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NONLINEAR SYSTEM VIBRATION
-THE APPEARANCE OF CHAOS

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Autobiography.

B.S. in Electrical Engineering from Mississippi State University in 1964. M.S. in Electrical Engineering from the University of New Mexico in 1966. Employed by Sandia National Laboratories in the Vibration Testing Division from 1964 through 1975. M. A. in Secondary Science Education from the University of New Mexico in 1976. High school mathematics teacher at St. Pius High School from 1975 through 1978. Employed by Los Alamos National Laboratory as either a Staff Member or Section Leader in the Dynamic Testing Section from 1979 through 1990.

Introduction

What we now consider to be chaotic motions were noted by Poincare in 1892 where he describes the motion of a system governed by differential equations as having a "sensitive dependence on initial conditions". More recently Lorentz described chaotic behavior in fluid dynamics and outlined what is now known as the Lorentz attractor[1]. Chaotic behavior has been observed in many fields including physics, chemistry, biology and sociology. The history of dynamical systems and chaotic behavior is described in very readable format by Gleick[2] and Abraham and Shaw[3].

Chaotic systems commonly occur in the context of vibration testing because in the testing of real structures we often encounter systems with significant nonlinearities driven by periodic or random forcing functions. Much of the classical chaos described in the literature occurs when strongly nonlinear systems are excited by periodic forces. Chaos like phenomena also occur in the context of systems driven by random forces.

This paper begins with an examination of the differential equation for a single degree of freedom force excited oscillator and considers the state space behavior of linear, nonlinear, and chaotic single degree of freedom systems. The fundamental characteristics of classical chaos are reviewed: sensitivity to initial conditions, positive Lyapunov exponents, complex Poincare maps, fractal properties of motion in the state space, and broadening of the power spectrum of the system response. Illustrated examples of chaotic behavior include motion in a two well potential - the chaos beam described in Moon[4] and a hardening base excited Duffing system. Chaos-like phenomenon which occur with nonperiodic forcing are examined in the context of the two well potential and hardening Duffing systems. The paper concludes with some suggestions for detecting and modelling nonlinear or chaotic behavior.

A Single Degree of Freedom Oscillator

Consider the single degree of freedom oscillator shown in Figure 1. A periodic driving force excites motion of the mass M. The differential equation describing this system is :

[illegible] $\zeta\omega Y'$ [illegible] $\omega^2 Y$ [illegible] M [illegible] ωt [illegible] (1)

where [illegible]

ω = [illegible]

$\zeta = \frac{C}{2M\omega}$ = viscous damping coefficient.

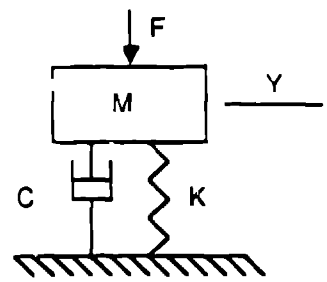


Figure 1
A Force Excited Single Degree of Freedom Oscillator

As an example choose $F/M = 1.10$, $\omega_n = 2\pi$, $\omega = 2\pi$, and $\zeta = 2.25\%$. The state variables describing the system are the displacement of the mass, Y, and the velocity of the mass Y'. This system is simulated using variable stepsize Runga Kutta algorithm implemented on a Sun 4 Sparkstation. The acceleration response of the system to sinusoidal forcing is shown in Figure 2. Sampled values of the velocity and displacement plotted in state space are shown in Figure 3.

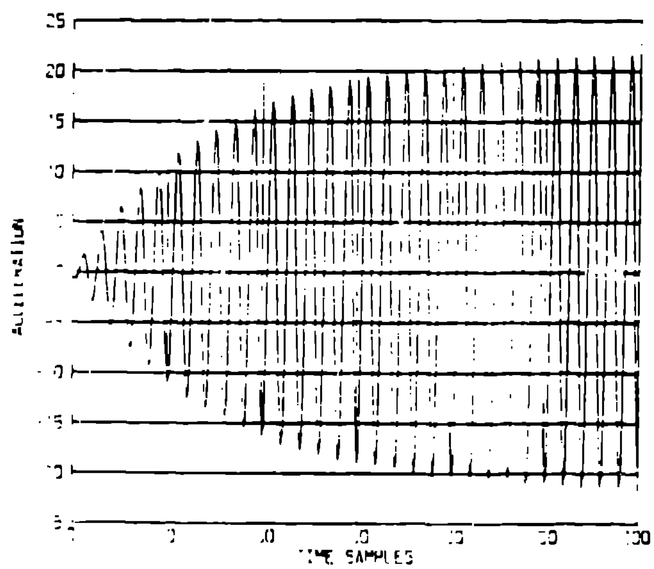


Figure 2
Acceleration Response of a Single Degree of Freedom Oscillator Driven by Sinusoidal Forcing.

The state of the system at any time t is represented by a point in the two dimensional Y Y' state space and the collection of all of the states resulting from a given sinusoidal drive form an ellipse when the system is in a steady state condition. In Figure 3 the sequence of states evolve from zero velocity and zero displacement to a final elliptical dynamic attractor. If we include the phase of the driving force as a variable in our

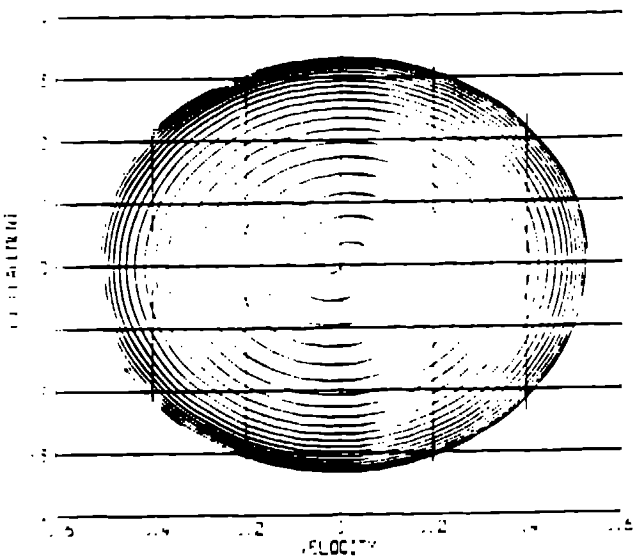


Figure 3
State Space Representation of The Transient and Steady State Responses of the Single Degree of Freedom Force Excited Oscillator.

state space representation the elliptical collection of states, in conjunction with the driving force, lie on a torus or doughnut in three dimensional space. This torus is a dynamic attractor of this system for this sinusoidal drive as all of the trajectories of system states eventually lie on the torus. If, at a given phase of the driving force, the value of the state is plotted in two dimensions the result is a point. For a linear system, the state of the system repeatedly returns to the same value with each cycle of the driving force. This plot of the state at a given phase of the driving force is known as a Poincare map. As the linear force excited system approaches steady state behavior the Poincare map approaches a single point as shown in Figure 4.

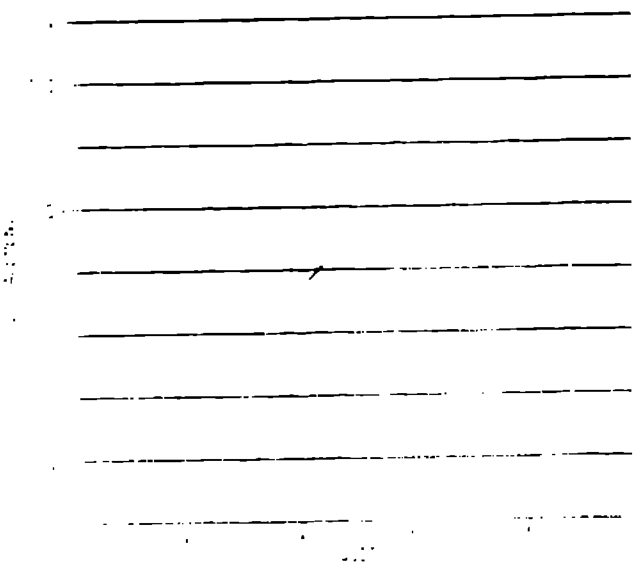
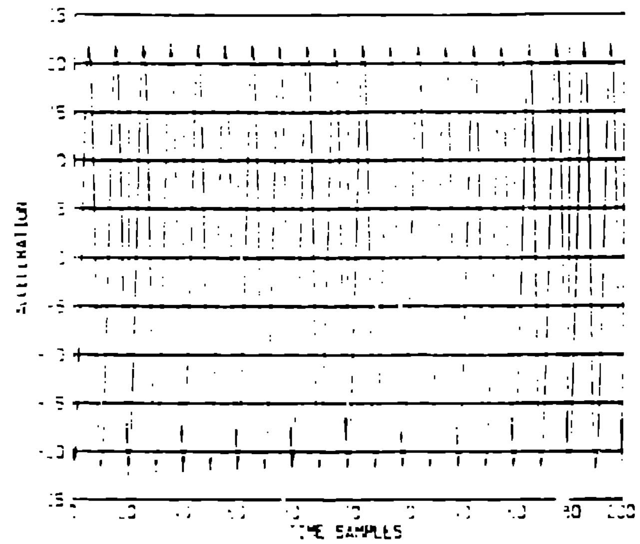


Figure 4
Poincare Map for the Linear Force Excited Oscillator

A further characteristic of a this linear system is worth noting. Consider two cases of system response. Following application of the excitation the transient response is allowed to decay and steady state response observed. At some arbitrary point a small perturbation to the system state (say 10 percent) is made. This perturbation is generated by taking the state values at a time t_0 during steady state response and multiplying each value by 1.10. Future perturbed and unperturbed responses are then compared. For this linear system the effect of the perturbation decays with time and the perturbed and unperturbed responses quickly become indistinguishable as shown in Figure 5, where acceleration is used as a measure of system response.



Perturbed Response .... Unperturbed Response ___

Figure 5
Output Convergence From Adjacent States for a Linear Force Excited System.
(10% Perturbation).

For a linear system the trajectories of nearby points in state space converge. More formally, the Lyapunov exponent measures the rate of convergence or divergence of trajectories in state space. The definition of The Lyapunov exponent takes the form:

$$d = d_0 e^{\lambda t} \quad (2)$$

Here d_0 is the initial distance between two nearby states and d s the distance a time t later. λ is the Lyaponov exponent. For this linear system we have a negative Lyapunov exponent as the trajectories of adjacent states in the state space converge. For a chaotic system the Lyaponov exponent is positive and represents an average divergence of nearby states with time. In general a system will have a spectrum of Lyapunov exponents which determine how the dimensions of lengths, areas, and volumes in state space change with time.

A Single Degree of Freedom Nonlinear System

For the linear system described by Equation 1 the sinusoidal response exhausts the possibilities for the steady state behavior of the system to a sinusoidal drive. In vibration testing we seldom have systems this simple, and further, real systems often have some component of nonlinearity in their forced behavior. This nonlinear force component may be minor or it may be a dominant factor in the forces involved in the system. A more realistic form for a driven oscillator is obtained by modifying equation 1 to include a nonlinear term

[illegible] (3)

In equation 3 the restoring force is formed from a linear component and a nonlinear component. The magnitude of the nonlinear component is adjusted by changing the magnitude of the coefficient α. For this cubic nonlinearity the nonlinear restoring force is symmetric with respect to displacement, and increases with increasing displacement. Equations of this type are forms of Duffing's equation, where a nonlinear stiffness term is added to a single degree of freedom oscillator. In this case stiffness increases with increasing response level so the system of equation 3 is referred to as a hardening Duffing oscillator. To simulate this system equation 3 is solved numerically using an adaptive stepsize Runga Kutta algorithm. In contrast to the linear system the response characteristics of the system vary with the magnitude of the drive. For low drive levels the system behaves in a nearly linear fashion. At increased drive levels the effects of the nonlinear term become increasingly severe. For $\alpha = 100.0$, $F = 200.0$, $\omega_n = 2\pi$, $\omega = 4\pi$, and $\zeta = 4.5\%$ a very distorted periodic acceleration response is observed as shown in Figure 6.

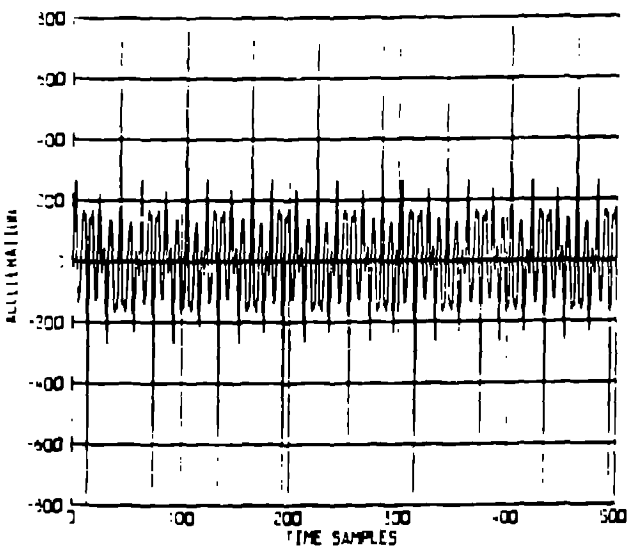


Figure 6
Acceleration Response of the Strongly Excited Hardening System Driven by Sinusoidal Excitation.

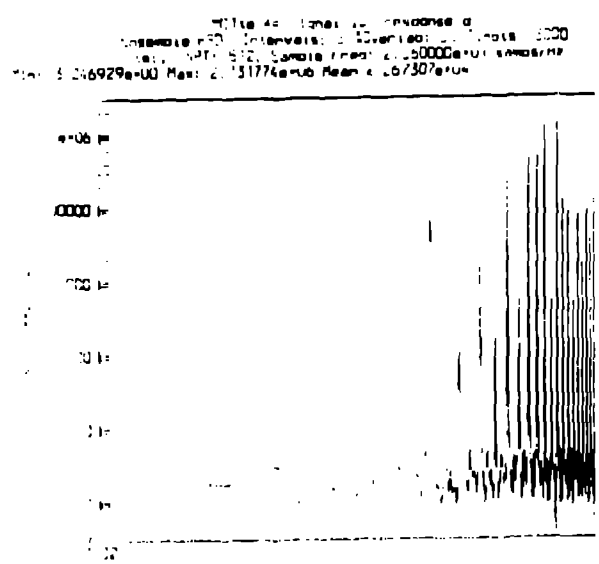


Figure 7
Response Power Spectrum for the Strongly Excited Hardening System Driven by Sinusoidal Excitation.

This response is quite distorted but is nonetheless typical of the responses observed at some frequencies when a real system is strongly excited by a sinusoidal drive. Figure 7 shows the Power spectrum of the time response shown in Figure 6. Note the fundamental response at the driving frequency of 1.0 Hz and the presence of numerous discrete harmonics. The Poincare map for this nonlinear system at driving phase zero consists of a discrete set of three points as shown in Figure 8.

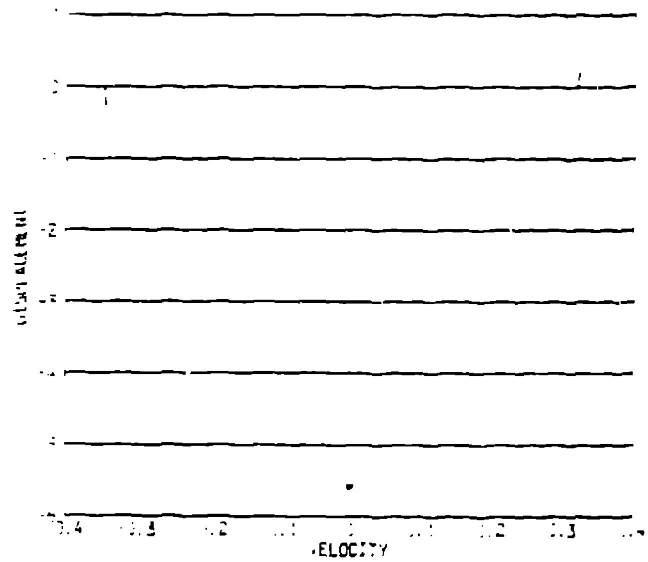
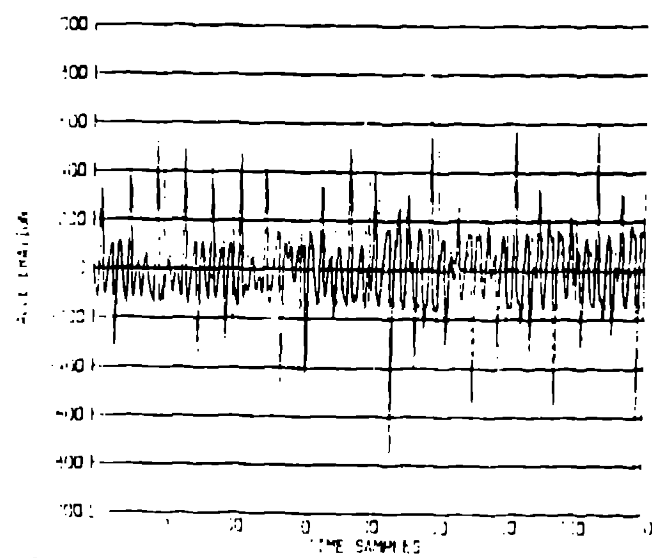


Figure 8
Poincare Map for the Strongly Excited Duffing Oscillator of Equation 3.

The sensitivity of this system to perturbations is tested by making a 1% perturbation to the system state 12.5 seconds after the force is applied to the system. Perturbed and unperturbed acceleration response are shown in Figure 9. Note that this system had not fully achieved a steady state response prior to the perturbation.



Perturbed Response .... Unperturbed Response
Figure 9
Output Divergence From Adjacent States for a Strongly Nonlinear Force Excited System. (1% Perturbation).

This perturbation of the state of the strongly nonlinear system does not decay but results in a global difference in the system response at all future times. In fact the system stabilizes at a different steady state response. This indicates that for some sets of initial conditions, the system described by Equation 3 is quite sensitive to perturbations of the initial conditions. The average Lyapunov exponent for this system and this excitation is not positive, however, as the system is not exhibiting chaotic behavior. At low drive levels the nonlinear system does not exhibit the sensitivity to initial conditions illustrated in Figure 9. The response of the relatively simple system of Equation 3 is in fact incredibly rich in behavior as drive level and drive frequency are varied. Nearly linear behavior, jump resonances, superharmonic responses, subharmonic responses, and chaotic behavior all occur[4].

Harmonic response generation is a common phenomenon in vibration testing and is often observed during sinusoidal sweeps. Using random excitation, peaks in the response spectrum are sometimes observed which are related to harmonic generation rather than directly to resonant behavior.

The Development of Chaotic Behavior

The Duffing Oscillator described in Equation 3 can exhibit chaotic behavior. Examples of such behavior are described in Abraham and Shaw[3], Moon[4], Thompson and Stewart[5], and Guckenheimer and Holmes[6]. Examples of chaotic behavior of this equation are shown in computer experiments in Kocak[7].

As an illustrative example of chaos the equation for a two well potential or buckled beam will be used. Consider the base excited system described by equation 4.

$$Y'' + C(Y'-Y_0') - K(Y-Y_0) + \alpha(Y-Y_0)^3 = 0 \qquad (4)$$

where Y = displacement of the driven mass.

Y_0 = displacement of the base mass.

Y_0'' = input base acceleration.

In contrast to equation 3, the linear stiffness term in equation 4 is negative and the cubic stiffness term is positive. This produces a situation in which the system oscillates in either of two potential wells. Transition between wells may not occur or may be either erratic or periodic, depending on the drive level. In a practical sense equation 4 represents the vibration of a base excited system with two stable equilibrium positions.

Consider excitation of this system with a sinusoidal base acceleration $Y_0'' = F \sin \omega t$ at a frequency close to that characteristic of oscillation in one potential well. For low input amplitudes F of base acceleration the system oscillates in one of the wells. As the peak base acceleration F is increased, an oscillation level is reached at which transitions between wells occur in an erratic manner. The relative displacement response $Y - Y_0$ of this system is shown in Figure 10 for a drive amplitude sufficient to excite transitions between potential wells. A Poincare map of this response triggered at a driving phase of zero degrees is shown in Figure 11. Unlike the linear system where the Poincare map was a single point and the nonlinear system where the Poincare map consisted of three discrete points, the Poincare map for this chaotic system consists of a complex fractal like structure of points. The system response does not repeat periodically yet a complex structure is exhibited in the state values with both densely populated and forbidden regions. This same sort of structure occurs in smaller regions of the

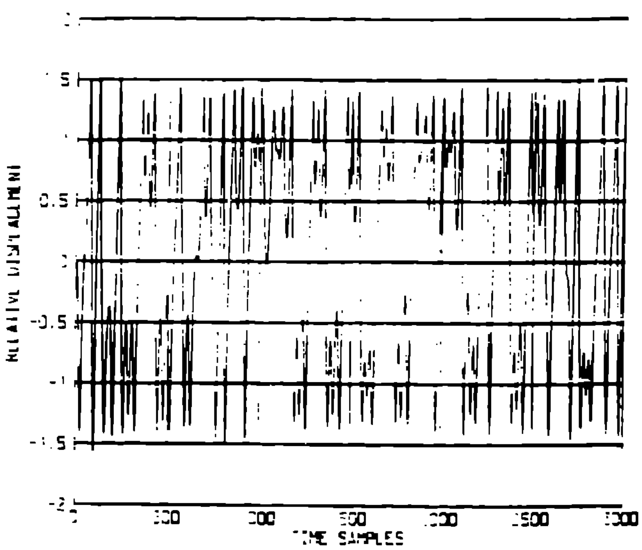


Figure 10
Output of the Two Well Potential System
Excited by a Sinusoidal Drive.

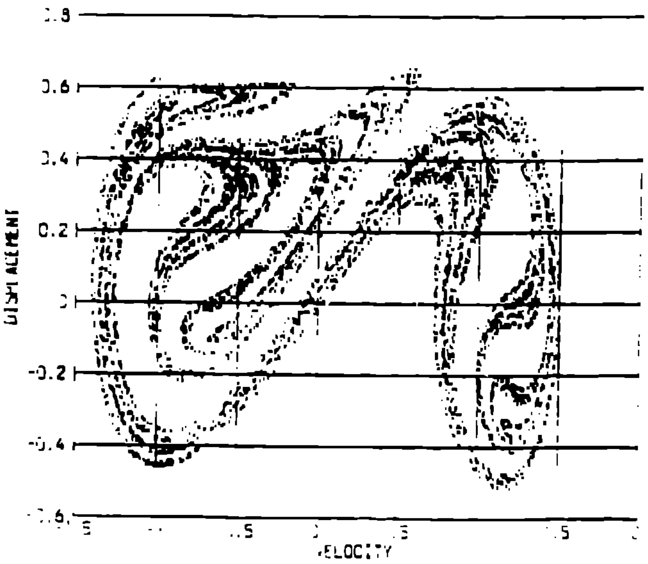


Figure 11
Poincare Map for a Two Well Potential
Chaotic System.

Poincare map. Figure 12 shows an expansion of a selected region of the Poincare map in Figure 11. Densely populated and forbidden regions occur on all length scales in the fractal like structures shown in Figures 11 and 12. On the average, this chaotic system exhibits a strong sensitivity to perturbations of the system state. Figure 13 shows the unperturbed and perturbed responses brought about by a 1 % perturbation of the system state.

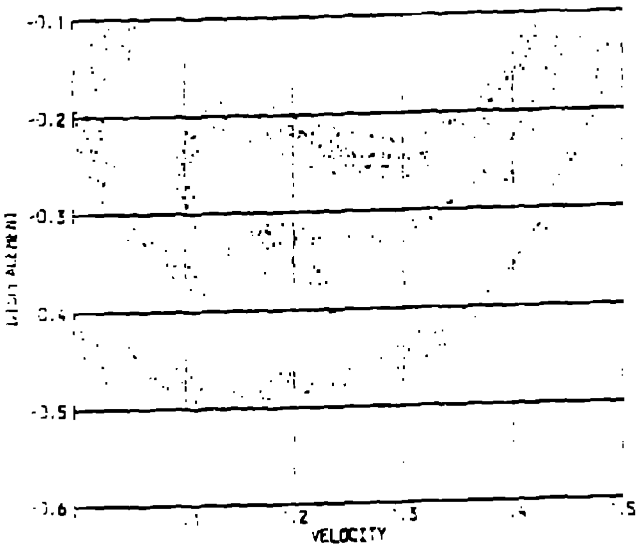
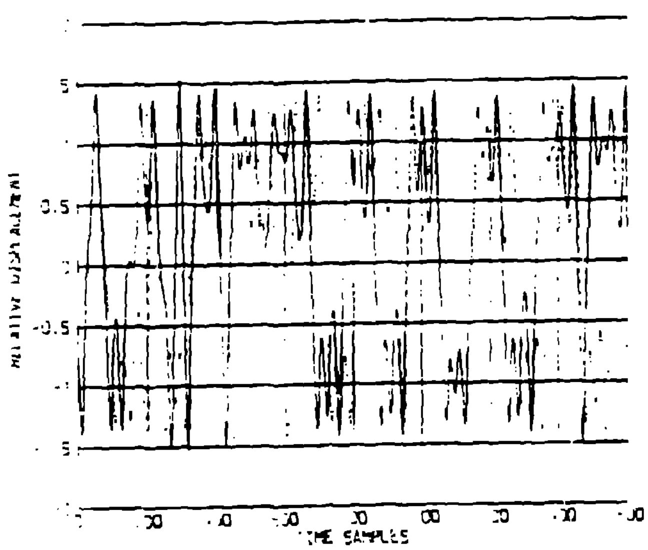


Figure 12
Expansion of a Region in the Poincare Map for the Two Well Potential Chaotic System.



Perturbed Response .... Unperturbed Response ___
Figure 13
Output Divergence From Adjacent States for a Two Well Potential Chaotic System.
(1% Perturbation).

While the initial perturbation is barely visible, its effect increases with time until the system follows a globally different response from about 150 time samples onward.

From a geometrical standpoint chaos is caused by a stretching and folding of trajectories in the phase space, a structure often referred to as a strange attractor in the literature. Three dimensional pictures of strange attractors are shown in a number of references[3,4,5,6]

The power spectrum of the response of this chaotic system is of interest and is shown in Figure 14

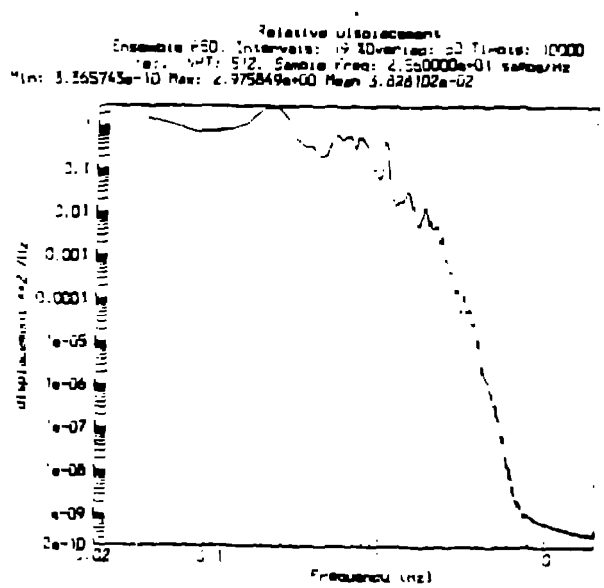


Figure 14
Power Spectrum of a Chaotic System

Note the broad range of frequencies exhibited in this response. This broadening of the spectral response is characteristic of chaotic systems. Periodic spikes in the response are also present. These periodic spikes do not indicate that chaos is not present but rather imply that response is more concentrated at certain frequencies. This power spectrum may be compared to the response spectrum of a nonlinear non-chaotic system shown in Figure 7.

Typical Characteristics of Chaos in Systems Driven by Sinusoidal Excitation.

From the examples shown above some typical characteristics of chaos have emerged. These characteristics include:

1. A sensitivity to changes in initial conditions. This means that, on the average, a small perturbation in initial conditions causes a large change in future behavior. For linear systems a small perturbation in initial conditions does not effect long term global behavior. In general, strongly nonlinear systems may exhibit a sensitivity to initial conditions in some regions of the state space. The definitive measure of the average sensitivity to changes in the initial conditions is the Lyapunov exponent. This exponent is positive for chaotic systems.

2. Fractal structure of the Poincare map. For a linear system the Poincare map is a single point. For nonlinear, nonchaotic systems the Poincare map consists of a discrete set of points. For chaotic systems the Poincare map consists of an infinite set of points arranged in a complex structure with both allowed and forbidden regions.

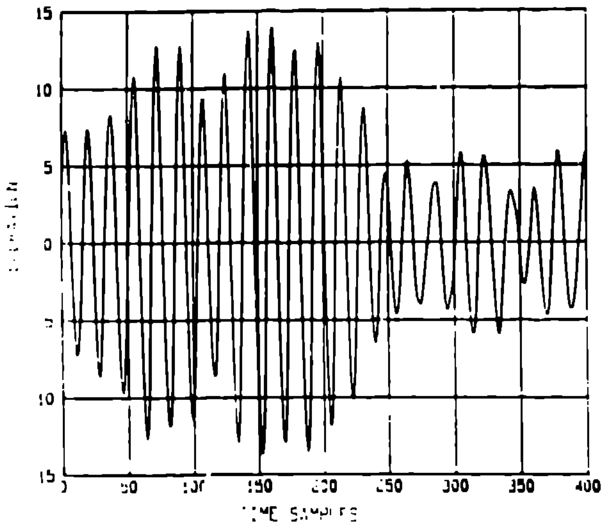
3. Broadening of the power spectrum of the response. The response of a chaotic system to a sinusoidal drive shows a broad power spectrum. For a linear system the power spectrum of the steady state response is a single line at the driving frequency. In general nonchaotic nonlinear systems show response at multiple subharmonics or superharmonics of the driving frequency.

Nonlinear Systems Excited by Random Forcing Functions.

The demonstration of chaotic behavior is clearly exhibited by the two well potential system described by equation 4. Sinusoidal excitation provides a convenient way of exhibiting a fractal structure in the Poincare map and illustrates that a chaotic system driven by a sine wave can exhibit response over a broad band of frequencies. In a practical sense, however, most structural vibration testing is conducted with random driving forces or accelerations. The natural question is then: What effect do random driving forces have on the response of nonlinear systems?

Since the system excitation is random noise the conventional Poincare map cannot be utilized though a three dimensional picture of the evolution of system states can be developed. A system driven by a given bandwidth, and a given realization of Gaussian random noise will have a particular deterministic response. The sensitivity of this response to perturbations of state forms a natural starting point for investigation.

Consider a linear base excited single degree of freedom system driven by band limited random noise. Using a Runge Kutta integration scheme a system with a resonant frequency of 0.1137 Hz. and 4.5% damping is driven by a base acceleration input with a flat power spectrum from 0.02 Hz. to 8 Hz. After the input has been applied for about 50 seconds the system state is perturbed by 1% and the perturbed and unperturbed responses examined. A plot of the perturbed and unperturbed responses is shown in Figure 15. Just as in the case of sinusoidal drive there is no detectable difference between the perturbed and unperturbed responses for this linear system.



Perturbed Response .... Unperturbed Response ____

Figure 15

Output Convergence From Adjacent States for a Linear System Driven by Band Limited Random Noise. (1% Perturbation).

To simulate a drastically nonlinear system the base excited two well potential system described by equation 4 is excited by band limited random noise, and following a stabilization period, the system state is perturbed by 1% and perturbed and unperturbed responses observed as shown in Figure 16.

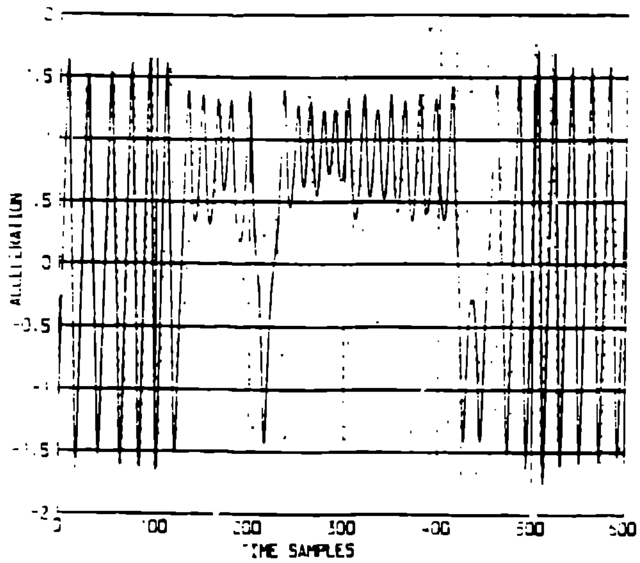
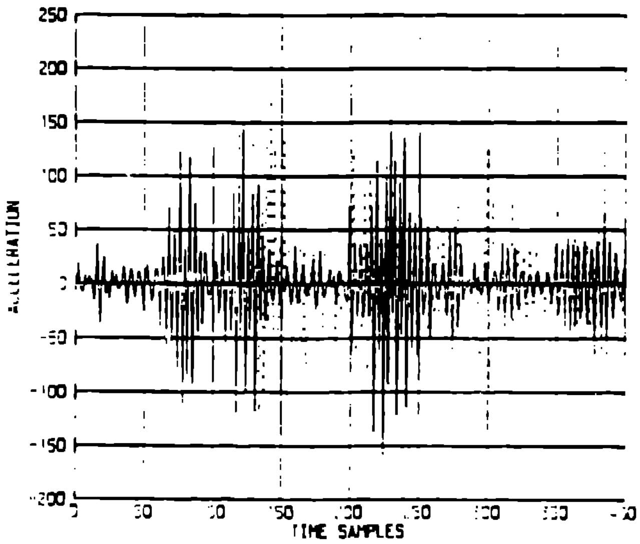


Figure 16

Output Divergence From Adjacent States for a Two Well Potential System Driven by Band Limited Random Noise. (1% Perturbation).

It is clear from Figure 16 that the same divergence from closely adjacent states occurs with random excitation as occurred in the chaotic system driven by sine excitation. A similar divergence of states occurs in the hardening Duffing system of equation 3 when the system is excited by random noise as shown in Figure 17.



Perturbed Response .... Unperturbed Response ____

Figure 17

Output Divergence From Adjacent States for a Hardening Duffing Oscillator Driven by Band Limited Random Noise. (1% Perturbation).

Similar divergence of responses caused by small perturbations in the state space occur then, for strongly nonlinear systems driven by random noise and by sinusoidal excitation. In fact, judging from the few examples which we have run, systems driven by random noise seem more sensitive to state perturbations than do systems driven by sinusoids assuming a given rms excitation level in each case. While the application of the term "chaos" to some nonlinear systems driven by random noise is perhaps premature it is clear that strongly nonlinear systems driven by random noise will, on the average, have positive Lyapunov exponents and that a chaos -like sensitivity to initial conditions is present.

Chaos-Like Behavior in Systems Driven by Random Excitation.

The above examples show that chaos-like sensitivities to initial conditions occur in strongly nonlinear systems driven by random excitation, including a hardening Duffing system driven by random noise. Consequently prediction of the response time series of such systems may be impractical as small errors in state estimates will magnify into different time history responses in finite time.

Modelling Nonlinear Systems from Experimental Data.

Given an experimentally observed response from a system undergoing a vibration test the input and response time histories may be used to determine the degree of nonlinearity present and, in some cases, to form an experimental system model. For example, for a system driven by sinusoidal excitation, generation of subharmonics or superharmonics is a clear indication of nonlinear behavior. The presence of strong nonlinearities in a system may well lead to chaotic behavior. For systems driven by random excitation one method of determining the degree of nonlinearity exhibited by the system is the construction of a "model" of the system. If detailed basic design information is available a finite element model or a "lumped mass" model of the system may be constructed prior to testing. Proper analysis using these models will predict nonlinear or chaotic behavior. In some cases, however, a model must be developed using experimental data.

One form of experimental model may be constructed by assuming that a given mass is associated with each accelerometer location[8]. The mass associated with each accelerometer location and the effective stiffness between locations may conceptually be computed using the "Force State Mapping Technique"[9,10,11,12]. This technique has been applied to numerous single degree of freedom systems and to several cases of three degree of freedom systems. If motion at the accelerometer locations is truly representative of the degrees of freedom present in the structure, and sufficient measurement locations are chosen, the method appears applicable to systems with many degrees of freedom.

A second form of experimentally derived model for nonlinear systems is described by Farmer[13], Casdagli[14], Crutchfield[15], and Billings[16]. In this method an approximate representation of the state space is constructed from delayed values of the system response. To illustrate this type of model consider the unknown system shown in Figure 18.

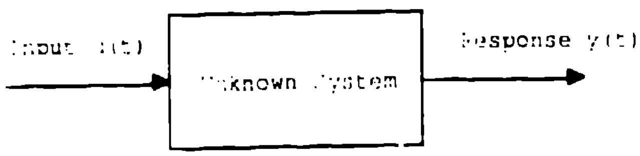
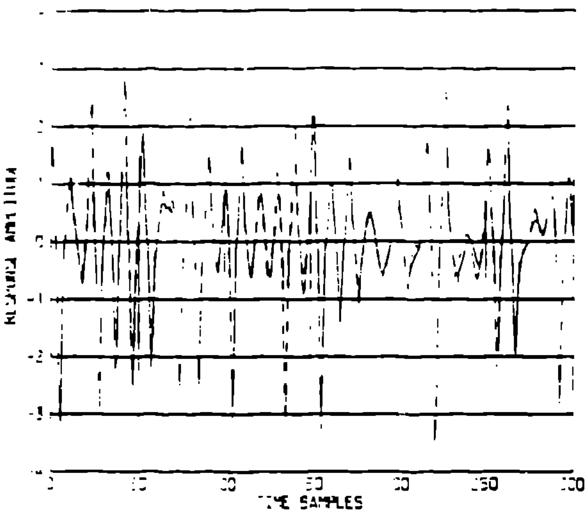


Figure 18
Unknown System Defined by Input-Output Behavior

Assuming that the input and response of this system are sampled at an appropriate sampling rate a delay coordinate model of the system is formed as shown in equation 5.

$$y(t) = f\{y(t-T), y(t-2T), \ldots, y(t-jT), u(t), u(t-T), \ldots, u(t-kT)\} \quad (5)$$

Here the u's represent current and delayed values of the system input and the y's current and delayed values of the system output. Equation 5 implies that the current output y(t) is a function of past outputs and current and past inputs. If the system is linear the function f is a linear combination of the delayed inputs and outputs. For nonlinear systems f may be any of numerous types of nonlinear functions. Application of delay coordinate (or ARMA) models to nonlinear systems is described in a paper by Hunter[16]. As an example of the application of such a model consider that we wish to predict future responses of a base excited hardening Duffing system to a band limited random noise excitation. Application of a nonlinear model of the form described in Equation 5 yields a predicted response to the random noise input. This model response is compared to the actual system response in Figure 19.



Measured Response ___ Predicted Response -----
Figure 19
Comparison of Actual and Predicted Responses for a Base Excited Hardening Duffing System Using a Nonlinear Delay Coordinate Model.

Nonlinear modelling may also be accomplished in the frequency domain using higher order transfer functions based on Volterra or Weiner series[17,18,19]. Basically these higher order transfer functions describe the energy transfer from pairs or triplets of frequencies to a given response frequency.

Many forms of nonlinear models exist. The model forms described in this section have all shown some success in modelling nonlinear systems but it must be admitted that the construction of experimental models for nonlinear vibration systems is in a very early stage of development.

Conclusion

The behavior of nonlinear dynamic systems is drastically and qualitatively different from that of linear systems. This is true for many of the systems typically encountered in vibration testing. Further, models used for linear systems often fail in predicting even the general form of the response for nonlinear systems. Specifically:

1. Many forms of nonlinearity typically encountered in vibration tests may lead to chaotic responses. These include systems where rattling, buckling, or nonlinear stiffness are encountered.

2. Nonlinear systems do not necessarily produce chaotic responses but such responses often occur when the nonlinearity is strong. Driving systems at high input levels may tend to emphasize nonlinear behavior and lead to chaotic responses.

3. Classical chaotic responses occur when sinusoidal excitation is used. Poincare maps and power spectral density plots provide means of detecting this type of chaotic behavior.

4. Chaos-like behavior occurs in systems driven by random noise. The responses of such systems are very sensitive to initial conditions as are the responses of chaotic systems driven by sinusoidal excitation.

5. Methods for the experimental modelling of nonlinear systems exist. These methods include force state mapping, delay coordinate models, and higher order frequency response functions. These models are all in the early stages of development.

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