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LAMINATED BEAMS - DEFLECTION AND STRESS AS A FUNCTION OF EPOXY SHEAR MODULUS

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Laminated Beams - Deflection and Stress as a Function of Epoxy Shear Modulus

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ABSTRACT

The large toroidal field coil deflections observed during the PLT power test are due to the poor shear behavior of the insulation material used between layers of copper. Standard techniques for analyzing such laminated structures do not account for this effect. This paper presents an analysis of laminated beams that corrects this deficiency. The analysis explicitly models the mechanical behavior of each layer in a laminated beam and hence avoids the pitfalls involved in any averaging technique. In particular, the shear modulus of the epoxy in a laminated beam (consisting of alternate layers of metal and epoxy) may span the entire range of values from zero to classical. Solution of the governing differential equations defines the stress, strain, and deflection for any point within a laminated beam. The paper summarizes these governing equations and also includes a parametric study of a simple laminated beam.

INTRODUCTION

The toroidal field coils in the Princeton Large Torus (PLT) Tokamak are subjected to a non uniform self generated magnetic field. This results in a force distribution that causes each toroidal field coil to bend in its own plane. The deflections observed on the PLT toroidal field coils[1,2] are not in agreement with calculations based on standard composite beam theory. In fact, the observed deflections are three to eight times larger than composite beam theory predicts. This paper presents a solution that models deflection and stress in straight laminated beams. The beams analyzed have mechanical properties similar to the PLT toroidal field coils.

The laminated beams that are considered in this paper are made of alternate layers of copper and epoxy. The unique mechanical behavior of these beams is due to the poor shear behavior of the epoxy layers. Experimental measurements[3] of effective shear modulus for an epoxy layer give values for the shear modulus thirty (30) times smaller than expected. This of course effects the stiffness and stress distribution within a laminated beam. The goal of this work was to develop a model that describes the behavior of a laminated beam as the shear modulus of the epoxy varies from classical values ($G_{epoxy} = E_{epoxy} / 2(1+\nu)$) to zero.

The differential equations governing deflection and stress in each layer of a straight laminated beam are presented. These equations treat every layer of the beam explicitly and do not depend on any averaging of the beam's cross sectional properties.

Hence this analysis provides an explicit statement of deflection and stress throughout a laminated beam and avoids the pitfalls of all averaging procedures. The solution of these equations for the particular problem of a simply supported uniformly loaded beam is also presented. Results are summarized for a parametric study of this problem and finally, the stiffness and stress distribution in a straight laminated beam having the same cross section as the PLT toroidal field coil is presented.

ANALYSIS

The displacement energy method is used to analyze this problem. First the total potential energy of the laminated beam is expressed in terms of the displacement of each layer of metal. Then, variational calculus is applied to the total potential energy expression to obtain the equations governing the displacement of every layer in the beam. These equations are the equations of static equilibrium and the stresses in the beam are obtained by taking derivatives of the displacement solution. The governing equations form a set of coupled ordinary differential equations and are solved by using the Ritz procedure. The small computer program written to obtain numerical results is listed in Appendix 1.

The total potential energy of a laminated beam (π^*) is given by

$$\pi^* = E - V \qquad (1a)$$

$$E = \int_{\text{volume}} \left[\frac{E_m \epsilon_{xx}^2)_m}{2} + \frac{G_m \gamma_{xz}^2)_m}{2} + \frac{E_e \epsilon_{zz}^2)_e}{2} + \frac{G_e \gamma_{xz}^2)_e}{2} \right] dV \qquad (1b)$$

$$V = \int_0^{\ell} q(x)w(x)dx \qquad (1c)$$

Where

E = Strain energy in the laminated beam

V = The potential energy of the applied load intensity "q".

E_m = Youngs modulus in the metal

G_m = Shear modulus in the metal

E_e = Youngs modulus in the epoxy

G_e = Shear modulus in the epoxy

$\epsilon_{xx})_m$ = xx component of strain in the metal

$\epsilon_{zz})_e$ = zz component of strain in the epoxy

$\gamma_{xz})_m$ = Shear angle in the metal

$\gamma_{xz})_e$ = Shear angle in the epoxy

Figure 1. describes the coordinate system used in the problem.

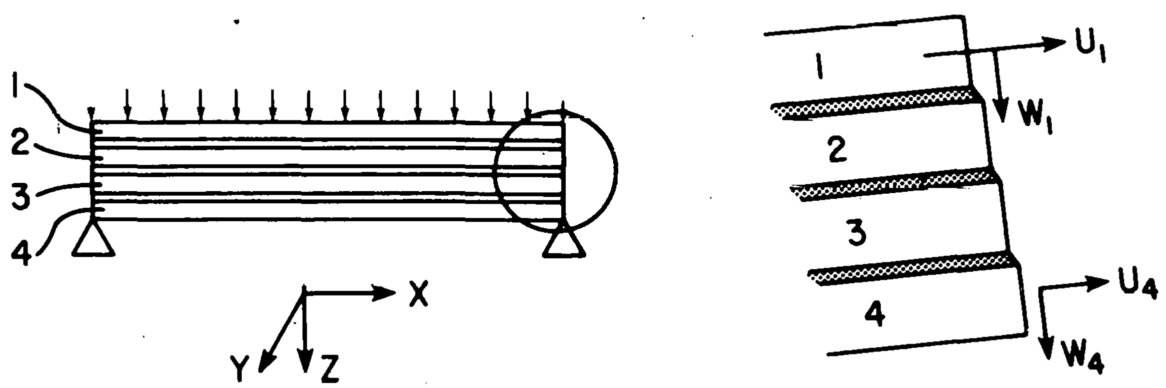

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Fig. 1. Coordinate system for laminated beam.

The Youngs moduli and shear moduli in Equation 1 are constants for the beam. The first and second terms in Equation 1 represent the axial bending and shearing behavior in each layer of metal. The third term models the lateral compression or tension in the epoxy bond and serves to transmit force in the z direction from layer to layer within the beam. The fourth term in Equation 1 represents the shear behavior in the epoxy layers. G_e in this term should be considered an effective shear modulus and independent of E_e. This fourth term transmits shear between adjacent layers of metal. As G_e becomes smaller, greater shear deflection in the epoxy bond is allowed, less shear exists between layers of metal, and the laminated beam becomes more flexible.

The strain terms in Equation 1 are defined in the usual manner. Within each metal layer the following relations are used:

$$\left.\begin{aligned} \varepsilon_{xx})_i &= \frac{\partial U_x)_i}{\partial x} && \text{(a)} \\ \gamma_{xz})_i &= \frac{\partial U_z)_i}{\partial x} + \frac{\partial U_x)_i}{\partial z} && \text{(b)} \end{aligned}\right\} \qquad (2)$$

where:

i = index referring to layer number in laminated beam $i = 1, 2, \ldots, n$ (see Fig. 1.)

$\varepsilon_{xx})_i$ = xx strain component in i^{th} metal layer

$\gamma_{xz})_i$ = shear angle in i^{th} metal layer $\gamma_{xz})_i = 2\varepsilon_{xz})_i$

$U_x)_i$ = x component of deflection field i^{th} metal layer

$U_z)_i$ = z component of deflection field i^{th} metal layer

The deflection field used in Equation 2a and 2b must be defined with care. If the deflection field is not simplified a system of partial differential equations will result. If the deflection field is too simple the equations of equilibrium will not be satisfied.[4] After considerable study, a deflection field that represents a typical Timoshenko beam was selected and found to be adequate. This deflection field maintains the strength of materials assumption that plane sections remain plane but this is restricted to hold only within each layer of metal. The total cross section of the beam will deform from a plane to a series of ridges and ramps. The shear deflection (γ_i) in the deflection field is also worthy of note. Although this term is a very small part of the deflection in each layer of the beam, it couples the equations together. If this term was not included, the equations would not satisfy static equilibrium.[4] Equation 3 summarizes the deflection field.

$$\left.\begin{aligned} U_x)_i &= u_i(x) - z\left(\frac{dw_i(x)}{dx} - \gamma_i(x)\right) && \text{(a)} \\ U_y)_i &\equiv 0 && \text{(b)} \\ U_z)_i &= w_i(x) && \text{(c)} \end{aligned}\right\} \quad (3)$$

where:

z = the value of the z coordinate within the i^{th} layer measured from a coordinate system with the $z = 0$ value located at the geometric center of the i^{th} metal layer

$u_i(x)$ = the x deflection of the center line of the i^{th} metal layer

$w_i(x)$ = the z deflection of the center line of the i^{th} metal layer

$\gamma_i(x)$ = the shear deflection of the center line of the i^{th} metal layer

Hence the strains in the metal are:

$$\left.\begin{aligned} \varepsilon_{xx})_i &= \frac{\partial U_x}{\partial x} = \frac{du_i}{dx} - z\left(\frac{d^2 w_i}{dx^2} - \frac{d\gamma_i}{dx}\right) && \text{(a)} \\ \gamma_{xz})_i &= \frac{\partial U_z}{\partial x} + \frac{\partial U_x}{\partial z} = \frac{dw_i}{dx} - \left(\frac{dw_i}{dx} - \gamma_i\right) = \gamma_i && \text{(b)} \end{aligned}\right\} \quad (4)$$

The 3n functions u_i, w_i, and γ_i ($i = 1, 2, \ldots, n$) are the unknowns of this problem. The first two terms in Equation 1 are now explicitly defined. The remaining two terms in Equation 1 will be defined in terms of the deflection field of the metal layers. By doing this, the layers of the beam are connected to each other, no new unknowns are introduced, the strain energy in the epoxy is described, and compatibility of the deflection fields is insured.

The lateral compressive or tensile strain in the epoxy layer i is defined by

$$\left.\varepsilon_{zz}\right)_i = \frac{w_{i+1} - w_i}{t_i} \tag{5}$$

where

$\left.\varepsilon_{zz}\right)_i$ = the zz strain component in the i^{th} epoxy layer (between metal laminates i and i+1)

t_i = the thickness of the i^{th} epoxy layer (i = 1, 2, ..., n-1)

The shear strain in the i^{th} epoxy layer (between metal layers i and i+1) is defined by the following argument.

Figure 2 describes the geometry of a typical epoxy layer in its deformed state.

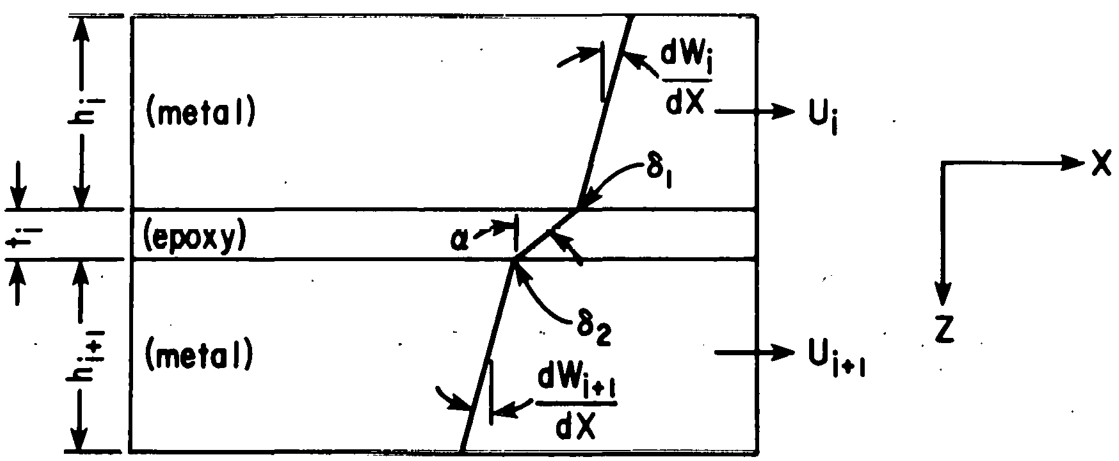

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Fig. 2. Shear in epoxy bond.

The angle α is defined by

$$\alpha = \frac{\delta_1 - \delta_2}{t} \tag{6}$$

where

$$\delta_1 = u_i - \frac{h_i}{2}\left(\frac{dw_i}{dx} - \gamma_i\right)$$

and

$$\delta_2 = u_{i+1} + \frac{h_{i+1}}{2}\left(\frac{dw_{i+1}}{dx} - \gamma_{i+1}\right) \tag{7}$$

The angle α includes both the shear angle $\gamma^*_{xz})_i$ and the average slope ($\bar{\alpha}$) of the epoxy layer

i.e. $$\alpha = \gamma^*_{xz})_i + \bar{\alpha} \tag{8}$$

The average slope of the i^{th} epoxy layer is simply a weighted average of the slopes for the i^{th} and $(i+1)^{st}$ metal layers.

$$\bar{\alpha} = \left[h_i\left(\frac{dw_i}{dx} - \gamma_i\right) + h_{i+1}\left(\frac{dw_{i+1}}{dx} - \gamma_{i+1}\right)\right]\frac{1}{(h_i + h_{i+1})} \tag{9}$$

where h_i is the thickness of the i^{th} metal layer.

Substitution of Eq. (9) into Eq. (8) yields

$$\gamma^*_{xz})_i = \frac{1}{t_i}\left[u_i - u_{i+1} - K_i\left(\frac{dw_i}{dx} - \gamma_i\right) - L_i\left(\frac{dw_{i+1}}{dx} - \gamma_{i+1}\right)\right] \qquad (10)$$

where

$$\left.\begin{array}{ll} K_i = \dfrac{h_i}{2} + \dfrac{h_i t_i}{(h_i + h_{i+1})} & \text{(a)} \\ & i = 1,\ 2,\ \ldots,\ n-1 \\ L_i = \dfrac{h_{i+1}}{2} + \dfrac{h_{i+1}\ t_i}{(h_i + h_{i+1})} & \text{(b)} \end{array}\right\} \qquad (11)$$

Then the total potential energy of the laminated beam π^* is:

$$\pi^* = \int_0^{\ell} \sum_{i=1}^{n} \frac{Eb}{2} \left[\left(\frac{du_i}{dx}\right)^2 h_i + \frac{h_i^3}{12}\left(\left(\frac{d^2w_i}{dx^2}\right) + \left(\frac{d\gamma_i}{dx}\right)^2 - 2\frac{d\gamma_i}{dx}\frac{d^2w_i}{dx^2}\right)\right] dx$$

$$+ \int_0^{\ell} \sum_{i=1}^{n} \frac{Gbh_i}{2}\gamma_i^2 dx + \int_0^{\ell} \sum_{i=1}^{n-1} \frac{E_e b}{2t_i}(w_{i+1}^2 + w_i^2 - 2w_i\, w_{i+1})\, dx$$

$$+ \int_0^{\ell} \sum_{i=1}^{n-1} \frac{G_e}{2}\frac{b}{t_i}\Bigg[u_i^2 + u_{i+1}^2 - 2u_i\, u_{i+1} + K_i^2\left(\left(\frac{dw_i}{dx}\right)^2 + \gamma_i^2 - 2\frac{dw_i}{dx}\gamma_i\right)$$

$$+ L_i^2\left(\left(\frac{dw_{i+1}}{dx}\right)^2 + \gamma_{i+1}^2 - 2\frac{dw_{i+1}}{dx}\gamma_{i+1}\right)$$

$$- 2u_iK_i\frac{dw_i}{dx} + 2u_iK_i\gamma_i - 2u_iL_i\frac{dw_{i+1}}{dx} + 2u_iL_i\gamma_{i+1}$$

$$+ 2u_{i+1}K_i\frac{dw_i}{dx} - 2u_{i+1}K_i\gamma_i + 2u_{i+1}L_i\frac{dw_{i+1}}{dx} - 2u_{i+1}L_i\gamma_{i+1}$$

$$+ 2K_iL_i\frac{dw_i}{dx}\frac{dw_{i+1}}{dx} - 2K_iL_i\frac{dw_i}{dx}\gamma_{i+1} - 2K_iL_i\gamma_i\frac{dw_{i+1}}{dx} + 2K_iL_i\gamma_i\gamma_{i+1}\Bigg] dx$$

$$- \sum_{i=1}^{n}\int_0^{\ell} q_i\, w_i\, dx \qquad (12)$$

where:

b = the uniform width of the laminated beam

ℓ = the length of the beam

q_i = the lateral load (lbs/in) on the i^{th} layer

Taking variational derivatives of this expression with respect to w_i, u_i, and γ_i gives the following differential equations and boundary conditions. These are the equations that describe the behavior of a laminated beam. In the following six expressions those terms labeled "(a)" are the natural boundary conditions that are defined by the variational procedure. The vertical line $\Big|_o^\ell$ indicates evaluation at the limits of integration. Those terms labeled "(b)" are the governing differential equations.

$$Ebh_i \frac{du_i}{dx} \delta u_i \Big|_o^\ell = 0 \tag{13a}$$

$$- Ebh_i \frac{d^2u_i}{dx^2}$$

$$+ (1-\delta_{1i}) \frac{G_e b}{t_{i-1}} \left(u_i - u_{i-1} + K_{i-1} \frac{dw_{i-1}}{dx} - K_{i-1}\gamma_{i-1} + L_{i-1} \frac{dw_i}{dx} - L_{i-1}\gamma_i\right)$$

$$+ (1-\delta_{ni}) \frac{G_e b}{t_i} \left(u_i - u_{i+1} - K_i \frac{dw_i}{dx} + K_i\gamma_i - L_i \frac{dw_{i+1}}{dx} + L_i\gamma_{i+1}\right) = 0 \tag{13b}$$

$$\frac{Ebh_i^3}{12}\left(\frac{d^2w_i}{dx^2} - \frac{d\gamma_i}{dx}\right)\frac{d\delta w_i}{dx}\Bigg|_o^{\ell} - \frac{Ebh_i^3}{12}\left(\frac{d^3w_i}{dx^3} - \frac{d^2\gamma_i}{dx^2}\right)\delta w_i\Bigg|_o^{\ell}$$

$$+ (1-\delta_{i1})\frac{G_e b}{t_{i-1}}\Bigg(L_{i-1}^2\frac{dw_i}{dx} - L_{i-1}^2\gamma_i - L_{i-1}u_{i-1}$$

$$+ L_{i-1}u_i + K_{i-1}L_{i-1}\frac{dw_{i-1}}{dx} - K_iL_i\gamma_{i-1}\Bigg)\quad \delta w_i\Bigg|_o^{\ell}$$

$$+ (1-\delta_{in})\frac{G_e b}{ti}\Bigg(K_i^2\frac{dw_i}{dx} - K_i^2\gamma_i - K_iu_i$$

$$+ K_iu_{i+1} + K_iL_i\frac{dw_{i+1}}{dx} - K_iL_i\gamma_{i+1}\Bigg)\quad \delta w_i\Bigg|_o^{\ell} = 0 \tag{14a}$$

$$\frac{Ebh_i^3}{12}\left(\frac{d^4w_i}{dx^4} - \frac{d^3\gamma_i}{dx^3}\right) + \frac{E_e}{t_{i-1}}(w_i - w_{i-1})(1-\delta_{i1}) + \frac{E_e}{t_i}(w_i - w_{i+1})(1-\delta_{in})$$

$$+ (1-\delta_{i1})\frac{G_e b}{t_{i-1}}\Bigg(- L_{i-1}^2\frac{d^2w_i}{dx^2} + L_{i-1}^2\frac{d\gamma_i}{dx} + L_{i-1}\frac{du_{i-1}}{dx}$$

$$- L_{i-1}\frac{du_i}{dx} - K_{i-1}L_{i-1}\frac{d^2w_{i-1}}{dx^2} + K_{i-1}L_{i-1}\frac{d\gamma_{i-1}}{dx}\Bigg)$$

$$+ (1-\delta_{in})\frac{G_e b}{t_i}\Bigg(- K_i^2\frac{d^2w_i}{dx^2} + K_i^2\frac{d\gamma_i}{dx} + K_i\frac{du_i}{dx}$$

$$- K_i\frac{du_{i+1}}{dx} - K_iL_i\frac{d^2w_{i+1}}{dx^2} + K_iL_i\frac{d\gamma_{i+1}}{dx}\Bigg)$$

$$- q_i = 0 \tag{14b}$$

$$\frac{Ebh_i^3}{12}\left(\frac{d\gamma_i}{dx} - \frac{d^2w_i}{dx^2}\right)\delta\gamma_i\Bigg|_o^{\ell} = 0 \tag{15a}$$

$$\frac{Ebh_i^3}{12}\left(\frac{d^3w_i}{dx^3} - \frac{d^2\gamma_i}{dx^2}\right) + Gbh_i\gamma_i$$

$$+ (1-\delta_{i1})\frac{G_e b}{t_{i-1}}\left(- L_{i-1}^2\frac{d\gamma_i}{dx} + L_{i-1}^2\frac{d^2w_i}{dx^2} - L_{i-1}\frac{du_{i-1}}{dx}\right.$$

$$\left. + L_{i-1}\frac{du_i}{dx} + K_{i-1}L_{i-1}\frac{d^2w_{i-1}}{dx^2} - K_{i-1}L_{i-1}\frac{d\gamma_{i-1}}{dx}\right)$$

$$+ (1-\delta_{in})\frac{G_e b}{t_i}\left(- K_i^2\frac{d\gamma_i}{dx} + K_i^2\frac{d^2w_i}{dx^2} - K_i\frac{du_i}{dx}\right.$$

$$\left. + K_i\frac{du_{i+1}}{dx} + K_iL_i\frac{d^2w_{i+1}}{dx^2} - K_iL_i\frac{d\gamma_{i+1}}{dx}\right) = 0 \tag{15b}$$

where δ_{ij} is the kronecker delta ($\delta_{ij} = 1$ for $i=j$, $\delta_{ij}=0$ otherwise)

Equations 13a, 14a, and 15a are boundary conditions that must always be satisfied. These equations may be satisfied in many different ways. The physical meaning of the solution is dependent on how the boundary conditions are satisfied. This is similar to the situation in a homogeneous beam problem where the choice of end condition determines whether a beam is simply supported, built in, cantelevered, etc.

Equations 13b, 14b, and 15b are solved by the Ritz technique. The particular physical problem considered in this paper is a uniformly loaded simply supported laminated beam. The approximating functions appropriate for this problem are:

$$\left.\begin{aligned} w_i(x) &= \sum_{j\ \mathrm{odd}}^{n} a_{ij} \ \mathrm{Sin}\,\frac{j\pi x}{\ell} && \text{(a)} \\ u_i(x) &= \sum_{j\ \mathrm{odd}}^{n} b_{ij} \ \mathrm{Cos}\,\frac{j\pi x}{\ell} && \text{(b)} \\ \gamma_i(x) &= \sum_{j\ \mathrm{odd}}^{n} c_{ij} \ \mathrm{Cos}\,\frac{j\pi x}{\ell} && \text{(c)} \end{aligned}\right\} \qquad (16)$$

These functions satisfy 13a, 14a, and 15a. However the constants a_{ij}, b_{ij} and c_{ij} are not known and need to be determined. This is accomplished by substituting the Equations 16a, 16b, and 16c into Equation 12 and solving for values of the a_{ij}, b_{ij}, and c_{ij} that will minimize π^*. After evaluating the integrals in Equation 12, the total potential energy of the laminated beam is expressed as a quadratic form in the a_{ij}, b_{ij}, and c_{ij}. Taking derivatives with respect to a_{ij}, b_{ij}, and c_{ij} yields a set of linear simultaneous algebraic equations in the a_{ij}, b_{ij}, and c_{ij}. After much algebraic manipulation the governing algebraic equations may be written as follows:

$$\frac{Ebh_i^3}{12}\left(\frac{j\pi}{\ell}\right)^4 a_{ij} - \frac{Ebh_i^3}{12}\left(\frac{j\pi}{\ell}\right)^3 c_{ij}$$

$$+ (1-\delta_{i1})\,\frac{E_e b}{t_{i-1}}\,(a_{ij}-a_{i-1\,j}) + (1-\delta_{in})\,\frac{E_e b}{t_i}\,(a_{ij}-a_{i+1\,j})$$

$$+ (1-\delta_{i1})\,\frac{G_e b}{t_{i-1}}\Bigg(L_{i-1}^2\left(\frac{j\pi}{\ell}\right)^2 a_{ij} - L_{i-1}^2\left(\frac{j\pi}{\ell}\right) c_{ij} - L_{i-1}\left(\frac{j\pi}{\ell}\right) b_{i-1\,j}$$

$$+ L_{i-1}\left(\frac{j\pi}{\ell}\right) b_{ij}$$

$$K_{i-1}\; L_{i-1}\left(\frac{j\pi}{\ell}\right)^2 a_{i-1\,j} - K_{i-1}L_{i-1}\left(\frac{j\pi}{\ell}\right) c_{ij}\Bigg)$$

$$+ (1-\delta_{in})\,\frac{G_e b}{t_i}\Bigg(K_i^2\left(\frac{j\pi}{\ell}\right)^2 a_{ij} - K_i^2\left(\frac{j\pi}{\ell}\right) c_{ij} - K_i\left(\frac{j\pi}{\ell}\right) b_{ij} + K_i\left(\frac{j\pi}{\ell}\right) b_{i+1\,j}$$

$$+ K_iL_i\left(\frac{j\pi}{\ell}\right)^2 a_{i+1\,j} - K_iL_i\left(\frac{j\pi}{\ell}\right) c_{ij}\Bigg) = q_i\,\frac{4}{j\pi} \quad (j\ \text{odd})$$

$$\text{for } i = 1, 2, \ldots, n \tag{17}$$

$$Ebh_i\left(\frac{j\pi}{\ell}\right)^2 b_{ij}$$

$$+ (1-\delta_{i1})\,\frac{G_e b}{t_{i-1}}\Bigg(b_{ij} - b_{i-1\,j} + K_{i-1}\left(\frac{j\pi}{\ell}\right) a_{i-1\,j} - K_{i-1}\,c_{i-1\,j} + L_{i-1}\left(\frac{j\pi}{\ell}\right) a_{ij}$$

$$- L_{i-1}\,c_{ij}\Bigg) + (1-\delta_{in})\,\frac{G_e b}{t_i}\Bigg(b_{ij} - b_{i+1\,j} - K_i\left(\frac{j\pi}{\ell}\right) a_{ij} + K_i\,c_{ij} - L_i\left(\frac{j\pi}{\ell}\right) a_{i+1\,j}$$

$$+ L_i\,c_{i+1\,j}\Bigg) = 0 \qquad \text{for } i = 1, 2, \ldots, n \tag{18}$$

$$\frac{Ebh_i^3}{12}\left(\frac{j\pi}{\ell}\right)^2 c_{ij} - \frac{Ebh_i^3}{12}\left(\frac{j\pi}{\ell}\right)^3 a_{ij} + Gbh_i\, c_{ij}$$

$$(1-\delta_{i1})\,\frac{G_e b}{t_{i-1}}\left(L_{i-1}^2\, c_{ij} - L_{i-1}^2 \left(\frac{j\pi}{\ell}\right) a_{ij} + L_{i-1}\, b_{i-1\,j} - L_{i-1}\, b_{ij}\right.$$

$$\left. - K_{i-1}\, L_{i-1}\left(\frac{j\pi}{\ell}\right) a_{i-1\,j} + K_{i-1}\, L_{i-1}\, C_{i-1\,j}\right)$$

$$(1-\delta_{in})\,\frac{G_e b}{t_i}\left(K_i^2\, c_{ij} - K_i^2\left(\frac{j\pi}{\ell}\right) a_{ij} + K_i\, b_{ij} - K_i\, b_{i+1\,j}\right.$$

$$\left. - K_i\, L_i\left(\frac{j\pi}{\ell}\right) a_{i+1\,j} + K_i\, L_i\, c_{i+1\,j}\right) = 0$$

$$\text{for } i = 1, 2, \ldots, n \qquad (19)$$

After close investigation of Equations 17, 18, and 19 it is apparent that the index j is decoupled from the j-1 and the j+1 terms. This means that the solution procedure for finding a_{ij}, b_{ij}, and c_{ij} ($i=1, \ldots, n$; $j = 1, 3, 5, \ldots N$) is a series of small problems (a 3n x 3n set of algebraic equations). In other words, the solution may be approximated to any degree of accuracy desired by solving the Equations 17, 18, and 19 repeatedly and making the value of N in Equations (16) as large as desired. Experience to date indicates that four or five terms (j = 1, 3, 5, 7) is adequate.

PARAMETRIC STUDIES

Numerical results obtained by solving equations 17, 18, and 19 give deflections and stresses that are in good agreement with expectations. When the shear modulus of the epoxy is set equal to zero the laminated beam acts as if the layers were bending independently. When the shear modulus is set equal to the classical $E_e/2(1+\nu)$ value the results agree with standard composite beam theory. In all of the following parametric studies the following items are kept constant.

E	(metal)	=	$16. \times 10^6$ psi
G	(metal)	=	$6. \times 10^6$ psi
E_e	(epoxy)	=	1.0×10^6 psi

The value of epoxy shear modulus is treated as a parameter in the following study. However, particular attention should be directed at the results for G_e = 10000. psi. Experimental work[3] at Princeton indicates that the PLT toroidal field coils have an epoxy shear modulus near this value.

The parametric studies are based on a beam with dimensions and end conditions shown in Figure 3.

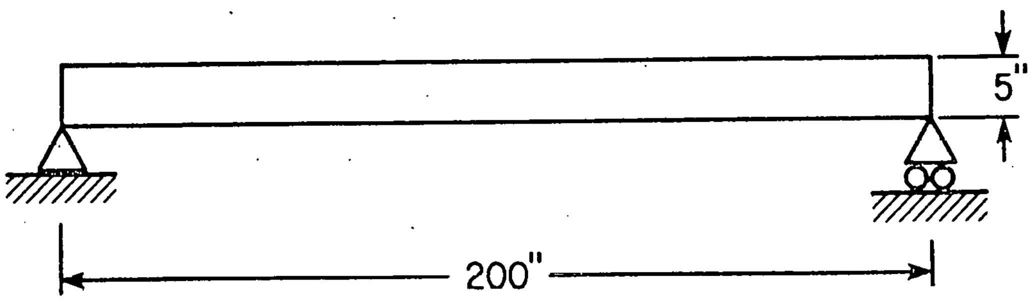

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Fig. 3.

The net loading intensity in all cases is 1.0 lb/in. The parameters that were varied are the epoxy shear modulus (G_e), the fraction of epoxy in the total cross section (R), and the number of layers in the beam. In all cases, the total height of the beam has remained a constant 5 inches. This means that as the number of layers in the beam changes, the height of each metal layer and each epoxy layer changes. (All metal layers in a calculation have the same thickness.) Table 1 contains a summary of the cross sections of all cases considered.

Figures 4, 5, and 6 illustrate the relationships among epoxy shear modulus, fraction of epoxy, and number of laminates. In all cases the figure labeled B presents the details of the lower part of the figure labeled A.

Figures 7, 8, and 9 illustrate dependence on shear modulus.

Figure 10 illustrates that maximum deflection of a laminated beam still varies linearly with ℓ^4.

Figures 11 and 12 illustrate strain development across a laminated beam as shear modulus varies. As G_e approaches larger values the strains approach classical results.

Table 2, 3, and 4 contain the numerical data presented in Figures 4 to 12. Those readers interested in maximum strains in epoxy and copper can find the results summarized in Tables 5, 6, and 7. These last tables present the maximum fiber strains in the outermost surface of copper and the maximum shear strain in the epoxy layer closest to the neutral axis.

The results presented so far have been for the beam illustrated in Figure 3. This beam was assumed to have a regular pattern of laminates. The equations developed in this paper can also model a straight beam made of laminates of different thicknesses. Figures 13 and 14 present the results for a straight laminated beam having the same cross section as the PLT toroidal field coil. This cross section was calculated to be 2.32 times more flexible in bending than a beam of solid copper having the same total build.

CONCLUSIONS

The equations presented here give a description of deflection and stress in straight laminated beams. The results obtained to date indicate that

1. The greater the fraction of epoxy in a cross section the more flexible that cross section becomes.

2. The lower the epoxy shear modulus the more flexible the cross section becomes.

3. The stresses in typical laminated beams are rather insensitive to the epoxy shear modulus. For values of the epoxy shear modulus greater than 10,000 psi the laminated beam is very similar to a solid beam of metal. However, once the shear modulus of the epoxy falls below some particular value the laminated beam becomes radically weaker. This result was not intuitively expected.

ACKNOWLEDGMENT

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APPENDIX 1

```
      IMPLICIT REAL*8(A-H,O-Z)
      REAL*8 ZZZ( 14400)
      REAL*8 A(120,120),ANS(120),AK(40),AL(40),H(40),T(40),Q(40)
      REAL*4  AIJ(40,10), BIJ(40,10), CIJ(40,10), TITLE(20)
      REAL*8 W(40),U(40),GAMMA(40),DGAMDX(40),ST1(40),ST2(40),GAMMAB(40)
      REAL*8 DUDX(40),DWDX(40),D2UDX2(40),D2WDX2(40)
      REAL*8  MOM(40),V(40) ,VB(40)     ,Z(40)
      EQUIVALENCE (A(1,1),ZZZ(1)),(ZZZ(1),W(1)),(ZZZ(101),U(1)),
     1 (ZZZ(201),GAMMA(1)),(ZZZ(301),DGAMDX(1)),(ZZZ(401),ST1(1)),
     2 (ZZZ(501),ST2(1)),(ZZZ(601),GAMMAB(1)),(ZZZ(701),DUDX(1)),
     3 (ZZZ(801),DWDX(1)),(ZZZ(901),D2WDX2(1)),(ZZZ(1001),D2UDX2(1)),
     4 (ZZZ(1101),MOM(1)),(ZZZ(1201),V(1)),(ZZZ(1301),VB(1))
      PIE = 3.141592654
C
    1 READ(5,100,END=1000)  TITLE
      READ(5,102)  EM,GM,EX,GX
      READ(5,102) ALEN,B,CENTZ
      READ(5,101)  N,M,IJK,KEY1,KEY2,IBUG
      N3 = 3*N
      IF ( KEY1 .NE. 0 )  GO TO 3
C     TEST IF  THE BEAM HAS A UNIFORM CROSS SECTION
C         IF KEY1 .EQ. 0  CROSS SECTION IS UNIFORM
C         IF KEY1 .NE. 0 CROSS SECTION VARIES
C     CODE FOR UNIFORM CROSS SECTION
      READ(5,102)  HA,TA
      DO 2  I = 1,N
      H(I)  = HA
    2 T(I)  = TA
      T(N)  = 0.
      GO TO 5
C     CODE FOR A VARIABLE CROSS SECTION
    3 DO 4  I = 1,N
      READ(5,102)  H(I) , T(I)
    4 CONTINUE
C        READ IN THE LOADING - IF KEY2=0 WE HAVE THE SAME LOAD ON ALL LAYERS
    5 IF ( KEY2 .NE. 0) GO TO 50
      READ(5,102)  QJ
      DO 48  I = 1,N
   48 Q(I)  = QJ
      QT = N*QJ
      GO TO 70
   50 READ(5,101)  NLOAD
      DO 6  I = 1,N
    6 Q(I)  = 0.
      QT = 0.
      DO 7  I = 1,NLOAD
      READ(5,103)  J,QJ
      QT = QT + QJ
    7 Q(J)  = QJ
C
C     DEFINE THE AK AND AL ARRAYS
   70 NL = N-1
      DO 8  I = 1,NL
      IP = I+1
      AK(I)  = H(I)/2. + H(I)*T(I)/( H(I)+H(IP))
    8 AL(I)  = H(IP)/2. + H(IP)*T(I)/( H(I)+H(IP))
```

```
      C
      C - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
      C                      SET UP ALL EQUATIONS
      C
            PRINT 110,   TITLE
            PRINT 111,  N,M,IJK
            PRINT 112, ALEN,B,CENTZ
            PRINT 113, EM,GM,EX,GX
            PRINT 114
            DO 88  I = 1,N
            PRINT 115,  I,Q(I)   ,H(I),T(I)
      88    CONTINUE
            PRINT 116,  QT
            DO 20  JJ = 1,M
      C
            DO 9  I = 1,120
            ANS(I) = 0.
            DO 9  J = 1,120
      9     A(I,J) = 0.
      C
            AJ = (JJ-1)*2 + 1
            PJ1 = AJ*PIE/ALEN
            PJ2 = PJ1*PJ1
            PJ3 = PJ2*PJ1
            PJ4 = PJ3*PJ1
      C
            DO 15  I = 1,N
            IL = I-1
            IP = I+1
            I1 = (I-1)*3 + 1
            I2 = I1 + 1
            I3 = I1 + 2
            IF ( I .EQ. N ) GO TO 10
      C     SOME TERMS THAT APPEAR FREQUENTLY IN THE EQUATIONS ARE
            CA1 = EM*B*H(I)*H(I)*H(I)/12.
            CA2 = EM*B*H(I)
            EA1 = EX*B/T(I)
            GA1 = GX*B/T(I)
            IF ( I .GT. 1 )  GO TO 10
      C
      C       SET UP THE EQUATIONS FOR THE FIRST LAYER
      C
            A(1,1) = CA1*PJ4 + EA1 + GA1*AK(1)*AK(1)*PJ2
            A(1,2) = -GA1*AK(1)*PJ1
            A(1,3) = -CA1*PJ3 - GA1*AK(1)*AK(1)*PJ1
      C
            A(1,4) = -EA1 + GA1*AK(1)*AL(1)*PJ2
            A(1,5) =  GA1*AK(1)*PJ1
            A(1,6) = -GA1*AK(1)*AL(1)*PJ1
            ANS(1) = 4.*Q(1)/(AJ*PIE)
      C
      C
      C
            A(2,1) = -GA1*AK(1)*PJ1
            A(2,2) = CA2*PJ2 + GA1
            A(2,3) = GA1*AK(1)
```

```
      C
             A(2,4) = -GA1*AL(1)*PJ1
             A(2,5) = -GA1
             A(2,6) = GA1*AL(1)
             ANS(2) = 0.
      C
      C
      C
             A(3,1) = -CA1*PJ3 - GA1*AK(1)*AK(1)*PJ1
             A(3,2) = GA1*AK(1)
             A(3,3) = CA1*PJ2 + GM*B*H(1) + GA1*AK(1)*AK(1)
      C
             A(3,4) = -GA1*AK(1)*AL(1)*PJ1
             A(3,5) = -GA1*AK(1)
             A(3,6) = GA1*AK(1)*AL(1)
             ANS(3) = 0.
             GO TO 15
      C        SOME MORE USEFULL CONSTANTS ARE
      10     EA2 = EX*B/T(IL)
             GA2 = GX*B/T(IL)
             IF ( I .EQ. N )  GO TO 14
      C
      C      SET UP THE EQUATIONS FOR THE MIDDLE LAYERS
      C
             A(I1,I1-3) = -EA2 + GA2*AK(IL)*AL(IL)*PJ2
             A(I1,I1-2) = -GA2*AL(IL)*PJ1
             A(I1,I1-1) =-GA2*AK(IL)*AL(IL)*PJ1
      C
             A(I1,I1) = CA1*PJ4 + EA2 + EA1 + GA2*AL(IL)*AL(IL)*PJ2 +
            1  GA1*AK(I)*AK(I)*PJ2
             A(I1,I1+1) = GA2*AL(IL)*PJ1 - GA1*AK(I)*PJ1
             A(I1,I1+2) = -CA1*PJ3 - GA2*AL(IL)*AL(IL)*PJ1
             A(I1,I1+2) = -CA1*PJ3 - GA2*AL(IL)*AL(IL)*PJ1 -GA1*AK(I)*AK(I)*PJ1
      C
             A(I1,I1+3) =-EA1 + GA1*AK(I)*AL(I)*PJ2
             A(I1,I1+4) = GA1*AK(I)*PJ1
             A(I1,I1+5) = -GA1*AK(I)*AL(I)*PJ1
             ANS(I1) = 4.*Q(I)/(AJ*PIE)
      C
      C
      C
             A(I2,I1-3) = GA2*AK(IL)*PJ1
             A(I2,I1-2) = -GA2
             A(I2,I1-1) = -GA2*AK(IL)
      C
             A(I2,I1) = GA2*AL(IL)*PJ1 - GA1*AK(I)*PJ1
             A(I2,I1+1) = CA2*PJ2 + GA2 + GA1
             A(I2,I1+2) = -GA2*AL(IL) + GA1*AK(I)
      C
             A(I2,I1+3) = -GA2*AL(I) * PJ1
             A(I2,I1+4) = -GA2
             A(I2,I1+5) = GA2*AL(I)
             ANS(I2) = 0.
      C
      C
      C
```

```
0121            A(I3,I1-3) = -GA2*AK(IL)*AL(IL)*PJ1
0122            A(I3,I1-2) = GA2*AL(IL)
0123            A(I3,I1-1) = GA2*AK(IL)*AL(IL)
          C
0124            A(I3,I1) = -CA1*PJ3 -GA2*AL(IL)*AL(IL)*PJ1
               1  -GA1*AK(I)*AK(I)*PJ1
0125            A(I3,I1+1) = -GA2*AL(IL) + GA1*AK(I)
0126            A(I3,I1+2) = CA1*PJ2 + GM*B*H(I) + GA2*AL(IL)*AL(IL)
               1 + GA1*AK(I)*AK(I)
          C
0127            A(I3,I1+3) = -GA1*AK(I)*AL(I)*PJ1
0128            A(I3,I1+4) = -GA1*AK(I)
0129            A(I3,I1+5) = GA1*AK(I)*AL(I)
0130            ANS(I3) = 0.
0131            GO TO 15
          C
          C     EQUATIONS FOR THE LAST LAYER
          C
0132      14    A(I1,I1-3) = -EA2 + GA2*AK(IL)*AL(IL)*PJ2
0133            A(I1,I1-2) = -GA2*AL(IL)*PJ1
0134            A(I1,I1-1) = -GA2*AK(IL)*AL(IL)*PJ1
          C
0135            A(I1,I1) = CA1*PJ4 + EA2 + GA2*AL(IL)*AL(IL)*PJ2
0136            A(I1,I1+1) = GA2*AL(IL)*PJ1
0137            A(I1,I1+2) = -CA1*PJ3 - GA2* AL(IL)*AL(IL)*PJ1
0138            ANS(I1) = 4.*Q(I)/(AJ*PIE)
          C
          C
          C
0139            A(I2,I1-3) = GA2*AK(IL)*PJ1
0140            A(I2,I1-2) = -GA2
0141            A(I2,I1-1) = -GA2*AK(IL)
          C
0142            A(I2,I1) = GA2*AL(IL)*PJ1
0143            A(I2,I1+1) = CA2*PJ2 + GA1
0144            A(I2,I1+2) = -GA2*AL(IL)
0145            ANS(I2) = 0.
          C
          C
          C
0146            A(I3,I1-3) = -GA2*AK(IL)*AL(IL)*PJ1
0147            A(I3,I1-2) = GA2*AL(IL)
0148            A(I3,I1-1) = GA2*AK(IL)*AL(IL)
          C
0149            A(I3,I1) =-CA1*PJ3 - GA2*AL(IL)*AL(IL)*PJ1
0150            A(I3,I1+1) = -GA2 *AL(IL)
0151            A(I3,I1+2) = CA1*PJ2 + GM*B*H(I) + GA2*AL(IL)*AL(IL)
0152            ANS(I3) = 0.
0153      15    CONTINUE
          C
          C   MATRIX COMPLETE  -  NOW PRINT MATRIX IF DEBUG REQUEST HAS BEEN MADE
          C
0154            IF ( IBUG .EQ. 0 )  GO TO 160
0155            DO 16  I =1,N3
0156            PRINT 119,  I,I,ANS(I)
0157            PRINT 120, (A(I,J),J=1,N3)
```

```
0158      16    CONTINUE
0159            PRINT 121
          C
0160      160   CALL SOLVE ( A,ANS,N3,KEY )
          C
0161            IF ( KEY .NE. 0 )  GO TO 1000
0162            DO 17  I = 1,N
0163            I1 = (I-1)*3 + 1
0164            I2 = I1 + 1
0165            I3 = I1 + 2
0166            AIJ(I,JJ) = ANS(I1)
0167            BIJ(I,JJ)= ANS(I2)
0168      17    CIJ(I,JJ)= ANS(I3)
0169      20    CONTINUE
          C
          C
          C
0170            DO 21  I=1,N
0171            PRINT 130,  I
0172            PRINT 131,(AIJ(I,J),J=1,M)
0173      21    CONTINUE
0174            DO 22  I = 1,N
0175            PRINT 132,  I
0176            PRINT 131, ( BIJ(I,J), J=1,M)
0177      22    CONTINUE
0178            DO 24  I = 1, N
0179            PRINT 133,I
0180            PRINT 131, (CIJ(I,J),J=1,M)
0181      24    CONTINUE
          C
0182            Z(1) = -CENTZ + H(1)/2.
0183            DO 25  J = 2,N
0184            Z(J) = Z(J-1) + T(I) + ( H(J)+H(J-1))/2.
          C
0185            IF ( DABS(Z(J)) .LT. 1.0E-6 )  Z(J) = 0.
0186      25    CONTINUE
0187            DUM = IJK
0188            XDEL = ALEN/DUM
0189            XDD = -XDEL
0190            IJK = IJK + 1
          C
0191            DO 40  II = 1, IJK
0192            DO 26  I = 1,40
0193            W(I) = 0.
0194            U(I) = 0.
0195            GAMMA(I) = 0.
0196            GAMMAB(I) = 0.
0197            DGAMDX(I) = 0.
0198            DUDX(I) = 0.
0199            DWDX(I) = 0.
0200            D2UDX2(I) = 0.
0201            D2WDX2(I) = 0.
0202            MOM(I) = 0.
0203            VB(I) = 0.
0204      26    V(I) = 0.
0205            TOTMOM = 0.
```

```
            TOTVM = 0.
            TOTVB = 0.
            XDD = XDD + XDEL
            DO 28  I = 1,N
            DO 27  J = 1,M
            AAJ = (J-1)*2 + 1
            ARG = AAJ*PIE*XDD/ALEN
            SSS =DSIN(ARG)
            CCC =DCOS(ARG)
            ARG = AAJ*PIE/ALEN
            W(I) = W(I) + AIJ(I,J) * SSS
            U(I) = U(I) + BIJ(I,J)*CCC
            GAMMA(I) = GAMMA(I) + CIJ(I,J)*CCC
            DWDX(I) = DWDX(I) + ARG*AIJ(I,J)*CCC
            DUDX(I) = DUDX(I) - ARG*BIJ(I,J) * SSS
            DGAMDX(I) = DGAMDX(I) - ARG*CIJ(I,J) * SSS
            D2WDX2(I) = D2WDX2(I) - ARG*ARG*AIJ(I,J) * SSS
   27       D2UDX2(I) = D2UDX2(I) - ARG*ARG*BIJ(I,J) * CCC
            ST1(I) = DUDX(I) + H(I)*(D2WDX2(I)-DGAMDX(I))/2.
            ST2(I) = DUDX(I) - H(I)*(D2WDX2(I)-DGAMDX(I))/2.
            AI = I
            Z1 = Z(I) - H(I)/2.
            Z2 = Z(I) + H(I)/2.
            ZAVG = (Z1+Z2)/2.
            MOM(I) =EM*B*(DUDX(I)*ZAVG*H(I) -(D2WDX2(I)-DGAMDX(I))*H(I)*
           1  H(I)*H(I) /12.)
            TOTMOM = TOTMOM + MOM(I)
            V(I) = GAMMA(I) * GM * B * H(I)
            TOTVM = TOTVM + V(I)
            IF ( I .EQ. 1 )  GO TO 28
            IL = I-1
            GAMMAB(I) = (U(IL)-U(I) -AK(IL)*(DWDX(IL)-GAMMA(IL)) -AL(IL)*
           1   (DWDX(I)-GAMMA(I)))/T(IL)
            VB(I) = -GAMMAB(I)*GX*B *T(IL)
            TOTVB = TOTVB + VB(I)
   28       CONTINUE
            RV = QT  *ALEN*(1.-2.*XDD/ALEN)/2.
            RMOM =  QT *   ALEN*(XDD-XDD*XDD/ALEN)/2.
            VTOT = TOTVM + TOTVB
            PRINT 140,  XDD
            DO 30  I = 1, N
            PRINT 142,I,W(I),U(I),GAMMA(I),DWDX(I),DUDX(I),DGAMDX(I),D2WDX2(I)
           1  , D2UDX2(I),ST1(I),ST2(I),GAMMAB(I)
   30       CONTINUE
            PRINT 141
            DO 31  I = 1 , N
            PRINT 143,  MOM(I),V(I),VB(I)
   31       CONTINUE
            PRINT 144,  RMOM,TOTMOM
            PRINT 145,  RV,      VTOT,  TOTVM,   TOTVB
   40       CONTINUE
            GO TO 1
 1000       CALL EXIT
  100       FORMAT(20A4)
  101       FORMAT(10I5)
  102       FORMAT(4F20.0)
```

```
 0259          103    FORMAT(I2,F25.0)
 0260          110    FORMAT(1H1,//20X,20A4)
 0261          111    FORMAT(//10X,'N =   ',I5,10X,'M =   ',I5,10X,'IJK =',I5)
 0262          112     FORMAT(/10X,'LENGTH OF BEAM(INCHES) = ',F15.2,
                     1 /10X,'WIDTH OF BEAM (INCHES) = ',F15.2,
                     2  /10X,'POSITION OF NEUTRAL AXIS = ', F15.2)
 0263          113    FORMAT( /10X,'EM =',D16.8,/10X,'GM =',D16.8,/10X,'EX =',D16.8,
                     1/10X,'GX =',D16.8)
 0264          114    FORMAT(//10X,'CROSS SECTION DESCRIPTION',/10X,'LAYER NO.', 2X,'LOAD
                     1DING INTENSITY (LBS/IN)',5X,'H(I)',20X,'T(I)' )
 0265          115    FORMAT(10X,I6,3D20.8 )
 0266          116    FORMAT(5X,'NET LOAD =',D20.8)
 0267          119    FORMAT(/5X,'ROW =',I5,  '   ANS(',I3,') = ',D20.8 )
 0268          120    FORMAT( 1X,ED20.8 )
 0269          121    FORMAT(1X,' - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
                     1- - - - - - - - - - - - - - - - - - - -')
 0270          130    FORMAT(/10X,'AIJ      I =',I5 )
 0271          131    FORMAT( 10X,5D20.8 )
 0272          132    FORMAT(/10X,'BIJ      I  = ',I5 )
 0273          133    FORMAT(/10X,'CIJ      I = ',I5 )
 0274          140    FORMAT(/10X,'X = ',F16.3,/5X,'    W          U        GAMMA        DWDX
                     1       DUDX       DGAMDX  D2WDX2       D2UDX2       ST1       ST2
                     2 GAMMAB')
 0275          141    FORMAT(12X,'MOM-M',10X,'V - M',13X,'V - B')
 0276          142    FORMAT(1X,I2,11F10.6 )
 0277          143    FORMAT(10X,F10.2,5X,F10.2,5X,F10.2 )
 0278          144    FORMAT(1X,'MOMENT SHOULD =',F10.2,5X,'MOMENT =',F10.2 )
 0279          145    FORMAT(1X,'SHEAR SHOULD = ',F10.2,5X,'SHEAR = ', F10.2,5X,
                     1' = (',  F10.2,'  + ',      F10.2, '   )'/)
 0280                 END
```

```
               SUBROUTINE SOLVE(A,ANS,N,KEY)
               REAL*8  A(120,120),ANS(1),C
               NM1 = N - 1
               KEY = 0
               DO 10  IR = 1, NM1
               C= A(IR,IR)
               IF ( C .EQ. 0.0 )  GO TO 200
               DUM = DABS(C)
               IF ( DUM .LT. 1.0E-10 )  PRINT 400,  IR
      C        PRINT 300,  IR,C
      C        PRINT 301,(A(IR,IC),IC=1,N)
               IRP1 = IR + 1
      C        PRINT 301, ( A(IRP1,IC),IC = 1,N)
               IRP2 = IR + 2
      C        PRINT 301, ( A(IRP2,IC),IC = 1,N)
      C        PRINT 301, ANS(IR),ANS(IRP1),ANS(IRP2)
               DO 9  I = IRP1 , N
               IF ( A(I,IR) .EQ. 0. )  GO TO 9
               C = A(I,IR)/A(IR,IR)
               A(I,IR) = 0.
               DO 8  J = IRP1, N
      8        A(I,J) = A(I,J) - C*A(IR,J)
               ANS(I) = ANS(I) - C*ANS(IR)
      9        CONTINUE
      C        PRINT 301, (A(IR,IC),IC = 1, N )
      C        PRINT 301, ( A(IRP1,IC),IC=1,N)
      C        PRINT 301,(A(IRP2,IC),IC=1,N)
      C        PRINT 301, ANS(IR),ANS(IRP1),ANS(IRP2)
      10       CONTINUE
      C        DO 15   I = 1,N
      C        PRINT 500,  I,ANS(I)
      C        PRINT 301,(A(I,J),J=1,N)
      C15      CONTINUE
      C        PRINT 301,  (ANS(IC),IC = 1,N)
               ANS(N) = ANS(N) / A(N,N)
               DO 20  I = 2,N
               IR = N+1 - I
               C = ANS(IR)
               IRP1 = IR + 1
               DO 18  J  = IRP1,N
      18       C = C - A(IR,J)*ANS(J)
      20       ANS(IR)    = C/ A(IR,IR)
      C        PRINT 301,  ( ANS(IC),IC=1,N)
               GO TO 9000
      200      KEY = IR
               PRINT 401,  KEY
               PRINT 301,(ANS(IC),IC=1,N)
      9000     RETURN
      300      FORMAT(//10X,I10,10X,D20.8 )
      301      FORMAT (1X,5D20.8 )
      400      FORMAT(//10X,'WARNING - DIAGONAL TERM ON ROW',I5,'   IS SMALLER THAN
              1AN 1.0E-10 ')
      401      FORMAT(//10X,'***** SINGULAR MATRIX - DIAGONAL TERM IS ZERO',10X,
              1    I5 )
      500      FORMAT(/10X,I5,10X,D20.8 )
               END
```

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TABLE 1

$nh + (n-1)t = H \qquad R = \frac{(n-1)t}{H}$

H = 5 inch

	R = 0.05		R = 0.10		R = 0.15	
n	h (ih)	t (in)	h (in)	t (in)	H (in)	t (in)
2	2.375	0.25	2.25	0.5	2.125	0.75
3	1.583333	0.125	1.5	0.25	1.416666	0.375
4	1.1875	0.083333	1.125	0.166666	1.0625	0.25
5	0.95	0.0625	0.9	0.125	0.85	0.1875
6	0.791666	0.05	0.75	0.10	0.7083333	0.15
8	0.59375	0.035714	0.5625	0.07142857	0.53125	0.107142857
12	0.3958333	0.0227272	0.375	0.04545454	0.35416666	0.068181818
20	0.2375	0.01315789	0.225	0.0263157	0.2125	0.03947368
30	0.158333	0.008620689	0.15	0.017241379	0.1465517	0.025862069

TABLE 2.

R = 0.05

Max Deflection vs. G_e and n (deflection in inches)

n→ G↓	2	3	4	5	6	8	12	20	30
375000.	0.125488	0.127656	0.128725	0.129366	0.129794	0.130330	0.130867	0.131298	0.131515
100000.	0.126306	0.128295	0.129299	0.129906	0.130314	0.130827	0.131343	0.131759	0.131968
10000.	0.136154	0.136096	0.136319	0.136523	0.136686	0.136918	0.137179	0.137408	0.137528
1000.	0.216330	0.209402	0.204369	0.201387	0.199484	0.197234	0.195144	0.193593	0.192858
100.	0.452480	0.636164	0.709579	0.736914	0.746631	0.749244	0.742575	0.732034	0.725423
50.	0.507046	0.839725	1.047946	1.153260	1.226186	1.280850	1.303911	1.300135	1.291920
10.	0.565747	1.173679	1.850724	2.500332	3.070120	3.927623	4.822034	5.359399	5.504592

TABLE 3. R = 0.10

Max Deflection vs. G_e and n (deflection in inches)

$G \downarrow$ \ $n \rightarrow$	2	3	4	5	6	8	12	20	30
375000.	0.125897	0.130462	0.132686	0.134019	0.134911	0.136029	0.137154	0.138060	0.138515
100000.	0.127509	0.131714	0.133812	0.135083	0.135936	0.137012	0.138099	0.138976	0.139418
10000.	0.146675	0.146943	0.147571	0.148090	0.148492	0.149053	0.149675	0.150214	0.150497
1000.	0.285919	0.284303	0.278109	0.273811	0.270914	0.267378	0.264017	0.261489	0.260284
100.	0.574668	0.911902	1.104365	1.203787	1.255214	1.297155	1.312042	1.306132	1.298246
50.	0.624237	1.137125	1.548667	1.832610	2.018578	2.219217	2.352114	2.394436	2.394517
10.	0.672494	1.437954	2.366305	3.353189	4.316438	5.992962	8.169095	9.856207	10.437098

TABLE 4. R = 0.15

Max. deflection vs. G_e and n (deflection in inches)

$G_\downarrow$ \ $n\rightarrow$	2	3	4	5	6	8	12	20	30
375000.	0.126432	0.133669	0.137150	0.139238	0.140637	0.142395	0.144168	0.145599	0.133568
100000.	0.128849	0.135507	0.138808	0.140808	0.142153	0.143852	0.145573	0.146967	0.134843
10000.	0.156805	0.157803	0.159033	0.159987	0.160709	0.161704	0.162796	0.163737	0.150475
1000.	0.349154	0.354363	0.348477	0.343720	0.340369	0.336195	0.332189	0.329173	0.304777
100.	0.696153	1.155737	1.453212	1.624639	1.722160	1.812594	1.859371	1.865118	1.748101
50.	0.749723	1.410151	1.988111	2.421609	2.726462	3.082598	3.348554	3.456865	3.269109
10.	0.800331	1.726827	2.880548	4.149118	5.434599	7.800255	11.167997	14.096046	14.299311

TABLE 5.

R = 0.05

Maximum tensile and shear strains
Applied moment = 5000. in lbs.
Applied shear = 100. lbs.

Key

Tensile (copper)
Shear (epoxy)

G \ n	2	3	4	5	6	8	12	20	30
375000.	0.000075 0.000076	0.000076 0.000067	0.000077 0.000075	0.000077 0.000072	0.000078 0.000075	0.000078 0.000076	0.000078 0.000076	0.000079 0.000076	0.000079 0.000076
100,000.0	0.000075 0.000282	0.000076 0.000250	0.000077 0.000282	0.000077 0.000271	0.000078 0.000283	0.000078 0.000283	0.000078 0.000283	0.000070 0.000284	0.000079 0.000284
10000.	0.000077 0.002668	0.000077 0.002483	0.000078 0.002777	0.000078 0.002683	0.00-078 0.002799	0.000078 0.002898	0.000079 0.002817	0.000079 0.002823	0.000070 0.002826
1000.	0.000092 0.020385	0.000087 0.021719	0.000085 0.025298	0.000084 0.025089	0.000084 0.026339	0.000084 0.026748	0.000084 0.027071	0.000084 0.027262	0.000084 0.027334
100.	0.000139 0.070400	0.000148 0.125477	0.000143 0.167338	0.000138 0.185013	0.000133 0.200591	0.000128 0.215236	0.000125 0.228324	0.000123 0.237119	0.000122 0.240554
50.	0.000150 0.081890	0.000179 0.174158	0.000183 0.233592	0.000179 0.301711	0.000174 0.337269	0.000167 0.376825	0.000159 0.413188	0.000155 0.438898	0.000154 0.449638
10.	0.000162 0.094242	0.000228 0.253808	0.000276 0.452708	0.000307 0.651362	0.000325 0.835059	0.000337 1.128399	0.000329 1.476938	0.000310 1.761533	0.000302 1.895361

TABLE 6.

R = 0.10

Maximum tensile and shear strains

Applied moment = 5000. in #

Applied shear = 100. #

key

Tensile (copper)
Shear (epoxy)

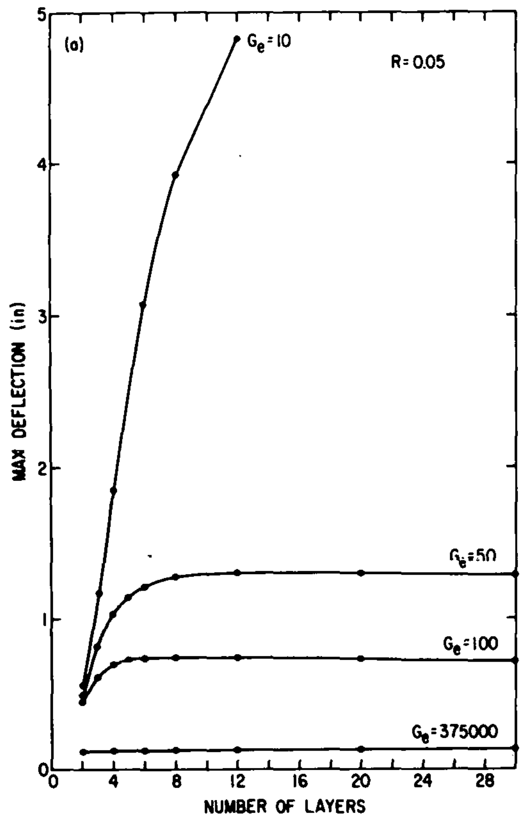
n	2	3	4	5	6	8	12	20	30
G 375000.	0.000075	0.000078	0.000079	0.000080	0.000081	0.000081	0.000082	9.000082	0.000083
	0.000075	0.000066	0.000075	0.000072	0.000075	0.000075	0.000075	0.000076	0.000076
100000.	0.000075	0.000078	0.000079	0.000080	0.000081	0.000081	0.000082	0.000083	0.000083
	0.000279	0.000247	0.000210	0.000269	0.000280	0.000281	01000282	0.000283	0.000284
10000.	0.000079	0.000080	0.000081	0.000081	0.000082	0.000082	0.000083	0.000083	0.000084
	0.002577	0.002398	0.002723	0.002642	0.002760	0.002776	0.002791	0.002803	0.002808
1000.	0.000105	0.000097	0.000094	0.000093	0.000093	0.000092	0.000092	0.000092	0.000093
	0.017837	0.020241	0.023819	0.023948	0.025219	0.025807	0.026297	0.026602	0.026721
100.	0.000163	0.000186	0.000187	0.000181	0.000176	0.000168	0.000161	0.000159	0.000158
	0.048541	0.096736	0.136526	0.158287	0.174895	0.192919	0.209522	0.221292	0.226198
50.	0.000173	0.000219	0.000236	0.000239	0.000236	0.000227	0.000215	0.000208	0.000206
	0.053793	0.123889	0.192793	0.242618	9.280679	0.327962	0.313377	0.406742	0.421496
10.	0.000182	0.000262	0.000328	0.000378	0.000414	0.000452	0.000463	0.000440	0.000424
	0.058903	0.160104	0.293950	0.441490	0.590683	0.861168	1.241119	1.595153	1.770150

TABLE 7. key R = 0.15

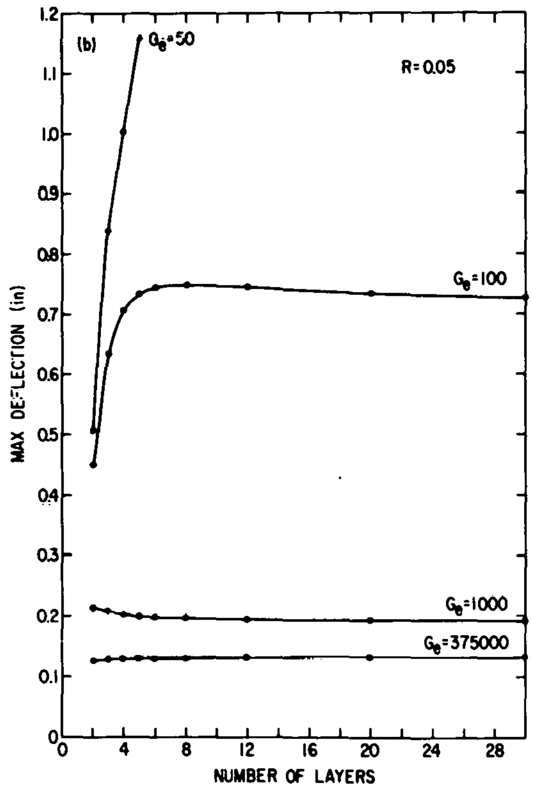
Maximum tensile and shear strains
Applied moment = 5000. in lbs.
Applied shear = 100. lbs

Tensile	(copper)
Shear	(epoxy)

h	2	3	4	5	6	8	12	20	30
G 375000.	0.000075 0.000074	0.000080 0.000065	0.000082 0.000074	0.000083 0.000071	0.000084 0.000074	0.000085 0.000075	0.000086 0.000075	0.000087 0.000075	0.000082 0.000073
100000.	0.000076 0.000275	0.000080 0.000245	0.000082 0.000276	0.000083 0.000266	0.000084 0.000278	0.000085 0.000280	0.000086 0.000281	0.000087 0.000282	0.000082 0.000275
10000.	0.000080 0.002517	0.000082 0.002357	0.000084 0.002680	0.000085 0.002608	0.000086 0.002727	0.000086 0.002749	0.000087 0.002770	0.000088 0.002786	0.000083 0.002713
1000.	0.000116 0.016652	0.000107 0.019422	0.000103 0.022914	0.000101 0.023218	0.000101 0.022547	0.000101 0.025159	0.000101 0.025747	0.000101 0.026128	0.000096 0.025494
100.	0.000184 0.041488	0.000217 0.085109	0.000221 0.122584	0.000216 0.145359	0.000210 0.162240	0.000200 0.181603	0.000192 0.199737	0.000189 0.212907	0.000180 0.211611
50.	0.000195 0.045310	0.000252 0.105772	0.000279 0.167908	0.000287 0.216627	0.000285 0.254807	0.000276 0.304734	0.000260 0.354211	0.000251 0.391196	0.000238 0.395030
10.	0.000205 0.048924	0.000296 0.131462	0.000273 0.242337	0.000437 0.368427	0.000484 0.500079	0.000541 0.750871	0.000569 1.131278	0.000544 1.513544	0.000499 1.650736

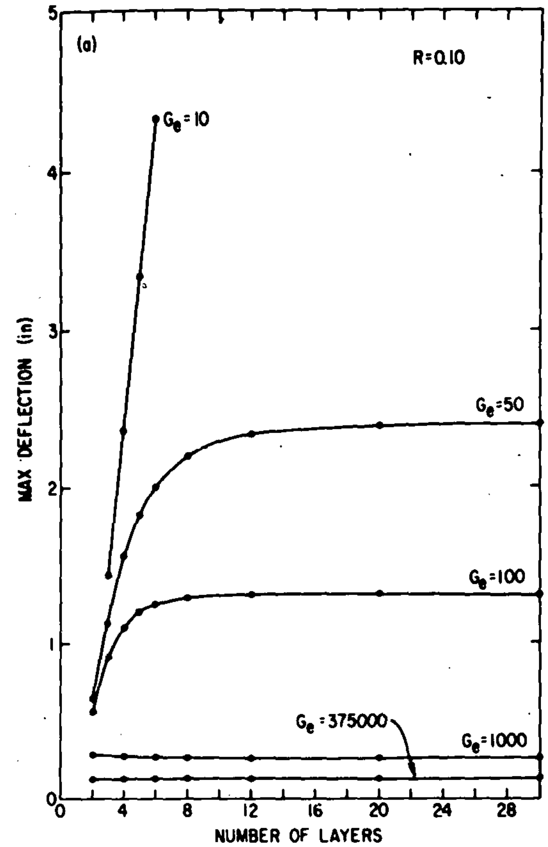


(a)

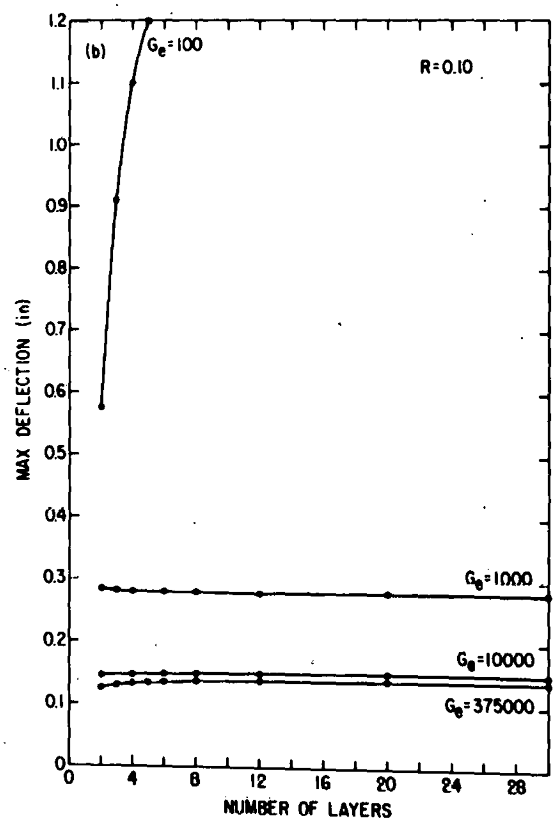


(b)

764770
Fig. 4(a) and (b).



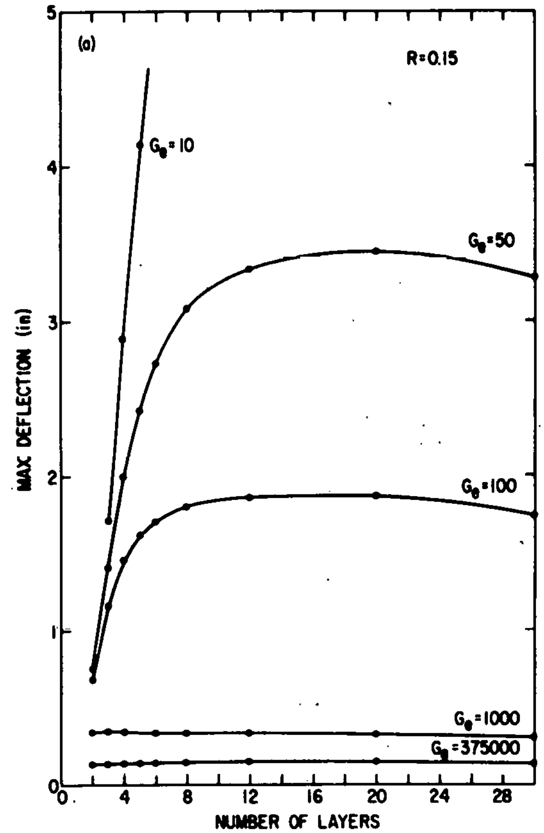
(a)



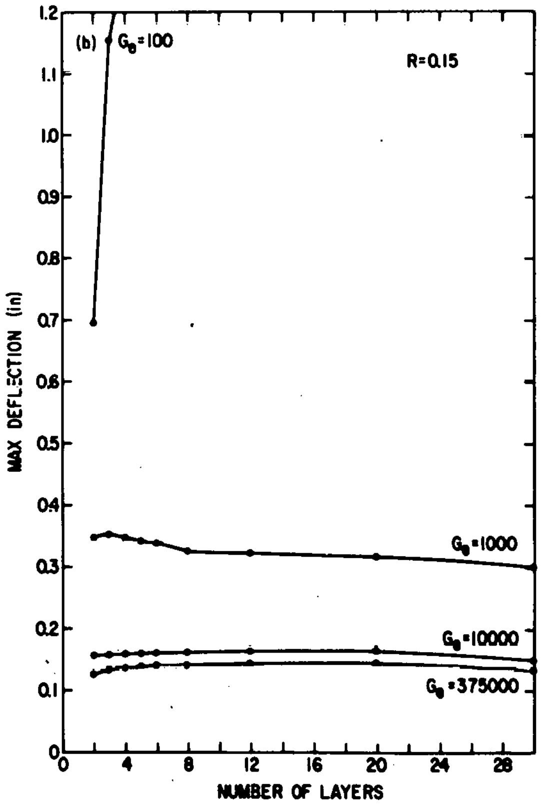
(b)

764789

Fig. 5(a) and (b).



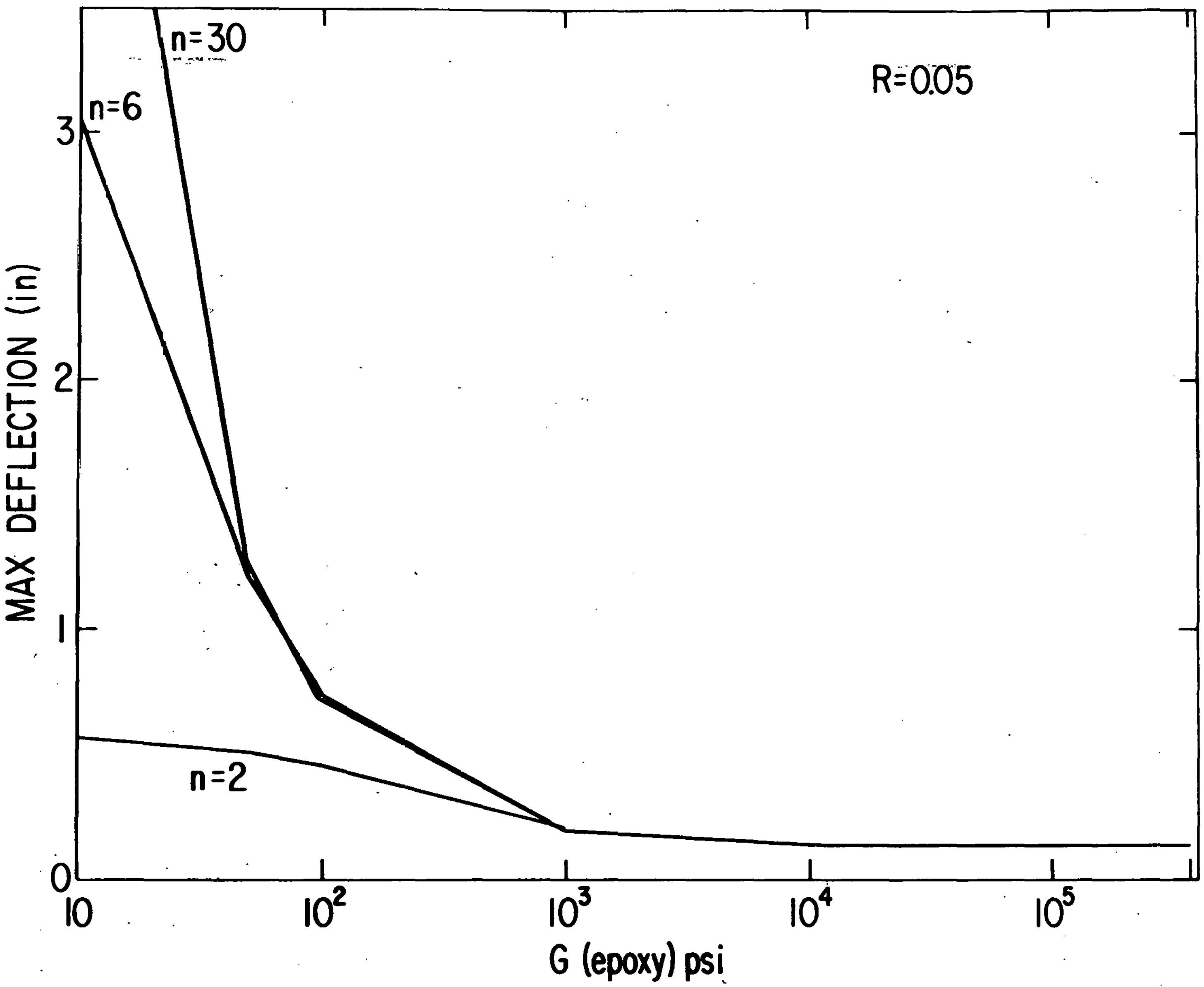
(a)



(b)

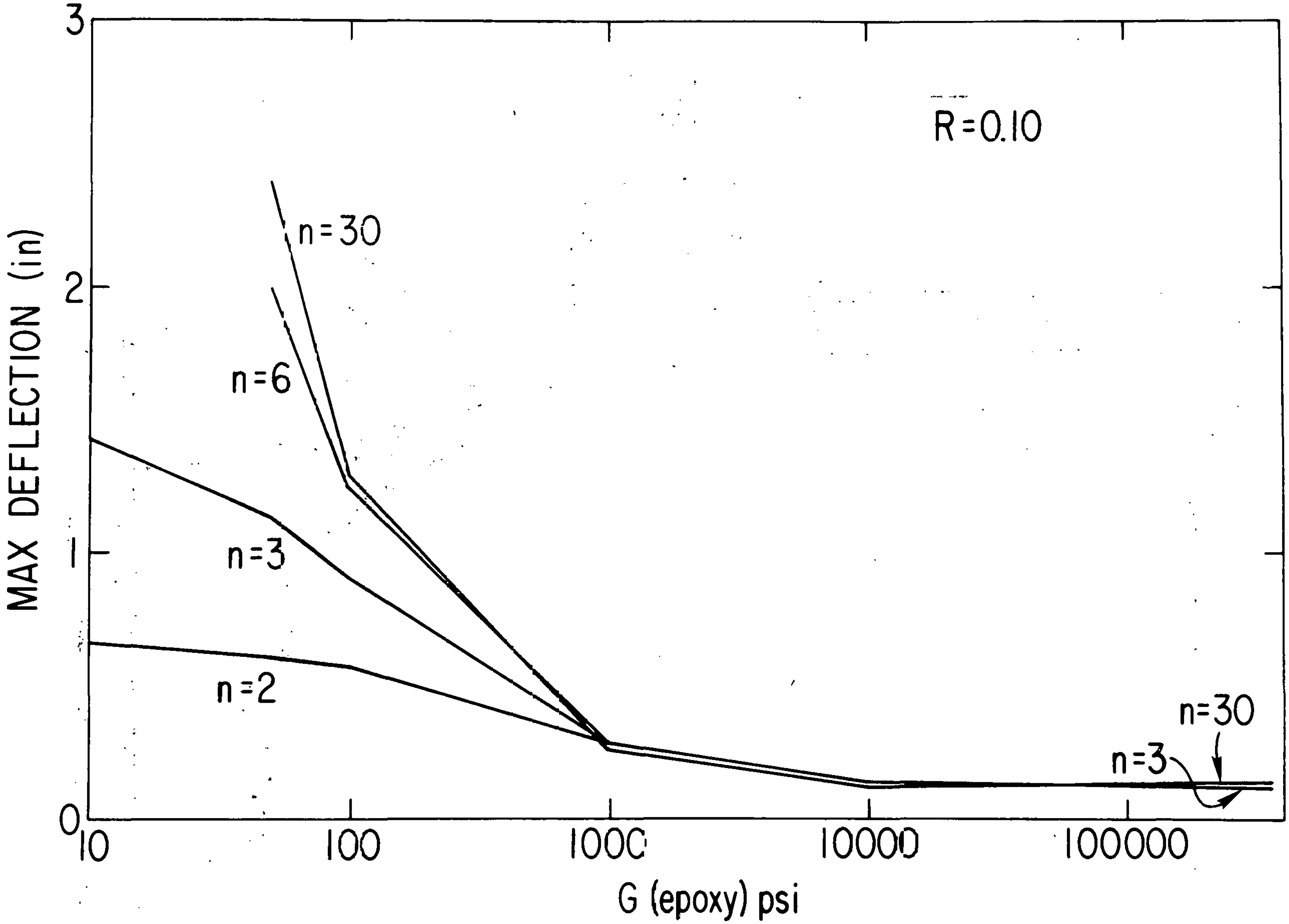
764768

Fig. 6(a) and (b).

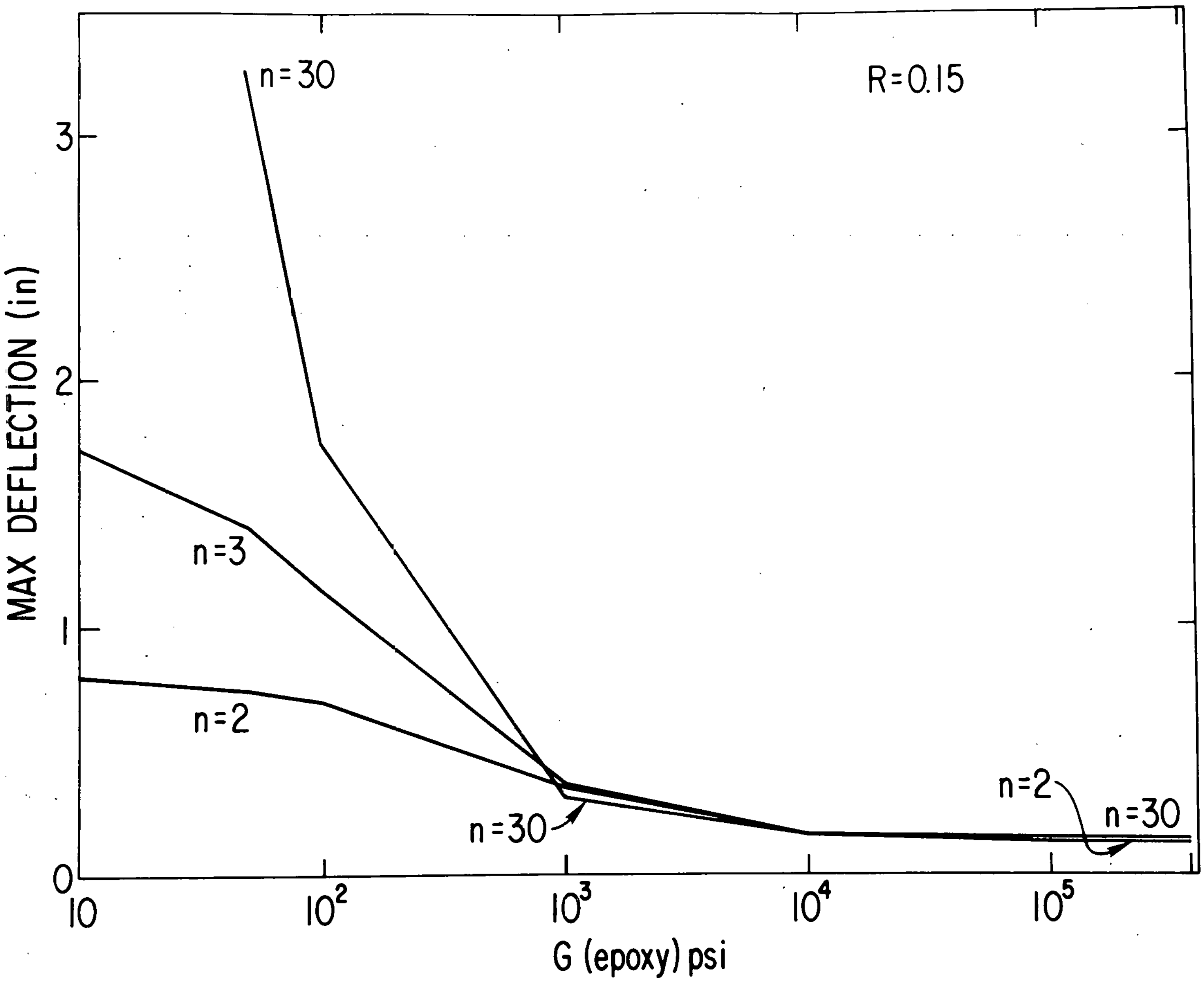


n=30
R=0.05
n=6
n=2
MAX DEFLECTION (in)
3
2
1
0
10
10^2
10^3
10^4
10^5
G (epoxy) psi

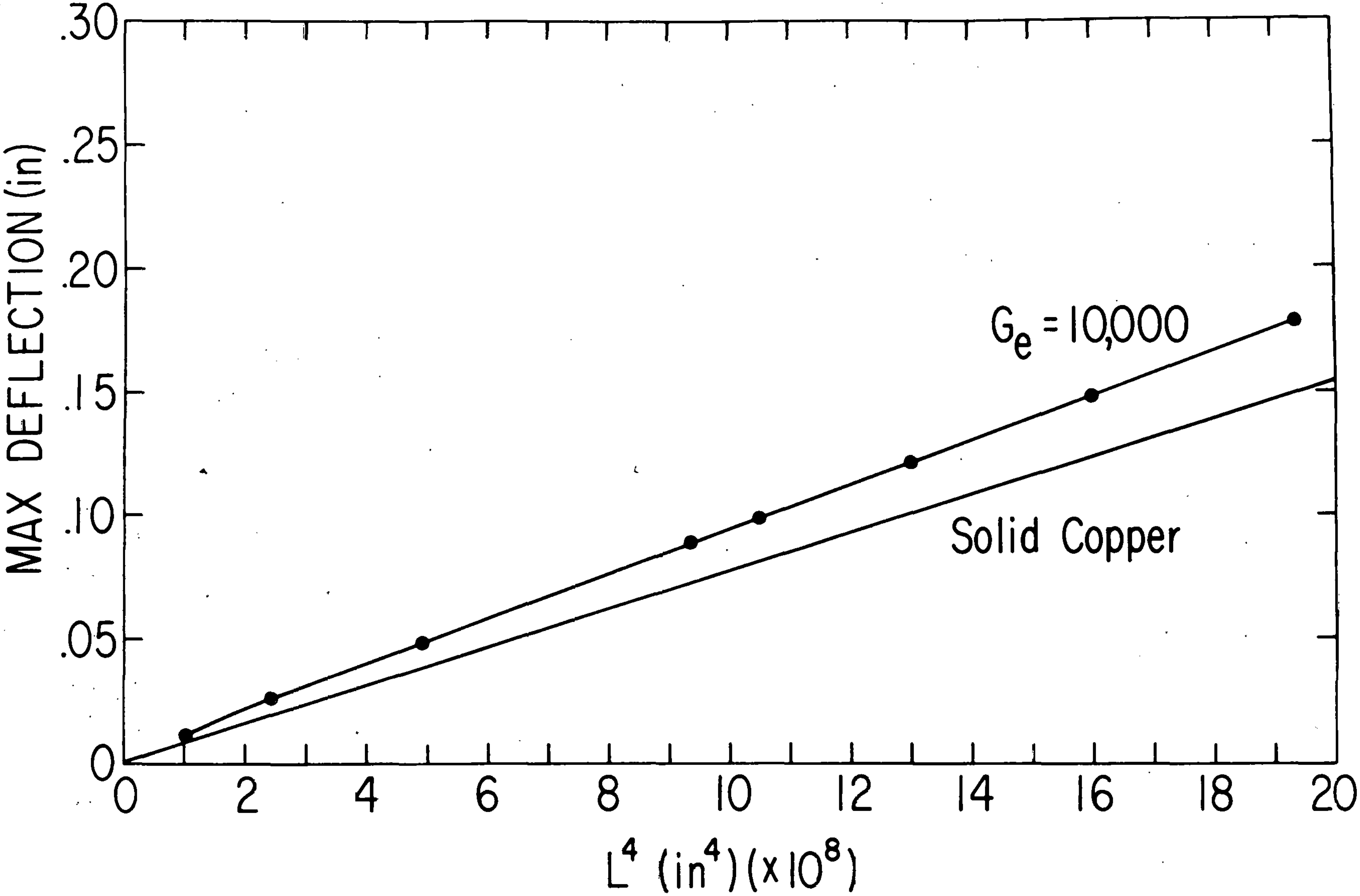
764780
Fig. 7.


R=0.10
n=30
n=6
n=3
n=2
n=30
n=3
MAX DEFLECTION (in)
3
2
1
0
10
100
1000
10000
100000
G (epoxy) psi

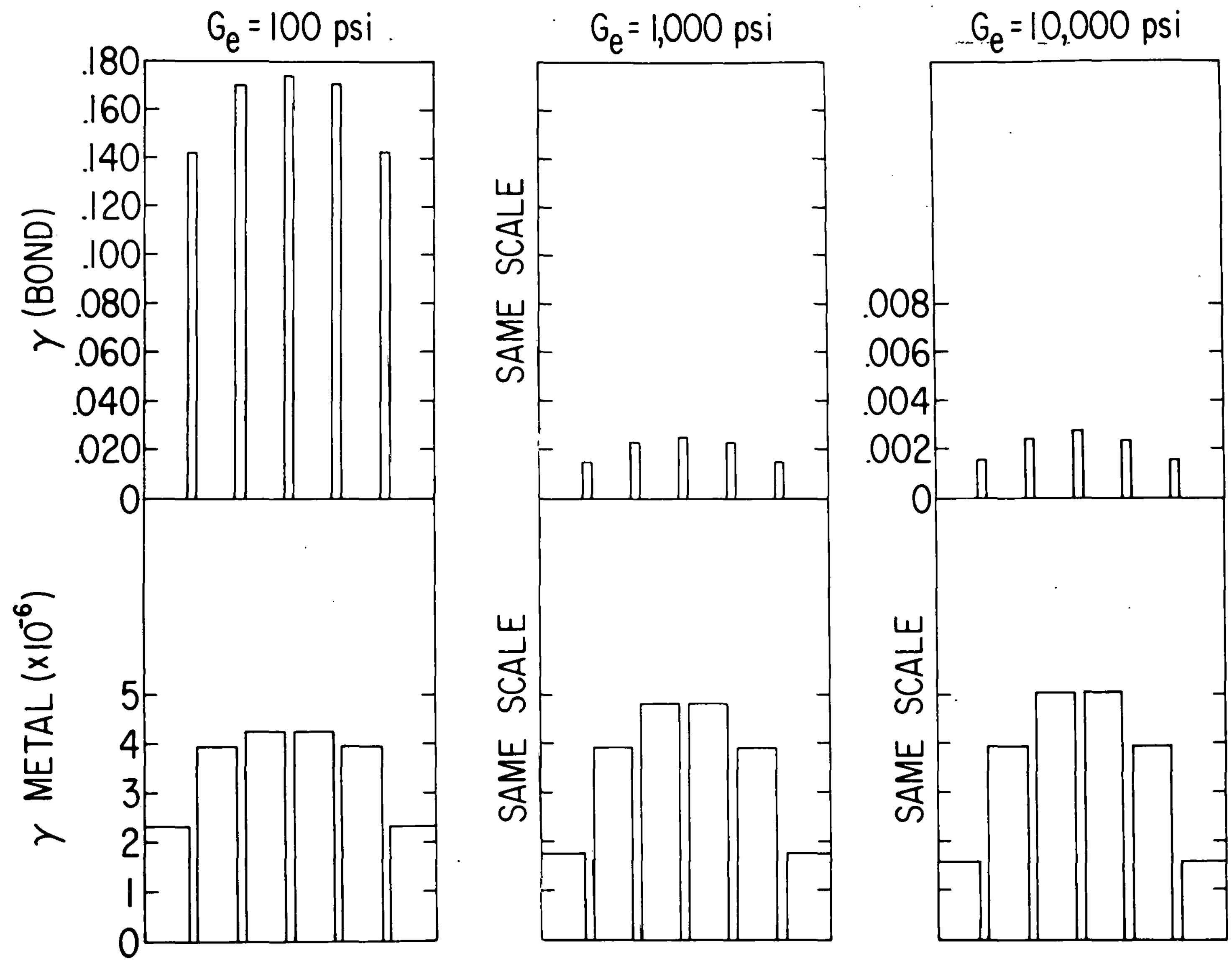
764777
Fig. 8.


n=30
R=0.15
MAX DEFLECTION (in)
3
2
1
0
n=3
n=2
n=30
n=2
n=30
10
10^2
10^3
10^4
10^5
G (epoxy) psi

764771
Fig. 9.

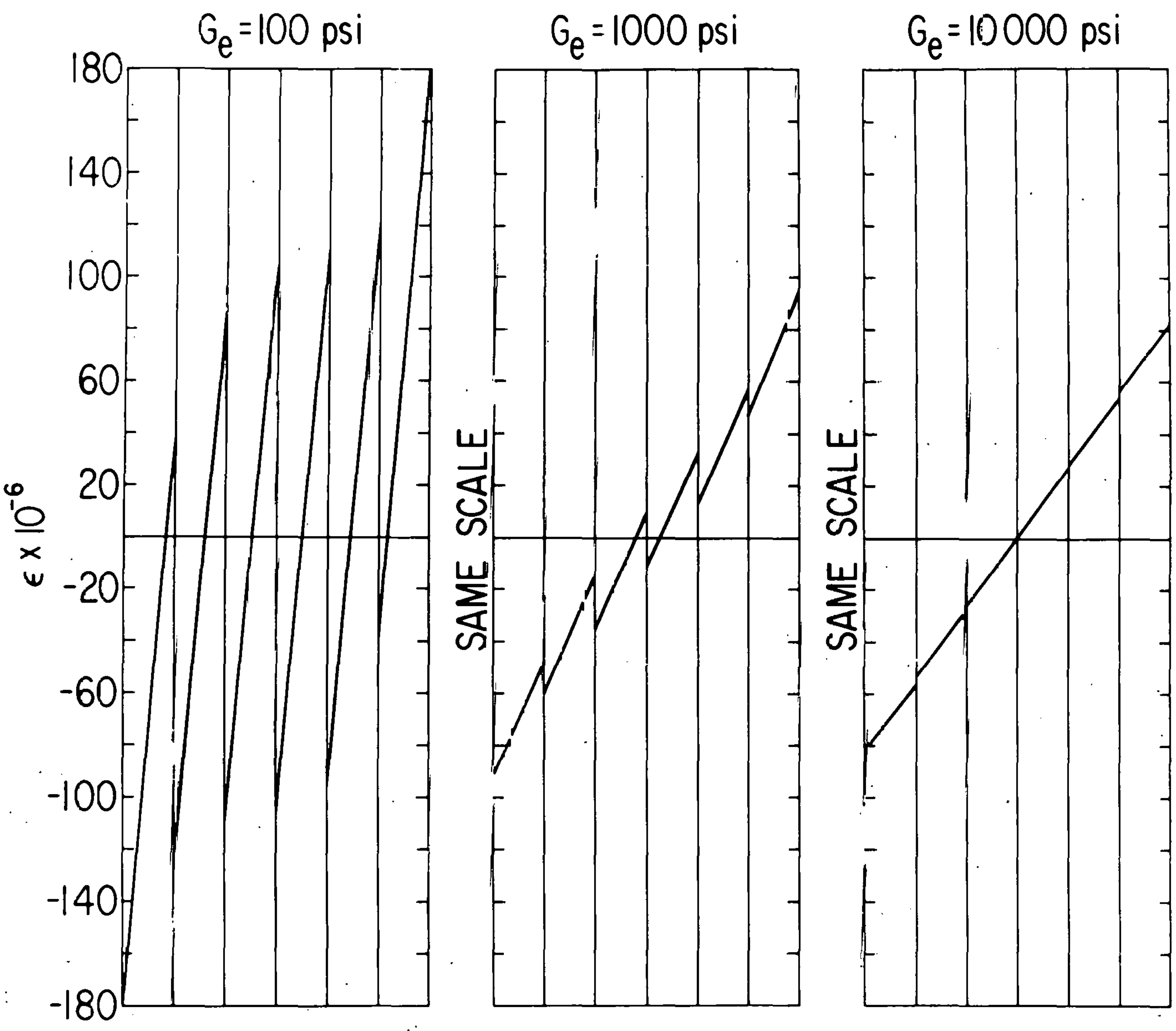


764779
Fig. 10.



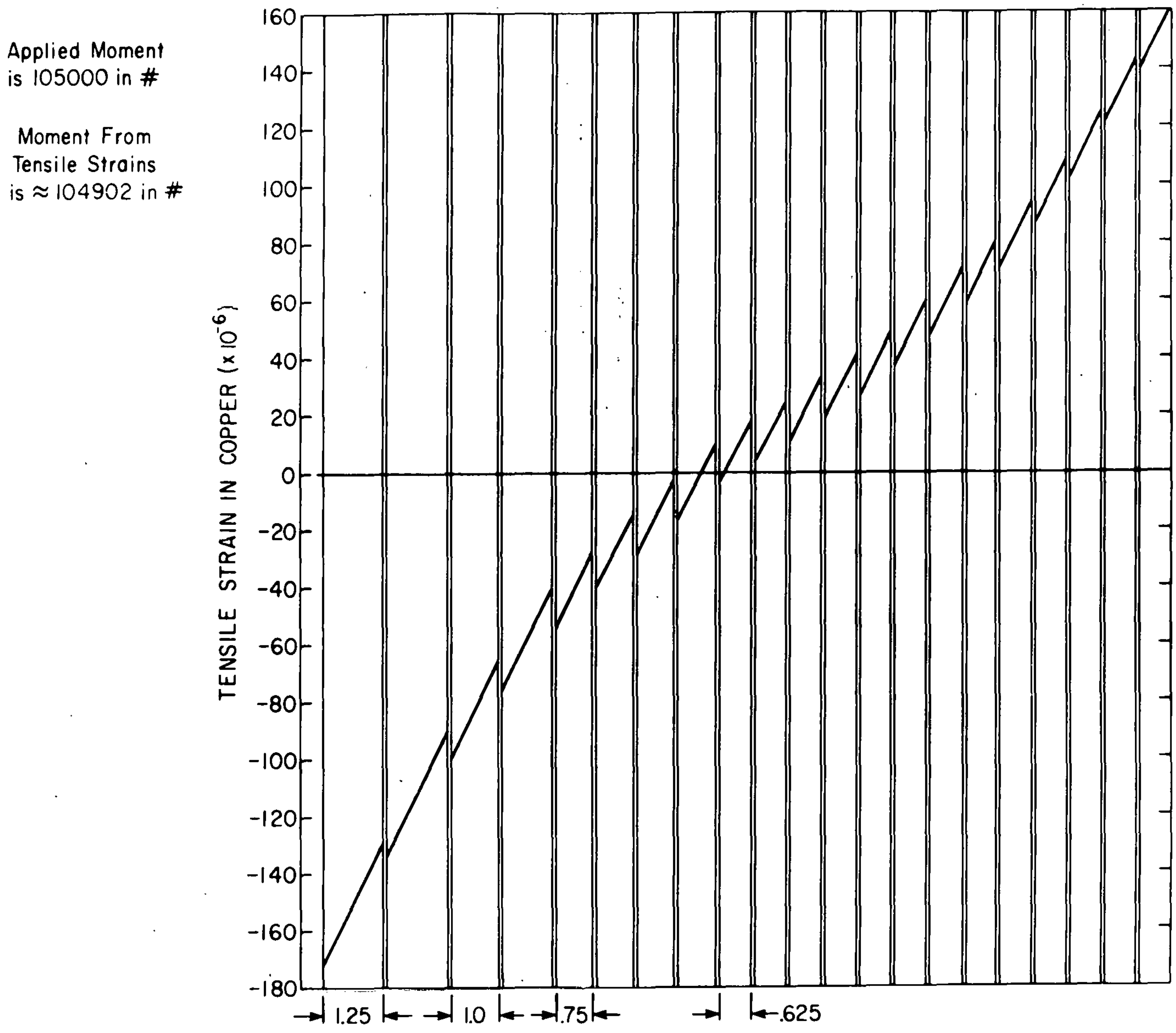
764774

Fig. 11. Shear strain in a six layer beam (R = 0.10).



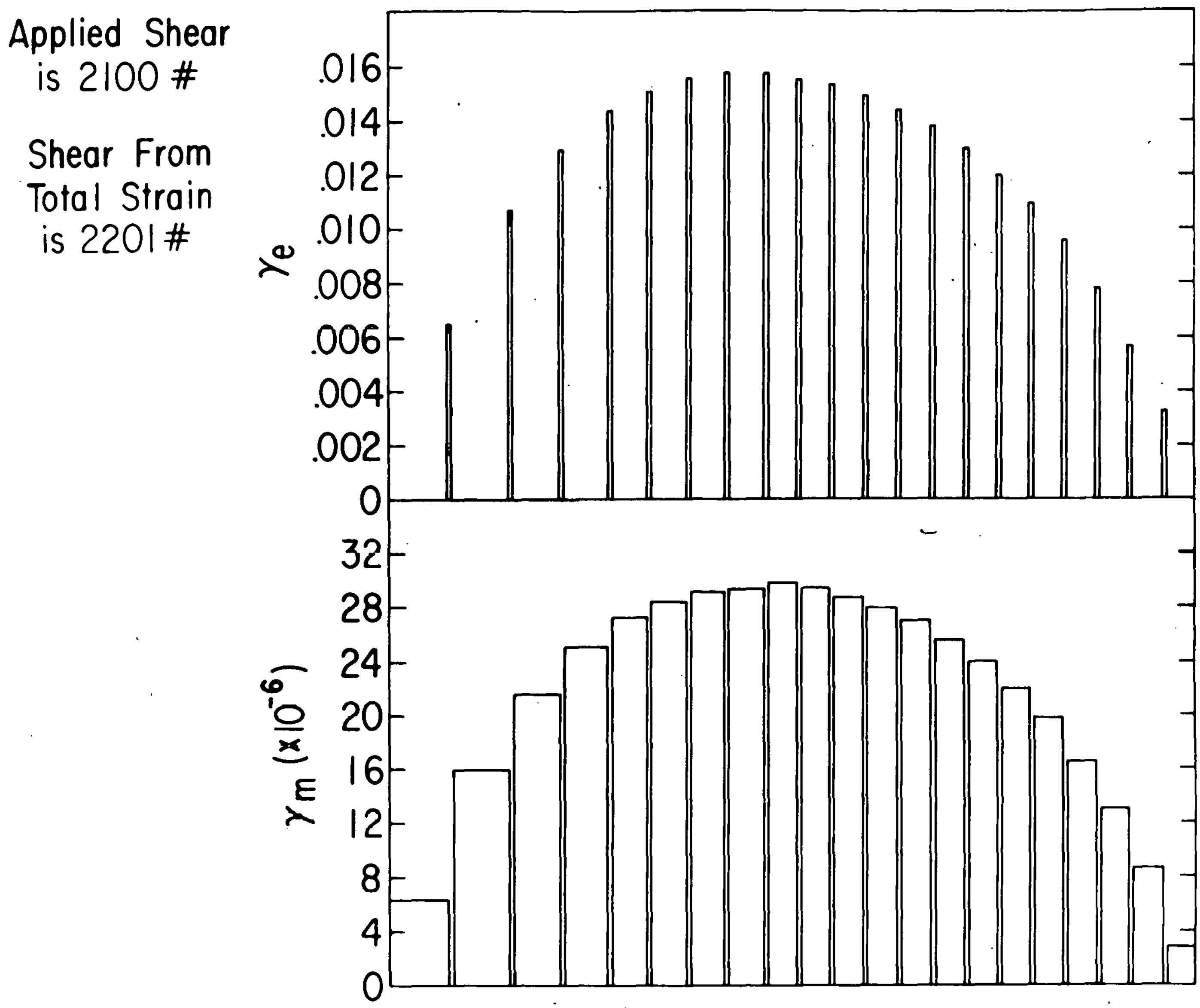
764775

Fig. 12. Tensile strain in a six layer beam (R = 0.10).



764773

Fig. 13. Tensile strain vs. layer in the PLT cross section.



764776

Fig. 14. Shear strain vs. layer in the PLT cross section.