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QUASILINEAR SYSTEMS, SEMIGROUPS, AND NONLINEAR COUPLING

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ABSTRACT

It is known that the semigroup $S(t)$ corresponding to the sum $A + B$ of two non-commuting generators, each having semigroups $S_A(t)$, respectively $S_B(t)$, is given by the Trotter product $S_A(t)*S_B(t) = \lim_{n\to\infty}\left[S_A\left(\frac{t}{n}\right)S_B\left(\frac{t}{n}\right)\right]^n$ provided the latter converges. We apply this principle in treating a quasilinear system with nonlinear coupling. The conjecture is that some hydrodynamic systems may have semigroups.

We try semigroup methods on the following perturbation:

$$\begin{array}{ll} \rho_t + \phi(\rho)_x + \sigma u_x = 0 \quad , & \sigma \geq 0,\ \phi' > 0,\ \Psi' > 0, \\ u_t + \sigma\gamma(\rho)_x + \Psi(u)_x = 0 \quad , & \phi(0) = \Psi(0) = \gamma(0) = 0 \quad , \\ \rho(x,0) = \rho_0(x),\ u(x,0) = u_0(x) \quad , & 0 < m \leq \gamma'(\rho) \leq M \quad , \end{array} \tag{1}$$

of a previously studied separated system of partial differential equations, (PDE's). We work in the Banach space $B_1 = L^1(R) \times L^1(R)$, where $R = (-\infty,\infty)$. The separated system, (put $\sigma = 0$ in (1)), is

$$\frac{dw}{dt} + A_1 w \ni 0 \ , \ w(0) = w_0 \ , \quad A_1 w \ni \begin{pmatrix} \phi(\rho)_x \\ \psi(u)_x \end{pmatrix} \ , \ w = \begin{pmatrix} \rho \\ u \end{pmatrix} \ , \tag{2}$$

in vector notation, where the set-valued operator A_1 has components given by Crandall, [Ref. 3, p. 111, (1.7)]. The perturbing system is

$$\frac{dw}{dt} + \sigma A_2 w \ni 0 \ , \ w(0) = w_0 \ , \quad A_2 w \ni \begin{pmatrix} u_x \\ \gamma(\rho)_x \end{pmatrix} \ , \ \sigma > 0 \tag{3}$$

where A_2, the coupling operator, also has a Crandall-type definition.

System (2) is uncoupled. Each component DE is solved globally using accretiveness principles, (Ref. 3, p. 110), which furnish a unique mild solution $w(t) = S_1(t)w_0$. $S_1(t)$ is the contracting semigroup operator generated by A_1. If $w_0 \in B_1 \cap B_\infty$ where $B_\infty = L^\infty(R) \times L^\infty(R)$, $w(t)$ is a generalized entropy solution of (2), [Ref. 3, p. 113, Theorem 1.2]. System (2) can be classically solved, up to a finite breakdown time, using the method of characteristics.

System (3) is nonlinear, coupled, and is essentially an unsolved nonlinear wave problem, (see [Ref. 8, pp. 306-320] for available material). To get a useful approximate problem, we compute the Yosida approximation $A_{2\lambda}$ of the coupling operator A_2, [Ref. 1, p. 65], by solving the system

$$\begin{aligned} &\rho + \lambda u_x = f \\ &u + \lambda \gamma(\rho)_x = g \ , \quad (f,g) \in B_1 \end{aligned} \tag{4}$$

so as to produce the resolvent $(I + \lambda A_2)^{-1}$. (Solving (4) is facilitated by a result of Brezis, [Ref. 9, p. 123] on the maximal monotonicity in Hilbert space of a set valued operator containing $-\gamma(\rho)_{xx}$; see [Ref. 7].) $A_{2\lambda} = \frac{1}{\lambda} [I - (I + \lambda A_2)^{-1}]$ turns out to have Lipschitz condition:

$$||A_{2\lambda} w_1 - A_{2\lambda} w_2|| \le \frac{1}{\lambda} \left[1 + \sqrt{M} \left(\frac{\sqrt{M}}{m} + 3 \right) \right] ||w_1 - w_2|| \ , \tag{5}$$

$$w_1, w_2 \in B_1 \ .$$

We solve the problem approximate to system (3):

$$\frac{dw}{dt} + \sigma A_{2\lambda} w = 0 \quad , \quad w(0) = w_0 \quad , \sigma > 0 \quad , \tag{3_λ}$$

using O.D.E. methods, to produce the quasi-contractive semigroup solution $w = S_{2\lambda}(t)w_0$ generated by $\sigma A_{2\lambda}$ in B_1 .

Using vector notation, we may rewrite system (1) as follows:

$$\frac{dw}{dt} + A_1 w + \sigma A_2 w \ni 0 \quad , \; w(0) = w_0 = \begin{pmatrix} \rho_0 \\ u_0 \end{pmatrix} \quad , \tag{6}$$

and the approximate problem we are going to solve is

$$\frac{dw}{dt} + A_1 w + \sigma A_{2\lambda} w \ni 0 \quad , \quad w(0) = w_0 \quad . \tag{6_λ}$$

The mild solution, $w_\lambda(t) = S_\lambda(t)w_0$ of (6_λ) ought now to be expressible as the Trotter product:

$$\begin{aligned} w_\lambda(t) = S_\lambda(t)w_0 &= \lim_{n\to\infty} \left[S_{2\lambda}\left(\frac{t}{n}\right)S_1\left(\frac{t}{n}\right)\right]^n w_0 \\ &= S_{2\lambda}(t) * S_1(t)w_0 \quad . \end{aligned} \tag{7}$$

The expeditious way to show that (7) converges is first to prove that $S_\lambda(t)$ exists, using a well known semigroup perturbation result: Theorem: Let A be maximal accretive [Ref. 3, p. 110, (1.2), (1.4)] in a Banach space X, and let F be a Lipschitz operator on X to itself, with Lipschitz constant L. Then for every $x_0 \in \overline{D(A)}$, the problem

$$\frac{du}{dt} + (A + F)\, u \ni 0 \quad , \quad u(0) = x_0$$

has the unique mild solution $u(t) = S(t)x_0 \in X$, where the operator $S(t)$, $t > 0$, has the semigroup property and is quasi-contractive:

$$S(t + s) = S(t)S(s) \quad , \quad S(0) = I$$

and

$$||S(t)x - S(t)y|| \leq e^{Lt}\, ||x - y|| \quad , \quad x, y \varepsilon X \quad .$$

The proof of this result is repeated in [Ref. 5, Theorem 3]. Also, the result is contained in a Corollary of Calvert and Gustafson, [Ref. 2, p. 152]. The theorem is directly applicable to the situation with Eq. (6_λ). The semigroup $S_\lambda(t)$ in (7) exists. The convergence of the product in (7) is then based on a result of Brezis and Pazy, [Ref. 1, pp. 68, 71; Theorems 3.2, 4.1], and the observation that A_1 in (2) is single valued on $D(A_1) \cap B_\infty$, [Ref. 3, p. 112, Lemma 1.1]. In the just-cited theorem of Brezis and Pazy, we put $T(t) = S_{2\lambda}(t)S_1(t)$. Tracing the logic of the cited theorems, we state our first main result:

Theorem 1: Approximate problem (6_λ) has the unique global mild solution given by (7). $S_\lambda(t)$ is a quasi-contractive semigroup with Lipschitz constant $e^{\beta_\lambda t}$, where $\beta_\lambda = \frac{\sigma}{\lambda}\left[1 + \sqrt{M}\left(\frac{\sqrt{M}}{m} + 3\right)\right]$. The Trotter product expression converges uniformly on bounded t-intervals.

Crandall states, [Ref. 3, p. 113, Theorem 1.2] that if $y \varepsilon B_1 \cap B_\infty$, then $S_1(t)y \varepsilon B_1 \cap B_\infty$, and that $||S_1(t)y||_\infty \leq ||y||_\infty$, where $||\cdot||_\infty = ||\cdot||_{L^\infty(R)} + ||\cdot||_{L^\infty(R)}$ is the norm of B_∞. Using methods like those leading to (5), [Ref. 7], it is shown that $A_{2\lambda}$ in Eq. (6_λ) is Lipschitz continuous in $B_1 \cap B_\infty$ in terms of the norm $||\cdot||_\infty$, with constant $\frac{1}{\lambda}\left[1 + \left(\frac{M}{m}\right)^{3/2} + \left(\frac{3M}{\sqrt{m}}\right)\right]$. Hence the semigroup $S_{2\lambda}(t)$ solving problem (3_λ) takes $B_1 \cap B_\infty$ into itself. It is not difficult then to deduce the following result:

Theorem 2: The semigroup $S_\lambda(t)$ generated by $A_1 + \sigma A_{2\lambda}$ in (6_λ), and which thereby gives the solution with initial data $y \varepsilon B_1 \cap B_\infty$, is such that $S_\lambda(t)y \varepsilon B_1 \cap B_\infty$. Moreover, we have $||S_\lambda(t)y||_\infty \leq e^{t\beta_\lambda^\infty}\, ||y||_\infty$, where $\beta_\lambda^\infty = \frac{\sigma}{\lambda}\left[1 + \left(\frac{M}{m}\right)^{3/2} + \left(\frac{3M}{\sqrt{m}}\right)\right]$.

The mild solution $w_\lambda(t) = S_\lambda(t)w_0$ given by Theorem 1 in B_1 has classical validity wherever $w_\lambda(t)$ is time-differentiable. Time-parametrized absolutely continuous locii in non-uniformly convex $B_1 = L^1(R) \times L^1(R)$ are not necessarily time-differentiable however. A mild solution is such a locus, and is the unique strong solution, when valid. Crandall faced this situation with semigroup $S_1(t)$ in [Ref. 3]. He showed that $S_1(t)$ gives the weak entropy solution for problem (2), whatever the value $t > 0$, [Ref. 3, p. 113, Theorem 1.2, (ii)], if $w_0 \epsilon B_1 \cap B_\infty$. We do likewise for our semigroup $S_\lambda(t)$, obtained for problem (6_λ).

<u>Theorem 3</u>: Trotter product expression (7), with $w_0 \epsilon B_1 \cap B_\infty$, is a generalized entropy solution for approximate problem (6_λ). Thus $w_\lambda(t) = (\rho_\lambda, u_\lambda)$ is such that, with $\rho_\lambda(0), u_\lambda(0) \epsilon L^1(R) \cap L^\infty(R)$,

$$\int_0^T\int_{-\infty}^{\infty}\{|\rho_\lambda - k_1|f_t + \text{sign}\ (\rho_\lambda - k_1)\ [\phi(\rho_\lambda) - \phi(k_1)]f_x$$

$$- \sigma\text{sign}\ (\rho_\lambda - k_1)\ [A_{2\lambda}(\rho_\lambda, u_\lambda)]_1\ f\}dxdt \geq 0 \qquad (8a)$$

and

$$\int_0^T\int_{-\infty}^{\infty}\{|u_\lambda - k_2|g_t - \sigma\text{sign}\ (u_\lambda - k_2)\ [A_{2\lambda}(\rho_\lambda, u_\lambda)]_2 g$$

$$+ \text{sign}\ (u_\lambda - k_2)\ [\psi(u_\lambda) - \psi(k_2)]g_x\}dxdt \geq 0 \quad , (8b)$$

for every pair $f(x,t), g(x,t)$ of non-negative twice differentiable test functions compactly supported in R, every pair k_1, k_2 of real constants, and arbitrary $T > 0$.

Here, the components $[A_{2\lambda}(\rho_\lambda, u_\lambda)]_1$, $[A_{2\lambda}(\rho_\lambda, u_\lambda)]_2$ of the Lipschitz operator $A_{2\lambda}$ in Eq. (3_λ) must be laboriously derived from $\begin{pmatrix} u_x \\ \gamma(\rho)_x \end{pmatrix} \epsilon A_2$ in Eq. (3), using $A_{2\lambda} = \frac{1}{\lambda}[I - (I + \lambda A_2)^{-1}]$.

The proof we give for Theorem 3 is a two-component version of a proof of Kruzkov, [Ref. 4, p. 236]. Starting with the problem

$$\frac{dw}{dt} + A_{1\mu}w + \sigma A_{2\lambda}w - \varepsilon\ \vec{\Delta}\ w = 0,\ w(0) = w_0 \ \epsilon\ B_1 \cap B_\infty\ , \qquad (9)$$

which has a unique classical solution expressible as a Trotter triple product:

$$w_{\lambda,\mu,\varepsilon}(t) = S_\varepsilon(t)*S_{2\lambda}(t)*S_{1\mu}(t)\, w_0 \quad ,$$

(where $A_{1\mu} = \frac{1}{\mu}[I - (I + \mu A_1)^{-1}]$ is the generator of semigroup $S_{1\mu}(t)$), we proceed in the manner of Kruzkov. After breaking (9) into coupled PDE's involving components of $A_{1\mu}$: $[A_{1\mu}(\rho)]_1$, $[A_{1\mu}(u)]_2$, and components of $A_{2\lambda}$: $[A_{2\lambda}(\rho,u)]_1$, $[A_{2\lambda}(\rho,u)]_2$, we form Kruzkov's non-negative integral expressions [Ref. 4, p. 236, (4.23)], using a pair of convex entropies and a pair of non-negative compactly supported test functions f,g. Letting $\mu \to 0$ first in these expressions preserves needed regularity. Then an application of the integration by parts recipe, and letting $\varepsilon \to 0$, brings us to our inequalities (8). This process is described in more detail in recent work of the author [Ref. 5].

For this specific problem of nonlinearly coupled PDE's, the convergence question, i.e., what happens when we refine the Yosida approximation, taking $\lambda \to 0$ in Eq. (6_λ), has not yet been dealt with. A recent work of the author, [Ref. 6], treats the case where the coupling operators in (6) and (6_λ), i.e., A_2, $A_{2\lambda}$, are linear. We foresee using similar methods at this time.

Namely, we shall study the λ-limit process by using an equivalent evolution equation:

$$\frac{dy}{dt} + A_1'(w_\lambda(t))y + \sigma A_{2\lambda} y = 0 \quad , \quad y(0) = w_0 \in B_1 \cap B_\infty \quad , \qquad (10)$$

where, in contrast with previous work [Ref. 6, Eq. (7)], the operator $A_{2\lambda}$ is now nonlinear: $A_{2\lambda} = \frac{1}{\lambda}[I - (I + \lambda A_2)^{-1}]$, satisfying Lipschitz condition (5).

Also, in (10) one sees the same linearized operator $A'(w_\lambda(t))$ as before, [Ref. 6, (9)],

$$A_1'(w_\lambda(t))w = \begin{pmatrix} \phi'(\rho_\lambda(x,t))\,\frac{\partial}{\partial x}\,\cdot & 0 \\ 0 & \psi'(u_\lambda(x,t))\,\frac{\partial}{\partial x}\,\cdot \end{pmatrix}\begin{pmatrix} \rho \\ u \end{pmatrix} \tag{11}$$

where $w_\lambda(t) = (\rho_\lambda(x,t),\ u_\lambda(x,t))$ is a vector known from (7). From (11) it can be seen that if we have uniqueness for the solution of (10), then the solution of (10) must be $w_\lambda(t)$ itself, the generalized entropy solution of (6_λ). Using (10), we can hope to study the convergence process as $\lambda \to 0$, as was done in the case of linear coupling, [Ref. 6, Eq. (7)]. It is important that the linearization and substitution of $w_\lambda(t)$ from (7), is done in a term not involved in the Yosida approximation $A_{2\lambda}$ in (6_λ) and (10).

The fact that $A_{2\lambda}$ is now nonlinear does not hinder our showing that $A_1'(w_\lambda(t))$ is Lipschitz continuous in t, [Ref. 6, Lemma 2], nor our selection of a subsequence $\{w_{\lambda_n}(t)\}$ by the weak $(*)$ compactness of the sets $\{\phi'(\rho_\lambda(x,t)),\ \lambda > 0\}$, $\{\psi'(u_\lambda(x,t)),\ \lambda > 0\}$ under the assumptions $0 < m \leq \phi' \leq M$, $0 < m \leq \psi' \leq M$, [Ref. 6, proof of Lemma 4]. It appears that other facets of our previous work with Eq. (10) will need a general overhaul.

The equations for one-dimensional isentropic flow, which are more or less physical, can be written in the form

$$\begin{aligned} &\hat{\rho}_t + u\hat{\rho}_x + \enclose{circle}{u_x} = 0 \quad , \quad \hat{\rho} = \log \rho \quad , \quad \rho = \text{density} \\ &u_t + \enclose{circle}{\gamma(\hat{\rho})_x} + \tfrac{1}{2}\,(u^2)_x = 0 \qquad\qquad u = \text{particle velocity} \end{aligned} \tag{12}$$

where γ is derived from an equation of state. Eq. (12) can be separated into the diagonal problem and the coupling problem. We encircle coupling terms in (12). Though Crandall [Ref. 3, p. 126] cites difficulties with the operator $\frac{1}{2}(u^2)_x$, one can hope that methods such as discussed herein might be applied ultimately to show a generalized entropy semigroup solution.

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