing on another. [5]

Incidence - The number of new cases of a disease in a population over a period of time. [5]

Incidence or incidence rate - The rate of occurrence of a disease within a specified period of time, often expressed as number of cases per 100,000 individuals per year. [2]

Individual risk - The risk to an individual rather than to a population. [5]

Individual susceptibility - The marked variability in the manner in which individuals will respond to a given exposure to a toxic agent. [5]

Linear (L) model - Also, linear dose-effect relationship; expresses the effect (e.g., mutation or cancer) as a direct (linear) function of dose. [2]

Linear-quadratic (LQ) model - Also, linear-quadratic dose-effect relationship; expresses the effect (e.g., mutation or cancer) as partly directly proportional to the dose (linear term) and partly proportional to the square of the dose (quadratic term). The linear term will predominate at lower doses, the quadratic term at higher doses. [2]

Logit model - A dose-response model which, like the probit model, leads to an S-shaped dose-response curve, symmetrical about the 50% response curve. The logit model leads to lower "very safe doses" than the probit model even when both models are equally descriptive of the data in the observable range. [7]

Lognormal distribution - A distribution of the frequency of a value plotted on a linear scale versus the value plotted on a logarithmic scale, which results in a bell-shaped curve. [1]

Lognormal distribution - If the logarithms of a set of values are distributed according to a normal distribution they are said to have a lognormal distribution, or be distributed "lognormally." [2]

Log-probit model - A dose-response model which assumes that each animal has its own threshold dose, below which no response occurs and above which a tumor is produced by exposure to a chemical. [7]

Maximally exposed individual - A hypothetical person who is exposed to a release of radioactivity in such a way that he receives the maximum possible individual radiation dose or dose commitment. For instance, if the release is a puff of contaminated air, the maximally exposed individual is a person at the point of the largest ground-level concentration and stays there during the whole time the contaminated-air cloud remains above. This term is not meant to imply that there really is such a person; it is used only to indicate the maximum exposure a person could receive. [6]

Maximum permissible concentration - The average concentration of a radionuclide in air or water to which a worker or member of the general population may be continuously exposed without exceeding regulatory limits on external or internal radiation doses. [6]

Mitigation - (1) Avoiding the impact altogether by not taking a certain action or parts of an action. (2) Minimizing impacts by limiting the degree or magnitude of the action and its implementation. (3) Rectifying the impact by repairing, rehabilitating, or restoring the affected environment. (4) Reducing or eliminating the impact over time by preservation and maintenance operations during the life of the action. (5) Compensating for the impact by replacing or providing substitute resources or environments. [6]

Model - A conceptual description and the associated mathematical representation of a system, component, or condition. It is used to predict changes in the system, component, or condition in response to internal or external stimuli as well as changes over time and space. An example is a hydrologic model to predict ground-water travel or

radionuclide transport from the waste-emplacement area to the accessible environment. [6]

Model - A simplified representation of some aspect of reality; either conceptual, visual, verbal, physical, mathematical, and/or logical.

Morbidity - A departure from a state of physical or mental well-being, resulting from disease or injury. Frequently used only if the affected individual is aware of the condition. Awareness itself connotes a degree of measurable impact. Frequently, but not always, there is a further restriction that some action has been taken such as restriction of activity, loss of work, seeking of medical advice, etc. [7]

Mortality (rate) - The rate at which people die from a disease, e.g., a specific type of cancer, often expressed as number of deaths per 100,000 per year. [2]

Mortality rate - The number of deaths that occur in a given population during a given time interval; usually deaths per  $10^3$  or  $10^5$  people per year. Can be age, sex, race, and cause specific. [7]

Normal distribution - A random variable  $X$  is said to be normally distributed if, for some number  $\mu$  and some positive number  $\sigma$ ,  $Y = (X-\mu)/\sigma$  has a standard normal distribution with probability density function

$$\phi(y) = (2\pi)^{-1/2} \exp(-Y^2/2) \quad [2]$$

One-hit model - The dose-response model based on the concept that a tumor can be induced by a single receptor that has been exposed to a single quantum or effective dose unit of a chemical. [7]

Population at risk - A limited population that may be unique for a specific dose-effect relationship; the uniqueness may be with respect to susceptibility to the effect or with respect to the dose or exposure itself. [5]

Population dose (population exposure) - The summation of individual doses received by all those exposed to the source or event being considered. [7]

Precision - A measure of how exactly the result is determined without reference to any "true" value. [5]

Precision - A measure of how consistently the result is determined by repeated determinations without reference to any "true" value. [7]

Premature death - A death that occurs before statistical expectation, usually attributable to a specific cause, and usually referring to deaths statistically estimated in a population rather than to individuals. [7]

Prevalence - The number of existing cases in a population who have the disease at a given point (or during a given period of time). [7]

Probability - A probability assignment is a numerical encoding of a state of knowledge. [5]

Probable error - The magnitude of error which is estimated to have been made in determination of results. [5]

Probit analysis - A statistical transformation which will make the cumulative normal distribution linear. In analysis of dose-response, when the data on response rate as a function of dose are given as probits, the linear regression line of these data yields the best estimate of the dose-response curve. The probit unit  $Y=5+Z(p)$ , where  $p$  = prevalence of response at each dose level and  $Z(p)$  = corresponding value of the standard cumulative normal distribution. [5]

Proportionate mortality ratio (PMR) - The fraction of all deaths from a given cause in the study population divided by the same fraction from a standard population. A tool for investigating cause-specific risks when only data on deaths are available. If data on the population at risk are also available, SMRs are preferred. [7]

Quality assurance - All the planned and systematic actions necessary to provide adequate confidence that a structure, system, or component is constructed to plans and specifications and will perform satisfactorily. [6]

Quality control - Quality-assurance actions that provide a means to control and measure the characteristics of an item, process, or facility to established requirements. [6]

Random error - Indefiniteness of result due to finite precision of experiment. Measure of fluctuation in result after repeated experimentation. [5]

Rate - In epidemiologic usage, the frequency of a disease or characteristic expressed per unit of size of the population or group in which it is observed. The time at or during which the cases are observed is a further specification. [7]

RAU - Risk analysis unit. [7]

Reasonably achievable - Mitigation measures or courses of action shown to be reasonable considering the costs and benefits in accordance with the National Environmental Policy Act of 1969. [6]

Relative risk - The ratio of the rate of the disease (usually incidence or mortality) among those exposed to the rate among those not exposed. [5]

Relative risk - An expression of excess risk relative to the underlying (baseline) risk; if the excess equals the baseline risk the relative risk is 2. [2]

Release limit - A regulatory limit on the concentration or amount of radioactive material released to the environment. [6]

Reliability - The probability a system performs a specified function or mission under given conditions for a prescribed time.

Residual uncertainty - Those inherent uncertainties in data, modeling, and assumed future conditions that cannot be eliminated. [6]

Response - The proportion or absolute size of a population that demonstrates a specific effect. May also refer to the nature of the effect. [7]

Risk - The potential for realization of unwanted, adverse consequences to human life, health, property, or the environment; estimation of risk is usually based on the expected value of the conditional probability of the event occurring times the consequence of the event given that it has occurred. [5]

Risk - Mathematically, expected loss; the probability of an accident multiplied by the consequence (loss converted into dollars) of the accident. [3]

Risk analysis - A detailed examination performed to understand the nature of unwanted, negative consequences to human life, health, property, or the environment; an analytical process to provide information regarding undesirable events; the process of quantification of the probabilities and expected consequences for identified risks. [5]

Risk analysis - The quantification of the degree of risk. [3]

Risk analysis - An analysis that combines or uses an uncertainty analysis along with the probability that the state evaluated in the analysis (geologic, biologic, etc.) exists. [4] Note that a risk analysis uses as an integral part an uncertainty analysis and an uncertainty analysis similarly contains a sensitivity analysis. [4]

Risk assessment - The process, including risk analysis, risk evaluation, and risk management alternatives, of establishing information regarding that risk and levels of risk for an individual, group, society, or the environment. [5]

Risk assessment - The combined functions of risk analysis and evaluation. [3]

Risk coefficient - A fitted constant in an equation that describes how an effect depends on dose. [2]

Risk estimation - The scientific determination of the characteristics of risks, usually in as quantitative a way as possible. These include the magnitude, spatial scale, duration and intensity of adverse

Risk estimate - Absolute - Risk estimate based on the assumption that there is some absolute number of deaths in a population exposed at a given age per unit of dose. [1] Relative - Risk estimate based on the assumption that the annual rate of radiation-induced excess cancer deaths is proportional to the ambient rate of occurrence of fatal cancer. [1]

Risk evaluation - A component of risk assessment in which judgments are made about the significance and acceptability of risk. [5]

Risk evaluation - The appraisal of the significance or consequences of a given quantitative measure of risk. [3]

Risk identification - Recognizing that a hazard exists and trying to define its characteristics. Often risks exist and are even measured for some time before their adverse consequences are recognized. In other cases, risk identification is a deliberate procedure to review and, it is hoped, anticipate possible hazards. [5]

Risk management - The process, derived through system safety principles, whereby management decisions are made concerning control and minimization of hazards and acceptance of residual risks. [3]

Rulemaking - Process of formulating specific regulations governing a particular matter. [6]

Safety - Relative protection from adverse consequences. [5]

Scenario - A particular chain of hypothetical circumstances often used in performance analysis to model possible events. [6]

Scenario analysis - Analytical process that attempts to quantify the probabilities and consequences of a postulated sequence of events. [6]

Sensitivity analysis - An analysis that defines quantitatively or semi-quantitatively the dependence of a selected performance assessment measure (or an intermediate variable) on a specific parameter or set of parameters. [4]

Standard deviation - A measure of dispersion or variation, usually taken as the square root of the variance. [5]

Standard geometric deviation - Measure of dispersion of values about a geometric mean; the portion of the frequency distribution that is one standard geometric deviation to either side of the geometric mean; accounts for 68% of the total samples. [5]

Standardized mortality ratio (SMR) - The ratio of observed deaths in a population to the expected number of deaths as derived from standard population rates with adjustment of age and possibly other factors such as sex or race. [7]

Standard normal deviation - Measure of dispersion of values about a mean value; the positive square root of the average of the squares of the individual deviations from the mean. [5]

Statistical significance - The statistical significance determined by using appropriate standard techniques of multivariate analysis with results interpreted at the stated confidence level and based on data relating species which are present in sufficient numbers at control areas to permit a valid statistical comparison with the areas being tested. [5]

Stochastic - A stochastic process is one in which the system incorporates an element of randomness, as opposed to a deterministic system. For example, in radiobiology stochastic effects are those in which the probability of an effect occurring rather than its severity is a function of dose, without threshold. [2]

Stochastic model - A model whose inputs are uncertain and whose outputs are therefore also uncertain and must be described by probability distributions. [6]

Surrogate - Something that serves as a substitute. In risk analysis, surrogates are often used when data on the item of interest (a chemical, an industry, an exposure, etc.) is lacking. As an example, underground mining of coal and hardrock minerals can be used as a surrogate for underground oil shale mining. [7]

Systematic error - A reproducible inaccuracy introduced by faulty equipment, calibration, or technique. [5]

Threshold - A pollutant concentration below which no deleterious effect occurs. [7]

Threshold dose - The minimum application of a given substance required to produce an observable effect. [7]

To the extent practicable - The degree to which an intended course of action is capable of being effected in a manner that is reasonable and feasible within a framework of constraints. [6]

Uncertainty - A lack of certainty about a quality, quantity, or model due to inherent randomness, artifactual variability, and/or incomplete knowledge.

Uncertainty analysis - A detailed examination of the systematic and random errors of a measurement or estimate; an analytical process to provide information regarding the uncertainty. [5]

Uncertainty analysis - The analysis that defines the dependence of a set of selected performance assessment measures on the set of uncertain input parameters. It includes the characterization of uncertainty in (1) the input parameters; (2) the evaluation methodology; and (3) the output performance assessment measures. [4]

Uncertainty assessment - The process of identifying, characterizing, analyzing, and evaluating the implications of uncertainties that are inherent to risk analysis.

Validation of computer codes and models - The process of obtaining assurance that a model as embodied in a computer program is a correct representation of the process or system for which it is intended. Ideally, validation is a comparison of predictions derived from the model with empirical observation. However, as this is frequently impractical or impossible owing to the large physical and time scales involved in HLW disposal, short-term testing supported by other avenues such as peer review are used to obtain this assurance. [4]

Verification of computer codes and models - Testing a code with analytical solutions for idealized boundary-value problems. A computer code will be considered verified when it has been shown to solve the boundary-value problems with sufficient accuracy. [6]

Worst-case analysis - An analysis based on assumptions and input data selected to yield a "worst impact" statement. [6]

Zero order analysis - The simplest approach to quantification of a risk with a limited treatment of each risk component (e.g., source terms, transport, health effects, etc.). [7]

## APPENDIX A

### **Proposed Procedure for Elicitation of Expert Judgments on Uncertain Quantities in a Risk Analysis**

## 1.0 Introduction

As part of the ORP's approach to uncertainty analysis, those factors of the risk analysis having sufficient levels of uncertainty to warrant treatment as probability distributions in the analysis are identified. As a first step to developing judgmental uncertainty distributions for these parameters, expert judgements are obtained to establish lower and upper bounds on the parameter values. This first step can be viewed as the first-order *a priori* judgment on the current state of certainty about these parameters. If the state of knowledge on a parameter were such that all we knew about it were its bounds, then the appropriate uncertainty characterization would be a uniform probability distribution between those bounds. (Thus, the primal uncertainty distribution is uniform between negative and positive infinity.) As more information and understanding about the parameter is included in the uncertainty judgment process, the uniform distribution may be upgraded to perhaps a 3-point subjective distribution, wherein the median value is specified as well as the bounds. There may even be enough knowledge to represent the parameter's uncertainty by a specific type of distribution, such as normal or lognormal. Ultimately, enough information and understanding of the parameter might become available to reduce the range of its distribution to the extent that it may be treated as a point value (as known with 100% uncertainty).

It is important to assure high quality in the input parameter uncertainty distributions used in an uncertainty analysis. The validity of the results depends directly on the quality of the input uncertainty characterizations and the results are quite sensitive to the type of distributions assumed. Careful consideration must be given to the implications of using a particular probability distribution to represent a measure of the state of knowledge. Accordingly, one approach to uncertainty analysis utilizes formal techniques for eliciting quantitative judgments of uncertainty from experts. There are substantial psychological and practical problems encountered in eliciting considered technical opinions from experts. These problems with judgmental error have been well-documented [Kahneman et al., 1982],

and methodologies for elicitation have been developed which attempt to counter these biases produced by common heuristics [Morgan et al., 1981]. The formal elicitation procedure proposed below is based upon consideration of these methodologies.

## 2.0 Elicitation Session Protocol

Elicitations are usually done in day long sessions with experts or surrogate experts. The protocol of the elicitation session is as follows:

1. introductory discussion of problem and objectives,
2. discussion of heuristics and biases involved in making subjective judgments,
3. discussion of technical issues and structural uncertainties,
4. structuring of elicitation questions, and
5. elicitation of judgmental probability distributions for specified uncertain parameters.

In order to provide a clearer understanding of each of these phases, a brief description of each phase, as performed for a real application, is given below. The problem was to elicit expert model structures and key parameter uncertainties in estimating annual average long-range sulfur budgets for the plumes of large coal-fired power plants [Morgan et al., 1984].

### PHASE 1: Introductory Discussion

Each elicitation session began with a discussion of the risk problem being addressed. It was explained that the primary objective was to obtain from the experts their best current professional judgments about the average oxidation rate of sulfur dioxide and the average

fraction of sulfur emitted as primary sulfate in the plumes of large coal-fired power plants located in the Northeastern United States. It was further explained that while the interviewers were ultimately interested in using these judged parameters in a model to estimate annual average impacts, they did not want to impose any structure upon the expert; so that if he desired to discuss these parameters as a function of time of day, season of year, or any other appropriate variables he was encouraged to do so.

#### PHASE 2: Discussion of Heuristics and Biases

The second phase of the elicitation session involved describing to the expert the types of heuristics and biases which are likely to be involved in making subjective judgments in the face of uncertainty. This discussion was organized around a briefing book which was prepared containing key concepts and evidence from experimental psychology studies documenting the existence and nature of these heuristics and biases. Informing the expert about the state of the elicitation field contributes to the establishment of rapport between the expert and the elicitator. While there are doubts that this briefing significantly affects the expert's answers, the expert better understands the approach taken to the elicitation, and the elicitation session assumes a more professional posture.

#### PHASE 3: Discussion of Technical Issues and Structural Uncertainties

The third phase of the session was an extended technical discussion by the experts of how they viewed the history and current status of the plume sulfur process field, their primary sources of information, what factors they viewed as controlling plume sulfur processes, and the physical and chemical mechanisms involved and their relative importance. If the expert stated something which in some way conflicted with evidence from the literature, reference would be made to the particular study and the experts were asked to elaborate.

It was during this phase of the session that much qualitative information was obtained from the experts reflecting their judgments about the structural uncertainties involved in plume sulfur modeling. A picture began to form of the expert's conceptual model of plume sulfur processes.

#### PHASE 4: Structuring of Elicitation Questions

The objective of this phase of the session was to structure the quantitative elicitation of judgmental probability distributions on the sulfur dioxide oxidation rates and the fraction of primary sulfate emission. This included determining which variables the experts' answers would be conditional upon (time-of-day, season, temperature, etc.), the units in which the parameter would be elicited (percent per hour, concentration versus transport time, etc.), and how the answers elicited should be combined to produce annual average results.

#### PHASE 5: Elicitation of Judgmental Probability Distribution

During the last phase of the session judgmental probability distributions were elicited from the experts on average sulfur dioxide oxidation rates under expert-specified conditions and of annual average fraction of sulfur emitted as sulfate. Judgmental probability distributions were elicited in the form of points on a cumulative probability distribution for the uncertain parameter in question. Figure A-1 shows an example of a set of elicited distributions. The experts were reminded that all questions pertained to average values of the parameter, and not to values which could occur at a given instant under certain conditions.

As an attempt to overcome an elicitation bias known as "anchoring," wherein the elicitree centers on his "best estimate," the expert was first asked for his absolute maximum and minimum limits on the value of the parameter in question. The experts were then asked for justification of these limits, and for convincing arguments as to their

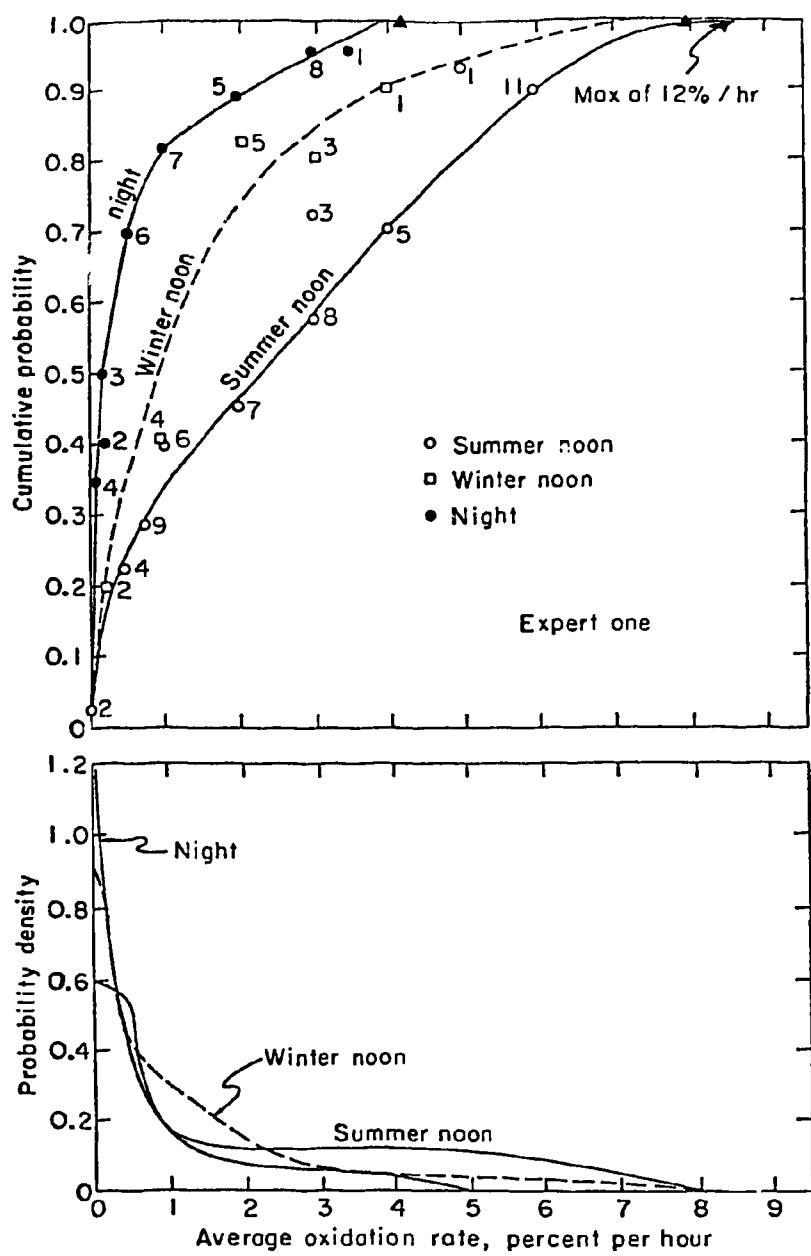


Fig. A-1. Example of elicited judgmental probability distributions. Points indicated in the upper plot are original elicited data points; numbers indicate the order in which they were obtained. Source: Morgan et al., 1982.

absoluteness. The expert was asked to imagine that he was separated from the field entirely for ten years, and that upon returning he found that it had been proven that the actual value of the parameter was greater or less than his limits. Could he think of any explanation that might justify such findings?

This done, the interviewers began to elicit actual points on the cumulative probability distribution for the uncertain parameter. The points were elicited in arbitrary order, and were kept hidden from the expert.

In the beginning of the sessions it was attempted, as the decision analysis literature suggests, to elicit points by having the expert make choices between sets of the two lotteries shown in Figure A-2 for different sets of odds (P's) given a value N of the parameter.

When the expert is presented with a set of odds where he is indifferent between the two lotteries, then P is equal to his or her judged probability that the value of the parameter in question is less than or equal to the given value, N. This combination of P and N represents a point of the cumulative probability distribution.

In order to assist the expert in thinking about the questions asked and as a motivation to think carefully about his or her answers, a "probability wheel" was utilized. This is a circular background of two colors, red and green, which has a spinner affixed to it. The portion of the background which is red or green is adjustable, thus shown in Figure A-3.

The lottery formulation of elicitation seemed to confuse the experts. They preferred to simply adjust the size of the green portion of the wheel so that it represented the probability that the parameter is less than or equal to the given value, N.

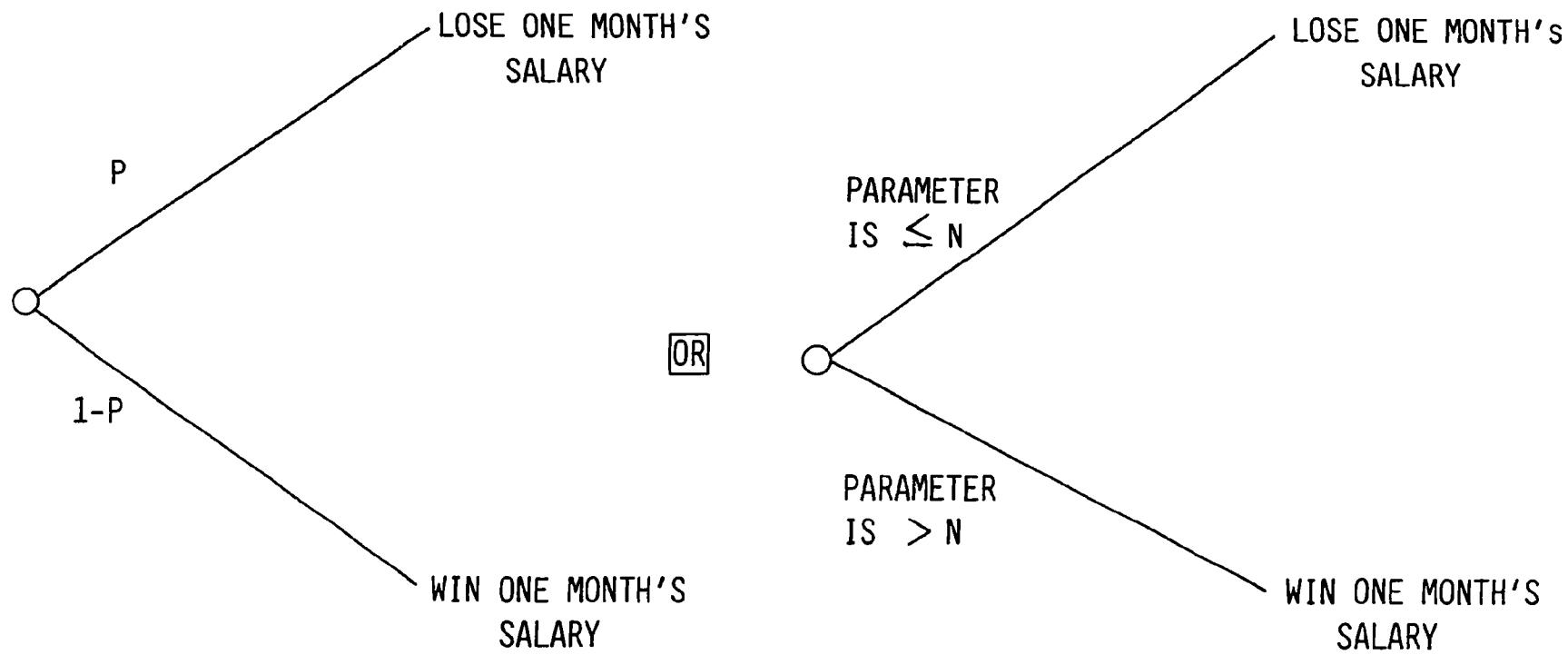


Fig. A-2. Lottery presented to experts to elicit points on their cumulative probability distributions for an uncertain parameter,  $N$ .

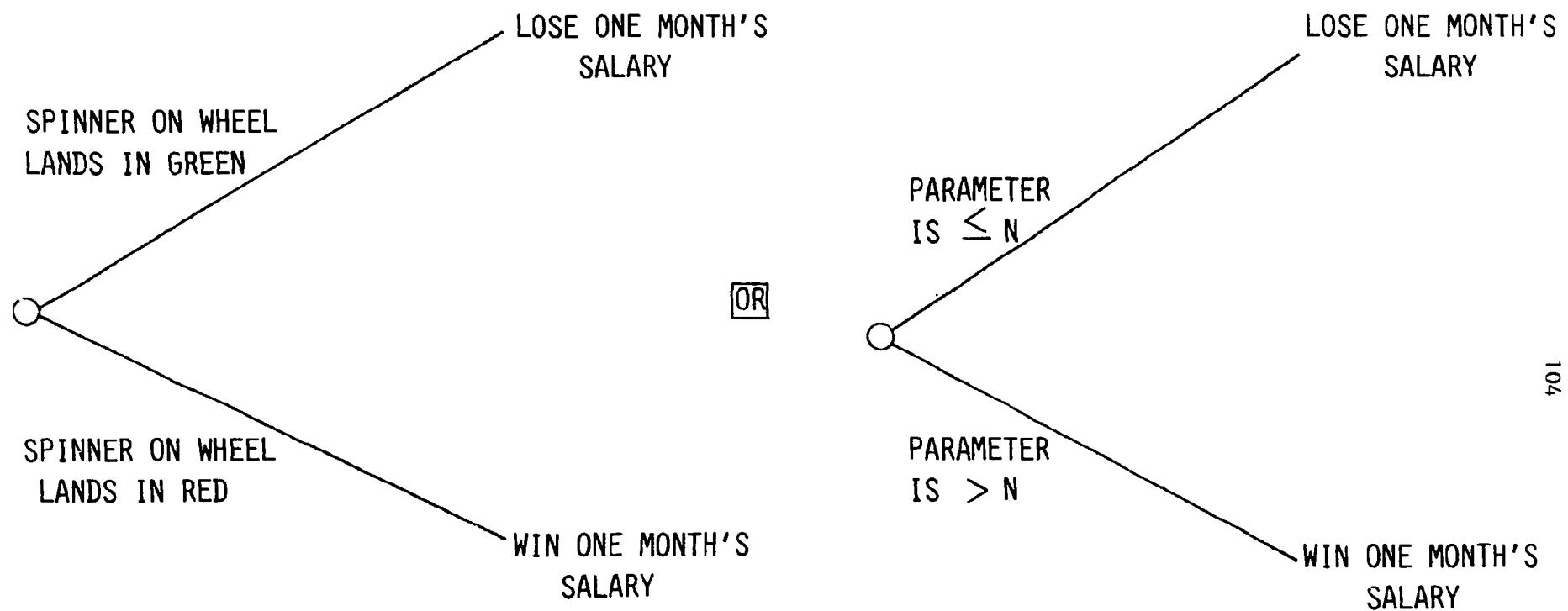


Fig. A-3. Alternative lottery presented to experts based upon the use of a probability wheel.

The experts were asked to think about each question separately without being concerned about consistency with previous answers by keeping the elicited points hidden and randomizing the order in which he was asked for points. After encoding the entire distribution, the expert was confronted with any inconsistencies and he was asked to explain them with respect to his original reasoning behind the points.

Appendix A: References

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## APPENDIX B

"Error Propagation for Large Errors" by Fritz A. Seiler  
Inhalation Toxicology Research Institute  
Lovelace Biomedical and Environmental Research Institute  
Albuquerque, New Mexico, 1986  
(submitted for publication in Risk Analysis)

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## Error Propagation for Large Errors

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An essential facet of a risk assessment is the correct evaluation of uncertainties inherent in the numerical results. Some uncertainties in the final results arise from errors in the input, others from deficiencies in the models used. If the calculation is based on an explicit algebraic expression, an analytical treatment of error propagation is possible, usually as an approximation valid for small errors. In many instances, however, the errors are large and uncertain. It is the purpose of this paper to demonstrate that despite large errors, an analytical treatment is possible in many instances. These cases can be identified by an analysis of the algebraic structure and a detailed examination of the errors in input parameters and mathematical models. From a general formula, explicit formulae for some simple algebraic structures that occur often in risk assessments are derived and applied to practical problems.

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KEY WORDS: Error Propagation, Analytical Treatment, Large Errors.

## 1. INTRODUCTION

Estimating uncertainties inherent in measurements or theoretical calculations is an integral part of scientific investigations. Depending on the type of calculation leading to the final result, numerical or analytical methods are indicated to determine the propagation of errors from input to result. Whereas some numerical methods can accomodate errors of any size, analytical methods are usually restricted to quantities with small relative errors. In this paper it will be shown, that in a considerable number of problems in risk assessment, an analytical treatment can be used even if the errors are relatively large.

The theory of error propagation by analytical methods is based on the algebra of stochastic variables, an area which has received increasing attention in the second half of this century (1,2). For the general case, integral transform methods are used to calculate the distribution function for the result of algebraic operations on random variables. Although this approach is somewhat complex and often results in considerable numerical computations, it has the advantage of providing the final probability distribution as a data base for statistical tests and confidence limits.

In risk assessments, interest is often limited to an estimate of the mean and its standard error. Then, a more direct method may be used, based on the Taylor series expansion of the risk function involved. A necessary condition for this approach is the existence of all the derivatives required in the expansion. Normally, this condition is met, since the functions most often used are well behaved and have derivatives which are either nonzero up to a certain order only, or converge rapidly to zero with increasing order.

To keep the formalism simple, symmetric probability distributions are assumed for the input parameters. The consequences of this decision are not as serious as might appear at first, especially if the errors are large. A large error implies not only that the sample mean is not well known, but also that the character of the distribution is even less precisely determined by the experimental data. In such a situation, the selection of either a normal or lognormal distribution is a reasonable approximation. Lognormal distributions can either be treated directly or transformed into logarithmic space and treated like normal distributions.

On the basis of these assumptions, it is the purpose of this paper to apply general analytical formulae derived elsewhere (3) for mean and standard error of the result of algebraic operations on random variables.

## 2. DISCUSSION OF ERRORS

### Random and Systematic Errors

In discussing errors and their propagation through a calculation, one of the most important distinctions is the one between random and systematic errors (3-6). This characterization governs the methods by which errors are treated and may affect the numerical values of the uncertainties.

Random or statistical errors of a measured quantity arise from many possible causes, the size and sign of the deviation cannot be predicted, nor can they be prevented. They can be decreased, however, by increasing the number of measurements taken. Systematic errors, on the other hand, if they are recognized at all, usually have one identifiable cause, affect every measurement by the same mechanism, and, if properly investigated, can often be

avoided or corrected for. They cannot, however, be decreased by increasing the number of measurements taken.

A typical example for random or statistical errors are the fluctuations in the count-rate of a radiation counter exposed to a constant particle flux. Typical systematic errors are those caused by a defect in a scale, resulting in the measurement of uniformly high masses, or by the use of a model that does not take into account a pertinent effect and, therefore, yields systematically distorted values. Errors encountered in practice, however, are often not as clearly random or systematic as those mentioned above. They require considerable thought and special treatment, such as an attempt at separating the systematic and random components of an error. This is possible only if the data are well documented.

It is of considerable importance to analyze the origin, magnitude, and sign of systematic errors as thoroughly as possible, since the logical course of action demands that, rather than quoting a systematic error, an appropriate correction be applied to the result whenever enough information is available. The uncertainty of that correction can then be treated mainly as a statistical error.

Large systematic uncertainties often arise when the values of crucial constants in a model are only poorly known. In trying to predict future levels of toxicants in the atmosphere, for instance, the projected energy consumption plays a critical role and influences the final result in a systematic manner. It may then be advantageous to declare this quantity a model or decision parameter without error, and the calculation is performed for different values of the parameter, covering the presumable range of variation. In this manner, the uncertainty is transferred from the parameter

itself to the decision of how to treat the result after the calculation has been done. In many ways that decision may be easier to make than the decision of which value and uncertainty to enter into the calculation.

### Total Uncertainty

In risk assessments it is desirable to determine a value for the total uncertainty of a quantity. There is, however, no accepted mathematical procedure for combining random and systematic errors into a total uncertainty. Indeed, one school of thought contends that two quantities of such different character should not be combined at all, whereas another school disputes the dissimilarity and advocates the combination of the two quantities as a matter of course (6,7).

In practice, there is yet another difference between statistical and systematic errors. This arises from the fact that the magnitude of a statistical error is a calculated value, however approximate, whereas the size of a systematic error is usually no more than an educated guess. Despite the differences between the two types of error, combinations can be made in a way that yields an interpretable result. The lack of an accepted combination procedure suggests the necessity of giving both errors separately as well as in a clearly stated combination. A step in this direction was taken in the last few years in the journal "Physical Review Letters," by giving estimates for both statistical and systematic errors together with the results.

Among the many suggestions for ways to combine systematic and random errors, two procedures stand out (6,7). One is the separate propagation of both errors through the calculation and subsequent quadratic combination according to:

$$S_{\text{tot}}^2 = S_{\text{stat}}^2 + S_{\text{syst}}^2 . \quad (1)$$

The other one advocates first the combination of both systematic and statistical errors according to Eq. (1) and then the simultaneous propagation through the calculation. Either way, the quantity calculated has again the character of a standard error.

### 3. PROPAGATION OF ERRORS

#### Selection of Appropriate Probability Distributions

In a discussion of errors and of error propagation, the assumption of a probability distribution for a stochastic variable is a decisive step, since it determines all properties of the probabilistic behavior of this quantity. However, the choice is usually made without much further thought and results mostly in the adoption of either a normal or a lognormal distribution.

Distributions of experimental data, which could be used to decide which distribution to apply, have generally one aspect in common: Evidence is abundant in the regions of high probability where the differences between distributions are small, but scant in the low probability areas, the tails, where the various distributions have widely different numerical values. Even experimental evidence is, therefore, often not conclusive.

As a consequence, it is much safer to perform mathematical operations in the high probability areas than in the tails of the distribution. Means and standard deviations can be determined to a good approximation, whereas calculations of 95% confidence levels or other operations involving the tails

are often questionable. In the evaluation of large uncertainties in experimental data and a possible discussion of confidence limits, this aspect should be borne in mind.

#### Functions of Stochastic Variables

The use of a stochastic variable in a mathematical function leads to a function value that is also a stochastic quantity. Its probability distribution does not usually remain the same as that of the variable, but is changed by the function. The values of most functions of normally distributed variables, for example, are no longer normally distributed. In Fig. 1, this relation is shown graphically for a simple function  $y = f(x)$ . The stochastic variable  $x$  is normally distributed and characterized by its mean  $x_0$  and its standard error  $\Delta x$ . Here, the function  $y = x^2$  is used as an illustration. The change in the distribution of the function value  $y$  is caused by the change in the slope or, more precisely, the second derivative of the function. The consequence is a non-normal distribution with a mean  $y^*$  that is different from the function value  $y_0 = f(x_0)$ . Thus a correction  $\delta y \equiv y^* - y_0$  has to be applied to the value  $y_0$ . The same conclusion is reached in an examination of most functions for one or several variables.

For functions of more than one variable, the aspect of independence has to be considered. In this paper, it will be assumed that all variables are independent of each other, i.e. that the variations in one stochastic variable do not prejudice the variations in any other variable. This is in agreement with the situation for the major factors in a risk assessment. In a particular factor, however, such as the health risk for a given exposure, correlations between two or three of the parameters are the norm, for instance

if the values were obtained in a fit to the same data. This situation has been dealt with extensively in the literature (4,8) and is discussed briefly in the next section. In the following, it will be assumed that such correlations within a calculation have been taken into account.

### The Gaussian Approximation For Small Errors

One algebraic approach to error propagation involves the expansion of the function  $y = f(\vec{x}) = f(x_1, x_2, \dots, x_n)$  in the neighborhood of point  $\vec{x}_0$  in a multi-dimensional Taylor series (9). Termination of the series after the first order terms results in the Gaussian formula for the propagation of errors. It is a good approximation as long as the relative errors are small, i.e., as long as  $(\Delta x_i/x_i) \ll 1$ . The standard error  $\Delta y$  of the mean is then given by

$$(\Delta y)^2 = \sum_{i=1}^n \left( \frac{\partial f(\vec{x})}{\partial x_i} \right)_0^2 (\Delta x_i)^2. \quad (2)$$

The index 0 denotes the numerical value of the derivatives evaluated at  $\vec{x} = \vec{x}_0$ . In this approximation, the shift of the mean is zero, that is

$$\delta y \equiv y^* - y_0 = 0. \quad (3)$$

The formulae for some simple algebraic structures which are encountered relatively often such as sums, differences, products, quotients, and products of power are given in Table I. The series used in their derivation terminates with the first term for sums, differences or a linear mixture of the two. For these cases the Gaussian approximation is exact, and therefore valid independent of the size of the errors. The formulae for all the other functions listed are applicable for small errors only.

The basic structure of all the formulae in Table I is similar. The square of the absolute error  $\Delta f$  for sums and differences is the sum of the

squares of the errors over all the terms. The square of the relative error  $\Delta y/y$  of products and quotients is equal to the sum of the squares of the relative errors, and in the case of power functions each term is multiplied by the square of the exponent.

In practical situations, this "sums of squares" structure affords an easy way of simplifying the expressions to be evaluated numerically. The typical terms in the sums on the right-hand side are essentially the squares of the absolute or relative errors. If one of the errors is 3 times smaller than the others, for example, its contribution to  $(\Delta y)^2$  is an order of magnitude less than those of the others and its contribution to the standard error  $\Delta y$  even smaller. This variable can therefore often be treated as a constant without error. For many variables the approximate range of errors is known in advance and the error formulae can be simplified accordingly.

For expressions more complex than those in Table I, the formulae can be assembled by parts as long as the latter are independent. As an example, the error of the function

$$f(x_1, \dots, x_n) = \frac{x_1 x_2^2 - x_3}{x_4 x_5} \quad (4)$$

can be calculated as that of a quotient between a difference and a product. The error of the first term in the difference can be calculated separately according to the last formula in Table I for the product of powers. Note that this method is not applicable when any variable appears more than once, since some of the parts are then no longer independent. In these cases the general Gaussian formula (2) has to be applied.

Also, that formula is only applicable for independence of all variables  $x_i$ . For correlated variables, a formula given in the literature must be

used for the standard error  $\Delta y$  (4,8). It can be written as the usual Gaussian terms plus a set of correction terms for the correlations,

$$(\Delta y)^2 = \sum_{i=1}^n \left( \frac{\partial f(\vec{x})}{\partial x_i} \right)_0^2 s_{ii}^2 + 2 \sum_{i=1}^n \sum_{j=i+1}^n \left( \frac{\partial f(\vec{x})}{\partial x_i} \right)_0 \left( \frac{\partial f(\vec{x})}{\partial x_j} \right)_0 s_{ij}^2. \quad (5)$$

Here, the parameters  $s_{ij}^2$  are the elements of the covariance matrix; the diagonal elements  $s_{ii}^2$  are the variances of the parameters, the off-diagonal elements  $s_{ij}^2$  are the covariances, a measure of the correlations. In practice this description is usually sufficient, since experimental correlations between three variables (triple correlations) are either unlikely or then rarely investigated well enough to be included in an error calculation.

#### 4. ERROR PROPAGATION IN FORMULAE TYPICAL OF BIOLOGY AND HEALTH RISK ASSESSMENT

##### Application of the General Formalism

In a report published elsewhere (3), general analytical formulae for the propagation of large errors have been derived under the assumption of normal distributions for the independent input parameters. The first terms of the multi-dimensional Taylor series used for this purpose (9) are given in Eq. (A.1) of the appendix and the corresponding shift  $\Delta y$  of the mean is presented in Eq. (A.2). In discussing these formulae from a practical point of view, it is important to understand that the Taylor series is as much a series in higher order differentials as a power series in the errors. Convergence is thus as much a question of the decrease of the higher order differentials as of the powers of the errors themselves.

A good example are the functions occurring in risk assessment. They are usually well behaved and the partial derivatives exist to any order of differentiation needed. Indeed, many derivatives go identically to zero at relatively low orders. A notable exception are functions containing exponentials which regenerate at every differentiation. However, in these cases the numerical values usually converge rapidly to zero.

The equations in the appendix are, therefore, given only to fourth order in the derivatives and to sixth order in the errors. If higher order terms are needed, they can be obtained from ref. 3, but it should be borne in mind that the complexity of the formula increases rapidly, making the analytical approach and its convergence somewhat questionable.

In this section, the application of the general formulae in the appendix to some typical algebraic structures will be discussed:

- linear combinations,
- powers of one variable,
- products linear in the normally distributed variables,
- products of powers of lognormally distributed variables,
- more complex composite forms.

For lognormally distributed variables, it is always possible to transform the function into logarithmic space and perform the error calculation for the normal distributions resulting there. For the final result, a transformation back to normal space is needed. This procedure is general, although sums and differences may lead to problems as they have to be transformed as a whole. As a practical example, the treatment of a product of powers of lognormally distributed variables will be given here.

### Linear Combinations

Many operations such as the combination of intermediate results and the calculation of weighted means lead to linear combinations of the kind

$$y = f(\vec{x}) = \sum_{i=1}^n a_i R_i . \quad (6)$$

If the coefficients  $a_i$  are known accurately and only the factors  $R_i$  have appreciable errors, the Gaussian approximation (Table I) yields an exact equation

$$(\Delta y)^2 = \sum_{i=1}^n (a_i \Delta R_i)^2 . \quad (7)$$

If both the factors  $a_i$  and  $R_i$  have large errors  $\Delta a_i$  and  $\Delta R_i$ , respectively, formula (A.1) in the appendix gives the exact solution

$$[\Delta y]^2 = \sum_{i=1}^n (a_i \Delta R_i)^2 + (R_i \Delta a_i)^2 + (\Delta a_i \Delta R_i)^2 . \quad (8)$$

The first two terms are the Gaussian approximation; the third is the correction term. It is always positive, thus increasing the final error.

### Powers of One Variable

The error propagation formulae for one variable are given in Eqs. (A.3) and (A.4) of the appendix. For positive, negative and fractional powers of one variable specialized formulae have been published and discussed elsewhere (3). The results will be summarized here briefly, because powers of a variable appear relatively often in risk assessments. They are usually discussed within the framework of the function  $y = f(\vec{x})$  in which they appear. In some cases, however, it is of interest to discuss the power of that variable all by itself.

Exact formulae can be derived for positive powers  $m$ . They are given in Table IV-2 and their numerical consequences are discussed in Table IV-5 of reference 3. It is shown that over a large range of errors, the Gaussian approximation is surprisingly accurate. This is not true for negative powers, which result in an error propagation formula which is an infinite series. Due to the proximity of the pole at  $x = 0$ , relative errors of 20-30% begin to result in a series of doubtful convergence or outright divergence (Tables IV-3 and IV-6 of ref. 3). This is an expression of the fact that for a large error in the denominator, the distribution of the argument includes the value zero, leading to an indeterminate result.

#### Products of Linear, Normally Distributed Factors

Products of linear factors are often encountered in practical problems. The partial derivatives of the function

$$y = f(\vec{x}) = x_1 x_2 \dots x_m , \quad (9)$$

terminate the series rapidly. A general formula is given in Eq. III-27 of reference 3; for up to four factors, explicit formulae are given in Table II. The expressions are exact and the shift of the means are found to be zero.

Because the complexity of the correction terms increases rapidly with the number of factors, calculations using this formula should be limited to as few factors as possible. This can be achieved readily due to the "sums of squares" structure of the formulae. All factors with smaller errors can be contracted into a single factor whose error is then computed with an appropriate approximation. If the error is considerably smaller than the larger ones, its contribution can even be neglected.

Product of Factors with Lognormal Distributions

Factors with lognormal distributions are often used when large uncertainties ranging over orders of magnitude have to be dealt with. For this distribution, the logarithm of a product of powers

$$y = f(\vec{x}) = x_1^{m_1} x_2^{m_2} \dots x_n^{m_n}, \quad (10)$$

is a weighted sum of normally distributed terms,

$$\log f(\vec{x}) = \sum_{i=1}^n m_i \log x_i, \quad (11)$$

and is, therefore, also normally distributed. In the Gaussian approximation (Table I), which is exact in this case, the error in logarithmic space is

$$s^2 = \sum_{i=1}^n m_i^2 s_i^2, \quad (12)$$

where  $s_i$  is the standard error of  $\log x_i$ . A transformation back to linear coordinate space yields unequal standard error limits given by the product of  $y$  and an exponential factor

$$\phi_{\pm} = y \exp [\pm (\sum_{i=1}^n m_i^2 s_i^2)^{1/2}], \quad (13)$$

where  $\phi_+$  and  $\phi_-$  represent the upper and lower error limits, respectively. The relative errors are given by

$$(\frac{\Delta y}{y})_{\pm} = \pm \{ \exp [\pm (\sum_{i=1}^n m_i^2 s_i^2)^{1/2}] - 1 \}. \quad (14)$$

It should be noted in this context, that other basic operations such as sums and differences of lognormally distributed quantities cannot be performed exactly in a simple analytical way (1). The applicability of the equations in this section for further calculations is therefore restricted.

In addition, situations involving both normally and lognormally distributed factors should be avoided if possible, since operations on parameters with different distributions, some symmetric, some asymmetric, introduce difficulties into an analytical treatment which lie beyond the scope of this paper. In this case the algebra of random variables (1) or numerical methods such as Monte Carlo techniques (10) or Latin Hypercube Sampling (11) should be used.

#### More Complex Algebraic Structures

A typical example of a more complex algebraic structure that occurs in risk assessments is the number  $H$  of health effects expected in a population exposed to a toxic agent with a nonlinear dose-effect relationship. For these cases, the elegant and convenient person-dose (e.g., person-rem) concept appropriate for linear dose-effect relationships is not valid, and the number of health effects has to be approximated by averaging over many sub-populations with nearly the same dose. This results in a general algebraic structure

$$H = F_1 F_2 \dots F_m \sum_{i=1}^N x_i y_i \dots z_i . \quad (15)$$

For a quadratic dose-effect relationship and a group with a dose  $D_i$  the risk for an individual  $r_i$  is

$$r_i = a_0 D_i^2 . \quad (16)$$

As an example, the expected number  $H$  of health effects will be written as

$$H = F_1 F_2 \sum_{i=1}^N n_i a_0 D_i^2$$

$$= a_0 F_1 F_2 \sum_{i=1}^N n_i D_i^2$$

$$= a_0 F_1 F_2 S, \quad (17)$$

where  $n_i$  is the number of individuals in group  $i$ , and

$$S \equiv \sum_{i=1}^N n_i D_i^2. \quad (18)$$

According to Eq. (A.2) in the appendix, the asymmetry of the distribution of  $H$  results in a relative shift  $\delta H$  between the mean  $H^*$  of the distribution and the calculated value  $H_0$  given by the exact expression

$$\frac{\delta H}{H_0} = \frac{H^* - H_0}{H_0} = \sum_{i=1}^N f_i \left( \frac{\Delta D_i}{D_i} \right)^2, \quad (19)$$

where

$$f_i \equiv \frac{n_i D_i^2}{S}. \quad (20)$$

The relative error  $\Delta H/H_0$  is obtained from Eq. (A.1). An elementary but somewhat lengthy calculation yields

$$\begin{aligned} \left( \frac{\Delta H}{H_0} \right)^2 &= \left\{ \left( \frac{\Delta F_1}{F_1} \right)^2 + \left( \frac{\Delta F_2}{F_2} \right)^2 + \left( \frac{\Delta a_0}{a_0} \right)^2 + \right. \\ &+ \sum_{i=1}^N f_i^2 \left[ \left( \frac{\Delta n_i}{n_i} \right)^2 + 4 \left( \frac{\Delta D_i}{D_i} \right)^2 \right] \left. \right\} + \\ &+ \left( \frac{\Delta a_0}{a_0} \right)^2 \left[ \left( \frac{\Delta F_1}{F_1} \right)^2 + \left( \frac{\Delta F_2}{F_2} \right)^2 \right] + \\ &+ \left[ 1 + \left( \frac{\Delta a_0}{a_0} \right)^2 \right] \left( \frac{\Delta F_1}{F_1} \right)^2 \left( \frac{\Delta F_2}{F_2} \right)^2 + \end{aligned}$$

$$\begin{aligned}
& + 2 \sum_{i=1}^n f_i \left( \frac{\Delta D_i}{D_i} \right)^2 \left\{ \left( \frac{\Delta a_0}{a_0} \right)^2 + \left[ 1 + \left( \frac{\Delta a_0}{a_0} \right)^2 \right] \left[ \left( \frac{\Delta F_1}{F_1} \right)^2 + \left( \frac{\Delta F_2}{F_2} \right)^2 \right] + \right. \\
& \quad \left. + \left( \frac{\Delta F_1}{F_1} \right)^2 \left( \frac{\Delta F_2}{F_2} \right)^2 \right\} + \\
& + \sum_{i=1}^n f_i^2 \left\{ 2 \left( \frac{\Delta D_i}{D_i} \right)^4 + \left[ \left( \frac{\Delta n_i}{n_i} \right)^2 + 4 \left( \frac{\Delta D_i}{D_i} \right)^2 \right] \left[ \left( \frac{\Delta a_0}{a_0} \right)^2 + \left( \frac{\Delta F_1}{F_1} \right)^2 + \right. \right. \\
& \quad \left. \left. + \left( \frac{\Delta F_2}{F_2} \right)^2 + \left( \frac{\Delta F_1}{F_1} \right)^2 \left( \frac{\Delta F_2}{F_2} \right)^2 + \left( \frac{\Delta a_0}{a_0} \right)^2 \left( \frac{\Delta F_2}{F_2} \right)^2 \right] + \right. \\
& \quad \left. + 6 \left( \frac{\Delta n_i}{n_i} \right)^2 \left( \frac{\Delta D_i}{D_i} \right)^2 \left[ 1 + \left( \frac{\Delta a_0}{a_0} \right)^2 + \left( \frac{\Delta F_1}{F_1} \right)^2 + \left( \frac{\Delta F_2}{F_2} \right)^2 \right] + \right. \\
& \quad \left. + 3 \left( \frac{\Delta D_i}{D_i} \right)^4 \left[ \left( \frac{\Delta a_0}{a_0} \right)^2 + \left( \frac{\Delta F_1}{F_1} \right)^2 + \left( \frac{\Delta F_2}{F_2} \right)^2 + \left( \frac{\Delta n_i}{n_i} \right)^2 \right] + \dots \right\} + \\
& + \dots . \tag{21}
\end{aligned}$$

The first three terms and the first sum are the usual Gaussian approximation, the next terms the higher order corrections. Due to the "sum of squares" structure, this equation can be simplified considerably, once numerical values are known. A lot depends on the number  $n$  of dose groups because the fractions  $f_i$  are roughly proportional to  $1/n$  and the second sum to Eq. (21) is thus roughly equal to the average term in the sum. In the third sum, however,  $f_i^2$  appears, which gives a sum of roughly  $1/n$  times the average term. With a value for  $n$  that is usually ten or more, this series expansion can thus be expected to have only a few sizeable terms of higher order.

The complete series in Eq. (21) also has nonzero terms of eighth and tenth order in the errors and of up to sixth order in the derivatives. A short inspection shows that all the corresponding terms

$$f_1 \left( \frac{\Delta D_1}{D_1} \right)^4 \left( \frac{\Delta x_k}{x_k} \right)^2 \left( \frac{\Delta x_1}{x_1} \right)^2 ,$$

$$f_1^2 \left( \frac{\Delta D_1}{D_1} \right)^4 \left( \frac{\Delta x_k}{x_k} \right)^2 \left( \frac{\Delta x_1}{x_1} \right)^2 ,$$

$$f_1^2 \left( \frac{\Delta D_1}{D_1} \right)^2 \left( \frac{\Delta x_k}{x_k} \right)^2 \left( \frac{\Delta x_1}{x_1} \right)^2 \left( \frac{\Delta x_m}{x_m} \right)^2 ,$$

$$f_1^2 \left( \frac{\Delta D_1}{D_1} \right)^4 \left( \frac{\Delta x_k}{x_k} \right)^2 \left( \frac{\Delta x_1}{x_1} \right)^2 \left( \frac{\Delta x_m}{x_m} \right)^2 ,$$

$$f_1^2 \left( \frac{\Delta D_1}{D_1} \right)^2 \left( \frac{\Delta a_0}{a_0} \right)^2 \left( \frac{\Delta F_1}{F_1} \right)^2 \left( \frac{\Delta F_2}{F_2} \right)^2 \left( \frac{\Delta n_1}{n_1} \right)^2 ,$$

where the indices k, 1, and m stand for parameters other than  $D_1$ , are very small, even for relative errors of 0.5 or so.

Another important aspect of calculating the uncertainty of the number  $H$  of health effects according to Eq. (17) is the fact that the uncertainties of  $n_1$  and  $D_1$  are connected, but not necessarily correlated. If the population is subdivided into groups with a narrow dose range, the group populations  $n_1$  are going to have large errors; if the group doses have large errors, the populations  $n_1$  are less uncertain. An inspection of Eq. (21) shows that it is preferable to have as many dose groups with narrowly defined doses as possible, given the quality of the data base.

The uncertainty of the expected number of health effects can thus be calculated for any reasonably large errors in the input data and the sensitivity for each parameter is established directly in analytical form.

5. DISCUSSION

Exact formulae or sufficiently accurate approximations can be derived for the propagation of large errors in some simple algebraic forms which occur frequently in risk assessment. For series that do not terminate analytically, enough terms can be given to allow a sufficient approximation for most error sizes. As shown in the last section, the number of terms given in the appendix is usually sufficient. If more terms are needed, they can be derived from the general formula (3). This decision involves a judgment of the convergence of the series (A.1). Conventionally, convergence can be established by estimating the remainder of the Taylor series (9) and inserting it into the error propagation formula. This procedure would, however, result in a highly complex form for the remainder of the error series. In most risk assessments, this procedure is not necessary because the functions are well behaved and their higher order derivatives quickly become negligibly small, if not identically zero.

An example was given in the last section, where the last terms decrease rapidly, not only because of the high powers of the relative errors ( $\Delta x_1/x_1$ ), but also because of the small values of the higher order derivatives which contain the small factor  $f_1^2$ . If the series does not converge promptly, the numerical situation should be investigated for an extraordinary environment such as the neighborhood of the pole [i.e.,  $f(x)$  is infinite] at  $x = 0$  in the function  $f(x) = 1/x$ . In this case, non-convergence could mean that the upper error limit is indeterminate due to the proximity of the pole.

The formulae presented can be simplified considerably by applying some a priori knowledge about the relative size and nature of the errors and

retaining only those terms that contribute noticeably to the final error. Because of the "sums of squares" structure of the error formulae, relatively small factors between errors can lead to the elimination of the smaller ones without significant loss of accuracy.

An analytical formulation of error propagation offers essentially three advantages and two disadvantages. Advantages are: One, the numerical calculations are rather modest once the algebra has been done. Two, the influence of various contributions to the final error can be discussed explicitly. Three, the range of applicability for the Gaussian approximation can also be judged explicitly. Numerical investigations show that over an often surprising range of error sizes, it still yields sufficiently accurate results (3). Disadvantages are: One, a rapid increase in the algebraic effort as the complexity of the function increases. Two, the standard errors thus determined do not allow an accurate calculation of upper and lower limits at 95% confidence level, because the distribution of the result is no longer normal. These limits are, therefore, just approximations, although for large errors that is a lesser problem of the uncertainty analysis.

Contrary to intuitive judgement, the analytical discussion of error propagation can thus be used to advantage even for large errors, complementing the more traditional numerical methods. In all cases investigated explicitly, the correction terms to the usual Gaussian approximation were positive, i.e. the Gaussian approximation underestimated the error. Thus, as errors grow larger, the method developed here becomes increasingly important in estimating errors realistically.

The formulae given for that purpose are often exact or very good approximations. In discussions involving the resulting errors and their

meaning, however, it should be remembered that the assumption of a certain probability distribution for a stochastic variable is in itself an approximation, often only tenuously supported by experimental data. It will thus be the uncertainties in the distributions and the standard errors of the input parameters that limit the determinative power of the error analysis.

It is in this area that the most difficult problems of an error calculation must be solved. Careful examination of the size and nature of the error for each variable and the assignment of an appropriate probability distribution are crucial to the success of the analysis. The propagation of these characteristics through the numerical calculation and the estimation of the final mean and its error are then accomplished readily, using the most appropriate method.

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## APPENDIX

Derivation of Error Formulae

The error propagation formulae used in this paper have been derived in a report published elsewhere <sup>(3)</sup>. They are based on the multidimensional Taylor series <sup>(9)</sup> for the function  $y = f(\vec{x})$ , where the vector  $\vec{x} = (x_1, x_2, \dots, x_n)$  is associated with a statistical error vector  $\vec{\Delta x} = (\Delta x_1, \Delta x_2, \dots, \Delta x_n)$ . Unless stated otherwise, the components of the error vector are assumed to be independent of each other and distributed normally. The general formulae and some tables for the construction of explicit formulae have been given in the original report <sup>(3)</sup>. In this paper, the first terms of the series are given for the case of one and several variables.

A.1. Functions of Several Variables

The series for the final error  $\Delta y$  of the function  $y = f(\vec{x})$  with terms of up to sixth order in the errors  $\Delta x_i$  and using derivatives of up to fourth order is

$$\begin{aligned}
 (\Delta y)^2 = & \sum_{i=1}^n \left( \frac{\partial f}{\partial x_i} \right)^2 (\Delta x_i)^2 + \frac{1}{2} \sum_{i=1}^n \left( \frac{\partial^2 f}{\partial x_i^2} \right)^2 (\Delta x_i)^4 + \\
 & + \sum_{i=1}^n \sum_{j=i+1}^n \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)^2 (\Delta x_i)^2 (\Delta x_j)^2 + \\
 & + \sum_{i=1}^n \left( \frac{\partial^3 f}{\partial x_i^3} \right) \left( \frac{\partial f}{\partial x_i} \right) (\Delta x_i)^4 + \\
 & + \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \left( \frac{\partial^3 f}{\partial x_i^2 \partial x_j} \right) \left( \frac{\partial f}{\partial x_j} \right) (\Delta x_i)^2 (\Delta x_j)^2 +
 \end{aligned}$$

$$\begin{aligned}
& + \frac{5}{12} \sum_{i=1}^n \left( \frac{\partial^3 f}{\partial x_i^3} \right)^2 (\Delta x_i)^6 + \\
& + \sum_{i=1}^n \sum_{j=i+1}^n \sum_{k=i+2}^n \left( \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k} \right)^2 (\Delta x_i \Delta x_j \Delta x_k)^2 + \\
& + \frac{3}{4} \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \left( \frac{\partial^3 f}{\partial x_i^2 \partial x_j} \right)^2 (\Delta x_i)^4 (\Delta x_j)^2 + \\
& + \frac{1}{4} \sum_{i=1}^n \sum_{j=i+1}^n \sum_{k=i+2}^n \left( \frac{\partial^3 f}{\partial x_i^2 \partial x_j} \right) \left( \frac{\partial^3 f}{\partial x_k^2 \partial x_j} \right) (\Delta x_i \Delta x_j \Delta x_k)^2 + \\
& + \frac{1}{2} \sum_{i=1}^n \left( \frac{\partial^4 f}{\partial x_i^4} \right) \left( \frac{\partial^2 f}{\partial x_i^2} \right) (\Delta x_i)^6 + \\
& + \sum_{i=1}^n \sum_{j=i+1}^n \sum_{k=i+2}^n \left( \frac{\partial^4 f}{\partial x_i^2 \partial x_j \partial x_k} \right) \left( \frac{\partial^2 f}{\partial x_j \partial x_k} \right) (\Delta x_i \Delta x_j \Delta x_k)^2 + \\
& + \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \left( \frac{\partial^4 f}{\partial x_i^3 \partial x_j} \right) \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right) (\Delta x_i)^4 (\Delta x_j)^2 + \\
& + \frac{1}{2} \sum_{i=1}^n \sum_{j=i+1}^n \left( \frac{\partial^4 f}{\partial x_i^2 \partial x_j^2} \right) \left( \frac{\partial^2 f}{\partial x_i^2} \right) (\Delta x_i)^4 (\Delta x_j)^2 + \dots \tag{A.1}
\end{aligned}$$

It should be kept in mind that in all of the formulae derivatives are evaluated numerically at the point  $\vec{x}_0$ .

The asymmetry in the distribution of the function  $y = f(\vec{x})$  leads to a mean  $y^*$  which is different from the function value  $y_0 = f(\vec{x}_0)$  calculated directly. The shift is given by

$$\delta y \equiv y^* - y_0 = \frac{1}{2} \sum_{i=1}^n \left( \frac{\partial^2 f}{\partial x_i^2} \right) (\Delta x_i)^2 +$$

$$\begin{aligned}
 & + \frac{1}{8} \sum_{i=1}^n \left( \frac{\partial^4 f}{\partial x_i^4} \right) (\Delta x_i)^4 + \\
 & + \frac{1}{4} \sum_{i=1}^n \sum_{j=i+1}^n \left( \frac{\partial^4 f}{\partial x_i^2 \partial x_j^2} \right) (\Delta x_i \Delta x_j)^2 + \dots
 \end{aligned} \tag{A.2}$$

Note that the formula for the shift of the mean is linear in the derivatives.

#### A.2. Functions of One Variable

The formulae derived for an  $n$ -dimensional function simplify considerably for one dimension. The standard error  $\Delta y$  of the function  $y = f(x)$  is then given by

$$\begin{aligned}
 (\Delta y)^2 &= \left( \frac{df}{dx} \right)^2 (\Delta x)^2 + \frac{1}{2} \left( \frac{d^2 f}{dx^2} \right)^2 (\Delta x)^4 + \\
 & + \left( \frac{d^3 f}{dx^3} \right) \left( \frac{df}{dx} \right) (\Delta x)^4 + \frac{5}{12} \left( \frac{d^3 f}{dx^3} \right)^2 (\Delta x)^6 + \\
 & + \frac{1}{2} \left( \frac{d^4 f}{dx^4} \right) \left( \frac{d^2 f}{dx^2} \right) (\Delta x)^6 + \dots
 \end{aligned} \tag{A.3}$$

An asymmetry in the distribution of the function value  $y = f(x)$  results in a nonzero first moment of the quantity  $\delta y$

$$\delta y = \frac{1}{2} \left( \frac{d^2 f}{dx^2} \right) (\Delta x)^2 + \frac{1}{8} \left( \frac{d^4 f}{dx^4} \right) (\Delta x)^4 + \dots \tag{A.4}$$

Again, all derivatives are evaluated at the point  $\vec{x}_0$ .

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Table I. Gaussian Approximation for Error Propagation in Simple Algebraic Structures<sup>a</sup>

$y = f(\vec{x})$ <sup>b</sup>	Error Formula	Level of Approximation
$y = x_1 \pm x_2 \pm \dots \pm x_n$	$(\Delta y)^2 = (\Delta x_1)^2 + (\Delta x_2)^2 + \dots + (\Delta x_n)^2$	exact
$y = x_1 x_2 x_3 \dots x_n$	$(\frac{\Delta y}{y})^2 = (\frac{\Delta x_1}{x_1})^2 + (\frac{\Delta x_2}{x_2})^2 + \dots + (\frac{\Delta x_n}{x_n})^2$	valid for small $(\Delta x_i/x_i)$
$y = \frac{x_1 x_2 \dots x_k}{x_{k+1} \dots x_n}$	$(\frac{\Delta y}{y})^2 = (\frac{\Delta x_1}{x_1})^2 + \dots + (\frac{\Delta x_k}{x_k})^2 + \dots + (\frac{\Delta x_n}{x_n})^2$	valid for small $(\Delta x_i/x_i)$
$y = x_1^{m_1} x_2^{m_2} \dots x_n^{m_n}$	$(\frac{\Delta y}{y})^2 = m_1^2 (\frac{\Delta x_1}{x_1})^2 + \dots + m_n^2 (\frac{\Delta x_n}{x_n})^2$	valid for small $(\Delta x_i/x_i)$

<sup>a</sup>Note: The symbols  $x_i$  do not denote variables, but stand for their numerical values.

<sup>b</sup> $m_i$  = arbitrary

Table II. Exact Formulae for the Relative Error of a Product of Linear Factors <sup>a,b,c</sup>

m	Gaussian Approximation	Correction Term for Exact Expression
2	$\sum_{i=1}^2 \left(\frac{\Delta x_i}{x_i}\right)^2$	$\left(\frac{\Delta x_1}{x_1}\right)^2 \left(\frac{\Delta x_2}{x_2}\right)^2$
3	$\sum_{i=1}^3 \left(\frac{\Delta x_i}{x_i}\right)^2$	$\left(\frac{\Delta x_1}{x_1}\right)^2 \left(\frac{\Delta x_2}{x_2}\right)^2 + \left(\frac{\Delta x_1}{x_1}\right)^2 \left(\frac{\Delta x_3}{x_3}\right)^2 + \left(\frac{\Delta x_2}{x_2}\right)^2 \left(\frac{\Delta x_3}{x_3}\right)^2 + \left(\frac{\Delta x_1}{x_1}\right)^2 \left(\frac{\Delta x_2}{x_2}\right)^2 \left(\frac{\Delta x_3}{x_3}\right)^2$
4	$\sum_{i=1}^4 \left(\frac{\Delta x_i}{x_i}\right)^2$	$\left(\frac{\Delta x_1}{x_1}\right)^2 \left(\frac{\Delta x_2}{x_2}\right)^2 + \left(\frac{\Delta x_1}{x_1}\right)^2 \left(\frac{\Delta x_3}{x_3}\right)^2 + \left(\frac{\Delta x_1}{x_1}\right)^2 \left(\frac{\Delta x_4}{x_4}\right)^2 + \left(\frac{\Delta x_2}{x_2}\right)^2 \left(\frac{\Delta x_3}{x_3}\right)^2 + \left(\frac{\Delta x_2}{x_2}\right)^2 \left(\frac{\Delta x_4}{x_4}\right)^2 +$ $+ \left(\frac{\Delta x_3}{x_3}\right)^2 \left(\frac{\Delta x_4}{x_4}\right)^2 + \left(\frac{\Delta x_1}{x_1}\right)^2 \left(\frac{\Delta x_2}{x_2}\right)^2 \left(\frac{\Delta x_3}{x_3}\right)^2 + \left(\frac{\Delta x_1}{x_1}\right)^2 \left(\frac{\Delta x_2}{x_2}\right)^2 \left(\frac{\Delta x_4}{x_4}\right)^2 + \left(\frac{\Delta x_1}{x_1}\right)^2 \left(\frac{\Delta x_3}{x_3}\right)^2 \left(\frac{\Delta x_4}{x_4}\right)^2 +$ $+ \left(\frac{\Delta x_2}{x_2}\right)^2 \left(\frac{\Delta x_3}{x_3}\right)^2 \left(\frac{\Delta x_4}{x_4}\right)^2 + \left(\frac{\Delta x_1}{x_1}\right)^2 \left(\frac{\Delta x_2}{x_2}\right)^2 \left(\frac{\Delta x_3}{x_3}\right)^2 \left(\frac{\Delta x_4}{x_4}\right)^2 .$

<sup>a</sup>Note: The symbols  $x_i$  do not denote variables, but their numerical values.

<sup>b</sup>For linear factors, no correction of the mean is necessary.

<sup>c</sup>Formulae for m=2:

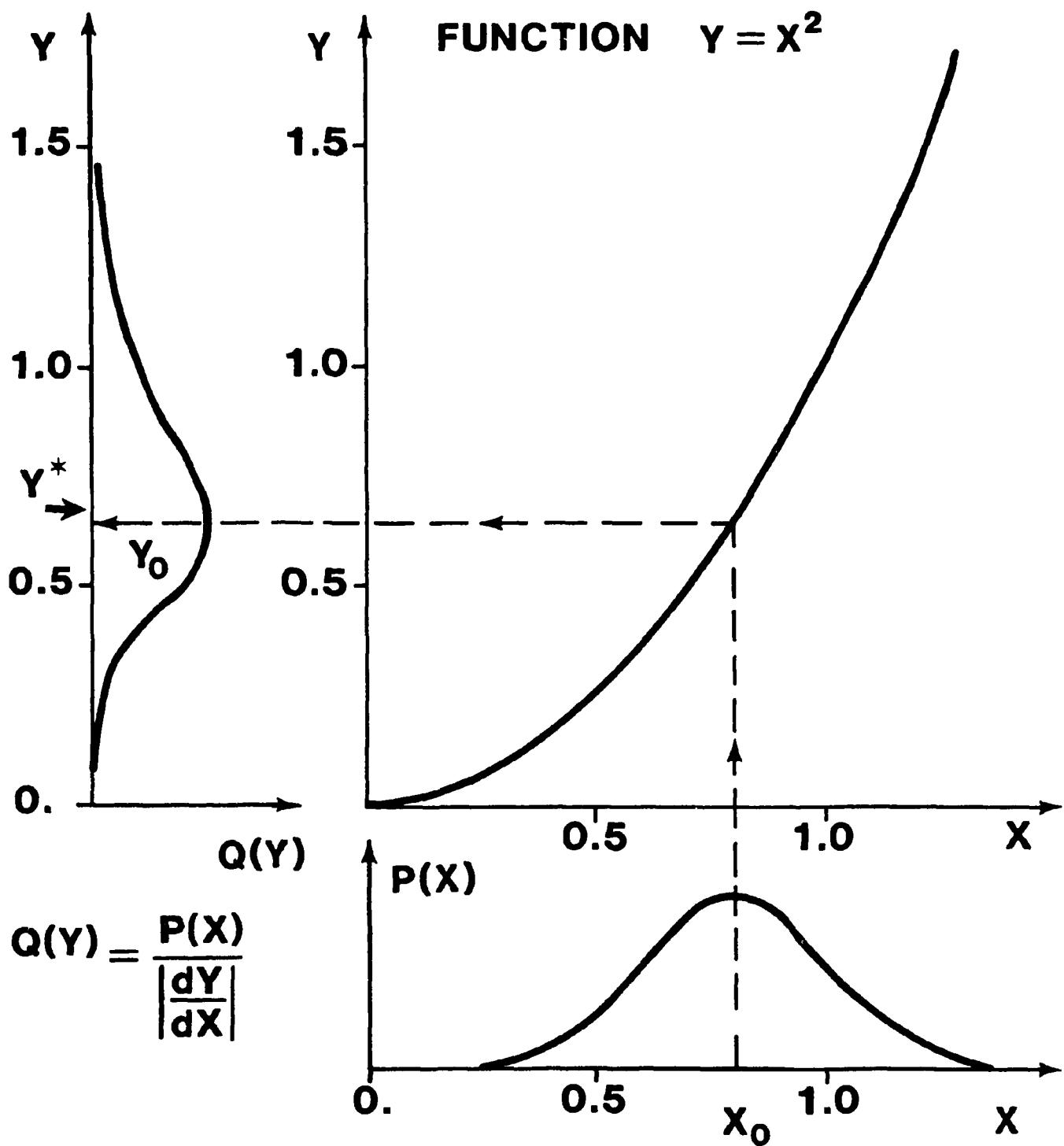
$$y = \prod_{i=1}^2 x_i = x_1 x_2 .$$

the table yields:

$$\left(\frac{\Delta y}{y}\right)^2 = \sum_{i=1}^2 \left(\frac{\Delta x_i}{x_i}\right)^2 + \left(\frac{\Delta x_1}{x_1}\right)^2 \left(\frac{\Delta x_2}{x_2}\right)^2 .$$

**Figure Caption**

Fig. 1. Composite plot of the function  $y = x^2$ , a normally distributed variable  $x$  ( $x_0 = 0.8$ ,  $\sigma = 0.2$ ) and the distribution of the result which is asymmetric and has a mean  $y^*$  larger than  $y_0 = 0.64$ .



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