INTEGRAL METHODS FOR SOLVING FOKKER-PLANCK-TYPE EQUATIONS*

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ABSTRACT

Fokker-Planck-type equations occur quite often in different domains of physics and applied mathematics as various realizations of a generic degenerate parabolic equation. Even in the simplest situations, the analysis of the general Fokker-Planck equation is difficult and has been mostly confined to the linear case, where partial results have been obtained in showing existence, uniqueness, regularity, and completeness of eigenfunctions. In the present paper, we present a canonical integral approach that solves, in principle, the most general linear or nonlinear Fokker-Planck-type equations. The method is formal in the sense that it does not provide *per se* the means to prove existence and uniqueness of the solution in an abstract setting. The formalism is based on the Green's functions and their natural extensions to nonlinear systems and allows one to compute the solution (assumed to exist uniquely), by using a canonical iterative scheme. We present several applications of the integral approach in connection with previously developed methods and results.

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1. INTRODUCTION

This paper is devoted to the presentation of an integral approach for solving Fokker-Planck-type equations.[1] The general Fokker-Planck equation is a degenerate parabolic equation of the form

$$u_t + \sum_i (b_i u)_{x_i} - \sum_{i,j} (a_{ij} u_{v_j})_{v_i} + \sum_i (a_i u)_{v_i} - au - f = 0 \tag{1.1}$$

where the probability distribution function $u \in \mathbf{R}_+$ depends on position $x \in \Omega_1 \subseteq \mathbf{R}^m$, velocity $v \in \Omega_2 \subseteq \mathbf{R}^n$, $m \leq n$, (a_{ij}) is a strictly positive definite matrix, and time $t > 0$. To specify the solutions of Eq. (1.1), additional restrictions have to be imposed in the form of: (i) initial condition at $t = 0$, and (ii) boundary conditions at the boundaries $\partial\Omega_1$ and $\partial\Omega_2$ of the domains Ω_1 and Ω_2, respectively.

Initial-boundary value problems involving Fokker-Planck-type equations occur frequently in reaction-diffusion systems, ecology and biology, electron scattering, kinetic theory, aerosol dynamics, and various stochastic phenomena. The corresponding problems may be linear or nonlinear, stationary or time-dependent, scalar or vector (multi-component). In the linear case, the functions b_i, a_{ij}, a_i, a, and f are independent of u. In the stationary case, none of the functions appearing in the equation depend on time. In the vector version, the distribution u has more than one component, $u \in (\mathbf{R}_+)^k$.

Several examples are listed below for $n = 1, m = 1$, $k = 1$.

1. Burgers' equation:

$$u_t + uu_x - u_{xx} = 0\ ; \tag{1.2}$$

2. Homogeneous Fokker-Planck equation:

$$u_t + vu_v - u_{vv} = 0\ ; \tag{1.3}$$

3. Nonhomogeneous time-dependent Fokker-Planck equations:

$$u_t + vu_x + vu_v - u_{vv} = 0\ , \tag{1.4}$$

$$u_t + vu_x - u_{vv} = 0\ ; \tag{1.5}$$

4. Nonhomogeneous stationary Fokker-Planck equations:

$$vu_x + vu_v - u_{vv} - au = 0\,, \tag{1.6}$$

$$vu_x - u_{vv} = 0\,. \tag{1.7}$$

Even in the linear case, the rigorous analysis of the Fokker-Planck equation is difficult and far from being complete. Partial results have been obtained in showing existence, uniqueness, regularity, and completeness of eigenfunctions, by using a host of standard and/or problem-tailored methods.[2-18]

In this paper, we present some aspects of the integral approach used in solving the Fokker-Planck equation. The first two sections are devoted to a canonical integral formalism that can be applied, in principle, to all the cases embodied in Eq. (1.1). The method is formal in the sense that it does not provide *per se* the means to prove existence and uniqueness of the solution in an abstract setting. These issues have to be handled separately, in a problem-determined context. The final section of the paper exemplifies some typical situations. Each example uses a different variant of the integral approach thus showing the versatility and potential of the method.

2. GENERAL INTEGRAL FORMALISM

The linear or nonlinear system can be represented in abstract form as

$$N(u) + \delta B(u) = f + \delta g \tag{2.1}$$

where $N(u)$ describes the linear or nonlinear equation, $B(u)$ represents linear or nonlinear boundary conditions, f is the volume source, and g represents the boundary sources (including initial conditions). The source terms f and g include all inhomogeneities, so we can consider, without loss of generality, that $N(0) = 0$ and $B(0) = 0$. The δ distributions multiplying the boundary terms in (2.1) allow a formally unified abstract treatment of both evolution equations and initial/boundary conditions. These δ distributions are associated with the direct boundary space of the problem,[19] being uniquely specified for each well-posed specific problem under consideration.

The system (2.1) represents an equation for the function u belonging to a linear functional space $\mathcal{L}(\Omega)$ endowed with an inner product denoted by $<,>$. For simplicity, we shall assume $\mathcal{L}(\Omega) = \mathcal{L}^*(\Omega)$; throughout this work, Ω denotes the set (including the time domain for time-dependent problems) that defines the phase space for Eq. (2.1). Since we include the boundaries in the formal treatment, Ω is a closed set, containing the boundaries of the phase space underlying the problem.

We assume that the first Gâteaux derivatives of the operators appearing on the left-hand side of Eq. (2.1) exist, and they are defined[20] by

$$[N'(u) + \delta B'(u)]h \stackrel{\text{def}}{=} \{(d/d\epsilon)[N(u + \epsilon h) + \delta B(u + \epsilon h)]\}_{\epsilon=0} . \tag{2.2}$$

In the linear case, the Gâteaux derivative is independent of u; in general, (i.e., in the nonlinear case), these operators depend nonlinearly on u but act linearly on the vector h. Thus, the operator adjoint to $N'(u) + \delta B'(u)$ is defined via the usual linear duality:

$$< [N'(u) + \delta B'(u)]h, v >=< h, [N'^*(u) + \delta^* B'^*(u)]v > , \tag{2.3}$$

where v is an arbitrary function from the dual space $\mathcal{L}^*(\Omega) = \mathcal{L}(\Omega)$. In Eq. (2.3), $N'^*(u)$ is the formal adjoint of $N'(u)$, and $B'^*(u)$ includes all surface terms required in defining the actual adjoint. Note in (2.3) that the operator δ^* is not the same δ distribution as in (2.1) or (2.2), but is a distribution associated with the adjoint boundary space of the problem.[19] To highlight this distinction, we use the symbolical notation δ^*. The exact significance and the differences between the δ and δ^* symbols will become clearer in the sections devoted to applications.

Following the theory developed in Refs. 21 and 22, we define the operators

$$L(u)h \stackrel{\text{def}}{=} \int_0^1 N'(\epsilon u)h d\epsilon \qquad \beta(u)h \stackrel{\text{def}}{=} \int_0^1 B'(\epsilon u)h d\epsilon \tag{2.4}$$

and

$$L^*(u)v \stackrel{\text{def}}{=} \int_0^1 [N'(\epsilon u)]^* v d\epsilon \qquad \beta^*(u)v \stackrel{\text{def}}{=} \int_0^1 [B'(\epsilon u)]^* v d\epsilon , \tag{2.5}$$

that still act linearly on h and v, respectively, while retaining a nonlinear parametric dependence on u. Like the variational operators, the integrated operators $L(u)$, $\beta(u)$, $L^*(u)$, $\beta^*(u)$, are still related by the linear duality relation

$$L^*(u) + \delta^* \beta^*(u) = [L(u) + \delta\beta(u)]^* . \tag{2.6}$$

We note also the important relationship satisfied by $L(u)$ and $\beta(u)$:

$$[L(u) + \delta\beta(u)]u = N(u) + \delta B(u) . \tag{2.7}$$

In the linear case, N and L coincide and do not depend on u. Equation (2.7) underscores the important role played by the integrated operators L, β: in contradistinction to the variational operators N', β', it is the pair of integrated operators $\{L(u), \beta(u)\}$ that replicates exactly the original nonlinear system (2.1) when applied to u. We define then

the backward (retarded) and forward (advanced) propagators, G_u and G_u^*, as the inverses of the operators $L(u) + \delta\beta(u)$ and $L^*(u) + \delta^*\beta^*(u)$, respectively:

$$[L(u) + \delta\beta(u)]G_u = 1 \tag{2.8}$$

and

$$[L^*(u) + \delta^*\beta^*(u)]G_u^* = 1 \tag{2.9}$$

where 1 denotes the unit operator. In the linear case, when L and β do not depend on u, the propagators are the ordinary Green's functions of the problem. Equations (2.8) and (2.9) can be written in terms of formal integral kernels as

$$[L(u(x)) + \delta\beta(u(x))]G(u(x); x, x') = \delta(x - x') \tag{2.10}$$

and

$$[L^*(u(x)) + \delta^*\beta^*(u(x))]G^*(u(x); x, x'') = \delta(x - x'') \tag{2.11}$$

where x is a shorthand notation for the generic variable in the phase space domain Ω (including its boundaries).

Since the operators $L(u) + \delta\beta(u)$ and $L^*(u) + \delta^*\beta^*(u)$ act linearly on the respective propagators, the relationships between the propagators and the expression for the solution u in terms of these propagators can be derived, as previously noted, in the same spirit as for the usual Green's functions formalism in linear theory.[23,24] Thus, forming the inner products of (2.10) and (2.11) with $G^*(u(x); x, x'')$ and $G(u(x); x, x')$, respectively, leads to the reciprocity relation[21,22]

$$G^*(u(x); x, x') = G(u(x'); x', x)\ , \tag{2.12}$$

which, in the linear case, reduces to the ordinary reciprocity relation between Green's functions.

The solution u of the original nonlinear system (2.1) is obtained in terms of the forward propagator G_u^* as follows by using Eqs. (2.5) and (2.7):

$$\begin{aligned} u &= < u, \delta > - < N(u) + \delta B(u), G_u^* > + < f + \delta g, G_u^* > \\ &= < u, [L^*(u) + \delta\beta^*(u)]G_u^* > - < [L(u) + \delta\beta(u)]u, G_u^* > + < f + \delta g, g_u^* > \\ &= < f + \delta g, G_u^* > = < G_u, f + \delta g > \ . \end{aligned} \tag{2.13}$$

In terms of integral kernels (2.13) reads

$$u(x) = \int_\Omega G(u(x); x, x')[f(x') + \delta g(x')]dx' = \int_\Omega G^*(u(x'); x', x)[f(x') + \delta g(x')]dx' \ , \tag{2.14}$$

that, once the propagators are expressed in terms of u, can be viewed as a nonlinear integral equation for u. In the linear case, G and G^* do not depend on u, and (2.13) and (2.14) simply express the solution as the propagation of the sources via the ordinary Green's functions.

3. INTEGRAL EQUATIONS FOR PROPAGATORS

By applying the formalism developed in the previous section, the original nonlinear problem has been reduced to a linear problem of finding G^* as the solution of Eq. (2.11), where u is assumed to be a known function. In the following, we shall sketch a general canonical method of carrying out such an inversion.

Consider (2.10) for a known vector u^0 and (2.11) for the actual solution $\tilde{u}$ of the original system (2.1), i.e.,

$$[L(u^0) + \delta\beta(u^0)]G_{u^0} = \delta(x - x') \tag{3.1}$$

and

$$[L^*(\tilde{u}) + \delta^*\beta^*(\tilde{u})]G^*_{\tilde{u}} = \delta(x - x'') \ . \tag{3.2}$$

Forming the inner product of (3.1) and (3.2) with $G^*_{\tilde{u}}$ and G_{u^0}, respectively, yields

$$< \ [L(u^0) + \delta\beta(u^0)]G_{u^0}, \ G^*_{\tilde{u}} \ > = G^*_{\tilde{u}} \tag{3.3}$$

and

$$< \ G_{u^0}, [L^*(\tilde{u}) + \delta^*\beta^*(\tilde{u})]G^*_{\tilde{u}} \ > = G_{u^0} \ . \tag{3.4}$$

Using the duality relationship (2.6) and subtracting (3.4) from (3.3) leads to

$$G^*_{\tilde{u}} = G_{u^0} + \ < \ G_{u^0}, [L^*(u^0) + \delta^*\beta^*(u^0) - L^*(\tilde{u}) - \delta^*\beta^*(\tilde{u})]G^*_{\tilde{u}} \ > \ . \tag{3.5}$$

Equation (3.5) is a closed-form nonlinear integrodifferential equation satisfied by the forward propagator $G^*_{\tilde{u}}$. A similar equation is satisfied by the backward propagator $G_{\tilde{u}}$.

Note also that (3.5) is *exact* and its nonlinear character occurs, not from closure approximations, but reflects *exactly* the nonlinearities of the original system (2.1). Moreover, it gives a practical recipe for finding the propagator via an iteration scheme.

Because it retains the full nonlinear information contained in the original problem (2.1), Eq. (3.5) may be rather cumbersome. Moreover, it yields only the propagator, so the solution to the original system must subsequently be computed from the convolution expressions given by (2.13) or (2.14).

An alternative, and sometimes more efficient approach, is to obtain an integrodifferential equation, similar to (3.5), for the solution $\tilde{u}$ itself. For this purpose, we note that since the function u^0 is known, Eq. (3.1) can, in principle, be solved to obtain G_{u^0} and thus $G^*_{u^0}$ as the inverse of the *linear* operators $L(u^0)+\delta\beta(u^0)$ and $L^*(u^0)+\delta^*\beta^*(u^0)$, respectively. Actually, the whole idea of this method is to choose u^0 in such a way that the propagators G_{u^0} and $G^*_{u^0}$ are readily available. Then, the solution $\tilde{u}$ can be obtained by using (3.1), (3.2), and (2.1), the linearity of the operators L^* and β^*, and performing the following sequence of operations:

$$\begin{aligned}
\tilde{u} &= < \tilde{u}, [L^*(u^0)+\delta^*\beta^*(u^0)]G^*_{u^0} > \\
&= < \tilde{u}, [L^*(\tilde{u})+\delta^*\beta^*(\tilde{u})]G^*_{u^0} > + < \tilde{u}, [L^*(u^0)+\delta^*\beta^*(u^0)-L^*(\tilde{u})-\delta^*\beta^*(\tilde{u})]G^*_{u^0} > \\
&= < [L(\tilde{u})+\delta\beta(\tilde{u})]\tilde{u}, G^*_{u^0} > + < \tilde{u}, [L^*(u^0)+\delta^*\beta^*(u^0)-L^*(\tilde{u})-\delta^*\beta^*(\tilde{u})]G^*_{u^0} > \\
&= < f+\delta g, G^*_{u^0} > + < \tilde{u}, [L^*(u^0)+\delta^*\beta^*(u^0)-L^*(\tilde{u})-\delta^*\beta^*(\tilde{u})]G^*_{u^0} > .
\end{aligned} \tag{3.6}$$

In the linear case, L^* and β^* do not depend on the unknown solution $\tilde{u}$ and thus (3.6) reduces to (2.14) with G^* independent of u, and solving (2.1) reduces to calculating the ordinary Green's function of the problem. In special situations, this can be done by using auxiliary problems[18] or eigenfunction expansions[16] (see the examples in Section 4.2 and 4.3).

In the general situation, the linear case may turn out to be more complicated from an iterative viewpoint than the nonlinear case. This is due to the fact that G^* is the actual Green's function of the problem and may be rather difficult to obtain. In this case, one computes first G^* via a perturbative scheme along the following lines:

If $L^*+\delta^*\beta^*$ is not readily invertible, we start with a simpler operator $L^*_0+\delta^*\beta^*_0$ that *is* readily invertible, namely $[L^*_0+\delta^*\beta^*_0]^{-1} = G^*_0$. Then, we write $L^*+\delta^*\beta^*$ as $L^*_0+\delta^*\beta^*_0+L^*_1+\delta^*\beta^*_1$ and compute $G^*(u^0)$ via the usual perturbative series

$$G^* = G^*_0 + G^*_0(1-(L^*_1+\delta^*\beta^*_1)G^*_0)^{-1}(L^*_1+\delta^*\beta^*_1)G^*_0 .$$

In the case of the linear Fokker-Planck equation, rigorous results for this scheme have been obtained by Weber,[4] Il'in,[5] and Eidel'man.[19]

4. ILLUSTRATIONS OF THE INTEGRAL APPROACH

4.1. A Nonlinear Equation: Applying the Propagator Method

In order to illustrate some basic aspects of the canonical formalism described in the previous section, we consider the equation

$$N(u) = u_{xx} - 2uu_x + 2u_x = 0 \qquad x \in \mathbf{R}_+ \tag{4.1}$$

with the boundary conditions

$$\begin{aligned} u(0) &= 0 \\ u_x(0) &= 1 \, . \end{aligned} \tag{4.2}$$

The solution is readily obtained in the form

$$u(x) = \frac{x}{x+1} \, . \tag{4.3}$$

We want to obtain this solution by applying the integral formalism based on propagators. It is easy to see that

$$\begin{aligned} L(u)G &= G_{xx} - uG_x - u_xG + 2G_x \\ L^*(u)G^* &= G^*_{xx} + uG^*_x - 2G^*_x \, . \end{aligned} \tag{4.4}$$

By choosing $u^0 = 2$, we solve the equation

$$L^*(2)(G^*_{u^0})_{xx} = (G^*_{u^0})_{xx} = \delta(x - x') \, ; \qquad G^*_{u^0}(x, x') = 0 \qquad x > x'$$

and get

$$G^*_{u^0}(x, x') = \begin{cases} x' - x & x' \geq x \\ 0 & x' < x \end{cases} \qquad G^*_{u^0}(x', x) = \begin{cases} x - x' & x \geq x' \\ 0 & x < x' \end{cases}$$

$$(G^*_{u^0})_{x'}(x', x) = \begin{cases} -\theta(x - x') & x \geq x' \\ 0 & x < x' \end{cases} \, .$$

Thus, (3.6) becomes

$$
\begin{aligned}
u(x) = & \int_0^x (f(x') + \delta(x') \cdot 0 + \delta'(x') \cdot 1)\, G^*_{u_0}(x', x) dx' = \\
& - \int_0^x u(x')(u(x') - 2)(-\theta(x - x'))dx' = \\
& - \int_0^x \delta(x')(-\theta(x - x'))dx' + \int_0^x u(x')(u(x') - 2)dx' = \\
& 1 + \int_0^x u(x')(u(x') - 2)dx' \, .
\end{aligned}
\tag{4.5}
$$

One verifies that this integral equation is solved by $\frac{x}{x+1}$.

4.2. Deriving the Green's Function from an Auxiliary Problem

As mentioned before when the auxiliary propagator $G^*_{u_0}$ cannot be computed, like in the previous section, one can apply a perturbative formalism for finding it. While having the advantage of being applicable to both nonlinear and linear problems, the perturbative scheme may turn out to be cumbersome, and alternative methods are desirable. One such method is described in the following example. Let us solve the linear one-dimensional boundary value problem

$$
Au \overset{\text{def}}{=} vu_x - u_{vv} = 0 \qquad v \in \mathbf{R} \tag{4.6}
$$

with boundary conditions

$$
u(0, v) = g(v) \qquad v \in \mathbf{R}_+ \, . \tag{4.7}
$$

Unlike the Green's function of the usual stationary Fokker-Planck equation, Eq. (1.6), the Green's function of the problem (4.6) cannot be found explicitly in terms of eigenfunctions by the method of Ref. 15. This is because the essential spectrum of the differential operator $u \to -u_{vv}$ is $\mathbf{R}_+$, and is not separated from zero, as required by the method in Ref. 15. Gorkov[10] was able to obtain an explicit solution of (4.6) by using a singular integral equation, but that approach is rather complicated.

An alternative method for deriving the Green's function for problem (4.6) has been proposed by Eidelman[25,26] and we apply it here to solve (4.6). Consider the forward Green's function $\mathcal{G}^*$ for the time-dependent problem

$$
\mathcal{G}^*_t + A^*\mathcal{G}^* = \mathcal{G}^*_t - v\mathcal{G}^*_x - \mathcal{G}^*_{xx} = 1 \, , \qquad x \in \mathbf{R} \, , \quad v \in \mathbf{R} \, , \tag{4.8}
$$

whose solution with kernel

$$\mathcal{G}^*(x,v,t,x',v',0) = \begin{cases} \dfrac{\sqrt{3}}{2\pi}\,\dfrac{1}{t^2}\,\exp\left\{-\dfrac{(v-v')^2}{4t} - \dfrac{3}{t^3}\left(x-x'+\dfrac{v+v'}{2}t\right)^2\right\} & t>0 \\ 0 & t<0 \end{cases} \tag{4.9}$$

was found by Kolmogoroff.[2]

The crucial observation is that the forward Green's function G^* whose kernel is defined as

$$G^*(x,v,x',v') \overset{\text{def}}{=} \int_0^\infty \mathcal{G}^*(x,v,t,x',v',0)dt = \frac{\sqrt{3}}{2\pi}\int_0^\infty \frac{dt}{t^2}\,\exp\left\{-\frac{(x-x')}{4t} - \frac{3}{t^3}\left(x-x'+\frac{v+v'}{2}t\right)^2\right\} \tag{4.10}$$

satisfies the equation $A^*G^* = 1$ and therefore solves the problem (4.6). Indeed, by applying formula (2.14) and letting $x \to 0_+$, one obtains an integral equation for the unknown part of the distribution $u(0,v)$, $v \in \mathbf{R}_-$:

$$u(0,v) - \int_{-\infty}^0 v'G^*(0,v',0,v)u(0,v')dv' = \int_0^\infty v'G^*(0,v',0,v)g(v')dv' \qquad v \in \mathbf{R}_- \,. \tag{4.11}$$

Once $u(0,v)$, $v \in \mathbf{R}_-$, the solution at $x > 0$ is obtained from (2.14)

$$u(x,v) = \int_{-\infty}^\infty v'G^*(0,v',x,v)u(0,v')dv' \,. \tag{4.12}$$

This method does not depend on the dimensions of the problem. Since for the three-dimensional time-dependent problem the Green function is also known,[3] this solves otherwise untreatable stationary three-dimensional problems. The application of this procedure to spherical geometry is reported in Ref. 18.

4.3. Integral Equations from Eigenfunction Expansions

In this section, we shall solve directly the integral equation for the solution itself, therefore avoiding the computation of the Green's function. We shall apply the formula (3.6) to the linear stationary one-dimensional problem (1.6) with $a \geq \epsilon > 0$ to show how to take advantage of the eigenfunction expansions when they exist.[12,16]

Adapting the results obtained in Ref. 12 for $a = 0$ to the present case ($a > 0$), we write the solution of the kinetic half-space problem in the form

$$u(0,v) = \sum_{n>0} c_n \varphi_n(v) \,, \tag{4.13}$$

$$u(0,v) = g(v) \,, \qquad v \in \mathbf{R}_+ \tag{4.14}$$

where we take into account[12] that: (i) the spectrum is discrete, (ii) zero does not belong to it, and (iii) $\varphi_n(v)$ are normalized, $\int_{-\infty}^{\infty} \varphi_n \varphi_m w dv = \delta_{nm}$ (w is the full range weight). The coefficients c_n can be computed by using full range formulas,[12,14-16]

$$c_n = \int_{-\infty}^{\infty} u(0,v)\varphi_n(v) w dv = \int_0^{\infty} g(v)\varphi_n(v) w dv - \int_0^{\infty} u(0,-v)\varphi_n(-v) w dv \,. \tag{4.15}$$

By using these coefficients in Eq. (4.13) and projecting the solution on positive and negative velocities, we obtain two (equivalent) integral equations for the unknown part of the distribution at the boundary, $u(0,v), v \in \mathbf{R}_-$, namely:

$$g(v) = \sum_{n>0} \varphi_n(v) \left\{ \int_0^{\infty} g(v')\varphi_n(v') w dv' - \int_0^{\infty} u(0,-v')\varphi_n(-v') w dv' \right\} \quad v \in \mathbf{R}_+ \tag{4.16}$$

$$u(0,v) = \sum_{n>0} \varphi_n(v) \left\{ \int_0^{\infty} g(v')\varphi_n(v') w dv' - \int_0^{\infty} u(0,-v')\varphi_n(-v') w dv' \right\} \quad v \in \mathbf{R}_- \tag{4.17}$$

These equations are the ones obtained by Mayya[14] for the Fokker-Planck equation and by Klaus et al. in the abstract formulation of the Fokker-Planck-type equations.[15]

To solve these equations, one applies the standard techniques, i.e., multiply (4.16) by $\varphi_m(v)$, $m > 0$, and integrate over the full range to get

$$c_m = \int_0^{\infty} g(v')\varphi_m(v') w dv' - \sum_{n>0} \int_0^{\infty} c_n \varphi_n(-v')\varphi_m(-v') w dv' \qquad m > 0 \,. \tag{4.18}$$

An equivalent system is obtained by multiplying (4.16) by $\varphi_m(-v)$, $m < 0$, to get

$$0 = \int_0^{\infty} g(v')\varphi_m(v') w dv' - \sum_{n>0} \int_0^{\infty} c_n \varphi_n(-v')\varphi_m(v') w dv' \qquad m < 0 \,, \tag{4.19}$$

from where one gets c_n via truncation. Once c_n are known, the distribution at the boundary is given by (4.13), $v \in \mathbf{R}$ and the full solution is obtained via a semigroup reconstruction.[12,16]

5. CONCLUSIONS

We have presented several aspects of an integral approach for solving Fokker-Planck type equations. The general formalism yields integral equations for the Green's functions (linear case) or propagators (nonlinear case) associated with the original problem. In general, these integral equations have to be solved iteratively, but compact solutions can be obtained sometimes by specific artifacts (like in Section 4.2).

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