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SELF-DUAL YANG-MILLS AS A TOTALLY INTEGRABLE SYSTEM

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ABSTRACT

The characteristics of a totally integrable system for the self-dual Yang-Mills equations are pointed out: the Parametric Bianchi-Bäcklund transformations, infinite conservation laws, the corresponding linear systems, and the infinite dimension Kac-Moody algebra.

Introduction

It has become increasingly clear that, besides its mathematical beauty, the Yang-Mills theory[1] may provide the key to our understanding of strong interactions. Despite many interesting theoretical and phenomenological observations such as confinement, asymptotic freedom[2], QCD (quantum chromodynamics) perturbative studies and lattice numerical analysis[3], the non-Abelian gauge theory is far from being solved.

In the past few years, with many colleagues of mine, we have investigated whether the beautiful and powerful techniques developed in solving the many so-called totally integrable systems in two dimensions can be used to solve the Yang-Mills fields in four dimensions. We have found, using the J formulation, that the self-dual Yang-Mills fields strikingly possess many of the characteristics of a totally integrable system[4,5]. The parametric Bianchi-Bäcklund transformations[6], infinite conservation laws[7], and the corresponding linear systems[4,5,8]. Recently we have added a new entry, the infinite-dimensional Lie algebra for the "hidden symmetry" of the self-dual Yang-Mills (SDYM) fields[9,10,11]. **It is the Lie algebra $s\ell(N,C)\otimes C(\lambda,\lambda^{-1})$ for the gauge group SL(N,C), and a** ***symmetric-space over the subalgebra*** $su(N)\otimes R(\lambda)$ ***for the real gauge*** **fields of SU(N).**

As for the full Yang-Mills system, it was hoped that similar progress can be made after the beautiful loop-space chiral equation was formulated for the Yang-Mills fields. However there have been many difficulties and not much progress has been made[5].

Because of limitation of space, in the following I shall mainly list the results.

I. The J Formulation of SDYM Field in Complexified E^4 Space.

In the complexified E^4 space, $\sqrt{2}\, y = x_1 + ix_2$, $\sqrt{2}\bar{y} = x_1 - ix_2$, $\sqrt{2}z = x_3 - ix_4$, $\sqrt{2}\bar{z} = x3 + ix_4$. The self-dual Yang-Mills equations $F_{\mu\nu} = 1/2\, \varepsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$, are

$$F_{yz} = 0 = F_{\bar{y}\bar{z}}, \qquad F_{y\bar{y}} + F_{z\bar{z}} = 0.$$

The first two equations imply that the gauge potential A_μ can always be written in the following form

$$A_y = D^{-1}\partial_y D, \quad A_z = D^{-1}\partial_z D, \quad A_{\bar{y}} = \bar{D}^{-1}\partial_{\bar{y}}\bar{D}, \quad A_{\bar{z}} = \bar{D}^{-1}\partial_{\bar{z}}\bar{D}.$$

For the gauge group SL(N,C), $\det D = \det\bar{D} = 1$. For real SU(N) potentials A_μ, one can show that D and $\bar{D}$ are related, in real coordinate space, as $D^\dagger = \bar{D}^{-1}$. Defining a matrix J by

$$J \equiv D\bar{D}^{-1} \quad , \tag{1.1}$$

which can be shown to be gauge invariant, and $\det J = 1$ for the gauge group SL(N,C), and that J can be made Hermitian, $J^\dagger \doteq J$ in the real coordinate space for real SU(N) gauge fields. Now the SDYM equation can be written as

$$\mathbf{B_y} \equiv \mathbf{J^{-1}\partial_y J}, \quad \mathbf{B_z} \equiv \mathbf{J^{-1}\partial_z J}, \text{ and} \tag{1.2a}$$

$$\partial_{\bar{\mathbf{y}}}\mathbf{B_y} + \partial_{\bar{\mathbf{z}}}\mathbf{B_z} = 0 \ , \tag{1.2b}$$

which we call the left SDYM-J equation; or equivalently the SDYM equation can be written as

$$\widehat{\mathbf{B}}_{\bar{\mathbf{y}}} \equiv \mathbf{J}\partial_{\bar{\mathbf{y}}}\mathbf{J^{-1}}, \quad \widehat{\mathbf{B}}_{\bar{\mathbf{z}}} \equiv \mathbf{J^{-1}\partial_z J} \quad \text{and} \tag{1.3a}$$

$$\partial_y\widehat{\mathbf{B}}_{\bar{\mathbf{y}}} + \partial_z\widehat{\mathbf{B}}_{\bar{\mathbf{z}}} = 0 \ , \tag{1.3b}$$

which we call the right SDYM-J equation.

II. Two Parameter Bianchi-Bäcklund Transformation.

One can easily show that the following transformation is a Bianchi-Bäcklund transformation

$$J^{-1}\partial_y J - J'^{-1}\partial_y J' = e^{i\alpha}\partial_{\bar{z}}(J^{-1}J'), \tag{2.1}$$

with the algebraic constraint $J'J^{-1} - J'^{-1}J = BI$, and α, β real, i.e. if J satisfies Eqs. (1.2, 1.3) so does J'.

III. Infinite Non-local Conservation Laws

Consider B_y and B_z of Eqs. (1.2, 1.3) being the first conserved currents,

$$V_y^{(1)} \equiv B_y = \partial_{\bar{z}}\chi^{(1)}, \quad V_z^{(1)} \equiv B_x = -\partial_{\bar{y}}\chi^{(1)}, \tag{3.1}$$

$\chi^{(1)}$ exists because Eqs. (1.2, 1.3). From this first current we can generate infinite number of them by the following iterative procedure

$$V_y^{(n+1)} = \mathcal{D}_y\chi^{(n)}, \quad V_z^{(n+1)} = \mathcal{D}_z\chi^{(n)}. \tag{3.2}$$

IV. The Linear System for the SDYM Fields

From the infinite non-local conservation law given in Section III, we can obtain the following linear differential equations

$$\partial_{\bar{z}}\chi = \lambda\,\mathcal{D}_y\chi \equiv \lambda(\partial_y + J^{-1}\partial_y J)\chi; \tag{4.1a}$$

$$-\partial_{\bar{y}}\chi = \lambda\,\mathcal{D}_z\chi \equiv \lambda(\partial_z + J^{-1}\partial_z J)\chi. \tag{4.1b}$$

The integrability of these equations gives the left SDYM equations of motion Eq. (1.2). Similarly, for the right SDYM equations Eq. (1.3) we have

$$\partial_x\hat{\chi} = -\frac{1}{\lambda}\,\hat{\mathcal{D}}_{\bar{y}}\hat{\chi} = -\frac{1}{\lambda}\,(\partial_{\bar{y}} + J\partial_{\bar{y}}J^{-1})\hat{\chi}, \tag{4.2a}$$

$$\partial_y\hat{\chi} = \frac{1}{\lambda}\,\hat{\mathcal{D}}_{\bar{z}}\hat{\chi} = \frac{1}{\lambda}\,(\partial_{\bar{z}} + J\partial_{\bar{z}}J^{-1})\hat{\chi}. \tag{4.2b}$$

V. The Kac-Moody Algebra for the Self-Dual Yang-Mills Fields

V.1) The Case of SL(N,C) SDYM Fields

We introduce the following two infinitesimal parametric transformations for the J-field

$$\delta_\alpha(\lambda)J = \alpha_a\delta_a(\lambda)J = -J\chi(\lambda)\,T_\alpha\chi(\lambda)^{-1} = \sum_{m=0}^{\infty}\lambda^m\alpha_a\delta_a^{(m)}J, \tag{5.1}$$

$$\hat{\delta}_\alpha(-\tfrac{1}{\lambda})J = \alpha_a\hat{\delta}(-\tfrac{1}{\lambda})J = \hat{\chi}(-\tfrac{1}{\lambda})\,T_\alpha\hat{\chi}(-\tfrac{1}{\lambda})^{-1}J = \sum_{m=0}^{\infty}\lambda^m\alpha_a\hat{\delta}_a^{(m)}J, \tag{5.2}$$

where $T_\alpha \equiv \alpha^a T_a$, α^a's are infinitesimal parameters and T_a's are traceless anti-Hermitian matrices satisfying $[T_a,T_b] = C^c{}_{ab}T_c$ with $C^c{}_{ab}$ the structure constants of su(N), . For complex α, T_α span the Lie algebra $s\ell(N,C)$, and for real α, T_α span the Lie algebra su(N). Using Eqs. (4.1, 4.2), it is easy to show that $J+\delta_\alpha J$, $J + \hat{\delta}_\alpha J$ satisfy the self duality equations, Eqs. (1.2, 1.3) respectively. Moreover, it follows from $T_r(T_a) = 0$ that $\det(J + \delta_\alpha J) = 1 = \det(J + \hat{\delta}_\alpha J)$. Therefore, these transformations are infinitesimal Bäcklund transformations. We can actually show that they satisfy the same Bäcklund transformations constructed previously in Ref. [6], as given in Section II.

After lengthy calculations, we can derive the following infinite algebraic relations

$$[\Delta_a^{(k)}, \Delta_b^{(\ell)}]J = - C_{ab}^{c}\Delta_c^{(k+\ell)}J, \quad -\infty \leq k, \quad \ell \geq \infty, \tag{5.3}$$

where

$\Delta_a^{(k)} = \delta_a^{(k)}$, for $k > 0$; $\Delta_a^{(k)} = \delta_a^{(0)} + \hat{\delta}_a^{(0)}$, for $k = 0$: and

$\Delta_a^{(k)} = (-)^k\, \hat{\delta}_a^{(-k)}$, for $k \leq 0$. This is the now well-known Kac-Moody algebra $s\ell(N,C) \otimes C(\lambda,\lambda^{-1})$. The important point to note is that is lacks the center of the algebra, which is of the form $c\delta_{ab}\delta_{k,-\ell}$, where c is a constant.

Since the indices m, n in Eq. (5.3) cover all integers, we can re-sum it into a single commutator in the complementary variables θ and θ'. Multiplying both sides of Eq. (5.3) by $e^{im\theta}\, e^{in\theta'}$, and summing, with the definition

$Q(\theta) \equiv \sum_{m=-\infty}^{\infty} e^{im\theta}\, Q_a^{(m)}$, we obtain

$$[Q_a(\theta), Q_b(\theta')] = C_{ab}^{c}\, Q_c(\theta)\, \delta(\theta - \theta'), \tag{5.4}$$

where the variable θ can be identified as (for unimodular λ) $e^{i\theta} = \lambda$, the CP^3 parameter[5,8].

V.2) **The Case of Real SU(N) SDYM Fields**

For J Hermitian and α_a real, we see that δJ, $\hat{\delta} J$ give new $J' \equiv J + \delta J$, $\hat{J}' = J + \hat{\delta} J$ respectively with $\det J' = 1 = \det\hat{J}'$; but J', $\hat{J}'$ are not Hermitian. From the condition $\hat{\chi}^{-1}(\bar{\lambda}) = \chi^{\dagger}(-1/\lambda)$ we can easily show $[\delta(-1/\lambda)J]^{\dagger} = \delta(\bar{\lambda})J$, therefore we can form two Hermitian transformations

$$\overset{(+)}{\delta}_a(\lambda)J \equiv \delta_a(\lambda)J + \hat{\delta}_a(-1/\lambda)J = \sum_{k=0}^{\infty} \lambda^k\, \overset{(+)}{\delta}{}_a^{(k)}J, \tag{5.5}$$

$$\overset{(-)}{\delta}_a(\lambda)J \equiv i[\delta_a(\lambda)J - \delta_a(-1/\lambda)J] = \sum_{k=0}^{\infty} \lambda^k \overset{(-)}{\delta}_a{}^{(k)}J, \qquad (5.6)$$

where J is restricted to be Hermitian. After lengthy derivation we find the algebra

$$[\overset{(+)}{d}_a{}^{(m)}, \overset{(+)}{d}_b{}^{(n)}] = C^c_{ab}\, \overset{(+)}{d}_c{}^{(m+n)},$$

$$[\overset{(+)}{d}_a{}^{(m)}, \overset{(-)}{d}_b{}^{(n)}] \equiv C^c_{ab}\, \overset{(-)}{d}_c\,(m+n), \qquad 0 < m,\ n < \infty \qquad (5.7)$$

$$[\overset{(-)}{d}_a{}^{(m)}, \overset{(-)}{d}_b{}^{(n)}] = C^c_{ab} \sum_{\ell=0}^{(m+n)} a_\ell\, \overset{(+)}{d}_c{}^{(\ell)}$$

where

$\overset{(+)}{d}_a{}^{(0)} \equiv \overset{(+)}{\delta}_a(0)$, $\overset{(\pm)}{d}_a{}^{(1)} \equiv \overset{(\pm)}{\delta}_a{}^{(1)}$, and $[\overset{(\pm)}{d}_a{}^{(m)}, \overset{(\pm)}{d}_b{}^{(1)}] \equiv C^c_{ab}\, \overset{(\pm)}{d}_c{}^{(m+1)}$

and the coefficients a_ℓ are completely determined in the calculation.

So we see that $\overset{(+)}{d}_a{}^{(m)}$, $\overset{(-)}{d}_b{}^{(n)}$ form a symmetric space-like algebra over the $su(N) \otimes R(\lambda)$; or it can be viewed as the Z_2 graded Lie algebra $[su(N) \otimes R(\lambda)] \oplus [su(N) \otimes R(\lambda)]$, where in $R_0(\lambda)$ the constants are not included. We can give still another description of this algebra. Using the fact that $(\Delta_a^{(k)}J)^\dagger = (-1)^k \Delta_a^{(-k)}J$ for real gauge fields, or by using the commutation relations (5.7), it can be shown that this algebra is given by $su(N) \otimes C_R(\lambda,\lambda^{-1})$. $C_R(\lambda,\lambda^{-1})$ is a subalgebra of $C(\lambda,\lambda^{-1})$, consisting of the polynomials invariant under the anti-linear transformations σ, $(\sigma^2 = 1)$, defined by $\sigma(\Sigma a_k\lambda^k) = \Sigma(-1)^k \bar{a}_k\lambda^{-k}$. $C_R(\lambda,\lambda^{-1})$ is the algebra over the real numbers generated by $(\lambda^k - (-\lambda)^{-k})$ and $i(\lambda^k - (-\lambda)^{-k})$, $k \geq 0$. Thus the algebra $su(N) \otimes C_R(\lambda,\lambda^{-1})$ for real SDYM fields can be viewed as a real form of the complex Kac-Moody algebra $s\ell(N,C) \otimes C(\lambda,\lambda^{-1})$.

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