MASTER

LA-UR--76-41
CONF-751250--1

LA-UR -76-41

TITLE: ESTIMATING AND FORECASTING FAILURE RATE PROCESSES BY MEANS OF THE KALMAN FILTER

AUTHOR(S): H. F. Martz, Jr., K. Campbell, and H. T. Davis

SUBMITTED TO: Conference on The Theory and Applications of Reliability, Tampa, Florida, December 15-18, 1975.

los alamos scientific laboratory
of the University of California
LOS ALAMOS, NEW MEXICO 87544

Form No. 836
St. No. 2629
1/75

UNITED STATES
ENERGY RESEARCH AND
DEVELOPMENT ADMINISTRATION
CONTRACT W-7405-ENG. 36

Estimating and Forecasting Failure-Rate Processes By Means of the Kalman Filter[†]

by

H. F. Martz, Jr.[*], K. Campbell[*], and H. T. Davis[**]

[*] University of California, Los Alamos Scientific Laboratory, Los Alamos, New Mexico.

[**] University of New Mexico, Albuquerque, New Mexico

[†] Presented at the AFOSR Conference on "The Theory and Applications of Reliability," University of South Florida, Tampa, Florida, December 15-18, 1975. Work performed under the auspices of the Energy Research and Development Adm.

ABSTRACT

A new non-parametric method is described for analyzing failure data. The approach used is to model the logarithm of the failure-rate process as a linear dynamic system with observations. This formulation permits the underlying failure-rate process to be corrupted by noise from various sources. In addition, the observations of the process are functions of simple non-parametric failure-rate estimates which are assumed to be noisy. The Kalman filter equations are used to provide the estimates and future forecasts. An example is provided.

1. INTRODUCTION

The failure-rate function, or hazard function, is of fundamental importance in both the theory and applications of reliability. Numerous parametric and non-parametric methods have been proposed for estimating the failure-rate function based on failure data. Parametric methods assume that the failure data arise from a specified distribution, but with unknown parameters which must be estimated from the data. A large portion of the book by Mann, Schafer, and Singpurwalla [14] is devoted to a discussion of such techniques. On the other hand, non-parametric methods do not require a distributional assumption. Barlow and Van Zwet [3] summarize and compare several non-parametric estimators for monotone failure-rate functions. Grenander [9] also discusses several non-parametric methods. Additional references may be found in [3].

As Singpurwalla [24] points out, a basic disadvantage of both approaches is the inflexibility due to the assumed model and lack of a theory for forecasting. Further, we cannot account for contamination of the failure-rate estimates from such sources as periodicities due to inspection, data recording or reporting errors, or maintenance policy effects. In an effort to account for such contamination and to provide a theory for forecasting, Castellino and Singpurwalla [7] and Singpurwalla [24] have presented new and novel approaches for estimating and forecasting failure-rate functions. In their approach, they think of the time-ordered sequence of certain non-parametric estimates of the failure-rate function as being generated by a time series process. The estimated failure-rate function is thus a stochastic process which they refer to as the *failure-rate process*. An appropriate Box-Jenkins time series model is then fitted to either the process itself [7], or a simple functional of the process [24]. The fitted model is then used to provide the required failure-rate estimates and forecasts. The approach is free of any assumptions regarding the failure distribution or the parametric form of its failure-rate function.

In this paper, we likewise consider the problem of estimating a failure-rate function, and then use Kalman Filtering techniques to forecast its future values based on failure and withdrawal data up to some point in time. The approach used is to consider a simple functional of the true failure-rate function which satisfies a certain linear random differential equation, referred to as the *state equation*. The unknown value of the specified functional of the true failure-rate function at any time is referred to as the *state of the system* (or *system state* or *state*) at the time. Consequently, a general parametric form for the failure-rate function will be assumed in order to identify and fit the state equation. However, this equation does include a random error (noise) component to account for errors in identifying and fitting the state equation. Likewise, this error accounts for the obvious fact that any such mathematical model is at best an imperfect representation of reality. A second equation is adjoined to the state equation in order to relate the state of the system at any time to a simple non-parametric estimate of the state at that time. This equation is referred to as the *observation equation*. This equation also includes a random error component to account for the statistical error associated with the non-parametric estimate. The set of both equations is referred to as a *linear dynamic system with observations*. Once the system has been identified (Section 3), the unknown parameters are then estimated from the failure data (Section 4). The Kalman filter equations (Section 2) are then used to generate minimum mean square error estimates and forecasts of the system state, which are then transformed to the required failure-rate estimates and forecasts. An expository introduction to the use of the Kalman filter in reliability is given by Breipohl [5].

The idea of using the Kalman filter in time series forecasting is not new. McWhorter [16], and McWhorter et. al. [17], [18] have all considered the use of the Kalman filter for forecasting certain economic time series in which structural regression models with randomly varying time-dependent coefficients are used. Belsley and Kuh [4] and Rosenberg [22] also discuss some of the theoretical research relating to the use of the Kalman filter in time series forecasting. McWhorter [16] empirically compared the performance of the Kalman filter and the BEA macroeconometric forecasting models [10] for five quarterly economic time series. Narasimham et. al. [20] have also compared the predictive performance of the BEA model with certain Box-Jenkins models. Kamat and Cox [12] also discuss the use of the Kalman filter in time series forecasting. Also, Duncan and Horn [8] examine linear dynamic estimation from a regression viewpoint.

The manner in which the Kalman filter is used here is entirely different from its previous use in forecasting economic time series. Regression models with randomly varying time-dependent coefficients are not used in estimating and forecasting the failure-rate function.

Rather, a completely different approach is taken, as discussed in detail in Section 3.

The Kalman filter method proposed here has advantages, as well as disadvantages, over the Box-Jenkins approach taken by Castellino and Singpurwalla. First, let us consider some of the advantages. One important advantage is that the failure-rate function can be estimated at any point in time and it is not necessary to consider equispaced time points. In certain instances, the necessity for considering equispaced time points may make inefficient use of the failure data (see Section 5).

A second advantage concerns model identification. To illustrate this, suppose that the failure data for the device in question have been obtained over periods of time in which different classes of underlying failure modes generate the observed failure data. For example, suppose that failure data are obtained during both the infant mortality and wear out regions of the useful life of the device. In such a case, it is reasonable to assume that either completely different models or the same model but with different parameters are likely to be required for the different regions. Since the standard Box-Jenkins models are non-dynamical, one is led to adaptive estimation or to consider the use of different models. Further, perhaps the environment in which the failure data have been obtained changes over time due to such things as changing policies regarding withdrawals, changing data recording policies, changing environmental test conditions, etc. This may also yield a situation in which a single non-dynamical model is inappropriate. On the other hand, the Kalman filter model is dynamical in nature and the parameters of the model may be time-varying. This advantage is also illustrated in Section 5.

A third advantage is that random model identification and fitting errors are more or less distinct and separate from the random statistical error accompanying the non-parametric failure-rate estimates. As illustrated in Section 5, this property may be used for controlling the extent to which the Kalman estimates are "smoothed" versions of the rather jagged non parametric estimates. This is further discussed in Section 5.

A fourth advantage is that the Kalman estimation and forecast error variances are conveniently computed in applying the Kalman filter equations. Consequently, probability limits can be readily and efficiently computed for the true failure-rate function and its forecasted values.

There is one major disadvantage in the Kalman filter approach as compared to the Box-Jenkins approach of Castellino and Singpurwalla. Briefly, the entire procedure is less non-parametric. Although there is no assumption regarding the failure distribution, the random error components in the state and observation equations will be assumed to follow specified distributions. The justification for these will be considered in Section 3. The proposed procedure may be thought of as lying somewhere between a completely non-parametric and a completely parametric approach.

A brief introduction to linear dynamic estimation will be presented in the next section. Failure-rate estimation within the framework of linear dynamic estimation will be considered in Section 3. Section 4 discusses procedures for fitting the proposed model to the data. A real-data example application will be presented in Section 5.

2. LINEAR DYNAMIC ESTIMATION

Linear dynamic estimation concerns the estimation of a physical process from observations of the process which may be corrupted by random "noise." The physical process is considered to be a random process which is linear in the state of the process.

As pioneers in this area, Wiener [26] and Kolmogorov [13] presented the basic theoretical solution to the problem of estimating the random process. The end result of their work was the specification of a weighting function for the optimal physically realizable estimator as the solution of a complicated integral equation. This estimator subsequently became known as the Wiener-Kolmogorov or Wiener filter. The details may be found in numerous modern textbooks on statistical control, communication, or information theory.

The practical problem of solving the integral equation of Wiener represented an additional degree of difficulty in applying the Wiener filter. Kalman [11] and Bucy and Kalman [6] recognized this shortcoming and proposed that the solution should be an algorithm which provides the numerical estimate from numerical observations with the aid of a digital computer. They converted the integral equation of Wiener into a non-linear differential equation which could be efficiently solved. Their basic method became known as the Kalman-Bucy or Kalman filter.

The mathematical statement of the general dynamic estimation problem will now be given. Consider the random linear differential equation (the state equation) given by

$$\frac{dx(t)}{dt} = A(t)\ x(t) + U(t) \tag{1}$$

where x(t) is an rx1 vector which represents the state of the system at time t, A(t) is a specified time-varying rxr matrix of coefficients, and U(t) is a rx1 vector-valued Wiener process representing the state error driving the system. From the Wiener process assumption it follows that $E[U(t)] = \emptyset$ and $E\left[U(t)U^T(t')\right] = K(t)\delta(t - t')$, where K(t) is a specified nonnegative definite matrix function of t and $\delta(t - t')$ is the Dirac delta function.

It is easily shown that the function given by

$$x(t) = \Phi(t,t_{i-1})x(t_{i-1}) + \int_{t_{i-1}}^{t} \Phi(t,\lambda)U(\lambda)\ d\lambda \tag{2}$$

satisfies (1) for initial condition $x(t_{i-1})$, where $\Phi(t,t_{i-1})$ satisfies the differential equation

$$\frac{\partial\Phi(t,t_{i-1})}{\partial t} = A(t)\Phi(t,t_{i-1}) \tag{3}$$

with initial condition $\Phi(t_{i-1},t_{i-1}) = I$. If we now define

$$u(t_{i-1}) = \int_{t_{i-1}}^{t_i} \Phi(t,\lambda)U(\lambda)\ d\lambda \quad , \tag{4}$$

then the discretized counterpart of (1) can be written as

$$x(t_i) = \Phi(t_i,t_{i-1})x(t_{i-1}) + u(t_{i-1}) \quad , \tag{5}$$

where $\{t_i\}$ is a specified sequence of time points. From the Wiener process assumption on U(t) it follows that $\{u(t_i)\}$ is a sequence of Gaussian random r-vectors with $E[u(t_i)] = \emptyset$ and $E\left[u(t_i)u^T(t_j)\right] = \delta_{ij}Q(t_i)$, where δ_{ij} is the Kronecker delta function. Thus, $\{u(t_i)\}$ is a sequence of independent Gaussian random vectors with mean $\emptyset$ and time-dependent covariance matrix

$$Q(t_i) \equiv \int_{t_{i-1}}^{t_i} \Phi(t,\lambda)K(\lambda)\Phi^T(t,\lambda)\, d\lambda \quad . \tag{6}$$

It is observed that the state equation in (5) is basically an autoregressive model with time-varying coefficients and non-stationary Gaussian white noise shock process.

Now consider the discrete-time linear *observation equation* given by

$$y(t_i) = H(t_i)x(t_i) + v(t_i) \quad , \tag{7}$$

where $y(t_i)$ is a px1 vector of observations, $H(t_i)$ is a specified pxr matrix relating $x(t_i)$ to $y(t_i)$, and $\{v(t_i)\}$ is a sequence of Gaussian random p-vectors with $E[v(t_i)] = \emptyset$ and $E\left[v(t_i)v^T(t_j)\right] = \delta_{ij}R(t_i)$. Thus $\{v(t_i)\}$ is also a non-stationary Gaussian white noise process known as the *observation error* within the system. Suppose we further assume that $u(t_i)$ is independent of $v(t_j)$ for all i and j. Equations (5) and (7) together are referred to as a *linear (discrete-time) dynamic system with observations*. The problem is to estimate $x(t_i)$ from the available sequence of observations $y(t_1), \ldots, y(t_i)$ and to forecast $x(t_m)$, where $t_m > t_i$.

Under the assumptions outlined above, as well as the assumption that $\Phi(t_i,t_{i-1})$, $H(t_i)$, $Q(t_i)$, and $R(t_i)$ are known for time points $t_1, \ldots, t_i$, the Kalman filter equations are known to provide the minimum mean square error estimate of $x(t_i)$. The Kalman estimates are also known to be MVU estimates as well. The details may be found in most modern textbooks on control theory such as Åström [2]. The Kalman filter equations are given by

$$\hat{x}_i = \bar{x}_i + \bar{P}_i H_i^T \left(H_i \bar{P}_i H_i^T + R_i\right)^{-1} \left(y_i - H_i \bar{x}_i\right) \tag{8}$$

$$P_i = \bar{P}_i - \bar{P}_i H_i^T \left(H_i \bar{P}_i H_i^T + R_i\right)^{-1} H_i \bar{P}_i \tag{9}$$

$$\bar{x}_i = \Phi_{i,i-1} \hat{x}_{i-1} \tag{10}$$

$$\bar{P}_i = \Phi_{i,i-1} P_{i-1} \Phi_{i,i-1}^T + Q_{i-1} \quad , \tag{11}$$

for $i = 1, 2, \ldots$, where for convenience in notation we have let $x_i \equiv x(t_i)$, $y_i \equiv y(t_i)$, $\Phi_{i,i-1} \equiv \Phi(t_i,t_{i-1})$, and so forth. Here $\hat{x}_i$ is the minimum mean square estimate of x_i and P_i is the covariance matrix of the estimation error $(\hat{x}_i - x_i)$. In a similar way, $\bar{x}_i$ is the minimum mean square estimate of x_i, given the observations $y_1, \ldots, y_{i-1}$, and $\bar{P}_i$ is its estimation error covariance matrix. The initial state estimate $\hat{x}_0$ and its error covariance matrix P_0 are also required to start the filtering process. Note that $\hat{x}_i$ is recursively computed and depends only upon $\hat{x}_{i-1}$ and y_i. Further note that P_i does not depend upon the observations, provided that Φ, H, Q, R, and P_0 are known. In practice, however, some or all of these quantities are unknown and must be estimated from the available data. This will be considered in Section 4. The matrix $\Phi_{i,i-1}$ is sometimes called the *state transition matrix* and the matrix $\bar{P}_i H_i^T \left(H_i \bar{P}_i H_i^T + R_i\right)^{-1}$ is often referred to as the *gain matrix (or gain)* of the Kalman filter.

3. FAILURE-RATE PROCESSES AS LINEAR DYNAMIC SYSTEMS

Let $h(t)$, the true failure-rate at time t, represent the state of the system at time t. Consider a lifetest experiment in which n items are initially placed on test and in which $r \leq n$ failures are recorded as they occur at times $0 \equiv T_0 < T_1 < T_2 < \cdots < T_r$. Let Z_i denote the total time on test between the $(i - 1)$st and the ith failure. In the case of either censored or truncated testing and no progressive withdrawals, we have that $Z_1 = nT_1$, $Z_2 = (n - 1)(T_2 - T_1)$, ..., $Z_i = (n - i + 1)(T_i - T_{i-1})$, ..., $Z_r = (n - r + 1)(T_r - T_{r-1})$. In the case of progressive withdrawals, Z_i is calculated by appealing directly to the definition of total time on test. The MLE, $\hat{h}(t)$, is a step function, constant between observations, and is given by

$$\hat{h}(t) = Z_i^{-1} \ , \ T_{i-1} < t \leq T_i \ , \ i = 1, \ldots, r \ . \qquad (12)$$

Unfortunately, the asymptotic variance of $\hat{h}(t)$ depends upon $h(t)$ [3] and this violates the conditions of (7). Since the logarithmic transformation is the appropriate variance stabilizing transformation, we consider the use of $\ell n\, h(t)$ here.

Correspondingly, we now consider the state equation for the logarithm of the failure-rate function, $\ell n\, h(t)$. Suppose that we define the state equation corresponding to a univariate version of (1) as

$$\frac{d\,\ell n\, h(t)}{dt} = \left[\frac{abt^{b-1} + cd^t\,\ell n d + e/t}{at^b + cd^t + e\,\ell n t + f} \right] \ell n\, h(t) + J(t) \ . \qquad (13)$$

Here A(t) of (1) represents the expression in brackets, where a, b, c, d, e and f are parameters of the expression. Motivation for this choice of A(t) is based on the following. Consider the deterministic counterpart of (13) obtained by ignoring U(t), the random error process driving the system. A solution to this deterministic equation is given by

$$\ell n\, h(t) = at^{b} + cd^{t} + e\, \ell n t + f \quad , \tag{14}$$

from which

$$h(t) = \exp\left[at^{b} + cd^{t} + e\ell n t + f\right] \quad . \tag{15}$$

Equations (14) and (15) will be referred to as the *nominal system model* for the log failure-rate and failure-rate process, respectively.

Several important parametric failure-rate functions are special cases of (15). The constant failure-rate (exponential) model $h(t) = \lambda$ is obtained by letting $a \equiv \ell n\, \lambda$, $b=c=d=e=f \equiv 0$. The linearly increasing (Rayleigh) model $h(t) = \alpha t$, $\alpha > 0$, is obtained by setting $a \equiv \ell n \alpha$, $b=c=f \equiv 0$, and $e \equiv 1$. The polynomial (Weibull) and exponential (extreme value) failure-rate models are likewise easily shown to be special cases of (15). Also, the first-order autoregressive log failure-rate model for equally spaced time points, $\ell n\, h(t + 1) = \alpha \ell n h(t)$, is obtained by letting $d \equiv \alpha$, $a=c=f \equiv 0$. It is noted here that higher-order autoregressive models can be obtained by considering a suitably dimensioned vector state variable consisting of an appropriate number of lags in the log failure-rate function. Each of the models and their combinations discussed above is, of course, assumed to be contaminated by Gaussian random noise input to the system as in (13). Thus, an actual log failure-rate process in practice is assumed to depart from the nominal system model in (14) depending upon the magnitude of the parameters in the assumed Wiener noise process contaminating (or corrupting) the system. The system state equation in (13) thus appears to be sufficiently flexible for use in many practical applications. It is also noted here that any assumed periodicities in the log failure-rate process could be accounted for by adding appropriate periodic terms to the nominal system model (14). Without such terms, it is impossible to forecast periodic behavior of the log failure-rate process when using the Kalman filter approach presented here. However, if there are periodicities in the MLE given in (12) as a result of periodically contaminated data, these will also likely be present to some extent in the Kalman filter estimates, even though (14) is used without adding periodic terms. The basis for this statement will be illustrated in Section 5.

The discrete-time version of (13) as given in (5) becomes

$$\ln h(t_i) = \left[\frac{a t_i^{b} + c d^{t_i} + e \ln t_i + f}{a t_{i-1}^{b} + c d^{t_{i-1}} + e \ln t_{i-1} + f} \right] \ln h(t_{i-1}) + u(t_{i-1}), \quad (16)$$

where $\Phi(t_i, t_{i-1})$ is obtained by solving (3) and is the expression given in brackets. Now $\Phi(t_i, t_{i-1})$ may be thought of as the time-dependent coefficient which maps the expected log failure-rate at time

t_{i-1} into the expected log failure-rate at time t_i. The additive Gaussian error $u(t_{i-1})$ in (16) accounts for potential modeling errors as well as other contaminants which are likely to perturb the nominal log failure-rate process model. The assumption of additive Gaussian error in (16) is equivalent to an assumed multiplicative log-Gaussian noise component in the failure-rate system model corresponding to (16). It is argued that a positively skewed distribution, such as the log-Gaussian, is appropriate for a multiplicative error in which the mapped nominal system state is more often expected to underestimate the true system state. That is, underestimation of the true failure-rate is perhaps more frequent than overestimation based on the assumed system model.

Now let us consider the observation equation (7) in a form tentatively given by

$$\ln \hat{h}(t_i) = \ln h(t_i) + v(t_i) \quad , \tag{17}$$

where $t_i \equiv T_i$, $i = 1, 2, \ldots, r$, and $\hat{h}(\cdot)$ is the MLE given in (12). For the case of testing without progressive withdrawals, let us determine the mean and variance of $\ln \hat{h}(t_i)$ in order to find the mean and variance of the observation error $v(t_i)$. It is well-known that, if x is an exponentially distributed random variable with mean μ, then $E[\ln x] = \ln \mu - \gamma$ and $V[\ln x] = \pi^2/6$, where γ is Euler's constant. Watson and Leadbetter [25] state that if $x_1, \ldots, x_n$ are iid failure time random variables then $(n - i + 1)[H(T_i) - H(T_{i-1})]$ are independent and exponentially distributed with mean 1, where $H(x)$ is the integrated failure-rate function corresponding to x. It then follows by use of the Mean

Value Theorem that $H(T_i) - H(T_{i-1}) = h(\xi)[T_i - T_{i-1}]$ for some ξ such that $T_{i-1} < \xi < T_i$. From this it is easily shown that

$$E\left[\ln \hat{h}(t)\right] = \ln h(t) + \gamma + O\left[\frac{h'(t)}{h(t)}\right] \qquad (18)$$

and

$$V\left[\ln \hat{h}(t)\right] = \frac{\pi^2}{6} + O\left[\frac{h'(t)}{h(t)}\right] \quad . \qquad (19)$$

Assuming the failure-rate function $h(t)$ to be reasonably smooth, the terms of $O[h'(t)/h(t)]$ can be neglected, thus yielding $E[\ln \hat{h}(t)] \doteq \ln h(t) + \gamma$ and $V[\ln \hat{h}(t)] \doteq \pi^2/6$. Thus, we redefine the observation equation (17) for use here as

$$\ln h^*(t_i) \equiv \ln \hat{h}(t_i) - \gamma = \ln h(t_i) + v^*(t_i) \quad , \qquad (20)$$

where now $v^*(t_i)$ is approximately normally distributed with mean 0 and variance $\pi^2/6$. Upon comparing (20) to (7), it is observed that $H(t_i) \equiv 1$ and $R(t_i) \equiv R = \pi^2/6$.

4. MODEL FITTING AND PARAMETER ESTIMATION

We now discuss the procedures to be used in fitting the linear dynamic system with observations to a given set of failure data. We begin by considering the nominal system model given in (14) which is used in calculating $\Phi(t_i, t_{i-1})$ in (16). Since all of the parameters a,b,...,f appearing in (14) will not likely be known *a priori*, a procedure for estimating these from the failure data is needed. We currently propose using nonlinear ordinary least squares (OLS) to estimate the unknown parameters in (14) according to the following scheme. Suppose that $N \leq r$ failures have been observed up to the present time, where N is large relative to the number of parameters a,b,...,f to be estimated. These N observations will be used to identify the nominal system model in (14). Consider the sum of squares function given by

$$S(a,b,\ldots,f) = \sum_{i=1}^{N} \left[\ln h^*(T_i) - aT_i^b + cd^{T_i} + e \ln T_i + f \right]^2$$

$$= \sum_{i=1}^{N} \left[- \ln Z_i - \gamma - aT_i^b + cd^{T_i} + e \ln T_i + f \right]^2 , \quad (21)$$

where $\gamma = 0.5772157 \cdots$ is Euler's constant. Use the techniques of constrained nonlinear least squares estimation to find the values of the parameters a,b,...,f which minimize S. These are the OLS estimates of a,b,...,f and will be labeled $\hat{a},\hat{b},\ldots,\hat{f}$. These estimates are then used in computing $\hat{\Phi}(t_i, t_{i-1})$, where the OLS estimates replace the unknown parameters.

Several aspects regarding this procedure should be mentioned. First, in many practical applications some of the parameters a, b,...,f are either likely to be known, or can be assumed to be known, *a priori*. For example, if the failure-rate process is justifiably believed to be basically a contaminated Weibull process, then b, c, and f can be set equal to zero and the general six-parameter nonlinear OLS problem reduces to a simple two-parameter linear OLS problem. Generally, it is unlikely that the full six-parameter model will be required.

Secondly, the form of $\Phi(t_i, t_{i-1})$ in (16) is such that a high degree of accuracy for $\hat{a},\hat{b},\ldots,\hat{f}$ is unnecessary. That is, slight changes in $\hat{a},\hat{b},\ldots,\hat{f}$ have relatively small effect on $\hat{\Phi}(t_i, t_{i-1})$ and thus somewhat "rough" estimates will usually be sufficient. In fact, due to the flexibility of a six-parameter model, different lower order subsets of these six parameters can sometimes be used to calculate $\hat{\Phi}(t_i, t_{i-1})$ values which do not have a significant effect on the estimates generated by the Kalman filter equations. This will be illustrated in the next section.

Thirdly, each of the terms appearing in (14) may be loosely interpreted as follows. The term (at^b) in (14) may be loosely thought of as accounting for exponential tendencies in the failure-rate process. The terms (cd^t) and $(e\,\ell n t)$ loosely account for autodependent and polynomial behavior, respectively, in the model.

Finally, the parameter f in (14) loosely acts as a scale parameter in the nominal failure-rate process model. Taken together, these terms can account for dissimilar process tendencies such as may occur during the infant mortality and wear-out regions of useful life of a device. This will also be illustrated in the next section.

Now consider estimation of the initial state $\ln h(t_0)$, initial state error variance P_0, and state error variance Q_i. For convenience we shall assume that the state error variance Q_i is constant over time and consider estimating the common value $Q_i \equiv Q$. Thus, the additive error contaminating the log failure-rate state equation (16) is assumed to be a stationary white noise Gaussian process with mean 0 and variance Q. For convenience we shall also set P_0 equal to Q, since the long-range performance of the Kalman filter is known to be rather insensitive to initial starting conditions.

Numerous estimation procedures have been proposed for estimating the state error variance and initial state of a linear dynamic system with observations. Pearson [21] provides an extensive bibliography and gives an excellent survey of available methods. Shellenbarger [23] and Abramson [1] develop maximum likelihood estimators of both the state and observation error covariance matrices. Mehra [19] also develops estimators of both error covariance matrices based on the use of residuals.

We consider estimators for Q and $E[\ln h(t_0)]$ based on the method of moments. By repeated use of (16), it is easily shown that the distribution of $\ln h_i^*$, conditional on the initial state $\ln h_0$, has mean and variance given by

$$E\left[\ln h_i^* \mid \ln h_0\right] = \Psi_{0i} \ln h_0 \tag{22}$$

and

$$V\left[\ln h_i^* \mid \ln h_0\right] = \Theta_i Q + \frac{\pi^2}{6} , \tag{23}$$

respectively, where we have defined

$$\Psi_{ki} = \prod_{j=1}^{i-k} \Phi_{j+k,j+k-1}, \quad i = 1, 2, \ldots; \; k = 0, 1, \ldots, i-1, \tag{24}$$

$$\Theta_i = \sum_{k=1}^{i-1} \Psi_{ki}^2 \quad 1, \quad i = 2, 3, \ldots \quad , \tag{25}$$

and

$$\Theta_1 = 1 \quad . \tag{26}$$

Now, by use of the fact that $E(X) = E[E(X|Y)]$ and $V(X) = V[E(X|Y)] + E[V(X|Y)]$, we find that

$$\frac{1}{N} \sum_{i=1}^{N} E\left[\ell n h_i^*\right] = E\left[\ell n h_0\right] \sum_{i=1}^{N} \Psi_{oi}/N \tag{27}$$

and

$$\frac{1}{N} \sum_{i=1}^{N} V\left[\ell n h_i^*\right] = Q \sum_{i=1}^{N} \left(\Theta_i + \Psi_{oi}^2\right)\Big/ N + \pi^2/6 \quad . \tag{28}$$

Define the first two sample moments of the sequence $\{\ell n h_i^*, \; i=1,\ldots,N\}$ as

$$u_1 \equiv \frac{1}{N} \sum_{i=1}^{N} \ell n \, h_i^* \tag{29}$$

and

$$u_2 \equiv \frac{1}{N} \sum_{i=1}^{N} \left(\ell n \, h_i^* - u_1\right)^2 \quad . \tag{30}$$

Equating (27) to (29) and (28) to (30) and solving for $E[\ln h_o]$ and Q yields the moment estimators given by

$$E\left[\widehat{\ln h_o}\right] = N u_1 \Big/ \sum_{i=1}^{N} \Psi_{oi} = \sum_{i=1}^{N} \ln h_i^{*} \Big/ \sum_{i=1}^{N} \Psi_{oi} \qquad (31)$$

and

$$\hat{Q} = N\left(u_2 - \pi^2/6\right) \Big/ \sum_{i=1}^{N} \left(\Theta_i + \Psi_{oi}^2\right) \quad . \qquad (32)$$

Two things should be pointed out here. First, since the nominal system model (14) is fitted to the data by means of OLS as in (21), $E[\widehat{\ln h_o}]$ will be identically equal to the value of (14), in which the appropriate OLS estimates have been inserted, at the initial time t_o. Thus, the initial Kalman filter estimate required to start the filter is the initial nominal system estimate. If setting $t = t_0 \equiv 0$ in (14) yields a value of $-\infty$, i.e., if $e \neq 0$, then an initial time t_0 should be selected such that $0 < t_0 < T_1$. Secondly, it may happen that $\hat{Q} < 0$, in which case an arbitrary nonnegative value must be selected for Q. It is noted here that the above estimates are considerably simpler to compute than corresponding maximum likelihood estimates which cannot be obtained in closed form.

Once $\hat{\Phi}(t_i, t_{i-1})$, $E[\widehat{\ln h_o}]$, and $\hat{Q}$ have been obtained the Kalman filter equations can be applied. The Kalman filter equations (8)-(11) now become

$$\ln \tilde{h}_i = \ln \bar{h}_i + 6\bar{P}_i\left(\ln h_i^* - \ln \bar{h}_i\right)/\left(6\bar{P}_i + \pi^2\right) \qquad (33)$$

$$P_i = \bar{P}_i - 6\bar{P}_i^2 /\left(6\bar{P}_i + \pi^2\right) \qquad (34)$$

$$\ln \bar{h}_i = \hat{\phi}_{i,i-1} \ln \tilde{h}_{i-1} \qquad (35)$$

$$\bar{P}_i = \hat{\phi}^2_{i,i-1} P_{i-1} + \hat{Q} , \qquad i = 1, 2, \ldots, N \qquad (36)$$

where $\ln \tilde{h}_0 \equiv E[\widehat{\ln h_0}]$ and $P_0 \equiv \hat{Q}$ are used as initial starting values. Recall that the subscript i denotes the ith observed failure time T_i. Kalman filter estimates of the log failure-rate function are thus calculated at each of the observed failure times. Future forecasts of the log failure-rate function at any time $t > t_N$ are calculated by means of

$$\ln \bar{h}(t) = \hat{\phi}(t,t_N) \ln \tilde{h}_N \quad , \qquad (37)$$

and the variance associated with the forecast error is estimated to be

$$\bar{P}(t) = \hat{\phi}^2(t,t_N)P_N + \hat{Q} \quad . \qquad (38)$$

5. AN EXAMPLE APPLICATION

We shall illustrate the Kalman filter procedure by the following example taken from NAILSC Report ILS 04-21-72. Castellino and Singpurwalla [7] used this same example. Singpurwalla [24] also considers this same example and Table 1 of that paper gives the failure and withdrawal times (in hours) for an A/C generator. A total of 55 failures were reported ranging from a minimum of 1.0 hour to a maximum of 1097.3 hours. However, only 53 generators failed at distinctly different times, since two generators failed at 3.0 hours and two generators failed at 252.8 hours.

Singpurwalla observed periodicity at lag 7 in the sequence of MLE's computed from (12) at 24-hour equispaced time points. This periodicity was the result of a weekly inspection policy wherein items soon expected to fail were withdrawn from the test. Castellino and Singpurwalla [7] fitted a Box-Jenkins ARIMA model of the form $(1,0,0) \times (2,1,0)_7 + \Theta_0$ directly to the MLE's of the failure-rate function. By use of this model, they were able to satisfactorily estimate and forecast the failure-rate function. Their estimates and forecasts preserved the periodicities in the data.

For simplicity and convenience, we shall deliberately ignore the periodic contaminants in the data and proceed directly to fit the general linear dynamic model presented in Sections 3 and 4. However, we are well aware that there is ample evidence that periodic-type terms should be included in the nominal system model (14).

Upon examining the plot of $(T_i, \ln h_i^*)$, i = 1, ..., 53, it was tentatively decided that parameters e and b in the nominal system model (14) could be set equal to 0 and 1, respectively. The resulting OLS curves fitted to the data $\{(T_i, \ln h_i^*), i = 1, 2, \ldots, 53\}$ tended to confirm this choice. The remaining parameters a, c, d, and f were estimated by use of the software program 205LSQS* using the method due to Marquardt [15]. The OLS estimates of these parameters were found to be $\hat{a} = 1.09113$, $\hat{c} = 2.61270$, $\hat{d} = 0.83065$ and $\hat{f} = -7.53429$. Thus, the nominal system model for the log failure-rate function was taken to be

$$\ln h(t) = 1.09113t + 2.61270(0.83065)^t - 7.53429 \quad . \qquad (39)$$

Both the input observation data $\ln \hat{h}(T_i)$ as well as the nominal system model (39) are plotted in Figure 1. It is observed that the apparent rapid decrease in the input data occurring during the brief infant mortality or break-in period is captured in the nominal model. It is further observed that the generators tend to begin wearing out sometime after the break-in period as evidenced by the increasing trend of the input data over time. The nominal model captures this increasing trend and is nearly linear during this region. Also, the chance-failure region is nominally estimated to be of fairly short duration. Finally, the input data marked with an asterisk (*) in Figure 1 represents the data used by Castellino and Singpurwalla [7] and Singpurwalla [24]

---

*Internal nonlinear OLS program available at the Los Alamos Scientific Laboratory, Los Alamos, New Mexico.

when considering 24-hour equispaced time intervals. It is clearly apparent that this particular subset of input data is nonrepresentative of all the data. Consequently, it must be remembered that failure-rate estimates and forecasts based on this subset should be interpreted *only* at 24-hour intervals and *not* at arbitrary points in time. On the other hand, the Kalman filter procedure effectively uses all of the input data, thus providing a composite view of the entire failure-rate function.

Moment estimates of $E[\ln h_0]$ and Q were computed to be -4.92 and -0.00055, respectively. Since $\hat{Q}$ is negative, we shall arbitrarily set $\hat{Q}$ equal to several nonnegative values and observe the corresponding performance of the filter. The Kalman filter estimates given by (33) are plotted in Figure 1. For these estimates, Q was taken to be 0.02. It is observed that the Kalman estimates are significantly smoother than the input MLE's and effectively represent a compromise between the input data and nominal system model. The Kalman estimates are observed to account nicely for the log failure-rate function during the break-in as well as the wear-out regions. In Figure 1 we have also plotted the Kalman estimates forecasted ahead at each observed failure to the time of the next failure. These estimates are given in (35) and illustrate the short-range forecasting ability of the Kalman filter procedure. These short-range forecasts are observed to be in good agreement with the Kalman estimates themselves.

The same estimates are plotted in Figure 2 except that now $\hat{Q} = 2.0$. In this case, less smoothing occurs and the ragged nature of the input data is largely preserved in the Kalman estimates.

This situation corresponds to somewhat imprecise knowledge of the nominal system model. It is interesting to observe that the Kalman estimates seem to be in phase with the input data. Periodicities in the input data are also likely to be preserved.

Figure 3 considers the case where $\hat{Q} = 0.001$. This corresponds to precise system knowledge. It is observed that the Kalman estimates nearly coincide with the nominal system model and are extremely smooth relative to the input data. In the case where $\hat{Q} \equiv P_0 = 0$, the Kalman estimates reduce to the nominal system estimates and no state noise is assumed to be driving the system.

In Figure 4 we have plotted the antilogs of corresponding estimates in Figure 1. This graph illustrates the Kalman filter's performance in estimating the failure-rate function. The performance appears to be satisfactory, and is analogous to Figure 1.

Figure 5 presents a plot of the Kalman estimates and forecasts of the failure-rate function for $\hat{Q} = 0.02$. The estimates are plotted up to time $T_{55} = 1097.3$ hours and the forecasts are given at 25-hour equispaced time intervals beginning with $t = 1100$ hours through $t = 1700$ hours. The forecasts were obtained by taking the antilog of (37). As pointed out in Section 3, periodicities cannot be forecast without the use of suitable periodic terms in the nominal system model. In Figure 5 we have also plotted approximate 95 percent probability limits for the underlying failure-rate function. At the observed failure times up to 1097.3 hours, the limits were computed according to $\exp[\ell n\, \tilde{h}_i \pm 1.96 \sqrt{P_i}]$. The 95 percent limits on the failure-rate forecasts were computed from $\exp[\ell n\, \bar{h}(t) \pm 1.96 \sqrt{\bar{P}(t)}]$, where

$\ln \bar{h}(t)$ and $\bar{P}(t)$ are given in (37) and (38).

In Figure 5, we have also plotted the estimates and forecasts given by the Box-Jenkins technique of Castellino and Singpurwalla. These estimates and forecasts are exclusively given at 24-hour intervals. Since the Kalman filter results are based on non-equispaced input data, a direct comparison cannot be made.

The sensitivity of the resulting Kalman filter estimates to the choice of a nominal system model (14) was investigated by fitting a different nominal model to the input data. When the nominal model

$$\ln h(t) = 0.03979t^{0.5} + 2.82400(0.85277)^{t} - 7.83338 \quad , \qquad (40)$$

was used rather than (39), the resulting estimates were not noticeably different from those in Figure 1 based on the use of (39). This is explained by noticing that $\Phi(t_i, t_{i-1})$ in (16) is a ratio and that changes in the nominal model tend to "cancel out" to a large extent.

Now the ent re procedure may be interpreted as a Bayesian technique for smoothing non-parametric failure-rate estimates. The degree of smoothing is governed by the magnitude of the state error variance estimate $\hat{Q}$, and the smoothing occurs in the direction of the nominal system model.

In conclusion, we have shown that the Kalman filter equations can be used to provide realistic failure-rate estimates and forecasts in the presence of imprecise knowledge of the actual form of the failure-rate function.

NAILSC DATA

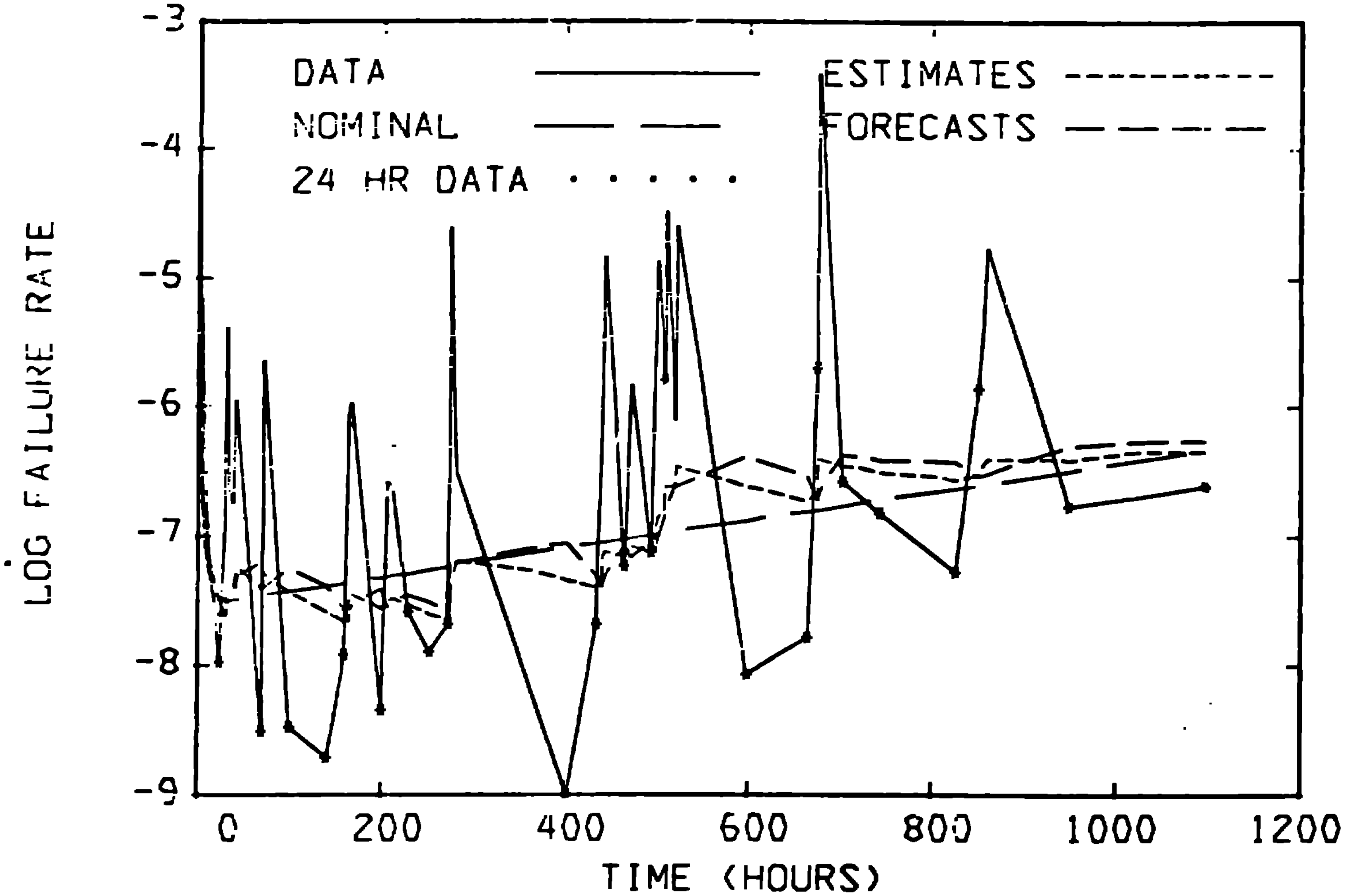

FIGURE 1. KALMAN FILTER ESTIMATION OF THE LOG FAILURE-RATE FUNCTION
Q=0 02

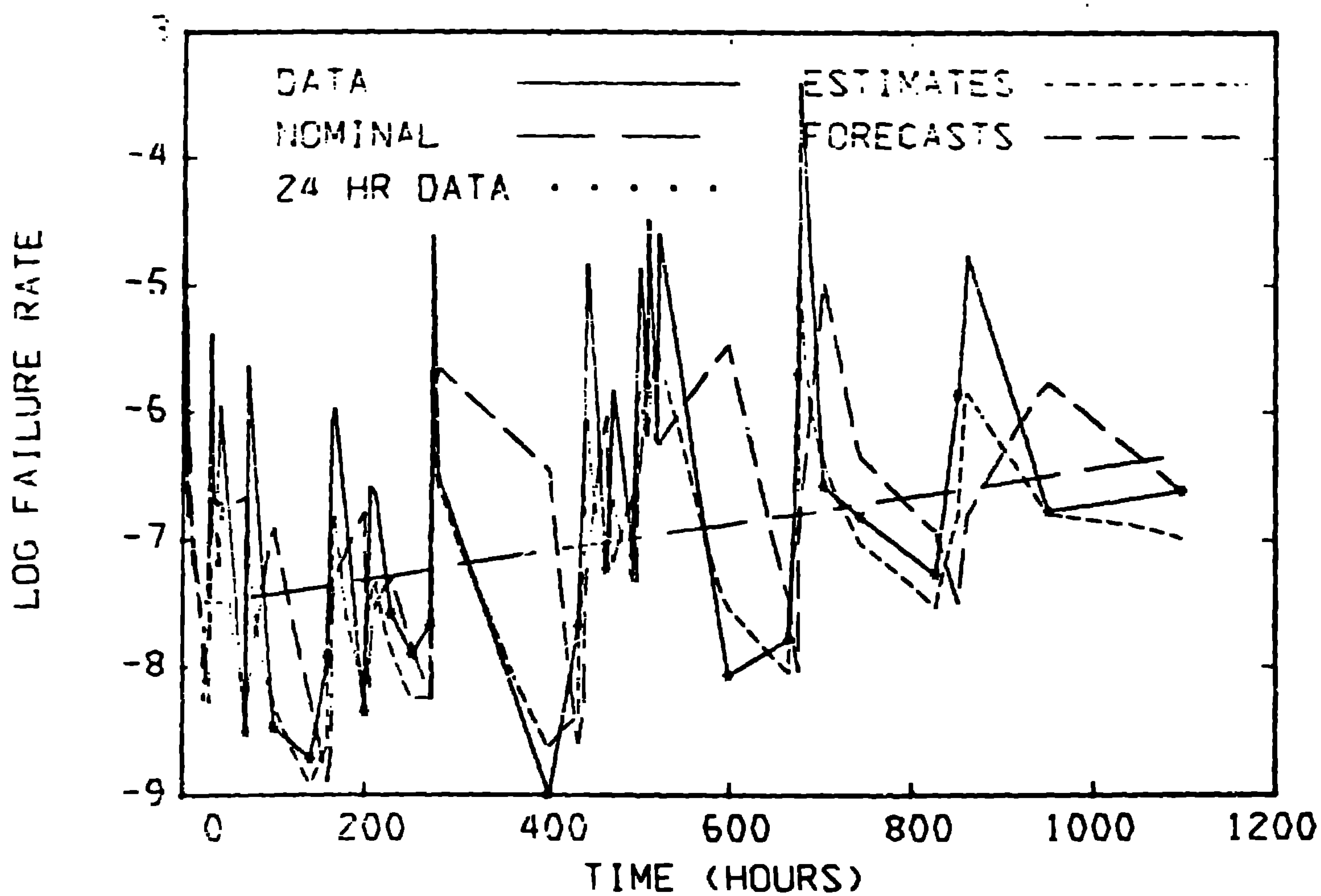

FIGURE 2 KALMAN FILTER ESTIMATION OF THE LOG FAILURE-RATE FUNCTION Q=2 0

NAILSC DATA

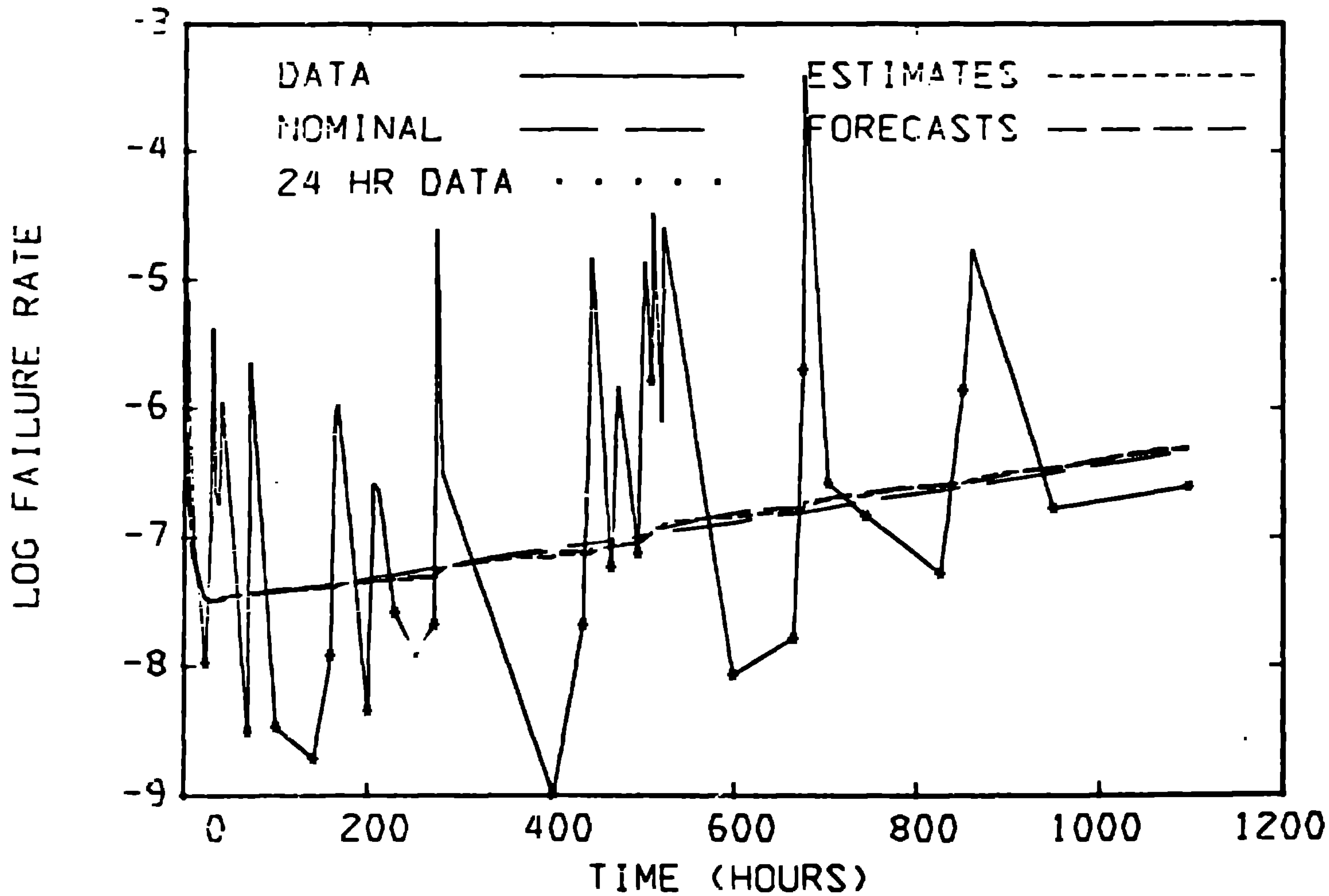

FIGURE 3. KALMAN FILTER ESTIMATION OF THE LOG FAILURE-RATE FUNCTION
Q=0 001

NAILSC DATA

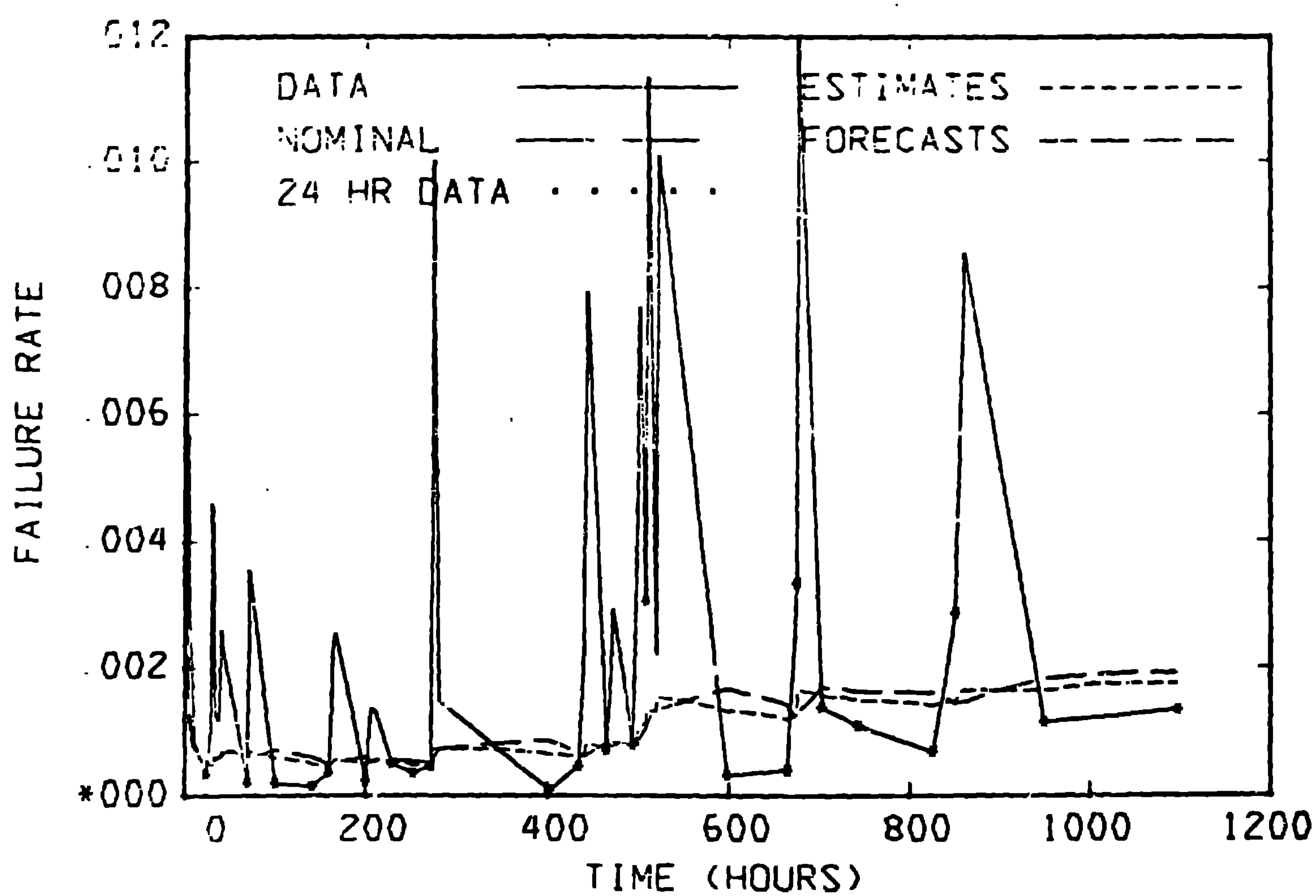

FIGURE 4 KALMAN FILTER ESTIMATION OF THE FAILURE-RATE FUNCTION

Q=0.02

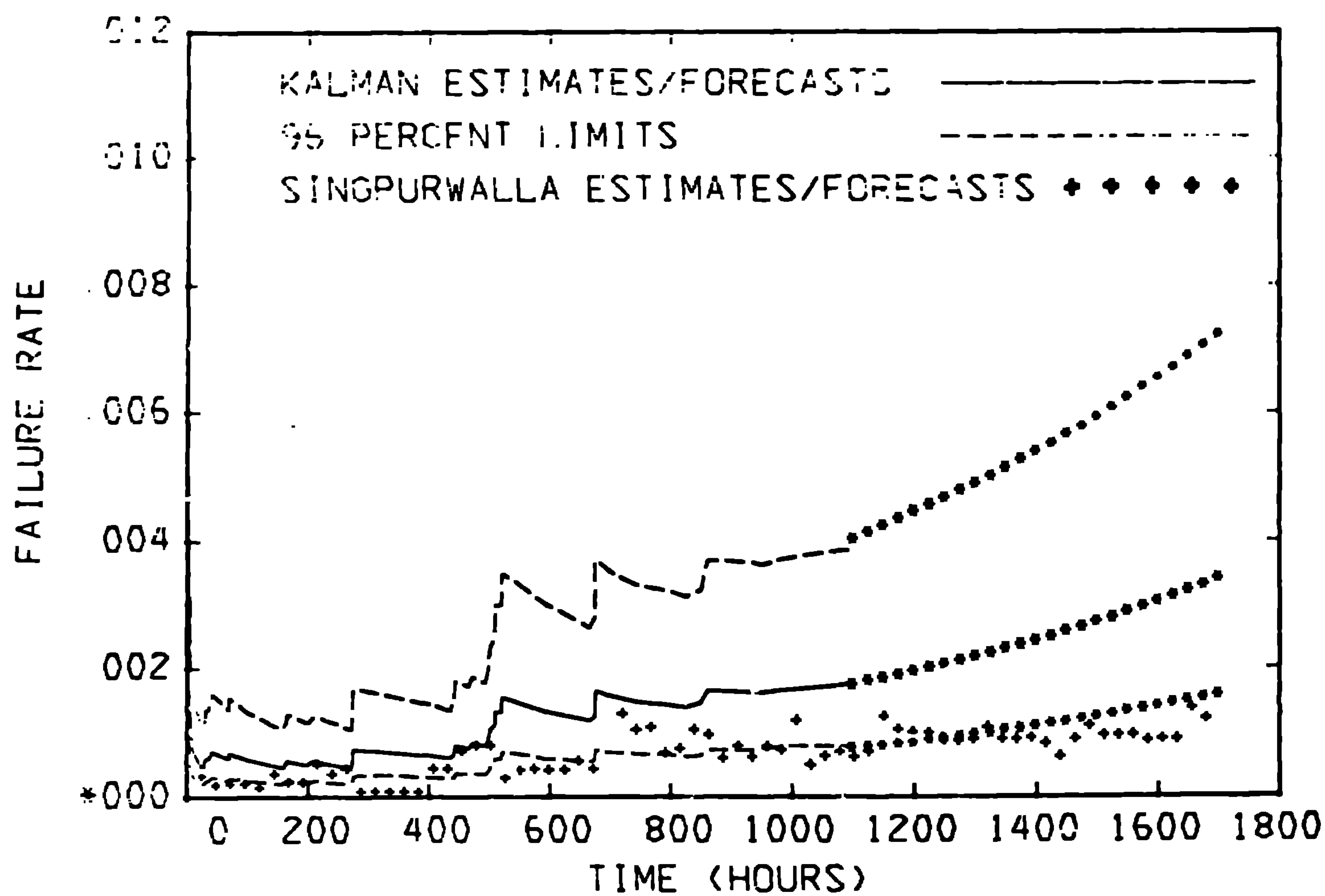

FIGURE 5 KALMAN FILTER ESTIMATES, FORECASTS AND PROBABILITY LIMITS OF THE FAILURE-RATE FUNCTION Q=0.02

REFERENCES

[1] P. D. Abramson, Jr., "Simultaneous Estimation of the State and Noise Statistics in Linear Dynamical Systems," NASA report R-332 (March 1970).

[2] K. J. Åström *Introduction to Stochastic Control Theory* (Academic Press, New York, 1970).

[3] R. E. Barlow and W. L. VanZwet, "Comparison of Several Nonparametric Estimators of the Failure Rate Function," Proc. of the N.A.T.O. Conference on Reliability at Turino, Italy, 30 June - 4 July, 1969 (Israel Publishing House).

[4] D. A. Belsley and E. Kuh, "Time-Varying Parameter Structures: An Overview," Annals of Econ. and Social Meas., 2 (October 1973).

[5] A. M. Breipohl, "Kalman Filtering and Its Application to Reliability," IEEE Trans. on Rel., R-18, 127-130 (August 1969).

[6] R. S. Bucy and R. E. Kalman, "New Results in Linear Filtering and Production Theory," Trans. of ASME, Series D: Journal of Basic Eng., 83, 95-108 (1961).

[7] V. F. Castellino and N. D. Singpurwalla, "On the Forecasting of Failure Rates -- A Detailed Analysis," Technical Memorandum Serial TM-62197, Program in Logistics, The George Washington University, Washington, D.C. (1973).

[8] D. B. Duncan and S. D. Horn, "Linear Dynamic Recursive Estimation from the Viewpoint of Regression Analysis," JASA, 67, 815-821 (December 1972).

[9] V. Grenander, "On the Theory of Mortality Measurement, Parts I and II," Skand. Aktuarietidskr., 39, 125-153 (1956).

[10] A. A. Hirsch, M, Liebenberg, and G. Green, "The BEA Quarterly Model," BEA Staff Paper No. 22, U.S. Dept. of Commerce, Washington, D.C. (1973).

[11] R. E. Kalman, "A New Approach to Linear Filtering and Prediction Problems," Basic Eng., 82, 35-45 (March 1960).

[12] S. J. Kamat and L. D. Cox, "Use of the Kalman Filter in Time-Series Forecasting," presented at the II Interamerican Conference on Systems and Informatics, Mexico City (November 1974).

[13] A. N. Kolmogorov, "Interpolation und Extrapolation von Stationaren Zufalligen Folgen," Bulletin of Academic Sciences, USSR, SER. MATH. (1941).

[14] N. R. Mann, R. E. Schafer, and N. D. Singpurwalla, *Methods for Statistical Analysis of Reliability and Life Data* (John Wiley and Sons, New York, 1974).

[15] D. W. Marquardt, "An Algorithm for Least Squares Estimation of Nonlinear Parameters," SIAM J., 11, 431-441 (1963).

[16] A. McWhorter, Jr., "Time Series Forecasting Using the Kalman Filter: An Empirical Study," presented at the 1975 Annual Meeting of ASA, Atlanta, Georgia, (August 1975).

[17] A. McWhorter, Jr., W. A. Spivey, and W. J. Wrobleski, "Varying Parameter Econometric Models for Interrelated Time Series," Proc. Bus. and Econ. Sect. of ASA, St. Louis, August 1974.

[18] A. McWhorter, W. A. Spivey, and W. J. Wrobleski, "Computer Simulation in Varying Parameter Regression Models," Proc. Bus. and Econ. Sect. of ASA, New York, December 1973.

[19] R. K. Mehra, "On the Identification of Variances and Adaptive Kalman Filtering," IEEE Trans. on Auto. Control, AC-15, 175-184 (April 1970).

[20] G. V. L. Narasimham, V. F. Castellino, and N. D. Singpurwalla. "On the Predictive Performance of the BEA Quarterly Economic Model and Box-Jenkins ARIMA Models," Proc. Bus. and Econ. Sect. of ASA, St. Louis, August 1974.

[21] J. O. Pearson, "Estimation of Uncertain Systems," Control and Dynamic Systems, 10, 255-343 (1973).

[22] B. Rosenberg, "A Survey of Stochastic Parameter Regression," Annals of Econ. and Social Meas., 2 (October 1973).

[23] J. C. Shellenbarger, "Estimation of Covariance Parameters for an Adaptive Kalman Filter," Proc. of the National Electronics Conf., Chicago, October 3-5, 1966, pp. 698-702.

[24] N. D. Singpurwalla, "Time Series Analysis and Forecasting of Failure-Rate Processes," in *Reliability and Fault Tree Analysis* (SIAM, Philadelphia, 1975), 483-507.

[25] G. S. Watson and M. R. Leadbetter, "Hazard Analysis I," Biometrika, 1 & 2, 175-184 (1964).

[26] N. Wiener, "The Extrapolation, Interpolation, and Smoothing of Stationary Time Series," OSRD 370, Report to the Services 19, Research Project DIC-6037, MIT (February 1942).