CONF - 770329 - - 1

PREPRINT UCRL- 79275

Lawrence Livermore Laboratory

NUMERICAL SOLUTION OF THE FOKKER-PLANCK EQUATIONS FOR A MULTI-SPECIES PLASMA

John Killeen and Arthur A. Mirin

March 11, 1977

Lecture to be presented at the COLLEGE IN THEORETICAL AND COMPUTATIONAL PLASMA PHYSICS

22 March 1977 - 9 April 1977 Trieste, Italy

NUMERICAL SOLUTION OF THE FOKKER-PLANCK EQUATIONS FOR A MULTI-SPECIES PLASMA*

John Killeen and Arthur A. Mirin

Lawrence Livermore Laboratory, Livermore, California 94550
and
Department of Applied Science, University
of California, Davis/Livermore, California 94550

ABSTRACT

Two numerical models used for studying collisional multi-species plasmas are described. The mathematical model is the Boltzmann kinetic equation with Fokker-Planck collision terms. A one-dimensional code and a two-dimensional code, used for the solution of the time-dependent Fokker-Planck equations for ion and electron distribution functions in velocity space, are described. The required equations and boundary conditions are derived and numerical techniques for their solution are given.

* [illegible]

I. INTRODUCTION

In the simulation of magnetically confined plasmas where the ions are not Maxwellian and where a knowledge of the distribution functions is important, kinetic equations must be solved. The proposition that a stable mirror plasma will yield net thermonuclear power depends on the rate at which particles are lost out the ends of the device. At number densities and energies typical of mirror machines, the end losses are due mainly to the scattering of charged particles into the loss cone in velocity space by classical Coulomb collisions. The kinetic equation describing this process is the Boltzmann equation with Fokker-Planck collision terms [1, 2].

The use of this equation is not restricted to mirror systems. The heating of plasmas by energetic neutral beams, the thermalization of α-particles in DT plasmas, the study of runaway electrons and ions in Tokamaks, and the performance of two-energy component fusion reactors are other examples where the solution of the Fokker-Planck equation is required [3].

The problem is to solve a nonlinear partial differential equation for the distribution function of each charged species in the plasma, as functions of seven independent variables (three spatial coordinates, three velocity coordinates, and time). Such an equation, even for a single species, exceeds the capability of any present computer so several simplifying assumptions are therefore required to treat the problem. Typical approximations that are made in present day codes [illegible] that the distribution [illegible] in velocity space (about the direction of the magnetic field). These assumptions [illegible] the number of

MASTER

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED independent variables: the three—two velocity space coordinates, v and θ, the speed and pitch angle, and the time, t. Even with these basic assumptions there has been an evolution of numerical Fokker-Planck calculations for the past fifteen years [2].

A multi-species code [3] was developed in order to study D-T and D-D mirror reactors, including the effects of reaction products. The principal assumptions of this code are that the "Rosenbluth potentials" (see Section 2) are isotropic and that the distribution functions can be represented by their lowest angular eigenfunctions. The resulting set of coupled equations for $f_a(v,t)$ are then solved numerically, using an implicit finite-difference scheme described in Section 3. An extensive parameter study [4] was conducted yielding values of the confinement parameter nτ and the figure of merit Q (the ratio of thermonuclear power to injected power) as a function of mirror ratio and injection energy.

In Section 5 we describe a highly versatile, two-dimensional multi-species code in which the ion kinetic equations are solved by finite-difference methods on a two-dimensional (v,θ) domain. The electron distribution function is still represented using the lowest angular eigenfunction (isotropic in a full velocity space); hence, a one-dimensional electron kinetic equation is solved. In Ref. [3] we give a variety of applications of this code and describe diagnostics that have been added for the special problems. For mirror systems with beam injection, we find values of the containment parameter nτ as much as 40% higher than those given by the one-dimensional Fokker-Planck code. We also consider two-component toroidal systems and calculate the energy multiplication factor for a number of different scenarios.

2. MULTI-SPECIES FOKKER-PLANCK EQUATIONS

The appropriate kinetic equations are Boltzmann equations with Fokker-Planck collision terms, often referred to simply as Fokker-Planck equations:

$$\frac{\partial f_a}{\partial t} + \underline{v} \cdot \frac{\partial f_a}{\partial \underline{r}} + \frac{\underline{F}}{m_a} \cdot \frac{\partial f_a}{\partial \underline{v}} = \left(\frac{\partial f_a}{\partial t}\right)_c + S_a + L_a \qquad (1)$$

where f_a is the distribution function in 6-dimensional phase space for particles of species a, S_a is the source term, $(\partial f_a/\partial t)_c$ is the collision term, and L_a contains loss terms.

The Fokker-Planck collision term for an inverse-square force was derived by Rosenbluth, et al. [4] in the form

$$\frac{1}{\Gamma_a}\left(\frac{\partial f_a}{\partial t}\right)_c = -\frac{\partial}{\partial v_i}\left(f_a \frac{\partial h_a}{\partial v_i}\right) + \frac{1}{2}\frac{\partial^2}{\partial v_i \partial v_j}\left(f_a \frac{\partial^2 g_a}{\partial v_i \partial v_j}\right) \qquad (2)$$

where $\Gamma_a = 4\pi e_a^4/m_a^2$. In the present work we write the "Rosenbluth potentials" [2]

$$g_a = \sum_b \left(\frac{Z_b}{Z_a}\right)^2 \ln \Lambda_{ab} \int f_b(\underline{v}')|\underline{v} - \underline{v}'| d\underline{v}' \qquad (3)$$

$$h_a = \sum_b \frac{m_a + m_b}{m_b} \left(\frac{Z_b}{Z_a}\right)^2 \ln \Lambda_{ab} \int f_b(\underline{v}')|\underline{v} - \underline{v}'|^{-1} d\underline{v}' \qquad (4)$$

The transformation of Eq. (2) to spherical polar coordinates (v, θ, ϕ) in velocity space has been given by Rosenbluth, et al. [4]. With our assumption of azimuthal symmetry, the resulting distribution functions are of the form $f_a(v, \mu, t)$, where $\mu = \cos\theta$ and $v = |\underline{v}|$. The equation for each species is

$$\frac{1}{f_n}\frac{\partial f_n}{\partial t} = -\frac{1}{v^2}\frac{\partial}{\partial v}\left[f_n v^2 \frac{\partial h_n}{\partial v}\right] - \frac{1}{v^2}\frac{\partial}{\partial \mu}\left[f_n(1-\mu^2)\frac{\partial h_n}{\partial \mu}\right] + \frac{1}{2v^2}\frac{\partial^2}{\partial v^2}\left[f_n v^2 \frac{\partial^2 g_n}{\partial v^2}\right]$$

$$+ \frac{1}{2v^2}\frac{\partial^2}{\partial \mu^2}\left\{f_n\left[\frac{1}{v^2}(1-\mu^2)^2\frac{\partial^2 g_n}{\partial \mu^2} + \frac{1}{v}(1-\mu^2)\frac{\partial g_n}{\partial v}\right.\right.$$

$$\left.\left. - \frac{1}{v^2}\mu(1-\mu^2)\frac{\partial g_n}{\partial \mu}\right]\right\} + \frac{1}{v^2}\frac{\partial^2}{\partial \mu \partial v}\left\{f_n(1-\mu^2)\left[\frac{\partial^2 g_n}{\partial \mu \partial v} - \frac{1}{v}\frac{\partial g_n}{\partial \mu}\right]\right\}$$

$$+ \frac{1}{2v^2}\frac{\partial}{\partial v}\left\{f_n\left[-\frac{1}{v}(1-\mu^2)\frac{\partial^2 g_n}{\partial \mu^2} - [illegible]\frac{\partial g_n}{\partial v} + \frac{2\mu}{v}\frac{\partial g_n}{\partial \mu}\right]\right\}$$

$$+ \frac{1}{2v^2}\frac{\partial}{\partial \mu}\left\{f_n\left[\frac{1}{v^2}\mu(1-\mu^2)\frac{\partial^2 g_n}{\partial \mu^2} + [illegible]\frac{\partial g_n}{\partial v}\right.\right.$$

$$\left.\left. + \frac{2}{v}(1-\mu^2)\frac{\partial^2 g_n}{\partial \mu \partial v} - \frac{2}{v^2}\frac{\partial g_n}{\partial \mu}\right]\right\} ,$$

The functions g_n and h_n, defined by Eqs. ([illegible]) and ([illegible]), can be represented by expansions in Legendre polynomials (Rosenbluth, et al., [illegible]). For this purpose we let

$$g_n(v,\mu,t) = \sum_{l=0}^{\infty} g_l^n(v,t)P_l(\mu), \qquad ([illegible])$$

where

$$g_l^n(v,t) = \frac{2l+1}{2}\int_{-1}^{1} g_n(v,\mu,t)P_l(\mu)\,d\mu \qquad ([illegible])$$

The expansions for g_μ and h_μ are

$$g_\mu(v,\mu,t) = \sum_{j=0}^{\infty} \sum_{b} \left(\frac{Z_b}{Z_\mu}\right)^2 \ln \Lambda_{\mu b} B_j^b(v,t) P_j(\mu) \tag{8}$$

$$h_\mu(v,\mu,t) = \sum_{j=0}^{\infty} \sum_{b} \frac{m_\mu + m_b}{m_b} \left(\frac{Z_b}{Z_\mu}\right)^2 \ln \Lambda_{\mu b} A_j^b(v,t) P_j(\mu) \tag{9}$$

where

$$A_j^b = \frac{4\pi}{2j+1}\left[\int_0^v \frac{(v')^{j+2}}{v^{j+1}} f_j^b(v',t)\,dv' + \int_v^\infty \frac{v^j}{(v')^{j-1}} f_j^b(v',t)\,dv'\right] \tag{10}$$

$$B_j^b = -\frac{4\pi}{4j^2-1}\left[\int_0^v \frac{(v')^{j+2}}{v^{j-1}}\left(1 - \frac{j-1/2}{j+3/2}\frac{(v')^2}{v^2}\right) f_j^b(v')\,dv'\right.$$

$$\left. + \int_v^\infty \frac{v^j}{(v')^{j-3}}\left(1 - \frac{j-1/2}{j+3/2}\frac{v^2}{(v')^2}\right) f_j^b(v')\,dv'\right]. \tag{11}$$

In the computations one takes a finite number of terms in the Legendre expansions of g_μ and h_μ.

Loss Cone Domain, Ambipolar Potential

Since examples studied are devoted to the problem of plasma confinement within systems of magnetic mirrors, we consider the mathematical description of such systems.

The loss cone angle is (Spitzer, [5])

$$\sin^2 \theta_{LC} = 1/R_m , \tag{12}$$

where $R_m = B_m/B(z)$; B_m is the magnetic field at the mirror, and $B(z)$ is the magnetic field at the interior point being considered. A particle whose angle in velocity space is less than θ_{LC} will be immediately lost from the mirror system. θ_{LC} is independent of velocity as well as particle mass and charge. Eq. (12) is derived under the assumption that no electrostatic potential exists, and θ_{LC} is the actual loss angle only under that condition.

However, because of their greater mobility, the scattering rate of electrons will be greater than that of ions, and more electrons than ions will tend to leak out of the ends of the device. Hence, an ambipolar potential will build up, being greatest at the center and decreasing towards the ends. The fact that this potential is established leads to a fundamental change in the loss characteristics for the two types of particles. The loss regions are then defined by a loss angle which is a function of speed and charge. If $Z_n e$ is charge and ϕ is electrostatic potential, the loss angle is given by

$$\sin^2 \theta_{LC} = 1/R_n , \tag{13}$$

where

$$R_n = \left(\frac{1 + Z_n e\phi / \frac{1}{2} m_n v^2}{R_m} \right)^{-1} \tag{14}$$

is the "effective" mirror ratio.

Equation (13) approaches Eq. (12) asymptotically as $v \to \infty$. For ions, the right hand side of Eq. (13) can exceed unity; no ion in such a velocity regime can be contained. Conversely, for electrons, this term can be less than zero; all electrons at such velocities will be electrostatically trapped. The loss region for ions is transformed from a cone into a hyperboloid of one sheet. Its minimum radius occurs at $v = B/c$, and is equal to the minimum ion velocity possible for confinement. The electron loss region is transformed into a hyperboloid of two sheets.

5. SOLUTION OF THE ONE-DIMENSIONAL MULTI-SPECIES EQUATIONS

In this section we describe a numerical model which has proved very useful [illegible], in which we assume that the Rosenbluth potentials given by Eqs. (8) and (9) are isotropic

i.e.,

$$\frac{\partial g_a}{\partial \mu} = \frac{\partial h_a}{\partial \mu} = 0. \tag{15}$$

With this assumption Eq. (5) becomes

$$\frac{1}{\Gamma_a}\frac{\partial f_a}{\partial t} = -\frac{1}{v^2}\frac{\partial}{\partial v}\left(f_a v^2 \frac{\partial h_a}{\partial v}\right) + \frac{1}{2v^2}\frac{\partial^2}{\partial v^2}\left(f_a v^2 \frac{\partial^2 g_a}{\partial v^2}\right)$$

$$+ \frac{1}{2v^3}\frac{\partial g_a}{\partial v}\left[(1-\mu^2)\frac{\partial^2 f_a}{\partial \mu^2} - 4\mu\frac{\partial f_a}{\partial \mu} - 2 f_a\right]$$

$$- \frac{1}{v^2}\frac{\partial}{\partial v}\left(f_a \frac{\partial g_a}{\partial v}\right) + \frac{1}{v^3}\frac{\partial g_a}{\partial v}\left(\mu\frac{\partial f_a}{\partial \mu} + f_a\right).$$

The above equation is separable, and if we let $f_a(v,\mu,t) = U_a(v,t)M_a(\mu)$ then

$$\frac{M_a}{\Gamma_a}\frac{\partial U_a}{\partial t} = -\frac{M_a}{v^2}\frac{\partial}{\partial v}\left(U_a v^2\frac{\partial h_a}{\partial v}\right) + \frac{M_a}{2v^2}\frac{\partial^2}{\partial v^2}\left(U_a v^2 \frac{\partial^2 g_a}{\partial v^2}\right) - \frac{M_a}{v^2}\frac{\partial}{\partial v}\left(U_a\frac{\partial g_a}{\partial v}\right)$$

$$+ \frac{U_a}{2v^3}\frac{\partial g_a}{\partial v}\left[(1-\mu^2)\frac{\partial^2 M_a}{\partial \mu^2} - 2\mu\frac{\partial M_a}{\partial \mu}\right].$$

We obtain an eigenvalue problem for $M_a(\mu)$

$$(1-\mu^2)\frac{d^2M_a}{d\mu^2} - 2\mu\frac{dM_a}{d\mu} + \Lambda_a M_a = 0 \tag{16}$$

which is Legendre's equation on the domain $-\cos\theta^a_{LC} \le \mu \le \cos\theta^a_{LC}$, where θ^a_{LC} is the loss cone angle for species a. For each eigenvalue Λ_a we have an equation of the form [in the following we have replaced $U_a(v,t)$ by $f_a(v,t)$]

$$\frac{1}{f_a}\frac{df_a}{dt} = -\frac{1}{v^2}\frac{\partial}{\partial v}\left[f_a\left(v^2\frac{\partial h_a}{\partial v} + \frac{\partial g_a}{\partial v}\right)\right] + \frac{1}{2v^2}\frac{\partial^2}{\partial v^2}\left\{v^2 f_a\frac{\partial^2 g_a}{\partial v^2}\right\}$$

$$-\frac{\Lambda_a}{2v^3}\frac{\partial g_a}{\partial v} f_a \tag{17}$$

The functions $h_a(v,t)$ and $g_a(v,t)$ are given by the equations

$$h_a(v,t) = 4\pi\sum_b\left(\frac{Z_b}{Z_a}\right)^2\frac{m_a+m_b}{m_b}\ln\Lambda_{ab}\left[\int_0^v f_b(v',t)\frac{v'^2}{v}\,dv'\right.$$

$$\left. + \int_v^\infty f_b(v',t)v'\,dv'\right] \tag{18}$$

$$g_a(v,t) = 4\pi\sum_b\left(\frac{Z_b}{Z_a}\right)^2\ln\Lambda_{ab}\left[\int_0^v f_b(v',t)v\left(1+\frac{1}{3}\frac{v'^2}{v^2}\right)v'^2\,dv'\right.$$

$$\left. + \int_v^\infty f_b(v',t)\left(1+\frac{1}{3}\frac{v^2}{v'^2}\right)v'^3\,dv'\right]. \tag{19}$$

The summations are taken over all the species of particles being considered, including type μ. If we use Eqs. (18) and (19) to evaluate the coefficients of Eq. (17), then the equation for $f_\mu(v,t)$ becomes

$$\frac{\partial f_\mu}{\partial t} = \frac{1}{v^2}\frac{\partial}{\partial v}\left[\alpha_\mu f_\mu + \beta_\mu \frac{\partial f_\mu}{\partial v}\right] - \frac{\gamma_\mu}{v^2} f_\mu \tag{20}$$

where

$$\alpha_\mu = 4\pi\Gamma_\mu \sum_b \left[\left(\frac{Z_b}{Z_\mu}\right)^2 \frac{m_\mu}{m_b} \ln \Lambda_{\mu b} \int_0^v f_b(v',t)v'^2 dv'\right]$$

$$\beta_\mu = 4\pi\Gamma_\mu \sum_b \left(\frac{Z_b}{Z_\mu}\right)^2 \ln \Lambda_{\mu b}\left[\frac{1}{3v}\int_0^v f_b(v',t)v'^4 dv' + \frac{v^2}{3}\int_v^\infty f_b(v',t)v' dv'\right]$$

$$\gamma_\mu = \frac{\Lambda_\mu}{2v} 4\pi\Gamma_\mu \sum_b \left(\frac{Z_b}{Z_\mu}\right)^2 \ln \Lambda_{\mu b}\left[\int_0^v f_b(v',t)v'^2\left(1 - \frac{1}{3}\frac{v'^2}{v^2}\right)dv'\right.$$

$$\left. + \frac{2v}{3}\int_v^\infty f_b(v',t)v' dv'\right] .$$

The number density of particles of type μ is

$$n_\mu(t) = 4\pi\int_0^\infty f_\mu(v,t)v^2 dv . \tag{21}$$

The last term on the right of Eq. (20) is the particle loss term due to scattering into the loss cone. If $\Lambda_\mu = 0$ then $n_\mu(t)$ should remain constant in time which gives the appropriate boundary conditions for the solution of Eq. (20).

$$(21) \qquad \frac{\partial f_\alpha}{\partial t} = [illegible]$$

is satisfied by the [illegible]

$$\left[a_\alpha f_\alpha + b_\alpha \frac{\partial f_\alpha}{\partial v} \right]_{v=0}^{v=v_{max}} = 0 \qquad (22)$$

for $t > 0$, where $0 \leq v \leq v_{max}$ is the domain of Eq. (20). At $v = 0$, Eq. (22) is consistent with $\partial f_\alpha/\partial v = 0$, since $\lim_{v \to 0} a_\alpha(v)/b_\alpha(v) = 0$. In multispecies problems with an ambipolar potential defined, the effective velocity domain will be the interval (v_{min}, v_{max}). In this case v_{min} will depend on the ion species and will equal zero for the electrons.

It is convenient to introduce dimensionless variables. Let $x = v/v_0$, where v_0 is a characteristic velocity, $\tau = t/t_H$ where t_H is a characteristic time, and let $F_\alpha = (4\pi v_0^3/K_\alpha)f_\alpha$, where

$$n_\alpha(0) = K_\alpha \int_0^\infty F_\alpha(x,0)x^2 dx .$$

(The normalizing constants K_α are determined by the initial density and distribution for each species.) We generally take t_H to be defined by the electron equation, i.e., $t_H = (2v_0^3/K_e)(m_e^2/\pi e^4)$.

We define the functionals

$$M(F) = \int_x^\infty F(y,\tau)y \, dy \qquad (23)$$

$$M(F) = \int_0^x F(y,\tau)\, y^4\, dy \tag{24}$$

$$E(F) = \int_0^x F(y,\tau)\, y^2\, dy \, . \tag{25}$$

In terms of the [illegible], Eq. ([illegible]) becomes

$$\frac{\partial F_a}{\partial \tau} = \frac{1}{x^2} \frac{\partial}{\partial x} \left[A_a F_a + B_a \frac{\partial F_a}{\partial x} \right] - \frac{C_a}{x^2} F_a + D_a \, , \tag{26}$$

where

$$A_a = \frac{4\pi\Gamma_a}{v_0^3} \sum_b \left[\left(\frac{Z_b}{Z_a}\right)^2 \frac{m_a}{m_b} F_b \ln \Lambda_{ab} M(F_b) \right]$$

$$B_a = \frac{4\pi\Gamma_a}{v_0^3} \sum_b \left(\frac{Z_b}{Z_a}\right)^2 K_b \ln \Lambda_{ab} \left[\frac{1}{3x} E(F_b) + \frac{x^2}{3} M(F_b) \right]$$

$$C_a = \frac{4\pi\Gamma_a}{v_0^3} \frac{\lambda_a}{2x} \sum_b \left(\frac{Z_b}{Z_a}\right)^2 K_b \ln \Lambda_{ab} \left[M(F_b) - \frac{1}{3x^2} E(F_b) + \frac{2x}{3} M(F_b) \right] .$$

To the right side of Eq. (26) we have added a source term $D_a(x,\tau)$ for each species.

Equation (26) is a system of coupled partial differential equations. For each particle species, a, we have an equation corresponding to λ_a, the eigenvalue of Eq. (16). In principle we could consider several eigenvalues for each species, but in this paper we have used only the lowest eigenvalue corresponding to the first normal mode. Hence we have one equation for each species. In solving the system given by Eq. (26) we consider it as a [illegible] equation of the form [illegible] (27)

[illegible]

$$[illegible] \quad AF \quad \frac{\partial F}{\partial x}$$

[illegible] matrices, and [illegible] the source vector, [illegible]

[illegible] (28)

[illegible] with [illegible] boundary conditions.

We [illegible] Eq. ([illegible] by finite-difference methods. On the domain [illegible] we have a finite-difference mesh denoted by x_i, [illegible] The x spacing [illegible] variable and we define [illegible] $= x_{i+1} - x_i$, $\Delta x_{i-1/2} = x_i - x_{i-1}$, $\Delta x_i = \frac{1}{2}(x_{i+1} - x_{i-1})$. We approximate Eq. [illegible] by the following implicit difference equation

$$\frac{F_i^{n+1} - F_i^n}{\Delta t} = \rho \frac{1}{x_i^{?}} \frac{G_{i+1/2}^{n+1} - G_{i-1/2}^{n+1}}{\Delta x_i} - \frac{C_i^{n+1}}{x_i^2} F_i^{n+1} + D_i^{n+1}$$

$$+ (1 - \rho) \frac{1}{x_i^2} \frac{G_{i+1/2}^{n} - G_{i-1/2}^{n}}{\Delta x_i} - \frac{C_i^n}{x_i^2} F_i^n + D_i^n \qquad (29)$$

[illegible]

$$\gamma_j^{n+1} = \frac{\rho}{x_j^2}\cdot\frac{1}{\Delta x_j}\left(\frac{A_{j+1/2}^{n+1}}{2}+\frac{B_{j+1/2}^{n+1}}{\Delta x_{j+1/2}}\right)$$

$$\delta_j^n = F_{j+1}^n\left[\frac{(1-\rho)}{x_j^2}\cdot\frac{1}{\Delta x_j}\left(\frac{A_{j+1/2}^n}{2}+\frac{B_{j+1/2}^n}{\Delta x_{j+1/2}}\right)\right] + F_j^n\Big\{\frac{1}{\Delta\tau}$$

$$+\frac{(1-\rho)}{x_j^2}\left[\frac{1}{\Delta x_j}\left(\frac{A_{j+1/2}^n - A_{j-1/2}^n}{2} - \frac{B_{j+1/2}^n}{\Delta x_{j+1/2}} - \frac{B_{j-1/2}^n}{\Delta x_{j-1/2}}\right) - C_j^n\right]\Big\}$$

$$+F_{j-1}^n\left[\frac{(1-\rho)}{x_j^2}\cdot\frac{1}{\Delta x_j}\left(-\frac{A_{j-1/2}^n}{2}+\frac{B_{j-1/2}^n}{\Delta x_{j-1/2}}\right)\right]$$

$$+\rho D_j^{n+1}+(1-\rho)D_j^n.$$

In order to linearize the system (30) we extrapolate the α, β, γ defined above to the new time step, $n + 1$, using their values at n and $n - 1$. The method used to solve Eq. (30) is the standard tridiagonal method given by RICHTMYER and MORTON [4].

At every time step we calculate the number density and average energy of each type of particle from the equations

$$n_\beta(\tau) = K_\beta I_2^\beta(\tau)$$

$$E_\beta(\tau) = \frac{3}{2}kT_\beta + \frac{1}{2}m_\beta v_0^2 I_4^\beta(\tau)/I_2^\beta(\tau)$$

where

$$I_2^\beta(\tau) = \int_0^\infty F_\beta(x,\tau)x^2dx$$

$$I_4^\beta(\tau) = \int_0^\infty F_\beta(x,\tau)x^4dx$$

... of the scattering loss term, S_{μ}, can be obtained ... from Eq. (16), for a given e_{μ}. For the ... equation corresponding to the first normal ... value

$$A_{\cdot} \quad (\quad R_{\alpha})^{-1} \qquad (31)$$

where R_{α} ... mirror ratio for particles of type a and ... the midpoint potential ϕ.

For electrons we have

$$R_e \quad R \quad 1 \quad \frac{\phi e}{\frac{1}{2} m_e v^2}^{-1} \qquad (32)$$

where R ... B_{max}/B_0, ... $\frac{1}{2} m_e v_{cr}^2$.

For ions we have

$$R_{\alpha} \quad R^{\frac{1}{2}} \; 1 + \frac{Z_{\alpha} \phi e}{\frac{1}{2} m_{\alpha} v^2}^{-1}, \qquad (33)$$

The procedure for determining ϕ follows ... that

$$Q^-(\tau) \quad n_e(\tau)$$

$$Q^+(\tau) \quad \sum_{\beta} Z_{\beta} n_{\beta}(\tau) \qquad (\beta \neq e)$$

(the sum taken over the ion species). At every time step Q^- and Q^+ are computed and the difference $(Q^+ - Q^-)/Q^-$ is compared to a specified

[illegible]

$$[illegible] \qquad (3.)$$

where [illegible]

[illegible]

[illegible]

where [illegible]

$$\frac{\partial f_k^n}{\partial t} = \sum_{\ell=1}^{N} \frac{1}{v^2} \frac{\partial}{\partial v}\left[\alpha_{k\ell}^n f_\ell^n + \beta_{k\ell}^n \frac{\partial f_\ell^n}{\partial v}\right] - \frac{\gamma_{k\ell}^n}{v^2} f_\ell^n \tag{37}$$

for $k = 1, 2, \ldots, N$.

A detailed description of this formulation appears in Killeen, et al. [illegible]

4. SOLUTION OF THE MULTI-SPECIES EQUATIONS IN A TWO-DIMENSIONAL VELOCITY SPACE

Equation [illegible] in (v,θ) coordinates, written in conservative form, is

$$\frac{1}{\Gamma_a} \left.\frac{\partial f_a}{\partial t}\right|_c = \frac{1}{v^2} \frac{\partial G_a}{\partial v} + \frac{1}{v^2 \sin\theta} \frac{\partial H_a}{\partial \theta} \tag{38}$$

where

$$G_a = A_a f_a + B_a \frac{\partial f_a}{\partial v} + C_a \frac{\partial f_a}{\partial \theta} \tag{39}$$

$$H_a = D_a f_a + E_a \frac{\partial f_a}{\partial v} + F_a \frac{\partial f_a}{\partial \theta} \tag{40}$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

The functions [illegible] and [illegible] are given by Eqs. [illegible] and (9). Equation [illegible] is integrated using the method of [illegible], or [illegible]. We first introduce

$$\frac{1}{\ldots_n}\frac{d\ldots_n}{\ldots} = \frac{1}{\ldots}\frac{d\ldots_n}{dr} \qquad [illegible]$$

using an implicit difference algorithm [illegible], and then advance

$$\frac{1}{\Gamma_n}\frac{\partial f_n}{\partial t} = \frac{1}{v^2 \sin\theta}\frac{\partial H_n}{\partial \theta} \tag{48}$$

in an analogous manner.

Equation (47) is differenced as follows:

$$\frac{f^{n+1}_{i,j} - f^{n}_{i,j}}{\Gamma_n \Delta t} = \frac{A^{n}_{i,j+1} f^{n+1}_{i,j+1} - A^{n}_{i,j-1} f^{n+1}_{i,j-1}}{2 v_j^2 \Delta v_j}$$

$$+ \frac{1}{v_j^2 \Delta v_j}\left[\frac{B^{n}_{i,j+1/2}\left(f^{n+1}_{i,j+1} - f^{n+1}_{i,j}\right)}{\Delta v_{j+1/2}} - \frac{B^{n}_{i,j-1/2}\left(f^{n+1}_{i,j} - f^{n+1}_{i,j-1}\right)}{\Delta v_{j-1/2}}\right]$$

$$+ \frac{1}{4 v_j^2 \Delta v_j}\left[\frac{C^{n}_{i,j+1}\left(f^{n}_{i+1,j+1} - f^{n}_{i-1,j+1}\right)}{\Delta\theta_i}\right.$$

$$\left. - \frac{C^{n}_{i,j-1}\left(f^{n}_{i+1,j-1} - f^{n}_{i-1,j-1}\right)}{\Delta\theta_i}\right] \tag{49}$$

Here, $f^{n}_{i,j} = f(v_j, \theta_i, n\Delta t)$.

By rearranging terms, Eq. (49) may be put into the tri-diagonal form:

$$-\alpha^{n}_{i,j} f^{n+1}_{i,j+1} + \beta^{n}_{i,j} f^{n+1}_{i,j} - \gamma^{n}_{i,j} f^{n+1}_{i,j-1} = \delta^{n}_{i,j} . \tag{50}$$

We see that the terms of mixed second derivative type may not be written fully implicitly if we wish to maintain a tri-diagonal form. Equation (48) is integrated in a similar manner, with the roles of v and θ reversed.

The direct two-dimensional approach for determining the [illegible] velocity-space distributions in mirror-confined plasmas can be used to analyze many different physical situations which cannot be studied with the one-dimensional Lorentzian mass-ratio approach [illegible]. In particular, transient plasmas [illegible] in mirror experiments may undergo rapid changes which do not allow the establishment of a Lorentzian mass-ratio distribution. Furthermore, the injection of well-collimated neutral [illegible] that are sharply peaked in [illegible] velocity space [illegible] would be required for adequate representation.

[illegible]

which serves as the figure of merit for a pulsed TCT system. [illegible]

Several features of the Fokker-Planck model are especially appropriate for representing physically significant effects in a TCT. The [illegible] of all species, including self-interaction [illegible] of the form of the distribution function [illegible] alpha particles and impurity ions are treated on an equal footing with [illegible] there is no inherent restriction [illegible] in the theory. Major [illegible] technique for implementing neutral beam [illegible] plasma and the two-dimensional nature of the velocity [illegible] for distortions of the distribution [illegible] forces.

[illegible] the constants of the particle [illegible]

$$\frac{[illegible]}{[illegible]} \quad [illegible] \tag{52}$$

$$[illegible] \tag{53}$$

[illegible] for a time-varying [illegible] through the time variation of these equations [illegible]

$$[illegible] \tag{54}$$

$$[illegible] \tag{55}$$

Since B is essentially just the toroidal field strength, it varies inversely with R, yielding

$$\frac{\dot B}{B} = -\frac{\dot R}{R} \tag{56}$$

With these results the [illegible] equation becomes

$$\frac{\partial f}{\partial t} + \frac{\dot R}{R}\left[[illegible] \right] [illegible] = \left(\frac{\partial f}{\partial t}\right)_c + S + L \tag{57}$$

where $(\partial f/\partial t)_c$ is given by Eq. ([illegible]). For isotropic electrons we have

$$\frac{\partial f_e}{\partial t} [illegible] \frac{\partial f_e}{\partial v} \frac{\dot R}{R} [illegible] \left(\frac{\partial f_e}{\partial t}\right)_c + S_e + L_e \tag{58}$$

The source term S_n in Eq. (1) is of the form

$$S_n(v,\theta,t) = \sum_{\ell} J_u^{\ell}(t)\ [illegible]_n(v,\theta)\ \delta_c^{u,\ell}(t) \tag{59}$$

where the shape function $[illegible]_n(v,\theta)$ is a Gaussian in v and $\cos\theta$ of [illegible] unity 1, $\delta_c^{u,\ell}(t)$ is either 0 or 1, and $J_u^{\ell}(t)$ is a current.

The loss term L_n may combine several loss processes. Losses due to charge exchange with the beam are expressed as

$$L_n^c = -\left[\sum_{i,\ell} L_{ui}^{\ell}\ [illegible]^{u,\ell}(t)\right] f_n(v,\theta,t) \tag{60}$$

where the quantities L_{ui}^{ℓ} are constant parameters. The presence of the vector $\delta_c^{u,\ell}$ allows the whole charge exchange process to be implemented [illegible]; that is, a given charge exchange term [illegible] with the corresponding source term [illegible] the [illegible] $[illegible]^{u,\ell}(t)$.

The effects of finite particle and energy confinement times in a Tokamak are incorporated by adding a contribution to L_a of the form

$$L_a^{\tau} = -\frac{f_a(v,\theta,t)}{\tau_p} + \frac{1}{v^2}\frac{\partial}{\partial v}\left[\left(\frac{1}{\tau_e} - \frac{1}{\tau_p}\right)\frac{v^3 f_a(v,\theta,t)}{2}\right] \quad (61)$$

where τ_p and τ_e are particle and energy confinement times, respectively. If we ignore all other terms in Eq. (1) and compute moments, we find

$$\frac{\partial n_a}{\partial t} = -\frac{n_a}{\tau_p} \quad (62)$$

$$\frac{\partial (n_a \bar{E}_a)}{\partial t} = -\frac{n_a \bar{E}_a}{\tau_e} \quad (63)$$

In problems where one wants to observe the relaxation of a distribution in the absence of a loss cone, or where one assumes that ions in a loss cone domain are not lost instantly, full velocity space boundary conditions are applied, namely:

(a) $f(v=0,\theta)$ is independent of θ (64)

(b) $\frac{\partial f}{\partial v}(v=0,\theta=\pi/2) = 0$ (65)

(c) $\frac{\partial f}{\partial \theta}(v,\theta=0) = \frac{\partial f}{\partial \theta}(v,\theta=\pi) = 0$. (66)

The first condition is a result of continuity at v=0. The second and third conditions are a result of the requirement that the distribution be azimuthally symmetric.

5. REFERENCES

1. J. Killeen and K. D. Marx, The Solution of the Fokker-Planck Equation for a Mirror-Confined Plasma, In *Methods in Computational Physics* (Academic Press, New York, 1970), Vol. 9, pp. 421-489.
2. A. H. Futch, Jr., J. P. Holdren, J. Killeen, and A. A. Mirin, Plasma Phys. 14, 211 (1972).
3. J. Killeen, A. A. Mirin, and M. E. Rensink, The Solution of the Kinetic Equations for a Multi-Species Plasma, In *Methods in Computational Physics* (Academic Press, New York, 1976), Vol. 16, Chapter VI.
4. M. N. Rosenbluth, W. M. MacDonald, and D. L. Judd, Phys. Rev. 107, 1 (1957).
5. L. Spitzer, *Physics of Fully Ionized Gases*, 2nd ed. (Wiley Interscience, New York, 1962).
6. R. D. Richtmyer and K. W. Morton, *Difference Methods for Initial-Value Problems* (Interscience-John Wiley, New York, 1967).
7. J. M. Dawson, H. P. Furth, and F. H. Tenney, Phys. Rev. Lett. 26, 1156 (1971).
8. H. P. Furth and D. L. Jassby, Phys. Rev. Lett. 32, 1176 (1974).