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MASTER

STATISTICAL MECHANICS OF LATTICE BOSON

FIELD THEORY*

by

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A lattice approximation to Euclidean, Boson quantum field theory is expressed in terms of the thermodynamic properties of a classical statistical mechanical system near its critical point in a sufficiently general way to permit the inclusion of an anomalous dimension of the vacuum. Using the thermodynamic properties of the Ising model, one can begin to construct non-trivial (containing scattering) field theories in 2, 3 and 4 dimensions. It is argued that, depending on the choice of the bare coupling constant, there are three types of behavior to be expected (i) the perturbation theory region, (ii) the renormalization group fixed point region, and (iii) the Ising model region.

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SECTION 1. INTRODUCTION AND SUMMARY

Since its inception quantum field theory has proven a challenging subject. It was not clear at first, and is only now becoming apparent, that the formal structure of this theory does in fact define an actual theory. In addition there has remained a practical problem of the calculation of the predictions of the theory. The only approach had been the perturbation theory in the coupling constant, but this series was asymptotic at best with a large value of the coupling constant, and the terms past the first few are extremely difficult to compute.

The start of the modern progress in the problem of constructing actual field theories can perhaps be traced to the discovery by Symanzik[1] that if the time, t, in at least certain model field theories, is replaced by i t, then the result was mathematically very similar to a classical statistical mechanical system. Nelson[2] made a fundamental step forward when he proved that if one had in Euclidean space a correlated random field obeying the Markov property[2,3] (roughly, if one has complete information about a system on the boundary of a region, no additional information about the interior is gained by further knowledge of the exterior) that one could construct from it a quantum field theory which satisfied all the Wightman axioms[4]. This equivalence leads to the idea that the Euclidean random field might be usefully approximated by a statistical mechanical system on a discrete lattice as was investigated by Guerra, et al[5] and Baker[6]. We continue this study in this article.

In the second section we review a small part of what is known of this approach (For a fuller account, see references [6] and [7]). We differ with the usual treatment in that we consider the free energy defined by

$$f = \beta^{-1} \ln \{\mathrm{Tr}\, [\exp(\beta A)]\} \tag{1.1}$$

as the fundamental quantity instead of the usual $\hbar = 1$. This extra freedom allows us to recover non-trivial quantum field theories from statistical mechanical systems that satisfy only the scaling relations but not the hyper-scaling relations. By this remark I mean the following. At the critical point there is singular behavior in various thermodynamic and statistical-mechanical properties. Each such singularity is characterized near the critical point, by a critical index which specifies how it diverges $[(1 - T_c/T)^{-\theta}]$ or vanishes, as the case may be. These indices are not all independent but are related by various equations[8]. Those equations which depend explicitly on the spatial dimension are called hyperscaling relations and are the only such relations not satisfied in the classical limit. It has been observed by Stell[9] that numerically at least all available data is consistent with the hyperscaling relations if d, the spatial dimension, is replaced by $d-\eta^*$, where η^* is the so-called anomalous dimension of the vacuum.

The material of the second section deals with the existence, analyticity, continuity of the free energy and the correlation functions in the infinite volume limit. The question of mass renormalizability is answered for the $g_0{:}\phi^4{:}_d$ theories.

In the third section we relate the behavior of the field theoretic renormalization constants to the size of the lattice spacing in the limit of small lattice spacing. This relation is given in terms of the usual thermodynamic functions, their amplitudes and indices of divergence as the critical temperature is approached from above.

In the final section, studying $g{:}\phi^4{:}$ theory, we use the representation obtained in the third section and the approximate, critical-phenomena, renormalization-group recursion-relations[10,11] as a guide to the intuition to argue that there are three regions of the bare coupling constant which show potentially different behavior. There is an intermediate range of the bare coupling constant in which the renormalization group[12,13,14] fixed point dominates the behavior. There is a range of weak bare coupling in which, at least for a finite renormalized mass, the convergence of the system to that fixed point is too slow for it to determine the behavior. We anticipate that this region is the one in which direct summation of the perturbation series will be possible (by appropriate methods). Finally there is the hyperstrong coupling region where the bare coupling constant is much larger than certain functions of the lattice spacing. In this region the behavior of the $g{:}\phi^4{:}_d$ theory is exactly controlled by that of the corresponding Ising model.

SECTION 2. LATTICE FIELD THEORY

Nelson[2] has taken an important step in the problem of constructing a self-interacting, Boson quantum field theory. He has shown under mild assumptions, that if one has given a Markov, random field defined over a d-dimensional, Euclidean space, one may construct from it a field theory which satisfies all the Wightman axioms[4] for a relativistic field theory in a Minkowski space of d-1 space and one time dimension. We will confine our attention to the problem of constructing a Markov random field in d-dimensional Euclidean space in such a way that the statistical mechanical methods developed to treat the problems of magnetic order on a crystal lattice can be used to advantage.

In terms of the axiomatic approach, the field theory is completely specified in terms of the set of Schwinger functions (complete Euclidean Green's functions), $S_{(N)}$. The statistical mechanical partition function $Z(H)$ is, in field theoretic language, their generating functional

$$Z(H) = \sum_{N=0}^{\infty} \frac{\beta^N}{N!} \int dx_1 \ldots dx_N \, H(x_1) \ldots H(x_N) \, S_{(N)}(x_1, \ldots, x_N) \tag{2.1}$$

It is usual to think of these Schwinger functions as given by the functional integral

$$Z(H) = K^{-1} \int [d\phi] \exp\left(-\beta \int d\tilde{x} [L(\phi) - \phi(x) H(x)]\right) \tag{2.2}$$

where the Lagrangian density L is a function of the field variables ϕ, and the integral in the exponents is now over the d-dimensional Euclidean space. The constant K is, of course, infinite or zero and is defined formally to impose the normalization condition,

$$Z(0) = 1 \tag{2.3}$$

The important quantities to be studied turn out to be most conveniently expressed in terms of the free-energy. It is given from the partition function by

$$\begin{aligned} f(H) &= \beta^{-1} \ln Z(H) \\ &= \sum_{N=1}^{\infty} \frac{\beta^{N-1}}{N!} \int dx_1 \ldots dx_N \, H(x_1) \ldots H(x_N) \, U_N (x_1, \ldots, x_N) \end{aligned} \tag{2.4}$$

where the U_N are the Ursell Functions (or the connected part of the Green's functions) and are defined by

$$\begin{aligned} U_1(x) &= S_1(x) \\ U_2(x_1,x_2) &= S_2(x_1,x_2) - S_1(x_1)\, S_1(x_2) \\ U_3(x_1,x_2,x_3) &= S_3(x_1,x_2,x_3) - S_2(x_1,x_2)\, S_1(x_3) \\ &\quad - S_2(x_1,x_3)\, S_1(x_2) - S_2(x_2,x_3)\, S_1(x_1) \\ &\ldots \end{aligned} \tag{2.5}$$

To make a lattice approximation to eq. (2.2) we replace the usual expression for a $:P(\phi):$ boson field theory

$$\int d\vec{x} L(\phi) = \tfrac{1}{2} \int_{-\infty}^{+\infty} d\vec{x} [(\nabla\phi)^2 + m_0^2 \phi^2 + g{:}P(\phi){:}] \tag{2.6}$$

by its direct, finite difference approximation. We define a field variable $\phi(x)$ only on the points of that portion of a hypercubic lattice of spacing Δ in a large box of length L on each edge. By $P(\phi)$ we mean a polynomial in ϕ of even degree with leading coefficient unity. With $N = L/\Delta$ we can then write

$$Z(H) = K^{-1} \int_{-\infty}^{+\infty}\!\!\cdots\!\int \prod_{\vec{j}=1}^{N} d\phi_{\vec{j}} \exp\left[-\tfrac{1}{2}\Delta^{d} \sum_{\vec{j}=1}^{N}\left(\frac{\sum_{\{\vec{\delta}\}}(\phi_{\vec{j}+\vec{\delta}} - \phi_{\vec{j}})^2}{\Delta^2} + m_0^2\, \phi_{\vec{j}}^2 + 2g_0\left(:P(\phi_{\vec{j}}):\right) - 2H_{\vec{j}}\, \phi_{\vec{j}}\right)\right] \tag{2.7}$$

where the sum over $\vec{\delta}$ is over half of the nearest-neighbors, i.e. $\vec{\delta} = (1,0,\ldots,0)$, $(0,1,\ldots,0)$, $\ldots$ $(0,0,\ldots,1)$.

If $g_0 = 0$, this action can be diagonalized directly in terms of the momentum transformed variables

$$q_{\vec{k}} = \Delta^{d} \sum_{\vec{j}=1}^{N} \exp(2\pi i\, \vec{k}\cdot\vec{j}\Delta)\, \phi_{\vec{j}} \tag{2.8}$$

as

$$\tfrac{1}{2} L^{-d} \sum_{\vec{k}} \left\{ \sum_{\{\vec{\delta}\}} [4\Delta^{-2} \sin^2(\pi \vec{k}\cdot\vec{\delta})] + m_0^{2} \right\} q_{\vec{k}}\, q_{-\vec{k}} \tag{2.9}$$

The double dots $:P(\phi):$ denote the usual normal-ordered product with respect to the free field ($g_0 = 0$) on a discrete lattice (For the details here and in the rest of this section the reader is referred to Baker[6]). This can be expressed in terms of the fields and their commutator as

$$:(\phi_{\vec{j}})^{p}: = \sum_{n=0}^{[p/2]} (-1)^n \frac{p!}{(p-2n)!\, n!}\, 2^{-n}\, C^n\, (\phi_{\vec{j}})^{p-2n} \tag{2.10}$$

where the commutator is a sum over the lattice Green's function and is

$$C = \frac{1}{L^d} \sum_{\vec{k}} [m_o^2 + 4\Delta^{-2} \sum_{\{\vec{\delta}\}} \sin^2(\pi \vec{k} \cdot \vec{\delta}\Delta)]^{-1}$$
$$\rightarrow \int_{-\frac{1}{2}\Delta}^{\frac{1}{2}\Delta} \cdots \int \frac{d\vec{k}}{m_o^2 + 4\Delta^{-2} \sum_{\{\vec{\delta}\}} \sin^2(\pi \vec{k} \cdot \vec{\delta}\Delta)} \qquad (2.11)$$

where the last line is the thermodynamic limit as $L \rightarrow \infty$. We can see easily from (2.11) that, if Δ is very small

$$C \propto \Delta^{2-d} \qquad d > 2$$
$$C \propto -\ln(\Delta m_o) \qquad d = 2 \qquad (2.12)$$
$$C \text{ is finite} \qquad d < 2.$$

This potential infinity is one of the troublesome problems of field theory. To return to (2.6) from our lattice approximation, there are two limits which must be taken. They are $L \rightarrow \infty$, and $\Delta \rightarrow 0$. We proceed in the way best suited to the techniques of statistical mechanics, namely we first take the thermodynamic limit $L \rightarrow \infty$ and then analyze the behavior as $\Delta \rightarrow 0$. This latter behavior will be intimately related to the problem of critical phenomenon, as we will see.

The first problem in our approach is taking the limit as the box size, L, becomes infinitely large, that is to say to take the thermodynamic limit. As long as $\Delta > 0$ the commutator C is finite and so the interaction in field theoretic language can be reduced by (2.11) to an ordinary, lower-semi-bounded polynomial; the coefficient of the highest power remains unchanged (and = 1 by convention). The main tool in establishing the existence and uniqueness of the thermodynamic limits is the extensive body of inequalities between correlation functions which have been established for the statistical mechanics of this type of problem. In order to use them we consider three types of boundary conditions for (2.7). First, if in the sum over $\vec{\delta}$, a field variable lies outside the box, we replace the whole term by zero. These are called free boundary conditions, and we use a subscript + to denote them. Secondly, if a field variable $\phi_{\vec{j}+\vec{\delta}}$ lies outside the box, we replace it by $\phi_{\vec{j}+(1-N)\vec{\delta}}$. These are called periodic boundary conditions. We use no subscript in this case. Finally for Dirichlet boundary conditions we replace any term in the sum over $\vec{\delta}$ in (2.7) which contains a field variable $\phi_{\vec{j}+\vec{\delta}}$ outside the box by ϕ_j^2. We use a subscript − here. Baker[6] has shown that $(0 \le H_i \le h)$

$$f_+(H) \geq f(H) \geq f_-(H) \tag{2.13}$$

$$S_{n-}(x_1,\ldots x_n) \leq S(x_1,\ldots,x_n) \leq S_{n,b} \tag{2.14}$$

where

$$S_{n,b} = \frac{\int_0^\infty x^n \exp[-\tfrac{1}{2}\beta m_0^2 \Lambda^d x^2 - g\beta :P(x): + \beta h x]dx}{\int_0^\infty \exp[-\beta(\tfrac{1}{2} m_0^2 \Lambda^d + d\Lambda^{d-2})x^2 - g\beta :P(x): + \beta h x]dx} \tag{2.15}$$

In addition, f_+ is monotonically decreasing as a function of L and f_- and S_- are monotonically increasing. These results are sufficient to show that

$$f = \lim_{L\to\infty} \beta^{-1} \ln Z_+ = \lim_{L\to\infty} \beta^{-1} \ln Z = \lim_{L\to\infty} \beta^{-1} \ln Z_- \tag{2.16}$$

independent of boundary conditions. Further the limit

$$S_{n,-} = \lim_{L\to\infty} S_{n,-}(x_1,\ldots, x_n) \tag{2.17}$$

exists and is well defined. When $h \to 0^+$ we can define the $S_{n,-}$ by continuity. Thus at least for Dirichet boundary conditions the theory is well defined in all dimensions for $\Delta > 0$.

In Euclidean boson field theories, the mass of the boson is defined in teims of the decay of the two particle correlation function. We have function of the coefficient p_2 of $:x^2:$ in $:P(x):$. By choosing that coefficient sufficiently large and positive, and sufficiently large and negative one can show,[6,15] that we can make m as large as we like and as small (positive) as we like. It can be shown that (2.20) is equivalent to (2.18), provided m is in a suitable, dimension-dependent range for the spin-½ Ising model. Rosen[16] has also investigated the equivalence of a slightly different definition.

The second half of the question of mass renormalizability is to show that the m of (2.20) is a continuous function of p_2. We have not succeeded in proving this result in general, however in the special case of $g_0:\phi^4:$ theory we have. The proof involves uniform (in L) bounds on the derivative $(\partial m/\partial p_2)$. (See Baker[6] for the details.) The reason this result can be established here is that certain correlation function inequalities hold here, but not in general. Thus, since $m(p_2)$ is a continuous, monotonically increasing function of p_2 in the range $0<m(p_2) < \infty$, we can always solve the equation

$$\mu = m(p_2) \tag{2.21}$$

for p_2 as a function of μ, g_0 and $\Delta > 0$.

In the case of $g_0:\phi^4:$ theory a number of additional useful properties can be established. Firstly the cluster property

$$0 \le S_-(\vec{r}_1,\ldots,\vec{r}_j,\vec{s}_1,\ldots,\vec{s}_h) - S_-(\vec{r}_1,\ldots,\vec{r}_j)\,S_-(\vec{s}_1,\ldots,\vec{s}_h) \le K'\exp(-\mu\rho) \tag{2.22}$$

where K' depends only on $\vec{j}$ and $\vec{h}$ and ρ is the shortest distance (Δ times distance in terms of lattice spacings) between the group of $\vec{r}$'s and the group of $\vec{s}$'s. This cluster property allows the demonstration that the mass-renormalized Schwinger functions S_- and free energy $f_- = f = f_+$ are continuous functions of g_0.

The motivating question of constructive quantum field theory has been, do the series expansions in g_0 (or the renormalized variable g) define a non-trivial field theory. We have seen that the limits $L \to \infty$ for real $0 \leq g_0 < \infty$ do exist and define a theory. In the complex region $Re(g_0) > 0$, $|g_0/m_0^4\Delta^d| << 1$ the expansion given by Baker[6] appendix E shows analyticity of S_- and f. Hence we can apply Carleman's theorem on the uniqueness of analytic functions to conclude that the perturbation expansion in g_0 does indeed define this theory as far as analytic continuation will take it. For $2 \leq d < 4$, we know that mass renormalization suffices to leave all terms of the perturbation series finite (they only diverge like n!). In two dimensions Eckmann et al[17] and Dimock[18] have shown that the angular wedge of analyticity does not shrink to zero as $\Delta \to 0$. In three dimensions the same results have been established by Magnen and Sénéor[19] and Feldman and Osterwalden[20]. The proofs have relied on cluster expansion methods patterned after those of statistical mechanics.

There remains to be investigated for general coupling constant the limit as $\Delta \to 0$. All the properties needed for a Euclidean field theory will hold if this limit can be well defined, with one possible exception. That exception is rotational invariance. It is believed to be a consequence of the approach to the critical point, and is exactly true in the 2-dimensional Ising model and also holds term by term in perturbation theory. Of course, it can not hold in lattice approximation with $\Delta > 0$.

SECTION 3. CRITICAL POINT BEHAVIOR AND THE ULTRA-VIOLET CUTOFF

As we noted in the previous section [eq. (2.19)], the removal of the ultra-violet cutoff ($\Delta \to 0$) implies that the correlation lengths, ξ, goes to infinity. In this section we will relate the behavior of the field theory renormalization constants to the behavior of various thermodynamic quantities[21,22] in the limit as the critical point is approached. This approach allows us to make use of the large body of information that has been derived about the latter problem by exact calculations and highly accurate numerical work based on Padé approximant summation of exact high-temperature and other series.

We will introduce one additional renormalization constant that can be thought of either as replacing coupling-constant renormalization, or as simply an additional renormalization constant. It does not effect any of the results for the free field (Gaussian model, $g_0 = 0$) case, since that action is homogeneous of degree two in the field variables. Our procedure is to attempt to construct non-trivial Euclidean field theories rather than to adhere to the precise procedures of the current presentations of field theory.

We will treat specifically the $g_0 {:}\phi^4{:}_d$ field theory. We will, as mentioned in section 2 maintain the commutator C with respect to the physical mass by treating only interactions of the form

$$g_0{:}\phi^4{:}_d + \delta m^2{:}\phi^2{:}_d \qquad (3.1)$$

with δm^2 adjusted so as to maintain m^2 at a preassigned value. As we observed in the previous section if $g_0 \ll \Delta^d m^4$, δm^2 may be formally expanded in powers of g_0, in accordance with usual theory. Thus the fundamental equation (2.2) becomes

$$Z(H) = K^{-1} \int_{-\infty}^{+\infty}\!\!\cdots\!\int \Pi d\phi_i \exp\Bigg[-\tfrac{1}{2}\beta\Delta^d \sum_i \Bigg\{ \sum_{\{\delta\}} \frac{(\phi_i - \phi_{i+\delta})^2}{\Delta^2} + m^2\phi_i^2$$
$$+ 2g_0[\phi_i^4 - 6C\phi_i^2 + 3C^2] + \delta m^2[\phi_i^2 - C] \; - \beta\sum_i H_i\,\phi_i \Bigg\}\Bigg] \qquad (3.2)$$

The usual procedure which we follow is also at this point to renormalize the external (magnetic) fields by the amplitude renormalization factor $Z_3^{1/2}$ so we replace

$$H_{\vec{i}} \rightarrow H_{\vec{i}} Z_3^{-1/2} \tag{3.3}$$

This change is conveniently absorbed by a change of variables

$$\psi_{\vec{i}} = Z_3^{-1/2} \phi_{\vec{i}} \tag{3.4}$$

Thus, multiply $Z(H)$ by a simple power of Z_3 and redefining K we have

$$\begin{aligned} Z(H) = K^{-1} \int \ldots \int \Pi d\psi_{\vec{i}} \exp\Bigg[- \tfrac{1}{2}\beta\Delta^d \sum_{\vec{i}} \Bigg\{ Z_3 \sum_{\{\vec{\delta}\}} \frac{(\psi_{\vec{i}} - \psi_{\vec{i}+\vec{\delta}})^2}{\Delta^2} + m^2 Z_3 \psi_{\vec{i}}^2 \\ + 2g_o Z_3^2 \left[\psi_{\vec{i}}^4 - 6CZ_3^{-1} \psi_{\vec{i}}^2 + 3C^2 Z_3^{-2} \right] + \delta m^2 Z_3 \left[\psi_{\vec{i}}^2 - C Z_3^{-1} \right] \Bigg\} \\ - \beta \sum_{\vec{i}} H_{\vec{i}} \psi_{\vec{i}} \Bigg] \end{aligned} \tag{3.5}$$

The conditions to be imposed to insure that we have a field theory are now most simply expressed in terms of derivatives of the free energy derived from (3.2). First the renormalized propragator is given in the usual way as

$$\Gamma_R^{(2)}(p,-p) = \left\{ \frac{\Delta^d}{(2\pi)^d} \sum_{\vec{j}=0}^{N-1} \frac{\partial^2 f(H)}{\partial H_o \partial H_{\vec{j}}} \Bigg|_{H=0} \exp\left[-2\pi i \vec{p} \cdot \vec{j} \Delta \right] \right\}^{-1} \tag{3.6}$$

The usual requirements are

$$\Gamma_R^{(2)}(p,-p) = m^2 + 4\pi^2 p^2 \tag{3.7}$$

for p near zero. Before we express (3.6) directly in terms of thermodynamic quantities, it is convenient to re-express (3.5) so that it looks like a continuous spin Ising model. To this end let

$$(\beta\Delta^{d-2}Z_3)^{\frac{1}{2}}\psi_{\vec{i}} = \sigma_{\vec{i}} \tag{3.8}$$

Then we have, neglecting the change in K as we are only concerned with derivatives of f(H),

$$Z(H) = K^{-1}\int_{-\infty}^{+\infty}\!\!\cdots\!\int \prod_{\vec{i}} d\sigma_{\vec{i}} \; \exp\Bigg[\sum_{\vec{i}}\Bigg\{\sum_{\{\delta\}}\sigma_{\vec{i}}\sigma_{\vec{i}+\vec{\delta}} \tag{3.9}$$

$$- \tfrac{1}{2}(2d + m^2\Delta^2 + \delta m^2\Delta^2 - 12C\Delta^2 g_o)\sigma_{\vec{i}}^{\,2} - g_o\beta^{-1}\Delta^{4-d}\sigma_{\vec{i}}^{\,4}$$

$$- \beta^{\frac{1}{2}}Z_3^{-\frac{1}{2}}\Delta^{(2-d)/2}H_{\vec{i}}\sigma_{\vec{i}}\Bigg\}\Bigg]$$

From eq. (3.9) we can see one evident, qualitative difference between the ordinary, $d = 2$ or 3 field theory and the ordinary spin ½ or continuous-spin Ising model. In the Ising model the spin distribution is two-peaked with peaks at about ±1. In the ordinary field theory one expects from lowest order perturbation theory that $\delta m^2 \approx (1+\Delta^{6-2d})g_o$ with a logarithmic divergence replacing Δ^0 in 3 dimensions. Thus the spin-weight distribution looks, to leading order in Δ like $\exp(-d\,\sigma^2)$ with corrections in the exponent $\approx -g_o\Delta^{4-d}(\sigma_{\vec{i}}^{\,4} - A\sigma_{\vec{i}}^{\,2})$ with A of order unity or $\ln\Delta$ in $d = 2$. Of course, these corrections ($d > 2$) are sufficient to significantly modify the longwavelength behavior so cannot be neglected, but qualitatively there may be differences in behavior.

If we now re-express (3.6) from (3.9) in terms of the expectation values of the $\sigma_{\vec{i}}$'s we get

$$\Gamma_R^{(2)}(p,-p) = \left\{\frac{\Delta^2}{(2\pi)^d Z_3}\sum_{\vec{j}=0}^{N-1}\langle\sigma_o\sigma_{\vec{j}}\rangle \exp\left[-2\pi i\,\vec{j}\cdot\vec{p}\Delta\right]\right\}^{-1} \tag{3.10}$$

$$\approx \frac{(2\pi)^d Z_3}{\Delta^2}\,\chi^{-1}\left(1 + \frac{(2\pi)^2}{2}\xi^2\Delta^2 p^2 + \ldots\right. \tag{3.11}$$

where we have used the thermodynamic variables

$$\chi = \sum_{j=0}^{N-1} \sigma_o \sigma_{\vec{j}} \qquad \xi^2 = \frac{\sum_{j=0} j^2 \langle \sigma_o \sigma_{\vec{j}} \rangle}{\sum_{j=0}^{N-1} \langle \sigma_o \sigma_{\vec{j}} \rangle} \tag{3.12}$$

in terms of the Ising-like σ variables. The definition of ξ is the moment definition. The renormalization equations come from comparing (3.7) with (3.11) and yield

$$m^2 = \frac{(2\pi)^d Z_3}{\Delta^2} \chi^{-1}, \quad 4m^2 \xi^2 \Delta^2 = 1 \tag{3.13}$$

The final renormalization equation comes from the zero momentum scattering amplitude. This quantity is expressed in terms of the one particle irreducible scattering amplitude which is defined as

$$\Gamma_R^{(4)}(0,0,0,0) = \frac{-\frac{\Delta^{3d}}{(2\pi)^{3d}} \sum_{\vec{j},\vec{k},\vec{\ell}=0}^{N-1} \left. \frac{\partial^4 f(H)}{\partial H_o \partial H_{\vec{j}} \partial H_{\vec{k}} \partial H_{\vec{\ell}}} \right|_{H=0}}{\left(\frac{\Delta^d}{(2\pi)^d} \sum_{j=0}^{N-1} \left. \frac{\partial^2 f(H)}{\partial H_o \partial H_{\vec{j}}} \right|_{H=0} \right)^4} \tag{3.14}$$

in the case where $\lim_{H \to 0^+} \langle \sigma_{\vec{i}} \rangle = 0$. Re-expressing (3.14) in terms of the σ variables we get

$$\Gamma_R^{(4)}(0,0,0,0) = \frac{-\frac{\beta\, \Delta^{4+d}}{(2\pi)^{3d} Z_3^{\,2}} \left(\frac{\partial^2 \chi}{\partial H^2} \right)}{\left(\frac{\Delta^2}{(2\pi)^d Z_3} \right)^4 \chi^4} \tag{3.15}$$

where we have defined

$$\frac{\partial^2 \chi}{\partial H^2} = \sum_{j,k,\ell} u_4(\sigma_o, \sigma_{\vec{j}}, \sigma_{\vec{k}}, \sigma_{\vec{\ell}}) \tag{3.16}$$

with u_4 as the fourth Ursell function as in (2.4) If we simplify (3.15) by using (3.13) we obtain

$$\Gamma_R^{(4)}(0,0,0,0) = \frac{\beta\, m^{4-d} \; \frac{\partial^2\chi}{\partial H^2}}{(2\pi)^d \;\; \chi^2\, \xi^d} = g \tag{3.17}$$

where g is the renormalized coupling constant.

Now, if following Wilson, we think of δm^2 as a temperature-like variable which we adjust to enforce mass renormalization (eq. 3.13), then in terms of the usual statistical mechanics notation (K is here a reciprocal dimensionless temperature)

$$\begin{aligned} \chi &\approx A_+(1-K/K_c)^{-\gamma}, \qquad \xi \approx D_+(1-K/K_c)^{-\nu} \\ \frac{\partial^2\chi}{\partial H^2} &\approx -B_+(1-K/K_c)^{-\gamma-2\Delta}, \quad \gamma = (2-\eta)\nu \end{aligned} \tag{3.18}$$

where the critical indices are those appropriate to the model described by (3.9).

Thus solving (3.11) and (3.17) we obtain

$$\begin{aligned} \Delta &\approx \frac{\sqrt{2}}{mD_+}\,(K_c-K)^{\nu} \\ Z_3 &\approx \frac{2A_+}{D_+^2(2\pi)^d}\,(K_c-K)^{\eta\nu} \approx A_+\left(\frac{\sqrt{2}}{D_+}\right)^{2-\eta}\frac{(m\Delta)^{\eta}}{(2\pi)^d} \\ \beta &\approx (gm^{d-4})\,\frac{A_+^2D_+^d}{B_+}\,(K_c-K)^{2\Delta-\gamma-d\nu} \\ &\approx (gm^{d-4})\,\frac{A_+^2D_+^d}{B_+}\left(\frac{\sqrt{2}}{mD_+\Delta}\right)^{\omega^*} \end{aligned} \tag{3.19}$$

where we have introduced the anomalous dimension of the vacuum by the relation

$$2\Delta = \gamma + (d-\omega^*)\nu \tag{3.20}$$

The renormalization group approach to critical phenomena as presently presented is based, in our context on the ideas that,

$$\omega^* = \max(0, d-4), \quad \beta \text{ is finite} \tag{3.21}$$

Thus, (gm^{d-4}) must be zero for $d \geq 4$. (The case $d = 4$ is special and $\Lambda^{-\omega^*}$ is replaced by $-\ell n\, \Lambda$.) In our approach we prefer to try to maintain a prescribed value of gm^{d-4} by allowing β to vary with lattice spacing just as δm^2, and Z_3 do. We will see in the next section that renormalization constants which have finite term-by-term perturbation expansions necessarily become lattice spacing dependent in the hyper-strong coupling, Ising model limit.

SECTION 4. BEHAVIOR FOR VARIOUS VALUES OF THE BARE COUPLING CONSTANT

Let us first examine the behavior of Euclidean field theory for the hyper-strong coupling limit,

$$g_0 >> \beta\Lambda^{d-4} \tag{4.1}$$

In equation (3 9) we see that this condition will make the coefficient of σ_i^4 much larger than unity. In this situation, we will exactly simulate a spin-½ Ising model and solve the mass renormalization equation (3.13) (the solution is, as we saw in Section 2, unique) by choosing the coefficient of σ_i^2 to be

$$2K\, g_0\, \beta^{-1}\Lambda^{4-d} = 6\, C\, \Delta^2\, g_0 - d - \tfrac{1}{2} m^2\Lambda^2 - \tfrac{1}{2}\delta m^2\Lambda^2 \tag{4.2}$$

by the adjustment of δm^2. The quantity K is, by(3.19)

$$K \approx K_c + \left(\frac{m\Lambda D_\pm}{\sqrt{2}}\right)^{1/\nu} \tag{4.3}$$

as $\Lambda \to 0$. We emphasize that δm^2 is necessarily of order Λ^{-2} to accomplish this result. In this situation, by choosing g_0 sufficiently large, the spin-weight factor becomes as closely proportional as we like, in the sense of distributions, to the Ising model one

$$\delta(\sigma_i^2 - K) \tag{4.4}$$

Thus the hyper-strong coupling limit, i.e., the bare coupling constant satisfying(4.1), is exactly given in terms of the solution to the corresponding Ising model problem.

Before reviewing the relevant Ising model data, let us as guidance to the intuition compute by the method of Wilson's approximate recursion relations[10,11] the expected behavior for very large and very small values of g_0. It is already well established that at least for $2 < d < 4$ and $g_0\Lambda^{4-d}\beta = O(1)$ that method predicts the convergence under iteration of the Hamiltonian to a fixed point with coefficients of σ^2 and σ^4 of order unity.

These recursion relation can be written (we use the hierarchical model) as

$$I_\mu(x) = \int_{-\infty}^{+\infty} dy \exp[-y^2 - \tfrac{1}{2} Q_\mu(x+y) - \tfrac{1}{2} Q_\mu(x-y)] \tag{4.5}$$

$$Q_{\mu+1}(x) = -2 \ln\left[I_\mu\left(2^{-(1-\sigma/d)/2} x\right) / I_\mu(0)\right] \tag{4.6}$$

where

$$Q_o(x) = g_o \beta^{-1} \Lambda^{4-d} \left(\frac{x^2}{2} - K\right)^2 \tag{4.7}$$

in our case. The functions $I_\mu(x)$ are approximations to the distribution function of x proportional to the mean of the 2^μ spins in a box roughly $2^{\mu/d}$ on an edge. The idea is that if the correlation length is long compared to $2^{\mu/d}$, then the distribution of 2^μ spins should not be very different from that of $2^{\mu-1}$ spins and so the iteration of (4.5) and (4.6) will lead to a fixed point distribution function which allows one to deduce the thermodynamic properties. Clearly this method only is expected to work at or near the critical point. The quantity σ in (4.6) is just $2-\eta$ in the notation of (3.18). If $g_o\beta^{-1}\Lambda^{4-d} << 1$, then we may linearize (4.5) and (4.6) to yield

$$\begin{aligned} r_{\ell+1} &= 2^{\sigma/d} (r_\ell + 3 u_\ell) \\ u_{\ell+1} &= 2^{(2\sigma/d-1)} u_\ell \\ Q_\ell(x) &\approx r_\ell x^2 + u_\ell x^4 \end{aligned} \tag{4.8}$$

If we iterate (4.8) our allowed $L = \log_2 \zeta = -d \log_2 m\Lambda$ steps, we find,

$$u_L \approx (\Lambda\omega)^{-(2\nu-d)}\,\Lambda^{4-d}\,\beta^{-1}\,g_0$$

$$= \frac{B_+\,(m\Lambda)^{2\eta}}{A_+^{\,2} D_+^{\,d}\,\beta}\left(\frac{m\Lambda D_+}{\sqrt{2}}\right)^{\omega^*} g_0 \tag{4.9}$$

where use was made of (3.19). Plainly, whatever the values of η, ω^*, etc. may be, eq. (4.9) defines a lower range, $g_0 \ll (m\Lambda)^{-2\eta-\omega^*}$ in which the fixed point ($u_L = O(1)$) cannot be reached. It would seem intuitively likely that the behavior in this region can be reached perturbatively from expansions about the free-field or gaussian model. We speculate that it is these solutions which are being constructed by highly rigorous methods in 2 and 3 dimensions.

For g_0 satisfying condition (4.1) we proceed in a different fashion. We start with (3.9) which we rewrite for $H = 0$ as

$$Z(0) = K^{-1}\int_{-\infty}^{+\infty}\cdots\int \prod_i d\sigma_i \exp\left[-\tfrac{1}{4}\sum_i \sum_{\{\delta\}}(\sigma_i - \sigma_{i+\delta})^2 + 8\,\tilde{g}\,(\sigma_i^{\,2} - \tfrac{1}{2}u)^2\right] \tag{4.10}$$

where

$$\tilde{g} = \tfrac{1}{4}\,g_0\,\beta^{-1}\Lambda^{4-d}\,, \qquad 2u = 3\,\beta\,C\,\Lambda^{d-2} - \tfrac{1}{4}\,\beta\,\Lambda^{d-2}(m^2 + \delta m^2)/g_0 \tag{4.11}$$

We know from (4.2) and (4.3) that by mass renormalization

$$u \approx K_c + \left(\frac{m\Lambda D_+}{\sqrt{2}}\right)^{1/\nu} + d\,\Lambda^{d-4}\beta/g_0 \tag{4.12}$$

which is of order unity. Thus the starting Q for the recursion relations (4.5) and (4.6) is

$$Q_0(x) = \tilde{g}(x^2 - u)^2 \tag{4.13}$$

We calculate directly from (4.5) that

$$I_o(x) = e^{-Q_o(x)}\int_{-\infty}^{+\infty} \exp[-\,y^2(1 + 6\,\tilde{g}\,x^2) - Q_o(y)]dy$$

$$= e^{-\tilde{g}\xi^2}\int_{-\infty}^{+\infty} \exp[-\,\tilde{g}\,y^4 - (1 + 4\,\tilde{g}\,u)\,y^2 - 6\tilde{g}\,\xi\,y^2 - \tilde{g}\,u^2]dy \qquad (4.14)$$

where we have used $\xi = x^2 - u$. For $\tilde{g}$ very large and ξ very small, we can approximately evaluate the integral by neglecting the $\tilde{g}y^4$ term as being of order $\tilde{g}^{-1}$. Thus

$$I_o(x) \approx \sqrt{\pi}\;\exp\left[-\,\tilde{g}\,\xi^2 - \tfrac{1}{2}\,\ln\left(1 - \frac{6\tilde{g}\xi}{4\tilde{g}u + 1}\right) - \tilde{g}\,u^2 - \tfrac{1}{2}\,\ln(4\tilde{g}u + 1)\right] \qquad (4.15)$$

Hence to order ξ^2 and dominate order in $\tilde{g}$ we have

$$I_o(x)/I_o(0) = \exp\left[-\left(\tilde{g} - \frac{3}{8u^2}\right)\left(x^2 - u - \frac{3u}{8u^2\tilde{g} - 3}\right)^2 + \frac{8u^4\tilde{g}^2}{8u^2\tilde{g} - 3}\right] \qquad (4.16)$$

which shows a fractional change in $\tilde{g}$ and u of only order $\tilde{g}^{-1}$. By use of (4.6) we compute

$$Q_1(x) = 2^{(2\sigma-d)/d}\left(\tilde{g} - \frac{3}{8u^2}\right)\left[x^2 - u\left(1 - \frac{3}{8u^2\tilde{g} - 3}\right)2^{(d-\sigma)/d}\right]^2 - 16\,u^4\tilde{g}^2/(8u^2\tilde{g} - 3) \qquad (4.17)$$

Thus as long as we are in the range of parameters $\tilde{g}$ large and u of order unity or larger we obtain,

$$Q_\ell(x) = \tilde{g}_\ell(x^2 - u_\ell)^2 - \tilde{g}_\ell\,u_\ell^2 \qquad (4.18)$$

with

$$\tilde{g}_{\ell+1} \approx 2^{(2\sigma-d)/d}\left(1 - \frac{3}{8u^2\tilde{g}_\ell}\right)\tilde{g}_\ell$$

$$u_{\ell+1} \approx 2^{(d-\sigma)/d}\left(1 - \frac{3}{8u^2\tilde{g} - 3}\right)u_\ell \qquad (4.19)$$

In the allowed number $L = [-d \log_2(m)]$ of iteration we expect the change of parameter of approximately

$$\tilde{g}_L \approx (\Lambda m)^{d-2\sigma}\,\tilde{g} = \frac{B_+(m^*)^{2\eta}}{4\,A_+^2\,D_*^{\,d}\,g}\left(\frac{m^* D_+}{\sqrt{2}}\right)^{\omega^*} g_o$$

$$u_L \approx \left(\frac{1}{\Lambda m}\right)^{d-\sigma} u \approx \left(\frac{1}{\Lambda m}\right)^{d-2+\eta} K_c \tag{4.20}$$

It will be observed that in this range of g the values of u_L and g_L do not move in the direction of the fixed point in spite of having correctly specified u. This behavior reflects one of the unrealistic aspects of the approximate recursion relations. If we vary u in (4.13) to allow the recursion relations to move towards the fixed point, we find that we must choose

$$u = \lambda\,\tilde{g}^{-\frac{1}{2}} \tag{4.21}$$

Then we have again to order ξ^2

$$I_o(x) \propto \exp[-\tilde{g}\,\xi^2 - a\,\tilde{g}^{\frac{1}{2}}\xi - b\tilde{g}\xi^2] \tag{4.22}$$

The recursion relations (4.19) are replaced by

$$\tilde{g}_{\ell+1} \approx 2^{(2\sigma-d)/d}(1-b)\tilde{g}_\ell$$

$$u_{\ell+1} \approx 2^{(d-\sigma)/d}\left(1 - \frac{a}{2\lambda(1-b)}\right) u_\ell \tag{4.23}$$

In order to maintain the relation (4.21) at the next iteration we select λ by holding $u_\ell^{\,2}\tilde{g}$ fixed, i.e. we must choose

$$1 = 2(1-b)\left(1 - \frac{a}{2\lambda(1-b)}\right)^2 \tag{4.24}$$

It is easy to show that (a & b are functions of λ) eq.(4.24) has a solution, $\lambda = O(1)$, which is valid and independent of $\tilde{g}$, for $\tilde{g}$ large. Using this solution for b, we have now for the L^{th} iteration

$$\tilde{g}_L \propto (\Delta m)^{\omega^*+2\eta-d\ \log_2(1-b)} g_o \qquad (4.25)$$

To the extent that the approximate recursion relations are useful as a guide to intuition, we can now conclude that while it may be that the critical behavior as $T \to T_c$ is described by a single renormalization group fixed point over the whole range $0 < g_o < \infty$, in the field theory context, except in a restricted range of bare coupling parameters, the initial Hamiltonian is too far from the fixed point one for the fixed-point behavior to become manifest on a length scale of order $(\Delta m)^{-1}$. Although the approximate recursion relations are clearly artificial in a number of ways, they give, for $2 - \eta < d < 4 - 2\eta$, three regions

I. $g_o \ll g_I \propto (m\Delta)^{-2\eta-\omega^*}$

II. $g_I \ll g_o \ll g_{III}$ fixed point region

III. $g_o \gg g_{III} \propto (\Delta m)^{-2\eta-\omega^* + d\ \log_2(1-b)}$ Ising region.

According to intuitive ideas, the behavior in regions I and III should show cross-over effects at very small $|K_c - K|$ corresponding to correlation lengths long compared to $(\Delta m)^{-1}$. In this interpretation, the spin-½ Ising model itself (limit $g_o \to \infty$) the cross-over point moves to the critical point and the "true" fixed point behavior is never seen. (An exception to this remark occurs for $d = 1$, because $K_c = \infty$ and regions II and III merge here. I am grateful to R. Schrader for a discussion of this point.) In region I, the apparent behavior could also be quite different from that of the fixed point region, as convergence to that fixed point has not yet occurred. For example, there could be, as indeed has now been proven in 2 and 3 dimensional ϕ^4 theory, a non-constant value of $g(g_o)$, as the Schwinger functions are analytic in an angular wedge for sufficiently small g_o.

In the region $d > 4 - 2\eta$, η_ℓ decreases with ℓ for all values of the bare coupling constant. This remark is in accord with the usual renormalization group theory that says that the fixed point becomes unstable relative to the gaussian $(g = 0)$ one. However, region III survives for sufficiently large g_0; it cannot depart in the required number of steps far from the high g_0 behavior. Consequently, the Ising model in this range of dimensions remains a potential source for the construction of a field theory.

By way of reference the best available numerical information for the spin-½ Ising model[21,23] for the critical indices by the high temperature series methods are given in Table I. These results will be seen to be different than those for the renormalization group fixed point where it is believed that $\omega^* = 0$ for $d = 3,4$. It is to be noted that the uncertainty in η for $d = 4$ is so large that a value $\eta = 0$ is not excluded. If Moore's estimate[23] of ν is used then $\eta = 0.02$ only.

The various amplitudes involved in (3.19) are available in the literature[21,23,24]. With their aid one can, from the available Ising model results, plus the computation of the momentum dependence of the propagator[25], and the scattering function, etc., begin the construction of a non-trivial field theories at least in 2, 3, and 4 dimensions with prescribed values of the renormalized coupling constant.

TABLE I

Critical Indices for the Ising Model

d	γ	η	ω^*
2	1.750	0.25	0
3	1.250 ± 0.003	0.041 ± 0.01	0.044 ± 0.004
4	1.065 ± 0.003	0.08 ± 0.04	0.54 ± 0.08

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