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ABSTRACT

The normal mode equation for coupled drift and Alfvén waves in a finite-β nonuniform plasma with a sheared magnetic field is solved, in the slab geometry, to investigate the instability of slow Alfvén waves. It is shown, that, besides having an appreciable growth rate, the instability also produces microscopic "tearing" of the rational surfaces which has important implications for anomalous transport.

Traditionally, anomalous transport in magnetically confined plasmas is ascribed to the presence of electrostatic drift-wave fluctuations, which are driven to large amplitudes by the expansion free energy associated with density and temperature gradients in the plasma. A mechanism of equally great interest, which seems to have received much less attention, is the excitation of low-frequency, short wavelength, predominantly electromagnetic fluctuations by the same free energy source. These two kinds of fluctuations can produce loss of confinement by fundamentally different processes. The former is limited to an enhancement of transports across the magnetic surfaces. The latter, on the other hand (in the presence of dissipation), may produce local "tearing" and break-up of magnetic surfaces and thus essentially permit transports along newly "re-connected" field lines. This process is especially significant for short-wavelength fluctuations, since the mode-rational surfaces are densely packed and considerable overlap of "magnetic islands" may readily occur.

In this letter, we examine the instability of slow Alfvén waves in a finite-β nonuniform plasma with sheared magnetic fields. The local dispersion relation for coupled drift and Alfvén waves (which is relevant to the shear-free case) is well known,[2] viz.

$$k_y^2\lambda_s^2 + \left(1 - \frac{\omega^2}{\omega_A^2}\right)\left(1 - \frac{\omega_e^*}{\omega}\right)\left(1 + i\delta\omega\right) = 0 \qquad (1)$$

where $\lambda_s^2 = T_e/M\omega_{ci}^2$, $\omega_A = k_{||}V_A$, $\omega_e^* = -(k_y cT_e/eB_o)(1/n_o)(dn_o/dx)$ with $-e < 0$ and $\delta > 0$ represents the dissipation effect on electrons. We have neglected the ion-sound and ion damping terms.

Equation (1) exhibits two instabilities:

$$\omega_1 \simeq \omega_e^* + \omega_e^* k_y^2 \lambda_s^2 (1 - i\omega_e^* \delta)/(\omega_e^{*2}/\omega_A^2 - 1)$$

$$\omega_2 \simeq \omega_A - \omega_A k_y^2 \lambda_s^2 (1 - i\omega_A \delta)/2(\omega_e^*/\omega_A - 1) . \qquad (2)$$

For $\omega_e^* < \omega_A$, ω_1 is unstable. This is the usual electrostatic drift instability. In this case ω_2 , which corresponds to slow Alfvén waves, is stable. For $\omega_e^* > \omega_A$, the Alfvén root ω_2 is unstable and the usual drift wave is stabilized. Note that the introduction of ω_A terms in Eq. (1) can be considered as a finite-β effect because $\omega_A/\omega_e^* \to \infty$ when $\beta \to 0$.

The normal mode equations for coupled drift and Alfvén waves in the presence of magnetic shear has been investigated previously.[3] However, this calculation restricts itself to an investigation of the finite-β stabilization of the usual electrostatic drift branch ω_1 . Nobody seems to have recovered the branch ω_2 from a nonlocal treatment, so far. It is this mode that we shall be discussing below.

Consider an inhomogeneous (gradient along $\hat{e}_x$) plasma in a sheared magnetic field $\underset{\sim}{B} = B_o(\hat{e}_z + \hat{e}_y x/L_s)$. We describe the electrons by a drift kinetic equation with a Krook-type density conserving collision operator. Ions are fully kinetic and collisionless. Since $\beta = 8\pi n_0(T_e + T_i)/B_0^2 << 1$ and we restrict our attention to slow Alfvén waves which only "bend" the field lines, the field perturbations can be described in terms of two scalar potentials ψ,ϕ defined by

$$\delta E_{\parallel} = -\nabla_{\parallel}\psi \; ; \quad \delta \underset{\sim}{E}_{\perp} = -\nabla_{\perp}\phi \; . \tag{3}$$

ψ, ϕ are related to the vector potential $A\,\hat{e}_z$ (defined by $\delta \underset{\sim}{B}_{\perp} = \nabla \times \underset{\sim}{A}$) through the relation $\psi = \phi - \omega A / k_{\parallel} c$. Linearizing the kinetic equations, solving for δf , integrating over the velocity space and using the quasineutrality condition $\delta n_e \simeq \delta n_i$, we obtain the equation

$$\lambda^2 \; \nabla_{\perp}^2 \; \phi = \varepsilon(\omega, x)\psi \tag{4}$$

where $\lambda^2 = \lambda_s^2 (1 + \tau) \dfrac{\omega + \omega_e^* \tau}{\omega - \omega_e^*}$, $\tau = \dfrac{T_i}{T_e}$, $\lambda_s^2 = \dfrac{T_e}{M\omega_{ci}^2}$,

$$\varepsilon(\omega, x) = 1 + \left[\frac{\omega}{|k_{\parallel} v_e|} \; Z\left(\frac{\omega + i\nu_e}{|k_{\parallel} v_e|}\right)\right]\left[1 + \frac{i\nu_e}{|k_{\parallel} v_e|} \; Z\left(\frac{\omega + i\nu_e}{|k_{\parallel} v_e|}\right)\right]^{-1}$$

$$+ \, i\pi^{1/2}\tau \, \frac{\omega + \omega_e^* \tau}{\omega - \omega_e^*} \; \frac{x_I}{x} \exp\left(-\frac{x_T^2}{x^2}\right) - \frac{x^2}{x_T^2} \, , \tag{5}$$

$k_{\parallel} = k_y x / L_s$; $x_I \equiv \omega L_s / k_y v_i$, $x_T^2 = \omega^2 L_s^2 (\omega - \omega_e^*) / k_y^2 c_s^2 (\omega + \omega_e^* \tau)$ and Z is the plasma dispersion function. We have assumed that for ions, $|\omega / k_{\parallel} v_i| > 1$ is satisfied in the whole x-region of interest and that $|\lambda_s^2 \tau \nabla_{\perp}^2| << 1$. Obtaining the perturbed $\delta j_{\parallel}$ from the first moments of $\delta f_{e,i}$ and using the Ampere's law we obtain a second equation between ψ and ϕ viz.

$$\frac{1}{k_{\parallel}} \; \nabla_{\perp}^2 [k_{\parallel} (\psi - \phi)] = - \frac{\omega(\omega + \omega_e^* \tau)}{\omega_A^2} \, \frac{1}{n_o} \, \nabla_{\perp} \cdot (n_o \nabla_{\perp} \phi) \tag{6}$$

where $\omega_A^2 = k_y^2 V_A^2$, V_A being the Alfvén speed. Equations (4) and (6) may be readily combined to give a fourth order differential equation in ϕ. We now assume $|d^2/dx^2| >> |k_y^2|$. Multiplying the fourth order equation by x^2 and integrating once, we obtain a second order differential equation for $E = d\phi/dx$:

$$\lambda^2 \frac{d}{dx}\left(\frac{dE/dx}{\varepsilon(\omega,x)}\right) - \left(1 - \frac{x_A^2}{x^2}\right) E = \frac{C}{x^2} \qquad (7)$$

where $x_A^2 = \omega(\omega + \omega_e^* \tau) L_s^2/k_y^2 V_A^2$ and C is the constant of integration. The physical meaning of this constant can be understood by passing to the large x limit; the two dominant terms are $E + C/x^2 \simeq 0$ which gives $\delta B_x \simeq$ constant. This is the "constant-Ψ" solution in usual resistive instability theories[5], which then matches on to the outside long-wavelength MHD solutions. The mode of present interest can be found without outside long-wavelength MHD support. So we let $C = 0$.

The mode-structure of the usual electrostatic drift branch is determined by the x_T term in $\varepsilon(\omega,x)$ so that one has to do the ordering $\lambda \sim x_T >> x_A$. Note that $\lambda \sim x_T$ gives $(\omega - \omega_*) \sim \omega_* L_n/L_s$, the well-known drift wave result. Note also that $x_T/x_A >> 1$ is always satisfied as long as $\omega/\omega_* > \beta$. On the other hand, to study the Alfvén branch, one has to order $\lambda \sim x_A << x_T$ so that the mode-structure is decided by the Alfvén term. It is this lack of proper ordering, which has prevented earlier workers[3] from recovering the Alfvén root. Note that $\lambda \sim x_A$ yields $\omega \sim k_y^2 V_A^2 (\lambda_s^2/L_s^2)/\omega_* < \omega_*$ as the root.

To the lowest order then, the Alfvén mode is simply described by the equation

$$\lambda^2 \frac{d^2}{dx^2} E = \left(1 - \frac{x_A^2}{x^2}\right) E \ . \tag{8}$$

In this limit, x_T and x_I are ordered large ($\rightarrow \infty$) and the electron inertia and electron dissipation are neglected. The solutions are given by $x^{1/2} J_n(ix/\lambda)$ and $x^{1/2} Y_n(ix/\lambda)$ where J_n and Y_n are the Bessel functions. The lowest order dispersion relation is given by

$$\left(\frac{x_A^2}{\lambda^2}\right)_0 = \frac{1}{4} - n^2 = \frac{\omega_0(\omega_0 - \omega_e^*)}{\omega_A^2} \frac{L_s^2}{\lambda_s^2} \tag{9}$$

where n is an integer. The eigenfrequencies are given by

$$\omega_{0,1} \simeq \omega_e^* \left[1 + \left(\frac{1}{4} - n^2\right) \frac{\omega_{A0}^2}{\omega_e^{*2}} \frac{\lambda_s^2}{L_s^2}\right]$$

and (10)

$$\omega_{0,2} \simeq \left(n^2 - \frac{1}{4}\right) \frac{\omega_{A0}^2}{\omega_e^*} \frac{\lambda_s^2}{L_s^2} \ .$$

The second root corresponds to the Alfvén branch and is the only unstable branch as shown below. To construct outward propagating (or decaying) solutions far away, we have to include combinations of J_n and Y_n functions. However, the Y_n solution blows up at $x = 0$. This pathology of the zeroth order eigenfunction is associated with the absence, in the zeroth order equations, of a "dissipative" force which may balance the parallel electric fields generated at $x = 0$ by time-dependent magnetic fluctuations. In a collisional plasma, resistivity provides such a force whereas in the collisionless problem, electron inertia takes up that role. We now show that it is possible to remove this pathology by a proper matching solution near $x = 0$.

We first consider the collisional case $\nu_e > \omega$ and $\nu_e/|k_{\|}|v_e >> 1$ in the x-region of interest. The equation we want to solve is

$$\lambda^2 \frac{d}{dx}\left[\left(1 - \frac{x_R^2}{x^2}\right)\frac{dE}{dx}\right] - \left(1 - \frac{x_A^2}{x^2}\right) E = 0 \tag{11}$$

where x_R is the resistive layer thickness defined by $x_R^2 = i\omega\nu_e L_s^2/k_y^2 v_e^2$. We assume $\lambda \sim x_A$ and treat $x_R/x_A << 1$ as a small parameter. Introducing $s = ix/\lambda$ as a new variable, the lowest order solution in the outer region, describing outgoing waves, is given by

$$E^{(0)} = C_1\, s^{1/2}\, H_\nu^{(2)}(s) \tag{12}$$

where $H_\nu^{(2)}$ is the Hankel function of second kind and

$$\nu^2 = (1/4) - (x_A^2/\lambda^2) \ . \tag{13}$$

The first order equation in the outer region can be solved with the boundary condition $E^{(1)}(s \to \infty) = 0$:

$$E^{(1)} = \frac{C_1}{W}\left[s^{1/2}J_\nu(s)\int_\infty^s \frac{d}{ds'}\left(s'^{1/2}\underline{J}_{-\nu}\right)\frac{d}{ds'}\left(s'^{1/2}H_\nu^{(2)}\right)\frac{x_R^2/\lambda^2}{s'^2}\,ds'\right.$$

$$\left. - s^{1/2}\underline{J}_{-\nu}(s)\int_\infty^s \frac{d}{ds'}\left(s'^{1/2}J_\nu\right)\frac{d}{ds'}\left(s'^{1/2}H_\nu^{(2)}\right)\frac{x_R^2/\lambda^2}{s'^2}\,ds'\right] \qquad (14)$$

where the Wronskian $W = -2\sin\pi\nu/\pi$. In the inner region, we introduce $\tilde{x} = x/x_R$ as a variable and write the lowest order equation as

$$\frac{d}{d\tilde{x}}\left[\left(1 - \frac{1}{\tilde{x}^2}\right)\frac{dE}{d\tilde{x}}\right] = -\frac{x_A^2/\lambda^2}{\tilde{x}^2}\,E\ .$$

Making the transformation $\Phi = [(\tilde{x}^2 - 1)/\tilde{x}](dE/d\tilde{x})$, we get an equation which has the solution

$$\Phi = (1 - \tilde{x}^2)^{1/2}\left(C_3 P^1_{\nu+1/2}(\tilde{x}) + C_4\, Q^1_{\nu+1/2}(\tilde{x})\right) \qquad (15)$$

where P^1_μ, Q^1_μ are the associated Legendre functions. The dispersion relation is obtained by imposing the parity conditions at $x = 0$ (viz. $\Phi'(0) = 0$ for even parity and $\Phi(0) = 0$ for odd parity), taking the asymptotic expansion of the Legendre functions ($|\tilde{x}| \to \infty$) and matching the resultant solution E to small - "s" limits of the outer region solution $E^{(0)} + E^{(1)}$. The dispersion relation takes the rather simple form

$$\nu^2 = n^2 - \frac{1}{2}\frac{x_R^2}{\lambda^2} = \frac{1}{4} - \frac{x_A^2}{\lambda^2} \qquad (16)$$

Equation (16) differs from the zeroth order Eq. (9) only in the first order correction term $\sim x_R^2/\lambda^2$. This result can also be obtained by a regular perturbation treatment. Equation (16) can be solved to give an Alfvén root with a frequency $\omega_{0,2} < \omega_*$ [defined in Eq. (10)] and a growth rate $= (m/M)(\nu_{ei}/2\beta)$. Note that this growth rate has an upper limit $\sim \nu_{ei}/2$ (because $\beta \gtrsim m/M$) and is larger by a factor β^{-1} from that of dissipative drift waves. Furthermore, unlike drift waves, there is no shear damping term involved because the wave function is localized deep inside the sound turning point. Note also that for even parity modes $\delta B_x|_{x=0} \neq 0$ i.e., the magnetic field perturbation has a tearing component at the rational surface. The instability will thus lead to the formation of microscopic magnetic islands on mode-rational surfaces.

The normal mode equation for the collisionless case $\omega > \nu_e$ is

$$\lambda^2 \frac{d}{dx}\left(\frac{dE/dx}{1 + x_e Z(x_e/|x|)/|x|}\right) - \left(1 - \frac{x_A^2}{x^2}\right) E = 0 \qquad (17)$$

where $x_e = \omega L_s/k_y v_e$. Assuming $x_e \ll x_A$, the outer region equation is the same as before. The inner region equation is harder to solve in this case. However, an approximate dispersion relation can be obtained by a regular perturbation theory. It is of the form

$$\nu^2 = 1/4 - x_A^2/\lambda^2 \simeq n^2 - in\pi^{-1/2}(x_e/\lambda)\,\ell n(x_I/\lambda) . \qquad (18)$$

Equation (18) differs from Eq. (16) in the form of the perturbation term $(x_e/\lambda)\,\ell n(x_I/\lambda)$ replacing x_R^2/λ^2. The reason for this difference is that the Landau damping term goes as x_e/x for large x (as against x_R^2/x^2 for resistive case). To avoid the logarithmic singularity at large x, we have to introduce a cutoff at the ion-Landau resonance point $x = x_I$. This explains the x_I in the logarithm. Equation (18) leads to the growth rate $(n/2)\pi^{-1/2}(L_n/\beta L_s)(m/M)^{1/2}\omega_* \ln(1/\sqrt{\beta})$ which is again a significantly large fraction of ω_*. Further, in this case also a "tearing" component of magnetic field fluctuation is finite for even modes, electron inertia removing the singularity at $x = 0$.

Two conditions necessary for the validity of above analysis were $\omega < \omega_*$ and $|d^2/dx^2| >> k_y^2$. The two may be combined to give $(n/m)(L_n/L_s)(L_n/\lambda_s) > \beta > n^2(L_n^2/L_s^2)$ where m is azimuthal mode number and n the integer in Eq. (9). We have also assumed x_e^2, $x_R^2 << x_A^2 << x_T^2$ which requires ω/ω_*, $\nu/\omega_* << (\beta M/m)$ and $\omega/\omega_* > \beta$ respectively, conditions which may be readily satisfied.

We now speculate on the consequences of microscopic magnetic islands of the type generated by this instability, on anomalous transport. The typical width w of an island is given by $(\delta B_x L_s/k_y B_0)^{1/2}$. Using the linear relation between δB_x and δn_e, we get

$$\frac{w}{L_n} \sim \beta^{1/2}\left(\frac{L_s}{L_n}\right)^{3/4}\left(\frac{\lambda_s}{L_n}\right)^{1/2}\left(\frac{\delta n}{n_0}\right)^{1/2} . \qquad (19)$$

For $L_s/L_n \sim 15$, $\lambda_s/L_n \sim 10^{-2}$, $\beta \sim 10^{-2}$, $\delta n/n_0 \sim 10^{-2}$, this gives $w/L_n \sim 10^{-2}$. Thus, the size of the island can become comparable to the Larmor radius for the typical density flcutuations observed in toroidal devices. We also know that the distance between mode rational surfaces is given by

$$\frac{\Delta r_s}{L_n} \sim (k_y\lambda_s)^{-1} \frac{\lambda_s}{L_n} \frac{qR}{L_s} \simeq (k_y\lambda_s)^{-1} 10^{-2} .$$

Thus for $k_y\lambda_s \sim 1$, we have $w \sim \Delta r_s$ i.e., there will be a strong overlap of magnetic islands on neighboring rational surfaces. It is quite likely then that the loss of heat and particles from the "confined" plasma is governed by a parallel flow along newly re-connected field lines. Taking a parallel collisional random walk model along field lines, which are themselves randomly going in and out due to ergodicity arising due to overlapping magnetic islands, one comes up with an energy confinement time $\tau_E \sim (\nu_e/v_e^2)(qR\ L_n^2/\lambda_s^2)^2$ which has the correct order of magnitude for typical tokamaks and also gives the observed scaling $\tau_E \sim n$.

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