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D. L. Bosley† *and J. Kevorkian*†

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† University of Washington
Seattle, WA 98195

SUSTAINED RESONANCE IN VERY SLOWLY VARYING OSCILLATORY HAMILTONIAN SYSTEMS*

D. L. Bosley† and J. Kevorkian†

Abstract. By formulating slowly varying oscillatory systems into Hamiltonian standard form, canonical averaging techniques can be performed automatically by symbolic manipulation programs to very high orders. For the very slow variation considered, these high orders are required to find uniformly valid solutions. When resonance is exhibited in these systems, the original system of 2N first order differential equations is reduced to two differential equations which embody the resonance behavior.

Sustained resonance, also referred to as phase locking, occurs when the leading order frequency of the reduced system oscillates about zero for long times. The general solution procedure is illustrated, and a highly accurate asymptotic solution is found explicitly for a frequently occurring class of problems, which results when only a single harmonic of the resonance is present. This solution was not possible for the same class of problems with the usual slow time. Two test cases are considered to numerically verify all results.

Key words. adiabatic invariants, averaging, Hamiltonian systems, near-identity transformations, phase-locking, sustained resonance.

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† Department of Applied Mathematics, FS-20, University of Washington, Seattle, Washington 98195.

1. Introduction. Slowly varying oscillatory systems occur frequently in physical applications, in both natural and man-made systems. The following are some examples. Planetary systems have an obvious oscillatory behavior which may be slowly changing due to various long-term effects such as precession, tidal dissipation, or variable mass (see Kevorkian [17]). The pancreatic beta cell is an example of a biological system exhibiting this behavior; a rapid alternation of the cellular membrane potential causes the release of insulin in response to slowly changing calcium concentration (see Pernarowski, Miura, and Kevorkian [22]). Spinning re-entry spacecraft are affected by the slowly increasing atmospheric density (see Kevorkian [13]). Many problems involving acoustic or electromagnetic waves have slowly varying features, such as the free-electron laser, designed so that relativistic electrons are passed through a slowly varying magnetic field to stimulate the high energy, low frequency emission of radiation (see Bosley and Kevorkian [4] and Li and Kevorkian [21]).

Perturbation techniques for oscillatory systems are discussed in Kevorkian and Cole [15], and more recently in Sanders and Verhulst [23]. Slowly varying oscillatory systems are discussed by Kevorkian in [17], who summarizes and extends his previous work in [12], [13], [14], and [16] and with Lewin [20], on averaging techniques for systems with slowly varying parameters. In particular the phenomenon of resonance is discussed, and uniformly valid expansions are found for a class of problems exhibiting transient resonance for linear and nonlinear oscillatory systems. Li and Kevorkian, in [18] and [21], explore sustained resonance, where the resonance persists for long times, with an emphasis on the application to free-electron lasers. Kath in [10] and [11] discusses a model for re-entry roll resonance proposed by Kevorkian in [13] and derives conditions necessary for sustained resonance to occur. Haberman in [9] also derives conditions for capture into sustained resonance based on energy bounds, and with Bourland in [5] extends and simplifies the Kuzmak-Luke procedure for slowly varying second order nonlinear oscillators with weakly nonlinear damping. In [6], Bourland and Haberman discuss the behavior of a slowly varying oscillator in a double potential well for motions near the separatrix and derive a solution in terms of a sequence of Melnikov functions to describe the transition region.

In all of these papers, the rate of the slow variation is considered to be equivalent to the strength of the weak damping or weakly nonlinear coupling. More precisely, if the ratio of the characteristic time associated with the oscillatory behavior to the characteristic time associated with the slow variation is specified as ε, a small parameter, we can refer to the slow time $\tilde{t} = \varepsilon t$. The strength of the damping or nonlinear effects are then usually considered $O(\varepsilon)$. However, in many problems the parameters vary more slowly relative to the order of the coupling. The case of the much slower time scale, $t^* = \varepsilon^2 t$, which we will call a *very slow* time scale, where the strength of the coupling is still $O(\varepsilon)$, is discussed qualitatively by Kevorkian as a generalization of the problems considered in [16] and [17] and is examined for a particular linear example in [1] by Ablowitz, Funk, and Newell.

However, many details remain unaddressed concerning both transient and sustained resonance for this case of very slow variation, as there are significant difficulties relative to the usual slow variation of parameters. The principal difficulty is that higher order expansions are required to find uniformly valid solutions for the very long times considered ($t = O(1/\varepsilon^2)$) due to the cumulative effect of averaged terms. Also, in the case of transient resonance (which is discussed in [2] and [3]) the $O(1)$ reduced problem which embodies the resonance behavior has no analytical solution in the resonance layer in general (as mentioned in [1], [16], and [17]).

Numerical integration of rapidly oscillating functions over long times is very difficult. For smaller values of ε, numerical integration not only yields inaccurate results, but requires an impractical amount of time. On the other hand, due to the very large number of terms, high order asymptotic expansions often cannot be calculated by hand. The solution technique presented here relies on the use of symbolic manipulation programs, which not only keep track of these terms but can be programmed to automatically execute a significant portion of the solution procedure. The final solution, while still containing a large number of terms, can nevertheless be easily evaluated with the symbolic manipulator's automatic generation of FORTRAN code directly from the derived expressions.

In this paper we examine the Hamiltonian system of 2N differential equations

$$\frac{dq_n}{dt} = \frac{\partial h}{\partial p_n} = \omega_n(p_i, t^*) + \varepsilon\, g_n(p_i, q_i, t^*; \varepsilon) \tag{1.1a}$$

$$\frac{dp_n}{dt} = -\frac{\partial h}{\partial q_n} = \varepsilon\, f_n(p_i, q_i, t^*; \varepsilon) \tag{1.1b}$$

$$n = 1, 2, \ldots, N$$

where $t^* = \varepsilon^2 t$ is the slow time scale, ε is a small parameter, $0 < \varepsilon << 1$, and the functions f_n and g_n are restricted to be 2π periodic in the q_i. This system is said to be in standard form, which assumes the transformation of the leading order system to action-angle variables. Although we do not consider the details here, our results generalize in a straightforward manner to non-Hamiltonian systems as discussed in [17].

Before we discuss the solution techniques for equations (1.1) we note that putting slowly varying oscillatory systems into standard form can be accomplished in several ways. Most often, asymptotic and Taylor series expansions in small parameters can be used, as can Fourier series expansions to make explicit the periodic nature of the functions involved. An action-angle transformation is commonly used to remove the dependence on oscillatory variables from the $O(1)$ Hamiltonian. In fact, many weakly dissipative systems can be written in standard form, by modifying the slow time dependence in the variational formulation (see Vujanovic and Jones [24]). As will be seen, finding a way to put the problem into standard form frequently has many advantages over solving the system in its original form. The methods of solution for (1.1) are relatively straightforward, permitting automation of the procedures in many cases, and solutions can be found accurately to high orders.

In section 2, we examine averaging techniques which eliminate the oscillatory behavior and the q_i dependence from the Hamiltonian and from the system of equations. In 2.1, we look at the averaging procedure for a general Hamiltonian system in the absence of any resonances. This procedure results in uniformly valid expansions for the solution of all the p_i and q_i for times $t = O(1/\varepsilon^2)$. Section 2.2 discusses the modifications necessary to this procedure when a resonance is present, and shows that the original problem of a system of 2N first order differential equations reduces to a system of two first order equations. The role of symbolic manipulation is indicated throughout.

In section 3, we examine the reduced problem for the case of sustained resonance where the leading order frequency oscillates about zero for long times. In 3.1, we define the procedure which generates the asymptotic solution for the general reduced problem. The procedure involves several transformations: removing the resonance, restoring the equations to standard form through an action-angle transformation, then averaging the system using a near-identity transformation. An additional adiabatic invariant is found as a result and the system is reduced to quadrature. In 3.2, we examine a model for a frequently occurring class of reduced problems. This class includes, for example, the reduced problems which result from the free-electron laser and the spin-orbital coupling of a planetary resonance. The details of these particular applications will be reported elsewhere. An explicit solution is discovered via the method of section 3.1. This solution does not exist for the same class of problems with the faster slow time $\tilde{t} = \varepsilon t$; it demonstrates the strength and practicality of the techniques used here. In 3.3 we consider two test cases to numerically verify all results.

2. Averaging for very slowly varying oscillatory Hamiltonian systems.

2.1 Solution for nonresonant Hamiltonian systems. In the absence of any resonances for the system, equations (1.1) permit a straightforward asymptotic solution. This is achieved by a canonical near-identity averaging transformation which eliminates the q_i from the Hamiltonian to any order desired while preserving the Hamiltonian form. This procedure is discussed in detail in section 4 of Kevorkian [17] for the case of slow variations depending on $\tilde{t} = \varepsilon t$. Therefore, we only summarize the results for the case of very slow variations depending on $t^* = \varepsilon^2 t$. The details are given in Bosley [2]. Consider a general Hamiltonian of the form

$$\begin{aligned} h(p_i, q_i, t^*; \varepsilon) = {} & \underline{h}_0(p_i, t^*) + \varepsilon [\underline{h}_1(p_i, t^*) + \hat{h}_1(p_i, q_i, t^*)] \\ & + \varepsilon^2 [\underline{h}_2(p_i, t^*) + \hat{h}_2(p_i, q_i, t^*)] + O(\varepsilon^3) \end{aligned} \tag{2.1}$$

where the averaged parts, $\underline{h}_j$, have been separated from the oscillatory parts, $\hat{h}_j$, of the Hamiltonian, and the oscillatory parts have zero average over the q_i.

$$\int_0^{2\pi}\int_0^{2\pi}\cdots\int_0^{2\pi} \hat{h}_j(p_i, q_i, t^*)\, dq_1\, dq_2 \cdots dq_N = 0 \tag{2.2}$$

We define a near-identity averaging transformation from the old variables (p_i, q_i) to a set of new variables (P_i, Q_i) to eliminate the q_i from the $O(\varepsilon)$ and $O(\varepsilon^2)$ parts of the Hamiltonian. This is achieved by the use of a generating function F depending on the new momenta, P_i, the old coordinates, q_i, and the slow time, t^*

$$F(P_i, q_i, t^*) = \sum_{i=1}^{N} P_i q_i + \varepsilon F_1(P_i, q_i, t^*) + \varepsilon^2 F_2(P_i, q_i, t^*) \tag{2.3}$$

where F_1 and F_2 are to be judiciously chosen later (for details on canonical transformations, see Goldstein [8] or Landau and Lifshitz [19]). The transformation resulting from the generating function (2.3) is given by

$$p_n = P_n + \varepsilon \frac{\partial F_1}{\partial q_n}(P_i, q_i, t^*) + \varepsilon^2 \frac{\partial F_2}{\partial q_n}(P_i, q_i, t^*) \tag{2.4a}$$

$$Q_n = q_n + \varepsilon \frac{\partial F_1}{\partial P_n}(P_i, q_i, t^*) + \varepsilon^2 \frac{\partial F_2}{\partial P_n}(P_i, q_i, t^*) \tag{2.4b}$$

These 2N mixed algebraic relations can be solved asymptotically for the old variables in terms of the new, $p_n = p_n(P_i, Q_i, t^*)$ and $q_n = q_n(P_i, Q_i, t^*)$. Note that if $\varepsilon = 0$, we simply have the identity transformation, $p_n = P_n$ and $q_n = Q_n$. This explains the "near-identity" nature of the transformation. For $\varepsilon \neq 0$, we find the asymptotic expansions for p_n and q_n

$$\begin{aligned} p_n = {} & P_n + \varepsilon \frac{\partial F_1}{\partial q_n}(P_i, Q_i, t^*) \\ & + \varepsilon^2 \left[\frac{\partial F_2}{\partial q_n}(P_i, Q_i, t^*) - \sum_{j=1}^{N} \frac{\partial^2 F_1}{\partial q_n \partial q_j} \frac{\partial F_1}{\partial P_j} \right] \\ & + \varepsilon^3 \left[\sum_{k=1}^{N} \sum_{j=1}^{N} \left(\frac{1}{2} \frac{\partial^3 F_1}{\partial q_n \partial q_j \partial q_k} \frac{\partial F_1}{\partial P_j} \frac{\partial F_1}{\partial P_k} + \frac{\partial^2 F_1}{\partial q_n \partial q_k} \frac{\partial^2 F_1}{\partial P_k \partial q_j} \frac{\partial F_1}{\partial P_j} \right) \right. \\ & \left. - \sum_{j=1}^{N} \left(\frac{\partial^2 F_2}{\partial q_n \partial q_j} \frac{\partial F_1}{\partial P_j} + \frac{\partial^2 F_1}{\partial q_n \partial q_j} \frac{\partial F_2}{\partial P_j} \right) \right] + O(\varepsilon^4) \end{aligned} \tag{2.5a}$$

$$\begin{aligned} q_n = {} & Q_n - \varepsilon \frac{\partial F_1}{\partial P_n}(P_i, Q_i, t^*) \\ & - \varepsilon^2 \left[\frac{\partial F_2}{\partial P_n}(P_i, Q_i, t^*) - \sum_{j=1}^{N} \frac{\partial^2 F_1}{\partial P_n \partial q_j} \frac{\partial F_1}{\partial P_j} \right] + O(\varepsilon^3) \end{aligned} \tag{2.5b}$$

where all partial derivatives of the generating functions have the old coordinates, q_i, replaced by the new coordinates, Q_i, and so are functions of only the new variables (P_i, Q_i) and the slow time t^*. The new Hamiltonian is then given in terms of the old Hamiltonian and the generating functions by the following expansion:

$$H(P_i, Q_i, t^*; \varepsilon) = H_0(P_i, t^*) + \varepsilon H_1(P_i, Q_i, t^*) + \varepsilon^2 H_2(P_i, Q_i, t^*) + \varepsilon^3 H_3(P_i, Q_i, t^*) + O(\varepsilon^4) \tag{2.6a}$$

where

$$H_0(P_i, t^*) = \underline{h}_0(P_i, t^*) \tag{2.6b}$$

$$H_1(P_i, Q_i, t^*) = \underline{h}_1(P_i, t^*) + \underset{\wedge}{h}_1(P_i, Q_i, t^*) + \sum_{n=1}^{N} \frac{\partial \underline{h}_0}{\partial p_n}(P_i, t^*) \frac{\partial F_1}{\partial q_n}(P_i, Q_i, t^*) \tag{2.6c}$$

$$\begin{aligned} H_2(P_i, Q_i, t^*) = {} & \underline{h}_2 + \underset{\wedge}{h}_2 + \sum_{i=1}^{N} \frac{\partial \underline{h}_0}{\partial p_i} \frac{\partial F_2}{\partial q_i} \\ & + \sum_{i=1}^{N} \frac{\partial \underline{h}_1}{\partial p_i} \frac{\partial F_1}{\partial q_i} + \sum_{i=1}^{N} \frac{\partial \underset{\wedge}{h}_1}{\partial p_i} \frac{\partial F_1}{\partial q_i} - \sum_{i=1}^{N} \frac{\partial \underset{\wedge}{h}_1}{\partial q_i} \frac{\partial F_1}{\partial P_i} \\ & + \sum_{i=1}^{N} \sum_{j=1}^{N} \left(\frac{1}{2} \frac{\partial^2 \underline{h}_0}{\partial p_i \partial p_j} \frac{\partial F_1}{\partial q_i} \frac{\partial F_1}{\partial q_j} - \frac{\partial \underline{h}_0}{\partial p_i} \frac{\partial^2 F_1}{\partial q_i \partial q_j} \frac{\partial F_1}{\partial P_j} \right) \end{aligned} \tag{2.6d}$$

$$\begin{aligned} H_3(P_i, Q_i, t^*) = {} & \underline{h}_3 + \underset{\wedge}{h}_3 + \frac{\partial F_1}{\partial t^*} + \sum_{i=1}^{N} \frac{\partial \underline{h}_1}{\partial p_i} \frac{\partial F_2}{\partial q_i} + \sum_{i=1}^{N} \frac{\partial \underset{\wedge}{h}_1}{\partial p_i} \frac{\partial F_2}{\partial q_i} \\ & - \sum_{i=1}^{N} \frac{\partial \underset{\wedge}{h}_1}{\partial q_i} \frac{\partial F_2}{\partial P_i} + \sum_{i=1}^{N} \frac{\partial \underline{h}_2}{\partial p_i} \frac{\partial F_1}{\partial q_i} + \sum_{i=1}^{N} \frac{\partial \underset{\wedge}{h}_2}{\partial p_i} \frac{\partial F_1}{\partial q_i} - \sum_{i=1}^{N} \frac{\partial \underset{\wedge}{h}_2}{\partial q_i} \frac{\partial F_1}{\partial P_i} \\ & - \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial \underline{h}_0}{\partial p_i} \left(\frac{\partial^2 F_1}{\partial q_i \partial q_j} \frac{\partial F_2}{\partial P_j} + \frac{\partial^2 F_2}{\partial q_i \partial q_j} \frac{\partial F_1}{\partial P_j} \right) + \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^2 \underline{h}_0}{\partial p_i \partial p_j} \frac{\partial F_1}{\partial q_i} \frac{\partial F_2}{\partial q_j} \\ & - \sum_{i=1}^{N} \sum_{j=1}^{N} \left(\left(\frac{\partial \underline{h}_1}{\partial p_i} + \frac{\partial \underset{\wedge}{h}_1}{\partial p_i} \right) \frac{\partial^2 F_1}{\partial q_i \partial q_j} - \frac{\partial \underset{\wedge}{h}_1}{\partial q_i} \frac{\partial^2 F_1}{\partial P_i \partial q_j} \right) \frac{\partial F_1}{\partial P_j} - \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^2 \underset{\wedge}{h}_1}{\partial p_i \partial q_j} \frac{\partial F_1}{\partial q_i} \frac{\partial F_1}{\partial P_j} \\ & + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left(\frac{\partial^2 \underline{h}_1}{\partial p_i \partial p_j} + \frac{\partial^2 \underset{\wedge}{h}_1}{\partial p_i \partial p_j} \right) \frac{\partial F_1}{\partial q_i} \frac{\partial F_1}{\partial q_j} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^2 \underset{\wedge}{h}_1}{\partial q_i \partial q_j} \frac{\partial F_1}{\partial P_i} \frac{\partial F_1}{\partial P_j} \end{aligned} \tag{2.6e}$$

$$+\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial \underline{h}_0}{\partial p_i}\left(\frac{1}{2}\frac{\partial^3 F_1}{\partial q_i \partial q_j \partial q_k}\frac{\partial F_1}{\partial P_j}+\frac{\partial^2 F_1}{\partial q_i \partial q_j}\frac{\partial^2 F_1}{\partial P_j \partial q_k}\right)\frac{\partial F_1}{\partial P_k}$$

$$-\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial^2 \underline{h}_0}{\partial p_i \partial p_j}\frac{\partial F_1}{\partial q_j}\frac{\partial^2 F_1}{\partial q_i \partial q_k}\frac{\partial F_1}{\partial P_k}+\frac{1}{6}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial^3 \underline{h}_0}{\partial p_i \partial p_j \partial p_k}\frac{\partial F_1}{\partial q_i}\frac{\partial F_1}{\partial q_j}\frac{\partial F_1}{\partial q_k}$$

All expressions in (2.6) are evaluated at the new coordinates, (P_i, Q_i).

As mentioned, one of the significant difficulties with the very slow time variation is the need to calculate high order expansions to ensure uniformly valid solutions for very long times. As can be seen from (2.6e), the $O(\varepsilon^3)$ Hamiltonian has an extremely large number of terms; an algebraic computation and simplification for any specific case is sure to take weeks of calculation by hand, and more significantly has a relatively high potential for undetected errors. In fact, one of the difficulties with perturbation techniques in general is the proliferation of higher order terms due to the Taylor expansions of even relatively simple $O(1)$ problems.

This difficulty can be resolved by using a symbolic computation program (e.g., SMP, MACSYMA, or Mathematica) to keep track of the higher order terms in the expansions. In particular, averaging is especially suited to symbolic computation since the mathematical procedures involved (e.g., Taylor series expansions, collection of like powers of a small parameter, and integrations of sinusoidal terms) are simple and fairly mechanical, but algebraically tedious. As a result, all of the derivations in this paper have relied heavily on SMP.

At this point in the procedure we choose F_1 and F_2 so that H_1 and H_2 have all Q_i dependencies removed. This is the part of the procedure referred to as *averaging*, as all oscillatory behavior is removed from the Hamiltonian (see [17] for details). The new Hamiltonian then has the following form:

$$\begin{aligned} H(P_i, Q_i, t^*; \varepsilon) = {} & \underline{H}_0(P_i, t^*) + \varepsilon\, \underline{H}_1(P_i, t^*) + \varepsilon^2\, \underline{H}_2(P_i, t^*) \\ & + \varepsilon^3\, \underset{\wedge}{H}_3(P_i, Q_i, t^*) + O(\varepsilon^4) \end{aligned} \tag{2.7}$$

The Q_i dependence has been removed to $O(\varepsilon^2)$ and the canonical differential equations become

$$\frac{dQ_n}{dt} = \frac{\partial H_0}{\partial P_n}(P_i, t^*) + \varepsilon \frac{\partial H_1}{\partial P_n}(P_i, t^*) + \varepsilon^2 \frac{\partial H_2}{\partial P_n}(P_i, t^*) + O(\varepsilon^3) \tag{2.8a}$$

$$\frac{dP_n}{dt} = -\varepsilon^3 \frac{\partial \hat{H}_3}{\partial Q_n}(P_i, Q_i, t^*) + O(\varepsilon^4) \tag{2.8b}$$

Equations (2.8b) indicate that the P_n are constants to $O(\varepsilon^2)$ for times $t \leq T$, with $T = O(1/\varepsilon^2)$. This constancy for very long times is known as *adiabatic invariance*, and the P_n are called the N *adiabatic invariants* for this problem. By asymptotically inverting the near-identity transformations, we find formulae which give these constants as functions of the original variables

$$\begin{aligned} P_n \equiv \mathcal{A}_n(p_i, q_i, t^*; \varepsilon) &= p_n - \varepsilon \frac{\partial F_1}{\partial q_n}(p_i, q_i, t^*) \\ &+ \varepsilon^2 \left(-\frac{\partial F_2}{\partial q_n}(p_i, q_i, t^*) + \sum_{j=1}^{N} \frac{\partial^2 F_1}{\partial q_n \partial P_j} \frac{\partial F_1}{\partial q_j} \right) = \text{constant} + O(\varepsilon^3) \end{aligned} \tag{2.9}$$

where the constant value of the adiabatic invariants can be computed directly from the initial conditions. Equations (2.8a) can be solved by quadrature since the equations uncouple due to the constancy of the P_i

$$\begin{aligned} Q_n &= \frac{1}{\varepsilon^2} \int^{t^*} \omega_n(\mathcal{A}_i, t^*)\, dt^* + \frac{1}{\varepsilon} \int^{t^*} \frac{\partial H_1}{\partial P_n}(\mathcal{A}_i, t^*)\, dt^* \\ &+ \int^{t^*} \frac{\partial H_2}{\partial P_n}(\mathcal{A}_i, t^*)\, dt^* + O(\varepsilon^3)\,(osc.) + O(\varepsilon^2)\,(avg.) \end{aligned} \tag{2.10}$$

These solutions can be substituted into the averaging transformations (2.5) to find $p_n(t)$ and $q_n(t)$, the solutions for the original variables

$$p_n(t) = \mathcal{A}_n + \varepsilon \frac{\partial F_1}{\partial q_n}(\mathcal{A}_i, Q_i(t), t^*)$$

$$+ \varepsilon^2 \left(\frac{\partial F_2}{\partial q_n}(\mathcal{A}_i, Q_i(t), t^*) - \sum_{j=1}^{N} \frac{\partial^2 F_1}{\partial q_n \partial q_j} \frac{\partial F_1}{\partial P_j} \right) + O(\varepsilon^3) \tag{2.11a}$$

$$q_n(t) = Q_n(t) - \varepsilon \frac{\partial F_1}{\partial P_n}(\mathcal{A}_i, Q_i(t), t^*)$$

$$- \varepsilon^2 \left(\frac{\partial F_2}{\partial P_n}(\mathcal{A}_i, Q_i(t), t^*) - \sum_{j=1}^{N} \frac{\partial^2 F_1}{\partial P_n \partial q_j} \frac{\partial F_1}{\partial P_j} \right) + O(\varepsilon^3) \tag{2.11b}$$

This procedure is exactly parallel to the derivation in [17] except for the $t^* = \varepsilon^2 t$ instead of the $\tilde{t} = \varepsilon t$ slow time dependence. The main difference is the explicit calculation of the $O(\varepsilon^3)$ terms to account for the very long times involved.

2.2 Reduction of Resonant Hamiltonian Systems. If a resonance occurs at any time in the system (1.1), the solution (2.11) becomes singular and is no longer valid. We now look at the case where a resonance is present and discuss the changes to the previous analysis necessary to account for this.

A resonance occurs in (1.1) when a critical combination of the $O(1)$ frequencies, $\sigma(p_i, t^*) = \sum_{n=1}^{R} r_n \, \omega_n(p_i, t^*)$ where the r_n are integers, vanishes at some time. The singularities in the solution (2.11) arise due to the presence of σ in the denominator of the near-identity transformation, which removes the q_i dependence from the Hamiltonian. Therefore, when a resonance is present in (1.1) we first isolate the resonance into a single angle variable before eliminating the rest of the q_i. We define a time-independent canonical transformation from the (p_i, q_i) to new variables $(\bar{p}_i, \bar{q}_i)$ given by

$$
\begin{aligned}
\overline{q}_1 &= q_1 + \frac{r_2}{r_1} q_2 + \ldots + \frac{r_R}{r_1} q_R & \overline{p}_1 &= p_1 \\
\overline{q}_2 &= q_2 & \overline{p}_2 &= p_2 - \frac{r_2}{r_1} p_1 \\
& & &\vdots \\
&\vdots & \overline{p}_R &= p_R - \frac{r_R}{r_1} p_1 \\
& & \overline{p}_{R+1} &= p_{R+1} \\
& & &\vdots \\
\overline{q}_N &= q_N & \overline{p}_N &= p_N
\end{aligned}
\tag{2.12}
$$

which isolates the resonance into a single variable, $\overline{q}_1$, allowing elimination of the rest of the $\overline{q}_i$ from the Hamiltonian. The new Hamiltonian is simply the old Hamiltonian with the (p_i, q_i) expressed in terms of the $(\overline{p}_i, \overline{q}_i)$. Because this transformation has isolated the resonance variable, this new Hamiltonian can be written in the form

$$
\begin{aligned}
\overline{h}(\overline{p}_i, \overline{q}_i, t^*; \varepsilon) &= \underline{\overline{h}}_0(\overline{p}_i, t^*) \\
&+ \varepsilon \left[\underline{\overline{h}}_1(\overline{p}_i, t^*) + \hat{\overline{h}}_{1c}(\overline{p}_i, \overline{q}_1, t^*) + \hat{\overline{h}}_{1s}(\overline{p}_i, \overline{q}_i, t^*) \right] \\
&+ \varepsilon^2 \left[\underline{\overline{h}}_2(\overline{p}_i, t^*) + \hat{\overline{h}}_{2c}(\overline{p}_i, \overline{q}_1, t^*) + \hat{\overline{h}}_{2s}(\overline{p}_i, \overline{q}_i, t^*) \right] + O(\varepsilon^3)
\end{aligned}
\tag{2.13}
$$

The underbar still represents the averaged part of the Hamiltonian, while the underhat indicates a zero average over the $\overline{q}_i$. The critical terms, indicated by a subscript c, contain all resonant behavior associated with $\overline{q}_1$, while the remainder of the oscillatory part, indicated by a subscript s, contains all nonresonant terms.

We now proceed in the same way as for the nonresonant case, using near-identity averaging transformations to eliminate the nonresonant portions of the Hamiltonian, $\hat{\overline{h}}_{js}$, transforming $(\overline{p}_i, \overline{q}_i)$ to (P_i, Q_i). The transformation formulas are altered slightly since we no longer are eliminating all the Q_i. We still use a generating function of the form (2.3) so the asymptotic expansions for the $(\overline{p}_i, \overline{q}_i)$ in terms of (P_i, Q_i) can still be found by (2.5). Since the oscillatory part of the

Hamiltonian has been split into resonant and nonresonant terms, equations (2.6) are now given by

$$H_0(P_i, t^*) = \bar{h}_0(P_i, t^*) \tag{2.14a}$$

$$\begin{aligned} H_1(P_i, Q_i, t^*) = {} & \bar{h}_1(P_i, t^*) + \bar{\hat{h}}_{1c}(P_i, Q_1, t^*) + \bar{\hat{h}}_{1s}(P_i, Q_i, t^*) \\ & + \sum_{n=1}^{N} \frac{\partial \bar{h}_0}{\partial \bar{p}_n}(P_i, t^*) \frac{\partial F_1}{\partial \bar{q}_n}(P_i, Q_i, t^*) \end{aligned} \tag{2.14b}$$

$$\begin{aligned} H_2(P_i, Q_i, t^*) = {} & \bar{h}_2 + \bar{\hat{h}}_{2c} + \bar{\hat{h}}_{2s} + \sum_{i=1}^{N} \frac{\partial \bar{h}_0}{\partial \bar{p}_i}\frac{\partial F_2}{\partial \bar{q}_i} \\ & + \sum_{i=1}^{N} \frac{\partial \bar{h}_1}{\partial \bar{p}_i}\frac{\partial F_1}{\partial \bar{q}_i} + \sum_{i=1}^{N} \left(\frac{\partial \bar{\hat{h}}_{1c}}{\partial \bar{p}_i} + \frac{\partial \bar{\hat{h}}_{1s}}{\partial \bar{p}_i}\right)\frac{\partial F_1}{\partial \bar{q}_i} - \sum_{i=1}^{N} \frac{\partial \bar{\hat{h}}_{1s}}{\partial \bar{q}_i}\frac{\partial F_1}{\partial P_i} \\ & - \frac{\partial \bar{\hat{h}}_{1c}}{\partial \bar{q}_1}\frac{\partial F_1}{\partial P_1} + \sum_{i=1}^{N}\sum_{j=1}^{N}\left(\frac{1}{2}\frac{\partial^2 \bar{h}_0}{\partial \bar{p}_i \partial \bar{p}_j}\frac{\partial F_1}{\partial \bar{q}_i}\frac{\partial F_1}{\partial \bar{q}_j} - \frac{\partial \bar{h}_0}{\partial \bar{p}_i}\frac{\partial^2 F_1}{\partial \bar{q}_i \partial \bar{q}_j}\frac{\partial F_1}{\partial P_j}\right) \end{aligned} \tag{2.14c}$$

and

$$\begin{aligned} H_3(P_i, Q_i, t^*) = {} & \bar{h}_3 + \bar{\hat{h}}_{3c} + \bar{\hat{h}}_{3s} + \frac{\partial F_1}{\partial t^*} + \sum_{i=1}^{N} \frac{\partial \bar{h}_1}{\partial \bar{p}_i}\frac{\partial F_2}{\partial \bar{q}_i} \\ & - \frac{\partial \bar{\hat{h}}_{1c}}{\partial \bar{q}_1}\frac{\partial F_2}{\partial P_1} - \sum_{i=1}^{N} \frac{\partial \bar{\hat{h}}_{1s}}{\partial \bar{q}_i}\frac{\partial F_2}{\partial P_i} + \sum_{i=1}^{N}\left(\frac{\partial \bar{\hat{h}}_{1c}}{\partial \bar{p}_i} + \frac{\partial \bar{\hat{h}}_{1s}}{\partial \bar{p}_i}\right)\frac{\partial F_2}{\partial \bar{q}_i} \\ & + \sum_{i=1}^{N} \frac{\partial \bar{h}_2}{\partial \bar{p}_i}\frac{\partial F_1}{\partial \bar{q}_i} + \sum_{i=1}^{N}\left(\frac{\partial \bar{\hat{h}}_{2c}}{\partial \bar{p}_i} + \frac{\partial \bar{\hat{h}}_{2s}}{\partial \bar{p}_i}\right)\frac{\partial F_1}{\partial \bar{q}_i} - \sum_{i=1}^{N} \frac{\partial \bar{\hat{h}}_{2s}}{\partial \bar{q}_i}\frac{\partial F_1}{\partial P_i} - \frac{\partial \bar{\hat{h}}_{2c}}{\partial \bar{q}_1}\frac{\partial F_1}{\partial P_1} \\ & - \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial \bar{h}_0}{\partial \bar{p}_i}\left(\frac{\partial^2 F_1}{\partial \bar{q}_i \partial \bar{q}_j}\frac{\partial F_2}{\partial P_j} + \frac{\partial^2 F_2}{\partial \bar{q}_i \partial \bar{q}_j}\frac{\partial F_1}{\partial P_j}\right) + \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial^2 \bar{h}_0}{\partial \bar{p}_i \partial \bar{p}_j}\frac{\partial F_1}{\partial \bar{q}_i}\frac{\partial F_2}{\partial \bar{q}_j} \\ & - \sum_{i=1}^{N}\sum_{j=1}^{N}\left(\frac{\partial \bar{h}_1}{\partial \bar{p}_i} + \frac{\partial \bar{\hat{h}}_{1s}}{\partial \bar{p}_i} + \frac{\partial \bar{\hat{h}}_{1c}}{\partial \bar{p}_i}\right)\frac{\partial^2 F_1}{\partial \bar{q}_i \partial \bar{q}_j}\frac{\partial F_1}{\partial P_j} \end{aligned} \tag{2.14d}$$

$$+ \sum_{i=1}^{N} \sum_{j=1}^{N} \left(\frac{\partial \bar{\underset{\wedge}{h}}_{1s}}{\partial \bar{q}_i} \frac{\partial^2 F_1}{\partial P_i \partial \bar{q}_j} - \frac{\partial^2 \bar{\underset{\wedge}{h}}_{1s}}{\partial \bar{p}_i \partial \bar{q}_j} \frac{\partial F_1}{\partial \bar{q}_i} \right) \frac{\partial F_1}{\partial P_j}$$

$$+ \sum_{i=1}^{N} \left(\frac{\partial \bar{\underset{\wedge}{h}}_{1c}}{\partial \bar{q}_1} \frac{\partial^2 F_1}{\partial P_1 \partial \bar{q}_i} \frac{\partial F_1}{\partial P_i} - \frac{\partial^2 \bar{\underset{\wedge}{h}}_{1c}}{\partial \bar{p}_i \partial \bar{q}_1} \frac{\partial F_1}{\partial \bar{q}_i} \frac{\partial F_1}{\partial P_1} \right)$$

$$+ \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left(\frac{\partial^2 \bar{\underline{h}}_1}{\partial \bar{p}_i \partial \bar{p}_j} + \frac{\partial^2 \bar{\underset{\wedge}{h}}_{1c}}{\partial \bar{p}_i \partial \bar{p}_j} + \frac{\partial^2 \bar{\underset{\wedge}{h}}_{1s}}{\partial \bar{p}_i \partial \bar{p}_j} \right) \frac{\partial F_1}{\partial \bar{q}_i} \frac{\partial F_1}{\partial \bar{q}_j}$$

$$+ \frac{1}{2} \frac{\partial^2 \bar{\underset{\wedge}{h}}_{1c}}{\partial \bar{q}_1^{\,2}} \left(\frac{\partial F_1}{\partial P_1} \right)^2 + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^2 \bar{\underset{\wedge}{h}}_{1s}}{\partial \bar{q}_i \partial \bar{q}_j} \frac{\partial F_1}{\partial P_i} \frac{\partial F_1}{\partial P_j}$$

$$+ \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{\partial \bar{\underline{h}}_0}{\partial \bar{p}_i} \left(\frac{1}{2} \frac{\partial^3 F_1}{\partial \bar{q}_i \partial \bar{q}_j \partial \bar{q}_k} \frac{\partial F_1}{\partial P_j} + \frac{\partial^2 F_1}{\partial \bar{q}_i \partial \bar{q}_j} \frac{\partial^2 F_1}{\partial P_j \partial \bar{q}_k} \right) \frac{\partial F_1}{\partial P_k}$$

$$- \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{\partial^2 \bar{\underline{h}}_0}{\partial \bar{p}_i \partial \bar{p}_j} \frac{\partial F_1}{\partial \bar{q}_j} \frac{\partial^2 F_1}{\partial \bar{q}_i \partial \bar{q}_k} \frac{\partial F_1}{\partial P_k} + \frac{1}{6} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{\partial^3 \bar{\underline{h}}_0}{\partial \bar{p}_i \partial \bar{p}_j \partial \bar{p}_k} \frac{\partial F_1}{\partial \bar{q}_i} \frac{\partial F_1}{\partial \bar{q}_j} \frac{\partial F_1}{\partial \bar{q}_k}$$

We again choose F_1 and F_2 to eliminate all nonresonant terms from the Hamiltonian to $O(\varepsilon^2)$. The new Hamiltonian is now of the form

$$\begin{aligned} H(P_i, Q_i, t^*; \varepsilon) = {} & \underline{H}_0(P_i, t^*) \\ & + \varepsilon \, [\, \underline{H}_1(P_i, t^*) + \underset{\wedge}{H}_{1c}(P_i, Q_1, t^*) \,] \\ & + \varepsilon^2 \, [\, \underline{H}_2(P_i, t^*) + \underset{\wedge}{H}_{2c}(P_i, Q_1, t^*) \,] \\ & + \varepsilon^3 \, [\, \underline{H}_3(P_i, t^*) + \underset{\wedge}{H}_3(P_i, Q_i, t^*) \,] \; + O(\varepsilon^4) \end{aligned} \qquad (2.15a)$$

where

$$\underline{H}_0(P_i, t^*) = \bar{\underline{h}}_0(P_i, t^*) \qquad (2.15b)$$

$$\underline{H}_1(P_i, t^*) = \bar{\underline{h}}_1(P_i, t^*) \qquad (2.15c)$$

$$\underset{\wedge}{H}_{1c}(P_i, Q_1, t^*) = \bar{\underset{\wedge}{h}}_{1c}(P_i, Q_1, t^*) \qquad (2.15d)$$

$$\underline{H}_2(P_i, t^*) = \bar{\underline{h}}_2(P_i, t^*) + \underline{Z}_2(P_i, t^*) \qquad (2.15e)$$

$$\underset{\wedge}{H}_{2c}(P_i, Q_1, t^*) = \underset{\wedge}{\bar{h}}_{2c}(P_i, Q_1, t^*) + \underset{\wedge}{Z}_{2c}(P_i, Q_1, t^*) - \frac{\partial \underset{\wedge}{\bar{h}}_{1c}}{\partial \bar{q}_1} \frac{\partial \underline{F}_1}{\partial P_1} \tag{2.15f}$$

and

$$\underline{H}_3(P_i, t^*) = \underline{\bar{h}}_3(P_i, t^*) + \underline{Z}_3(P_i, t^*) + \frac{\partial \underline{F}_1}{\partial t^*} \equiv 0 \tag{2.15g}$$

where $\underline{Z}_2$ and $\underset{\wedge}{Z}_{2c}$ are the averaged and resonant parts respectively of

$$Z_2(P_i, Q_i, t^*) = \sum_{i=1}^{N} \frac{\partial \underset{\wedge}{\bar{h}}_{1s}}{\partial \bar{p}_i} \frac{\partial F_1}{\partial \bar{q}_i} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^2 \underline{\bar{h}}_0}{\partial \bar{p}_i \partial \bar{p}_j} \frac{\partial F_1}{\partial \bar{q}_i} \frac{\partial F_1}{\partial \bar{q}_j} \tag{2.16}$$

and where $\underline{Z}_3$ is the averaged part of

$$Z_3(P_i, Q_i, t^*) = -\frac{1}{2} \frac{\partial^2 \underset{\wedge}{\bar{h}}_{1c}}{\partial \bar{q}_1^{\,2}} \left(\frac{\partial \underset{\wedge}{F}_1}{\partial P_1}\right)^2 + \sum_{i=1}^{N} \frac{\partial \underset{\wedge}{\bar{h}}_{2s}}{\partial \bar{p}_i} \frac{\partial F_1}{\partial \bar{q}_i} \tag{2.17}$$

$$+ \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left(\frac{\partial^2 \underline{\bar{h}}_1}{\partial \bar{p}_i \partial \bar{p}_j} + \frac{\partial^2 \underset{\wedge}{\bar{h}}_{1c}}{\partial \bar{p}_i \partial \bar{p}_j} + \frac{\partial^2 \underset{\wedge}{\bar{h}}_{1s}}{\partial \bar{p}_i \partial \bar{p}_j} \right) \frac{\partial F_1}{\partial \bar{q}_i} \frac{\partial F_1}{\partial \bar{q}_j}$$

$$+ \frac{1}{6} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{\partial^3 \underline{\bar{h}}_0}{\partial \bar{p}_i \partial \bar{p}_j \partial \bar{p}_k} \frac{\partial F_1}{\partial \bar{q}_i} \frac{\partial F_1}{\partial \bar{q}_j} \frac{\partial F_1}{\partial \bar{q}_k}$$

Notice that $\underset{\wedge}{H}_{2c}$ contains reference to an arbitrary averaged function $\underline{F}_1$. The choice of $\underline{F}_1$ therefore is significant in two ways: in accounting for averaged terms of $O(\varepsilon^3)$ which become $O(\varepsilon)$ terms over long times (t^* of $O(1)$), and as a contribution to the resonant terms in the $O(\varepsilon^2)$ Hamiltonian.

Because $Q_2, Q_3, \ldots, Q_N$ are removed from the Hamiltonian to $O(\varepsilon^2)$, the associated conjugate momenta, $P_2, P_3, \ldots, P_N$ are constants to $O(\varepsilon^2)$. In the case of resonance, these are the N–1 adiabatic invariants which remain valid through the resonance; they are constant to $O(\varepsilon^2)$ for times $t \le T$, with $T = O(1/\varepsilon^2)$. By inverting the near-identity and isolating transformations, we find the adiabatic invariants as functions of the original variables (p_i, q_i)

$$P_n \equiv \mathcal{A}_n(p_i, q_i, t^*; \varepsilon) = p_n + \frac{r_n}{r_1} p_1 - \varepsilon \frac{\partial F_1}{\partial \overline{q}_n}(\overline{p}_i, \overline{q}_i, t^*)$$

$$+ \varepsilon^2 \left(-\frac{\partial F_2}{\partial \overline{q}_n}(\overline{p}_i, \overline{q}_i, t^*) + \sum_{j=1}^{N} \frac{\partial^2 F_1}{\partial \overline{q}_n \partial P_j} \frac{\partial F_1}{\partial \overline{q}_j} \right) = \text{constant} + O(\varepsilon^3) \tag{2.18}$$

for $n = 2, 3, \ldots, N$, where the evaluations of the partial derivatives at $(\overline{p}_i, \overline{q}_i)$ are replaced by the substitutions (2.12) (note that $r_n = 0$ for $n > R$). This function of the original variables is a constant of the motion to $O(\varepsilon^2)$ along solution trajectories. Several different systems have been examined by the averaging procedure and the original systems numerically integrated; the results are then substituted into expression (2.18), verifying that the adiabatic invariants are indeed constant to $O(\varepsilon^2)$.

One of the most notable advantages of symbolic manipulation programs is their extensive programmability. The entire procedure just outlined in section 2.2 can be fully automated on SMP. After entering the Hamiltonian in standard form, the programmed steps include isolating the resonance, eliminating all nonresonant terms by solving for the $O(\varepsilon)$ and $O(\varepsilon^2)$ generating functions, solving for the averaged $O(\varepsilon^3)$ terms to determine $\underline{E}_1$, simplifying the final Hamiltonian, and inverting all the transformations for an asymptotic solution of the adiabatic invariants in terms of the original variables. On a VAXserver 3500 running Ultrix-32 v3.0 with 12 MB of RAM and 51 MB of virtual memory, SMP version 1.5 took approximately 20 minutes of CPU time to execute this procedure for a 2-D problem. The peak amount of memory required was approximately 12 MB. This procedure probably would have taken less time and have required less memory if run on later versions of SMP or perhaps on another symbolic manipulator; the factorization algorithms (which are time and memory intensive operations) have been much improved.

In section 2.1, for the case where no resonances were present, we found a solution to the original problem in terms of the N adiabatic invariants, which were constants dependent on initial conditions, and N quadratures for the $Q_n(t)$. When a resonance is present only N–1 constants are found, and the system of differential equations of order 2N is reduced to a system of order two, for the variables P_1 and Q_1

$$\frac{dQ_1}{dt} = \sigma(P_i, t^*) + \varepsilon\,[\frac{\partial H_1}{\partial P_1}(P_i, t^*) + \frac{\partial \hat{H}_{1c}}{\partial P_1}(P_i, Q_1, t^*)] + \varepsilon^2\,[\frac{\partial H_2}{\partial P_1}(P_i, t^*) + \frac{\partial \hat{H}_{2c}}{\partial P_1}(P_i, Q_1, t^*)] + O(\varepsilon^3) \tag{2.19a}$$

$$\frac{dP_1}{dt} = -\varepsilon\,\frac{\partial \hat{H}_{1c}}{\partial Q_1}(P_i, Q_1, t^*) - \varepsilon^2\,\frac{\partial \hat{H}_{2c}}{\partial Q_1}(P_i, Q_1, t^*) + O(\varepsilon^3) \tag{2.19b}$$

This is the reduced problem we consider in the section 3 for sustained resonance and in [2] and [3] for transient resonance. The remaining P_n are all constants and the remaining Q_n, $n = 2, 3, \ldots$, N are found as quadratures dependent on the solution for the system (2.19)

$$\begin{aligned} Q_n(t^*) = {} & \frac{1}{\varepsilon^2}\int^{t^*} \frac{\partial H_0}{\partial P_n}(P_1(t^*), P_2, \ldots, P_N, t^*)\,dt^* \\ & + \frac{1}{\varepsilon}\int^{t^*} \frac{\partial H_1}{\partial P_n}(P_1(t^*), P_2, \ldots, P_N, t^*)\,dt^* \\ & + \frac{1}{\varepsilon}\int^{t^*} \frac{\partial \hat{H}_{1c}}{\partial P_n}(P_1(t^*), P_2, \ldots, P_N, Q_1(t^*), t^*)\,dt^* \\ & + \int^{t^*} \frac{\partial H_2}{\partial P_n}(P_1(t^*), P_2, \ldots, P_N, t^*)\,dt^* \\ & + \int^{t^*} \frac{\partial \hat{H}_{2c}}{\partial P_n}(P_1(t^*), P_2, \ldots, P_N, Q_1(t^*), t^*)\,dt^* + O(\varepsilon) \end{aligned} \tag{2.20}$$

All that remains is to find uniformly valid solutions to the two coupled differential equations (2.19) throughout the resonance when $\sigma(P_i, t^*) \approx 0$.

3. The reduced problem for sustained resonance.

3.1 General solution procedure. In this section we examine the solution for the reduced problem (2.19) generated by the averaging procedure given in section 2.2. This system of two differential equations isolates the resonance behavior present in the system of 2N equations (1.1). If the leading order frequency makes a slow passage through zero, then transient resonance is indicated, the results for which are presented in [2] and [3]. However, should the leading order frequency remain near zero for long times, the system then exhibits behavior known as sustained resonance or phase-locking. Sustained resonance is possible only when the frequency σ is a function of P_1, indicating that it is a strictly nonlinear phenomenon.

Since the sustained resonance condition is that $\sigma(P_1, t^*) \approx 0$ for long times, t^* of $O(1)$ or longer, it makes sense to solve for the critical resonant momentum $P_c(t^*)$, which satisfies

$$\sigma(P_c(t^*), t^*) \equiv 0 \tag{3.1}$$

and assume that the actual momentum P_1 is near $P_c(t^*)$. It is easy to show that the appropriate rescaled dependent variables P^* and Q^* are

$$P_1 = P_c(t^*) + \varepsilon^{1/2} P^*(\hat{t}; \varepsilon) \tag{3.2a}$$

$$Q_1 = Q^*(\hat{t}; \varepsilon) \tag{3.2b}$$

and that the rescaled time $\hat{t}$ is

$$\hat{t} = \varepsilon^{1/2} t \tag{3.2c}$$

This gives the expansion for the equations in the new variables

$$\frac{dQ^*}{d\hat{t}} = P^* \frac{\partial \sigma}{\partial P}(P_c(t^*), t^*)$$
$$+ \varepsilon^{1/2} \left[\frac{\partial^2 \sigma}{\partial P^2}(P_c(t^*), t^*) \frac{P^{*2}}{2} + \frac{\partial \underline{H}_1}{\partial P}(P_c(t^*), t^*) + \frac{\partial \hat{\underline{H}}_{1c}}{\partial P}(P_c(t^*), Q^*, t^*) \right] \tag{3.3a}$$

$$+ \varepsilon \left[\frac{\partial^3 \sigma}{\partial P^3} \frac{P*^3}{6} + \frac{\partial^2 H_1}{\partial P^2} P* + \frac{\partial^2 \hat{H}_{1c}}{\partial P^2} P* \right]$$

$$+ \varepsilon^{3/2} \left[\frac{\partial^4 \sigma}{\partial P^4} \frac{P*^4}{24} + \frac{\partial^3 H_1}{\partial P^3} \frac{P*^2}{2} + \frac{\partial^3 \hat{H}_{1c}}{\partial P^3} \frac{P*^2}{2} + \frac{\partial H_2}{\partial P} + \frac{\partial \hat{H}_{2c}}{\partial P} \right] + O(\varepsilon^2)$$

and

$$\frac{dP*}{d\hat{t}} = - \frac{\partial \hat{H}_{1c}}{\partial Q} (P_c(t*), Q*, t*) - \varepsilon^{1/2} \frac{\partial^2 \hat{H}_{1c}}{\partial Q \, \partial P} (P_c(t*), Q*, t*) \, P*$$

$$- \varepsilon \left[\frac{dP_c(t*)}{dt*} + \frac{\partial^3 \hat{H}_{1c}}{\partial Q \, \partial P^2} (P_c(t*), Q*, t*) \frac{P*^2}{2} + \frac{\partial \hat{H}_{2c}}{\partial Q} (P_c(t*), Q*, t*) \right] \tag{3.3b}$$

$$- \varepsilon^{3/2} \left[\frac{\partial^4 \hat{H}_{1c}}{\partial Q \, \partial P^3} (P_c(t*), Q*, t*) \frac{P*^3}{6} + \frac{\partial^2 \hat{H}_{2c}}{\partial Q \, \partial P} (P_c(t*), Q*, t*) \, P* \right] + O(\varepsilon^2)$$

(the subscript 1 is dropped from the partial derivatives). One effect of this rescaling is that the order of the ratio of the slow time $t*$ to the new fast time $\hat{t}$ is even higher relative to the order of the first perturbation term, $\hat{\varepsilon} = \varepsilon^{1/2}$, since $t* = \hat{\varepsilon}^3 \hat{t}$. This will necessitate a still higher order expansion in $\hat{\varepsilon}$ due to the relatively slower time variation than with the $t* = \varepsilon^2 t$ scaling. A second result of the rescaling is that the equations for $P*$ and $Q*$ are not in standard form. This question will be addressed presently.

Since the $O(1)$ problem is linear in $P*$, we can differentiate (3.3a) again with respect to $\hat{t}$ to derive a single second order equation for $Q*$:

$$\frac{d^2 Q*}{d\hat{t}^2} + \frac{\partial \sigma}{\partial P} (P_c(t*), t*) \frac{\partial \hat{H}_{1c}}{\partial Q} (P_c(t*), Q*, t*) = \hat{\varepsilon} \, D(Q*, \frac{dQ*}{d\hat{t}}, t*; \hat{\varepsilon}) \tag{3.4}$$

where $t* = \hat{\varepsilon}^3 \hat{t}$. Equation (3.4) represents a very slowly varying nonlinear oscillator with weak damping term $\hat{\varepsilon} D$. In general, (3.4) is not solvable explicitly, even with $\hat{\varepsilon} = 0$, as the solution involves the inversion of the integral

$$\hat{t} + \phi = \frac{1}{\sqrt{2}} \int^{Q^*} \frac{dQ}{\sqrt{E - \sigma_P \underset{\wedge}{H}_{1c}(Q)}} \tag{3.5}$$

where

$$\sigma_P = \frac{\partial \sigma}{\partial P}(P_c(t^*), t^*) \tag{3.6}$$

which is expressible in terms of known functions only for certain $\underset{\wedge}{H}_{1c}(Q)$. For example, if $\underset{\wedge}{H}_{1c}(Q)$ is a polynomial of degree less than or equal to four, then the integral can be expressed in terms of circular and/or Jacobian elliptic functions. This is also the case if $\underset{\wedge}{H}_{1c}(Q)$ contains a single $\sin(\omega Q)$ or $\cos(\omega Q)$ term.

Bourland and Haberman in [5] have extended the procedure of Kuzmak-Luke to solve the nonlinear oscillator (3.4) for a general potential when the slow time is $\tilde{t} = \varepsilon t$ instead of $t^* = \hat{\varepsilon}^3 \hat{t}$. However, the procedure of Kuzmak-Luke proves impractical for the $t^* = \hat{\varepsilon}^3 \hat{t}$ slow time, as computation of the higher order terms (terms which are necessary for a uniformly valid $O(1)$ solution) requires finding particular solutions to a sequence of linear but non-constant coefficient second order differential equations whose forcing functions contain increasingly large numbers of terms. The use of symbolic computation in finding these solutions is limited, as the integrals prove difficult in general and the integration library may or may not contain all the necessary forms.

The procedure of averaging, however, can still be used by casting (3.3) back into standard form (1.1) through an action-angle transformation, since (3.3) is derivable from the Hamiltonian

$$\begin{aligned} H^*(P^*, Q^*, t^*; \hat{\varepsilon}) &= \frac{P^{*2}}{2} \frac{\partial \sigma}{\partial P}(P_c(t^*), t^*) + \underset{\wedge}{H}_{1c}(P_c(t^*), Q^*, t^*) \\ &+ \hat{\varepsilon} \left[\frac{\partial^2 \sigma}{\partial P^2} \frac{P^{*3}}{6} + \frac{\partial \underline{H}_1}{\partial P} P^* + \frac{\partial \underset{\wedge}{H}_{1c}}{\partial P} P^* \right] \end{aligned} \tag{3.7}$$

$$+ \hat{\varepsilon}^2 \left[\frac{dP_c(t^*)}{dt^*} Q^* + \frac{\partial^3 \sigma}{\partial P^3} \frac{P^{*4}}{24} + \frac{\partial^2 \underline{H}_1}{\partial P^2} \frac{P^{*2}}{2} + \frac{\partial^2 \hat{H}_{1c}}{\partial P^2} \frac{P^{*2}}{2} + \hat{H}_{2c} \right]$$

$$+ \hat{\varepsilon}^3 \left[\frac{\partial^4 \sigma}{\partial P^4} \frac{P^{*5}}{120} + \frac{\partial^3 \underline{H}_1}{\partial P^3} \frac{P^{*3}}{6} + \frac{\partial^3 \hat{H}_{1c}}{\partial P^3} \frac{P^{*3}}{6} + \frac{\partial \underline{H}_2}{\partial P} P^* + \frac{\partial \hat{H}_{2c}}{\partial P} P^* \right] + O(\hat{\varepsilon}^4)$$

The action-angle transformation can be found by solving the Hamilton-Jacobi equation for the $O(1)$ Hamiltonian

$$\frac{1}{2} \frac{\partial \sigma}{\partial P} \left(\frac{\partial W}{\partial Q^*} \right)^2 + \hat{H}_{1c}(Q^*) = E_0(p) \tag{3.8}$$

The generating function W, a function of the old coordinate and the new momentum, is found to be

$$W(Q^*, p) = \sqrt{\frac{2}{\sigma_P}} \int^{Q^*} \sqrt{E_0(p) - \hat{H}_{1c}(Q)}\, dQ \tag{3.9}$$

and the old momentum P^* is given by

$$P^* = \frac{\partial W}{\partial Q^*} = \sqrt{\frac{2\,(E_0 - \hat{H}_{1c}(Q^*))}{\sigma_P}} \tag{3.10a}$$

The new coordinate, q, is given by

$$q = \frac{\partial W}{\partial p} = \frac{1}{\sqrt{2\,\sigma_P}} \frac{\partial E_0}{\partial p} \int^{Q^*} \frac{dQ}{\sqrt{E_0(p) - \hat{H}_{1c}(Q)}} \tag{3.10b}$$

Relations (3.10) are mixed algebraic relations which must be solved simultaneously to find the explicit transformation

$$Q^* = Q^*(p, q) \qquad\qquad P^* = P^*(p, q) \tag{3.11}$$

Notice that an inversion of the integral (3.10b) is necessary for this calculation. This is exactly the inversion required by (3.5) in solving the single second order differential equation. As one might expect, this means that finding the transformation to action-angle variables is equivalent to solving the $O(1)$ problem.

An additional condition is necessary to solve for the transformation. The functional form $E_0(p)$ is determined by requiring p to be the action, defined by

$$p = \oint P^* \, dQ^* \tag{3.12}$$

where t^* is held fixed and the integral is over one period of the motion in the P^*-Q^* plane. Equivalently, the periodicity in the new variable q of the old Q^* can be constrained to be independent of p (i.e., $Q^* = Q^*(p, q)$ has fixed period in q independent of p). It has not always been recognized that this restriction is *essential* for the transformation to standard form. Although we can choose E_0 to have any functional dependence on the new p and thereby eliminate q from the $O(1)$ Hamiltonian, the resulting differential equations will have terms of $O(\hat{\varepsilon})$ and higher orders which are secular (i.e., grow linearly in q) for any choice other than the one specified by a constant multiple of (3.12). Although this formulation cannot be done explicitly in general, an implicit definition from the integral

$$p = \sqrt{\frac{2}{\sigma_p}} \oint \sqrt{E_0(p) - \underset{\wedge}{H}_{1c}(Q^*)} \, dQ^* \tag{3.13}$$

can be used.

The relations (3.10) and (3.13) give the explicit substitutions of the form (3.11) which transform the Hamiltonian (3.7) to a new Hamiltonian which is a function of (p, q, t^*)

$$h(p, q, t^*; \hat{\varepsilon}) = H^*(P^*(p, q, t^*), Q^*(p, q, t^*), t^*; \hat{\varepsilon}) + \hat{\varepsilon}^3 \frac{\partial W}{\partial t^*} \tag{3.14}$$

When expanded this expression has the form

$$\begin{aligned} h(p, q, t^*; \hat{\varepsilon}) &= E_0(p, t^*) + \hat{\varepsilon} \, h_1(p, q, t^*) \\ &+ \hat{\varepsilon}^2 h_2(p, q, t^*) + \hat{\varepsilon}^3 h_3(p, q, t^*) + O(\hat{\varepsilon}^4) \end{aligned} \tag{3.15}$$

Since we have chosen (p, q) to satisfy (3.13), both P^* and Q^* have fixed period in q equal to one and independent of p. Also, the $O(1)$ Hamiltonian is independent of q, which means that (3.15) is in standard form (1.1). (In section 1 we stated that standard form (1.1) should be 2π periodic in the q_i, but it is only necessary to have a fixed periodicity. If we desire a 2π periodicity, we simply divide the integral in (3.12) by 2π.)

The transformations (3.2) and (3.10) have removed the resonance from the system and as a result, the averaging procedure given in section 2.1 can be applied to (3.15) to remove the q dependence and to solve for an additional adiabatic invariant $\mathcal{A}(p, q, t^*; \hat{\varepsilon})$.

To proceed with the averaging we split the Hamiltonian (3.15) into its oscillatory and averaged parts

$$
\begin{aligned}
h(p, q, t^*; \hat{\varepsilon}) = \underline{h}_0(p, t^*) &+ \hat{\varepsilon}\, [\underline{h}_1(p, t^*) + \underset{\wedge}{h}_1(p, q, t^*)] \\
&+ \hat{\varepsilon}^2\, [\underline{h}_2(p, t^*) + \underset{\wedge}{h}_2(p, q, t^*)] \\
&+ \hat{\varepsilon}^3\, [\underline{h}_3(p, t^*) + \underset{\wedge}{h}_3(p, q, t^*)] + O(\hat{\varepsilon}^4)
\end{aligned}
\tag{3.16}
$$

where the underhat denotes the oscillatory part with zero average over q and the underbar denotes the averaged part. We define a near-identity transformation using the generating function

$$
\begin{aligned}
F(P, q, t^*) = Pq &+ \hat{\varepsilon}\, F_1(P, q, t^*) \\
&+ \hat{\varepsilon}^2\, F_2(P, q, t^*) + \hat{\varepsilon}^3\, F_3(P, q, t^*)
\end{aligned}
\tag{3.17}
$$

by which we transform (p, q) to (P, Q). The only modification from the procedure in section 2.1 is that an $O(\hat{\varepsilon}^3)$ generating function has been included and the new Hamiltonian should be calculated to $O(\hat{\varepsilon}^4)$; this is solely due to the change in the slow time to fast time ratio ($t^* = \hat{\varepsilon}^3 \hat{t}$ rather than $t^* = \varepsilon^2 t$). It is necessary to find

$p = p(P, Q)$ asymptotically to $O(\hat{\varepsilon}^4)$ and $q = q(P, Q)$ to $O(\hat{\varepsilon}^3)$ since average terms of $O(\hat{\varepsilon}^4)$ must be found. Extending the procedure which generated (2.5) to these orders we find

$$\begin{aligned}
p = P &+ \hat{\varepsilon}\frac{\partial F_1}{\partial q}(P, Q, t^*) + \hat{\varepsilon}^2\left[\frac{\partial F_2}{\partial q} - \frac{\partial^2 F_1}{\partial q^2}\frac{\partial F_1}{\partial P}\right] \\
&+ \hat{\varepsilon}^3\left[\frac{\partial F_3}{\partial q} - \frac{\partial^2 F_2}{\partial q^2}\frac{\partial F_1}{\partial P} - \frac{\partial^2 F_1}{\partial q^2}\frac{\partial F_2}{\partial P} + \frac{1}{2}\frac{\partial^3 F_1}{\partial q^3}\left(\frac{\partial F_1}{\partial P}\right)^2 + \frac{\partial^2 F_1}{\partial q^2}\frac{\partial^2 F_1}{\partial q \partial P}\frac{\partial F_1}{\partial P}\right] \\
&+ \hat{\varepsilon}^4\left[-\frac{\partial^2 F_1}{\partial q^2}\frac{\partial F_3}{\partial P} - \frac{\partial^2 F_2}{\partial q^2}\frac{\partial F_2}{\partial P} - \frac{\partial^2 F_3}{\partial q^2}\frac{\partial F_1}{\partial P} + \frac{1}{2}\frac{\partial^3 F_2}{\partial q^3}\left(\frac{\partial F_1}{\partial P}\right)^2\right. \\
&\quad + \frac{\partial^2 F_1}{\partial q^2}\frac{\partial^2 F_1}{\partial q \partial P}\frac{\partial F_2}{\partial P} + \frac{\partial^2 F_2}{\partial q^2}\frac{\partial^2 F_1}{\partial q \partial P}\frac{\partial F_1}{\partial P} + \frac{\partial^2 F_1}{\partial q^2}\frac{\partial^2 F_2}{\partial q \partial P}\frac{\partial F_1}{\partial P} \\
&\quad + \frac{\partial^3 F_1}{\partial q^3}\frac{\partial F_1}{\partial P}\frac{\partial F_2}{\partial P} - \frac{\partial^2 F_1}{\partial q^2}\left(\frac{\partial^2 F_1}{\partial q \partial P}\right)^2\frac{\partial F_1}{\partial P} - \frac{\partial^3 F_1}{\partial q^3}\frac{\partial^2 F_1}{\partial q \partial P}\left(\frac{\partial F_1}{\partial P}\right)^2 \\
&\quad \left. - \frac{1}{6}\frac{\partial^4 F_1}{\partial q^4}\left(\frac{\partial F_1}{\partial P}\right)^3 - \frac{1}{2}\frac{\partial^2 F_1}{\partial q^2}\frac{\partial^3 F_1}{\partial q^2 \partial P}\left(\frac{\partial F_1}{\partial P}\right)^2\right] + O(\hat{\varepsilon}^5)
\end{aligned} \tag{3.18a}$$

and

$$\begin{aligned}
q = Q &- \hat{\varepsilon}\frac{\partial F_1}{\partial P}(P, Q, t^*) - \hat{\varepsilon}^2\left[\frac{\partial F_2}{\partial P} - \frac{\partial^2 F_1}{\partial q \partial P}\frac{\partial F_1}{\partial P}\right] \\
&- \hat{\varepsilon}^3\left[\frac{\partial F_3}{\partial P} - \frac{\partial^2 F_2}{\partial q \partial P}\frac{\partial F_1}{\partial P} - \frac{\partial^2 F_1}{\partial q \partial P}\frac{\partial F_2}{\partial P}\right. \\
&\quad \left. + \frac{1}{2}\frac{\partial^3 F_1}{\partial q^2 \partial P}\left(\frac{\partial F_1}{\partial P}\right)^2 + \left(\frac{\partial^2 F_1}{\partial q \partial P}\right)^2\frac{\partial F_1}{\partial P}\right] + O(\hat{\varepsilon}^4)
\end{aligned} \tag{3.18b}$$

where all partial derivatives are evaluated at the new variables (P, Q). Substituting (3.18) into

$$H(P, Q, t^*; \hat{\varepsilon}) = h(p(P, Q, t^*), q(P, Q, t^*), t^*; \hat{\varepsilon}) + \hat{\varepsilon}^4\frac{\partial F_1}{\partial t^*} \tag{3.19}$$

then expanding as in section 2.1 gives the new Hamiltonian and we can solve sequentially for F_1, F_2, and F_3 to eliminate Q to $O(\hat{\varepsilon}^3)$.

A simplification of the results occurs when the Hamiltonian is a function of just one p and one q. In this case all of the generating functions can be found explicitly in terms of the Hamiltonian (3.16). This gives

$$F_1(P,q,t^*) = \frac{-1}{\Omega_0(P,t^*)} \int^q \tilde{h}_1(P,q,t^*)\, dq \tag{3.20}$$

where we have defined the $O(\hat{\varepsilon}^j)$ frequency

$$\Omega_j(p,t^*) = \frac{\partial \underline{h}_j}{\partial p}(p,t^*) \tag{3.21}$$

Defining the intermediate quantity

$$Z_2(p,q,t^*) = -\frac{1}{2}\frac{\partial}{\partial p}\left(\frac{\tilde{h}_1^2(p,q,t^*)}{\Omega_0(p,t^*)} \right) \tag{3.22}$$

which can be split into oscillatory and averaged parts

$$Z_2(p,q,t^*) = \underline{Z}_2(p,t^*) + \tilde{Z}_2(p,q,t^*) \tag{3.23}$$

we then find

$$\begin{aligned} F_2(P,q,t^*) &= \frac{-1}{\Omega_0(P,t^*)} \int^q [\tilde{h}_2(P,q,t^*) + \tilde{Z}_2(P,q,t^*)]\, dq \\ &\quad + \frac{\Omega_1(P,t^*)}{\Omega_0^2(P,t^*)} \int^q \tilde{h}_1(P,q,t^*)\, dq \end{aligned} \tag{3.24}$$

We define the second intermediate quantity

$$\begin{aligned} Z_3(p,q,t^*) &= -\frac{\partial}{\partial p}\left(\frac{\tilde{h}_1(p,q,t^*)\, \tilde{h}_2(p,q,t^*)}{\Omega_0(p,t^*)} \right) \\ &\quad + \frac{1}{6}\frac{\partial}{\partial p}\left(\frac{1}{\Omega_0(p,t^*)} \frac{\partial}{\partial p}\left(\frac{\tilde{h}_1^3(p,q,t^*)}{\Omega_0(p,t^*)} \right)\right) \\ &\quad + \sqrt{\frac{\Omega_1(p,t^*)}{\Omega_0(p,t^*)}}\; \frac{\partial}{\partial p}\left(\tilde{h}_1^2(p,q,t^*) \sqrt{\frac{\Omega_1(p,t^*)}{\Omega_0^3(p,t^*)}} \right) \end{aligned} \tag{3.25}$$

which is also split into oscillatory and averaged parts

$$Z_3(p, q, t^*) \;=\; \underline{Z}_3(p, t^*) \;+\; \underset{\sim}{Z}_3(p, q, t^*) \tag{3.26}$$

to give a solution for $F_3(P, q, t^*)$

$$\begin{aligned}
F_3(P, q, t^*) \;=\; & \frac{-1}{\Omega_0(P, t^*)} \int^q [\underset{\wedge}{h}_3(P, q, t^*) + \underset{\wedge}{Z}_3(P, q, t^*)]\, dq \\
& + \left[\frac{\Omega_2(P, t^*)}{\Omega_0^{\,2}(P, t^*)} - \frac{\Omega_1^{\,2}(P, t^*)}{\Omega_0^{\,3}(P, t^*)} + \frac{\underline{Z}_2(P, t^*)}{\Omega_0^{\,3}(P, t^*)} \frac{\partial \Omega_0}{\partial p}(P, t^*) \right] \int^q \underset{\wedge}{h}_1(P, q, t^*)\, dq \\
& + \frac{\Omega_1(P, t^*)}{\Omega_0^{\,2}(P, t^*)} \int^q \underset{\wedge}{h}_2(P, q, t^*)\, dq \\
& - \frac{\underline{Z}_2(P, t^*)}{\Omega_0^{\,2}(P, t^*)} \int^q \frac{\partial \underset{\wedge}{h}_1}{\partial p}(P, q, t^*)\, dq
\end{aligned} \tag{3.27}$$

The $O(\hat{\varepsilon}^4)$ Hamiltonian H_4 which results from this transformation has also been found in terms of h_0, h_1, h_2, h_3, h_4 and $\underline{Z}_2$, $\underline{Z}_3$ but is too long to include here. The final Hamiltonian with all Q dependence removed to $O(\hat{\varepsilon}^3)$ is given by

$$\begin{aligned}
H(P, t^*; \hat{\varepsilon}) \;=\; & \underline{h}_0(P, t^*) \;+\; \hat{\varepsilon}\; \underline{h}_1(P, t^*) \\
& + \hat{\varepsilon}^2\, [\underline{h}_2(P, t^*) + \underline{Z}_2(P, t^*)] \\
& + \hat{\varepsilon}^3\, [\underline{h}_3(P, t^*) \;+\; \underline{Z}_3(P, t^*) \;+\; \frac{\underline{Z}_2(P, t^*)\, \Omega_1(P, t^*)}{\Omega_0(P, t^*)}] \;+\; O(\hat{\varepsilon}^4)
\end{aligned} \tag{3.28}$$

The new action P is an adiabatic invariant to $O(\hat{\varepsilon}^3)$

$$\frac{d\underline{P}}{d\hat{t}} \;=\; -\,\hat{\varepsilon}^4\, \frac{\partial \underset{\wedge}{H}_4}{\partial Q} \;+\; O(\hat{\varepsilon}^5) \tag{3.29a}$$

Thus the equation for Q is uncoupled

$$\begin{aligned}
\frac{d\underline{Q}}{d\hat{t}} \;=\; & \Omega_0(P, t^*) \;+\; \hat{\varepsilon}\, \Omega_1(P, t^*) \;+\; \hat{\varepsilon}^2\, [\Omega_2(P, t^*) + \frac{\partial \underline{Z}_2}{\partial p}(P, t^*)] \\
& + \hat{\varepsilon}^3\, [\Omega_3(P, t^*) + \frac{\partial \underline{Z}_3}{\partial p}(P, t^*) + \frac{\partial}{\partial p} \frac{\underline{Z}_2\, \Omega_1}{\Omega_0}(P, t^*)] \;+\; O(\hat{\varepsilon}^4)
\end{aligned} \tag{3.29b}$$

and the solution is given by

$$P(t^*) = P_0 + O(\hat{\varepsilon}^4) \tag{3.30a}$$

$$\begin{aligned} Q(t^*) = Q_0 &+ \frac{1}{\hat{\varepsilon}^3}\int_0^{t^*} \Omega_0(P_0, t^*)\, dt^* + \frac{1}{\hat{\varepsilon}^2}\int_0^{t^*} \Omega_1(P_0, t^*)\, dt^* \\ &+ \frac{1}{\hat{\varepsilon}}\int_0^{t^*} [\Omega_2(P_0, t^*) + \frac{\partial Z_2}{\partial p}(P_0, t^*)]\, dt^* \\ &+ \int_0^{t^*} [\Omega_3(P_0, t^*) + \frac{\partial Z_3}{\partial p}(P_0, t^*) + \frac{\partial}{\partial p}\frac{Z_2\,\Omega_1}{\Omega_0}(P_0, t^*)]\ dt^* + O(\hat{\varepsilon}) \end{aligned} \tag{3.30b}$$

valid for times t^* of $O(1)$ where Q_0 and P_0 are constants found from initial conditions. The near-identity transformation (3.18) can be inverted to find P as a function of the original variables to calculate the value of the adiabatic invariant

$$P = \mathcal{A}(p, q, t^*; \hat{\varepsilon}) \equiv \text{constant} + O(\hat{\varepsilon}^4) \tag{3.31}$$

which again can be expressed in terms of the original Hamiltonian (3.16)

$$\begin{aligned} \mathcal{A}(p, q, t^*; \hat{\varepsilon}) = p &+ \hat{\varepsilon}\frac{1}{\Omega_0(p, t^*)}\, h_1(p, q, t^*) \\ &+ \hat{\varepsilon}^2 [\frac{1}{\Omega_0}(h_2 - Z_2) - \frac{\Omega_1}{\Omega_0^2} h_1 + \frac{1}{4}\frac{\partial}{\partial p}\left(\frac{1}{\Omega_0^2}\right) h_1^2] \\ &+ \hat{\varepsilon}^3 [\frac{1}{\Omega_0}(h_3 - Z_3) + (\frac{\Omega_1^2}{\Omega_0^3} - \frac{\Omega_2}{\Omega_0^2} - \frac{1}{\Omega_0}\frac{\partial}{\partial p}\left(\frac{Z_2}{\Omega_0}\right))\, h_1 \\ &\quad - \frac{\Omega_1}{\Omega_0^2} h_2 + \frac{1}{2}\frac{\partial}{\partial p}\left(\frac{1}{\Omega_0^2}\right) h_1 h_2 - \frac{1}{2}\frac{\partial}{\partial p}\left(\frac{\Omega_1}{\Omega_0^3}\right) h_1^2 \\ &\quad + \frac{1}{12\,\Omega_0}\frac{\partial^2}{\partial p^2}\left(\frac{1}{\Omega_0^2}\right) h_1^3] + O(\hat{\varepsilon}^4) \end{aligned} \tag{3.32}$$

where all functions are evaluated at the old variables (p, q).

The asymptotic solutions (3.30) are easily substituted back into the transformations (3.18) then (3.11) to get asymptotic solutions for P^* and Q^* as functions of time. As seen, the procedure described by equations (3.16) to (3.32) gives an easily programmed algorithm for a symbolic manipulator to do averaging for a Hamiltonian system. This is facilitated by the fact that all results can be expressed in terms of the original Hamiltonian, so that solutions can be found directly. The general formulas themselves have been derived on SMP to reduce them to simplest form.

3.2 A model problem. In [17], Kevorkian examines several applications which exhibit sustained resonance: the motion of an asymmetrical planet in a slowly varying gravitational field, the high altitude motion of a spinning re-entry vehicle, and the electron dynamics of free-electron lasers. In all of these examples, the resonance occurs in a single harmonic to $O(\varepsilon)$. We therefore note that a model Hamiltonian for a commonly occurring class of reduced problems can be written in the form

$$H(P_1, Q_1, t^*; \varepsilon) = H_0(P_1, t^*) + \varepsilon A(P_1, t^*) \sin Q_1 + \varepsilon^2 B(P_1, t^*) \tag{3.33}$$

where $t^* = \varepsilon^2 t$, and for sustained resonance to be possible, H_0 must be a nonlinear function of P_1. The system of equations for the Hamiltonian (3.33) is

$$\frac{dQ_1}{dt} = \sigma(P_1, t^*) + \varepsilon \frac{\partial A}{\partial P_1}(P_1, t^*) \sin Q_1 + \varepsilon^2 \frac{\partial B}{\partial P_1}(P_1, t^*) \tag{3.34a}$$

$$\frac{dP_1}{dt} = - \varepsilon A(P_1, t^*) \cos Q_1 \tag{3.34b}$$

where $\sigma(P_1, t^*) = \partial H_0 / \partial P_1$. Following the procedure outlined in section 3.1, we obtain the following equations corresponding to (3.3) governing the rescaled variables P^* and Q^* as functions of $\hat{t} = \varepsilon^{1/2} t$.

$$\frac{dQ^*}{d\hat{t}} = \frac{\partial\sigma}{\partial P}(P_c(t^*), t^*)\ P^* + \hat{\varepsilon}\ [\frac{\partial^2\sigma}{\partial P^2}\frac{P^{*2}}{2} - \frac{\partial A}{\partial P}\cos Q^*\ ]$$

$$+ \hat{\varepsilon}^2\ [\frac{\partial^3\sigma}{\partial P^3}\frac{P^{*3}}{6} - \frac{\partial^2 A}{\partial P^2}\ P^*\ \cos Q^*\ ] \tag{3.35a}$$

$$+ \hat{\varepsilon}^3\ [\frac{\partial^4\sigma}{\partial P^4}\frac{P^{*4}}{24} - \frac{\partial^3 A}{\partial P^3}\frac{P^{*2}}{2}\cos Q^* + \frac{\partial B}{\partial P}] + O(\hat{\varepsilon}^4)$$

$$\frac{dP^*}{d\hat{t}} = -A(P_c(t^*), t^*)\sin Q^* - \hat{\varepsilon}\frac{\partial A}{\partial P}\ P^*\ \sin Q^*$$

$$-\ \hat{\varepsilon}^2\ [\frac{\partial^2 A}{\partial P^2}\frac{P^{*2}}{2}\sin Q^* + \frac{dP_c(t^*)}{dt^*}] \tag{3.35b}$$

$$-\ \hat{\varepsilon}^3\ \frac{\partial^3 A}{\partial P^3}\frac{P^{*3}}{6}\sin Q^* + O(\hat{\varepsilon}^4)$$

(the subscript 1 has been dropped from the partial derivatives). This new system of differential equations is associated with the Hamiltonian (3.7) which becomes

$$H^*(P^*, Q^*, t^*; \hat{\varepsilon}) = \sigma_P(t^*)\ \frac{P^{*2}}{2} - A(t^*)\cos Q^*$$

$$+ \hat{\varepsilon}\ [\sigma_{PP}(t^*)\ \frac{P^{*3}}{6} - A_P(t^*)\ P^*\ \cos Q^*\ ]$$

$$+ \hat{\varepsilon}^2\ [\sigma_{PPP}(t^*)\ \frac{P^{*4}}{24} - A_{PP}(t^*)\ \frac{P^{*2}}{2}\cos Q^* + \frac{dP_c}{dt^*}(t^*)\,Q^*\ ] \tag{3.36}$$

$$+ \hat{\varepsilon}^3\ [\sigma_{PPPP}(t^*)\frac{P^{*5}}{120} - A_{PPP}(t^*)\ \frac{P^{*3}}{6}\cos Q^* + B_P(t^*)\,P^*\ ] + O(\hat{\varepsilon}^4)$$

The notation of subscripting with respect to P indicates partial differentiation and all the partial derivatives of σ, A, and B are evaluated at $P_c(t^*)$ and t^*, so are functions of the slow time only.

If we eliminate P^* from (3.35) we find the following special case of (3.4)

$$\frac{d^2Q^*}{d\hat{t}^2} + \sigma_P(t^*)\ A(t^*)\ \sin Q^* = \hat{\varepsilon}\ D(Q^*, \frac{dQ^*}{d\hat{t}}, t^*; \hat{\varepsilon}) \tag{3.37}$$

a very slowly varying pendulum equation with weak nonlinear damping. For the case $\hat{\varepsilon} = 0$, t^* fixed, we have the phase plane for Q^* shown in Figure 3.1a. The region inside the heavy line is characterized by periodic motion of Q^* about a center; this is the region of sustained resonance. Outside the heavy line, Q^* increases or decreases monotonically. When $\hat{\varepsilon} = 0$ there is no way to move from one region to the other. However if $\hat{\varepsilon} \neq 0$ and slow variation is permitted, these two effects can cause capture into sustained resonance (Figure 3.1b) or escape from sustained resonance (Figure 3.1c). The rate at which either of these processes occurs is very slow compared to a single oscillation. For this reason, higher order terms can have a significant effect on the long term behavior of the system.

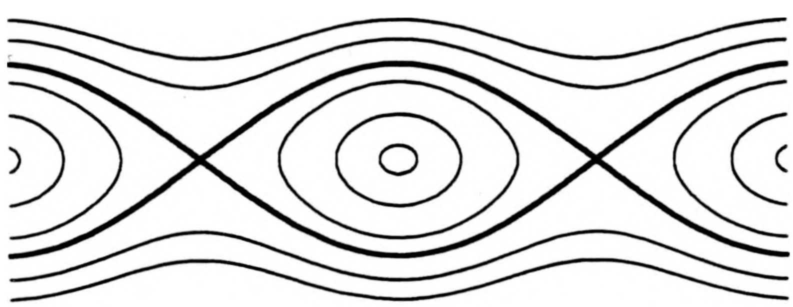

Fig. 3.1a

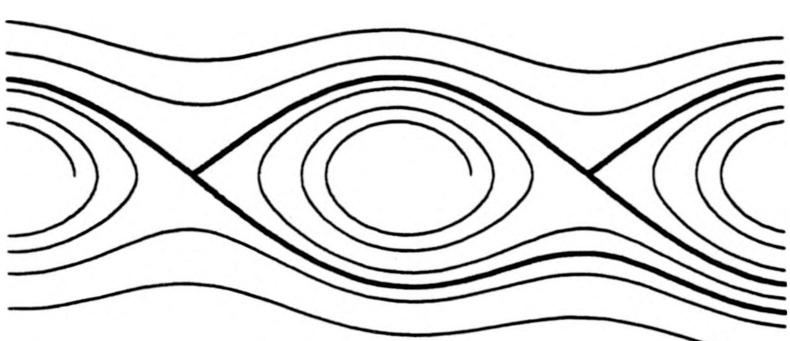

Fig. 3.1b

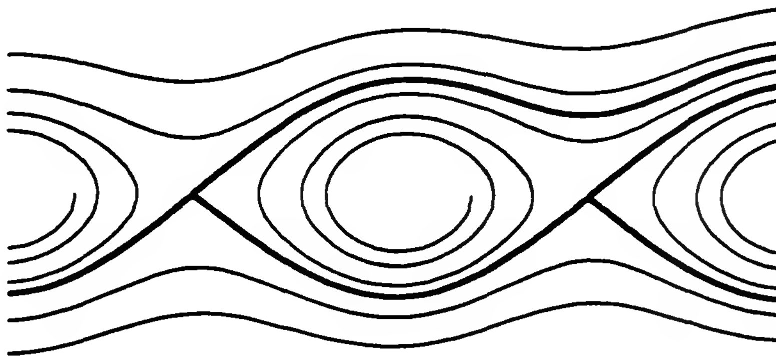

Fig. 3.1c

Figure 3.1 Slowly Varying Phase Planes

Rather than solving the problem as a single second order O.D.E. in Q^*, the Hamiltonian (3.36) can be transformed to standard form via an action-angle transformation, generated by (3.9)

$$W(Q^*, p, t^*) = \sqrt{\frac{2}{\sigma_P}} \int^{Q^*} \sqrt{\mathcal{E}_0(p) + A \cos Q}\, dQ$$

$$= 4\sqrt{\frac{A}{\sigma_P}} \left\{ E(\sin^{-1}(\frac{1}{k}\sin\frac{Q^*}{2}), k) - (1-k^2)\, F(\sin^{-1}(\frac{1}{k}\sin\frac{Q^*}{2}), k) \right\} \tag{3.38}$$

Here F and E are the elliptic integrals of first and second kinds respectively, and k is the modulus ($0 \le k < 1$). The reader will find many useful results for elliptic integrals in Byrd and Friedman [7]. The generating function W depends on p only through $k(p, t^*)$, defined implicitly by

$$p = 16\sqrt{\frac{A(t^*)}{\sigma_P(t^*)}} \left\{ E_c(k) - (1-k^2)\, K(k) \right\} \tag{3.39}$$

This relation is found by specifying p to be the action for the system using (3.13). The functions K and E_c are the complete elliptic integrals of the first and second kinds respectively. Because (3.39) is invertible only in principle, the functional dependence of the transformation (and hence the new Hamiltonian) on p will be defined implicitly only through the functional dependence on k. From the generating function (3.38) we obtain the old variables as functions of the new in the following form

$$Q^* = 2\sin^{-1}(k \;\mathrm{sn}(4\,K(k)\,q,\,k)) \tag{3.40a}$$

$$P^* = 2\,k\sqrt{\frac{A}{\sigma_P}}\;\mathrm{cn}(4\,K(k)\,q,\,k) \tag{3.40b}$$

For this transformation, a necessary condition is that σ_P and A have the same sign. If not, we simply shift Q^* by π. The new Hamiltonian $h\,(p,\,q,\,t^*;\,\hat{\varepsilon}\,)$ is defined by (3.14) where now

$$E_0(p,\,t^*) \;=\; A(t^*)\,(2k^2-1) \tag{3.41a}$$

$$\begin{aligned} h_1(p,\,q,\,t^*) \;=\; & \beta_{11}(t^*)\;\;k\,(2\,k^2-1)\;\;\mathrm{cn}(4\,K(k)\,q,\,k) \\ & + \;\beta_{12}(t^*)\;\;k^3\,\mathrm{cn}^3(4\,K(k)\,q,\,k) \end{aligned} \tag{3.41b}$$

$$\begin{aligned} h_2(p,\,q,\,t^*) \;=\; & \beta_{21}(t^*)\;\;\sin^{-1}(k\,\mathrm{sn}(4\,K(k)\,q,\,k)) \\ & + \;\beta_{22}(t^*)\;\;k^2\,(2\,k^2-1)\;\;\mathrm{cn}^2(4\,K(k)\,q,\,k) \;+\; \beta_{23}(t^*)\;\;k^4\,\mathrm{cn}^4(4\,K(k)\,q,\,k) \end{aligned} \tag{3.41c}$$

$$\begin{aligned} h_3(p,\,q,\,t^*) \;=\; & \beta_{31}(t^*)\;\;k\;\;\mathrm{cn}(4\,K(k)\,q,\,k) \\ & + \;\beta_{32}(t^*)\;\;k^3\,(2\,k^2-1)\;\;\mathrm{cn}^3(4\,K(k)\,q,\,k) \;+\; \beta_{33}(t^*)\;\;k^5\,\mathrm{cn}^5(4\,K(k)\,q,\,k) \\ & + \;\beta_{34}(t^*)\;\;E_{osc}(4\,K(k)\,q,\,k) \end{aligned} \tag{3.41d}$$

and k is a function of p and t^* as defined by (3.39). The function $E_{osc}(u,\,k)$ is shorthand notation for the oscillatory part of the elliptic integral of the second kind; it is defined to have zero average when integrated over the first argument u.

$$E_{osc}(u,\,k) = E(\mathrm{am}(u,\,k),\,k) \;-\; \frac{E_c(k)}{K(k)}\;u \tag{3.42}$$

The β_{ij} are functions of t^* alone through σ, A, and B and their partial derivatives, and are given by

$$\beta_{11}(t^*) = 2\, A_P(t^*)\sqrt{\frac{A(t^*)}{\sigma_P(t^*)}} \tag{3.43a}$$

$$\beta_{12}(t^*) = \frac{4}{3}\left(A(t^*)\,\frac{\sigma_{PP}(t^*)}{\sigma_P(t^*)} - 3\, A_P(t^*)\right)\sqrt{\frac{A(t^*)}{\sigma_P(t^*)}} \tag{3.43b}$$

$$\beta_{21}(t^*) = 2\,\frac{dP_c}{dt^*}(t^*) \tag{3.43c}$$

$$\beta_{22}(t^*) = \frac{2\, A(t^*)\, A_{PP}(t^*)}{\sigma_P(t^*)} \tag{3.43d}$$

$$\beta_{23}(t^*) = \frac{2}{3}\,\frac{A(t^*)}{\sigma_P(t^*)}\left(A(t^*)\,\frac{\sigma_{PPP}(t^*)}{\sigma_P(t^*)} - 6\, A_{PP}(t^*)\right) \tag{3.43e}$$

$$\beta_{31}(t^*) = 2\sqrt{\frac{A(t^*)}{\sigma_P(t^*)}}\; B_P(t^*) \tag{3.43f}$$

$$\beta_{32}(t^*) = \frac{4}{3}\left(\frac{A(t^*)}{\sigma_P(t^*)}\right)^{3/2} A_{PPP}(t^*) \tag{3.43g}$$

$$\beta_{33}(t^*) = \frac{4}{15}\left(\frac{A(t^*)}{\sigma_P(t^*)}\right)^{3/2}\left(A(t^*)\,\frac{\sigma_{PPPP}(t^*)}{\sigma_P(t^*)} - 10\, A_{PPP}(t^*)\right) \tag{3.43h}$$

$$\beta_{34}(t^*) = 2\sqrt{2\, A(t^*)}\;\frac{d}{dt^*}\left(\sqrt{\frac{A(t^*)}{\sigma_P(t^*)}}\right) \tag{3.43i}$$

The Hamiltonian (3.41) is periodic in q with period one and has no dependence on q to $\mathcal{O}(1)$; also, all resonance has been removed so that the $\mathcal{O}(1)$ frequency will not vanish. We therefore use the formulas derived in the averaging technique of section 3.1 to eliminate the oscillatory behavior to the desired order. Transforming (p, q) to (P, Q) according to (3.17) and using results (3.20) to (3.27), we solve for F_1, F_2, and F_3. The resulting Hamiltonian with Q dependence removed to $\mathcal{O}(\hat{\varepsilon}^3)$ is

$$H = \underline{H}_0 + \hat{\varepsilon}^2 \underline{H}_2 + \hat{\varepsilon}^4\,[\underline{H}_4 + \underset{\wedge}{\underline{H}}_4] \tag{3.44a}$$

where

$$\underline{H}_0 = A\,(2\bar{k}^2 - 1) \tag{3.44b}$$

$$\underline{H}_2 = \frac{(2\bar{k}^2 - 1)\,(E_c(\bar{k}) - (1-\bar{k}^2)\,K(\bar{k}))}{12\,A\,K(\bar{k})}\Big\{4\,A\,(3\,\beta_{22} + 2\,\beta_{23}) - (\beta_{11} + \beta_{12})(12\,\beta_{11} + 5\,\beta_{12})\Big\} \tag{3.44c}$$

$$- \frac{1}{24\,A}\Big\{3\,(2\,\bar{k}^2 - 1)^2\,\beta_{11}^2 - \bar{k}^2\,(1-\bar{k}^2)\,(8\,A\,\beta_{23} - 8\,\beta_{11}\,\beta_{12} - 5\,\beta_{12}^2)\Big\}$$

and $\underline{H}_4$ has been calculated, but is not shown here. The variable $\bar{k}$ is used to indicate dependence on the new averaged variable P defined by

$$P = 16\sqrt{\frac{A}{\sigma_P}}\;\Big\{E_c(\bar{k}) - (1-\bar{k}^2)\,K(\bar{k})\Big\} \tag{3.45}$$

We can now easily find the solutions for (P, Q) from the canonical differential equations.

$$\frac{dQ}{d\hat{t}} = \frac{\partial \underline{H}_0}{\partial P} + \hat{\varepsilon}^2\,\frac{\partial \underline{H}_2}{\partial P} + \hat{\varepsilon}^4\,[\,\frac{\partial \underline{H}_4}{\partial P} + \frac{\partial \underset{\wedge}{H}_4}{\partial P}\,] + O(\hat{\varepsilon}^5) \tag{3.46a}$$

$$\frac{dP}{d\hat{t}} = -\,\hat{\varepsilon}^4\,\frac{\partial \underset{\wedge}{H}_4}{\partial Q} + O(\hat{\varepsilon}^5) \tag{3.46b}$$

Equation (3.46b) implies that P is a constant to $O(\hat{\varepsilon}^3)$ for times t^* of $O(1)$, since all terms in $\dot{P}$ are oscillatory in the Q. Q can then be found as a simple quadrature of (3.46a)

$$Q(t^*) = \frac{1}{\hat{\varepsilon}^3}\int^{t^*} \frac{\partial \underline{H}_0}{\partial P}(P, t^*)\,dt^* + \frac{1}{\hat{\varepsilon}}\int^{t^*} \frac{\partial \underline{H}_2}{\partial P}(P, t^*)\,dt^* + O(\hat{\varepsilon}) \tag{3.47}$$

The near-identity transformation defined by the generating functions (3.17) can be inverted to find P as a function of the original variables

$$P = \mathcal{A}(p, q, t^*; \hat{\varepsilon}) \equiv \text{constant} + O(\hat{\varepsilon}^4) \tag{3.48}$$

which for this problem gives

$$\mathcal{A}(p, q, t^*; \hat{\varepsilon}) = 16\sqrt{\frac{A}{\sigma_P}}\ \left\{E_c(k) - (1-k^2)\,K(k)\right\}$$

$$+\hat{\varepsilon}\,\frac{4\,K(k)}{\sqrt{A(t^*)\,\sigma_P(t^*)}}\left\{\beta_{11}(t^*)\ k\,(2\,k^2-1)\ \mathrm{cn}(4\,K(k)\,q, k) + \beta_{12}(t^*)\ k^3\,\mathrm{cn}^3(4\,K(k)\,q, k)\right\}$$

$$-\hat{\varepsilon}^2\,\Big[\frac{K(k)\,(8\,A\,\beta_{21}\sin^{-1}(\bar{k}\,\mathrm{sn}) + \beta_{11}^2)}{2\sqrt{A^3\,\sigma_P}}$$

$$+\frac{(E_c(k)\,(1-2\,k^2) - K(k)\,(1-k^2))}{3\sqrt{A^3\,\sigma_P}} * \left\{4\,A\,(3\,\beta_{22}+2\,\beta_{23}) - (\beta_{11}+\beta_{12})(12\,\beta_{11}+5\,\beta_{12})\right\}$$

$$+\frac{(E_c(k) - (1-k^2)\,K(k))}{2\sqrt{A^3\,\sigma_P}\,(1-k^2)}\ \mathrm{cn}^2\,((2\,k^2-1)\,\beta_{11} + k^2\,\beta_{12}\,\mathrm{cn}^2)^2$$

$$+\frac{4\,k^2\,K(k))}{\sqrt{A\,\sigma_P}}\ \mathrm{cn}^2\,((2\,k^2-1)\,\beta_{22} + k^2\,\beta_{23}\,\mathrm{cn}^2)$$

$$+\frac{k^2\,(1-k^2)\,K(k))}{2\sqrt{A^3\,\sigma_P}}\left\{8\,A\,(2\,\beta_{22}+\beta_{23}) - 5\,(2\,\beta_{11}+\beta_{12})^2\right\}\Big] \tag{3.49}$$

The $O(\hat{\varepsilon}^3)$ term has been calculated via SMP but, again, is too long to include here. The entire expansion to $O(\hat{\varepsilon}^3)$ is used in the following numerical verification of the constancy of $\mathcal{A}$.

3.3 Numerical results and final asymptotic solution. We determine the accuracy of the analytical expression for the adiabatic invariant by numerically integrating the original differential equations derivable from the Hamiltonian (3.41), then substituting the results for (p, q) into formula (3.49). Because h does not explicitly depend on p, however, but only on p through k, it is easier to set up the system of differential equations to be integrated using the variables q and k, which we have done for purposes of numerical verification.

We consider two test problems to numerically verify our results. Both are modifications of examples resulting from physical systems which exhibit sustained resonance. Problem I is specified by the functions

$$\sigma(P_1, t^*) = 1 - \frac{1 + (5/2 - 2t^*)^2}{P_1^2} \tag{3.50a}$$

$$A(P_1, t^*) = \frac{2(5/2 - 2t^*)}{P_1} \tag{3.50b}$$

$$B(P_1, t^*) = \frac{(1 + (5/2 - 2t^*)^2)^2}{8P_1^3} \tag{3.50c}$$

which correspond to a slight modification of expressions used for the free-electron laser problem considered in detail in [2] and [4]. The equations which result for P_1 and Q_1 are

$$\frac{dQ_1}{dt} = 1 - \frac{1 + (5/2 - 2t^*)^2}{P_1^2} - \varepsilon \frac{2(5/2 - 2t^*)}{P_1^2} \sin Q_1 - \varepsilon^2 \frac{3(1 + (5/2 - 2t^*)^2)^2}{8P_1^4} \tag{3.51a}$$

$$\frac{dP_1}{dt} = -\varepsilon \frac{2(5/2 - 2t^*)}{P_1} \cos Q_1 \tag{3.51b}$$

The second test case, Problem II, is specified by the functions

$$\sigma(P_1, t^*) = \sqrt{2}\, P_1 - (P_1^2 + \omega^2(t^*))^{1/2} \tag{3.52a}$$

$$A(P_1, t^*) = \frac{\omega^2(t^*)}{2(P_1^2 + \omega^2(t^*))^{1/4}} \tag{3.52b}$$

$$B(P_1, t^*) = 0 \tag{3.52c}$$

where we let

$$\omega(t^*) = 2e^{-2t^*} \tag{3.52d}$$

These functions give rise to the differential equations

$$\frac{dQ_1}{dt} = \sqrt{2}\, P_1 - (P_1^{\,2} + \omega^2(t^*))^{1/2} - \varepsilon \, \frac{\omega^2(t^*)\, P_1}{4\,(P_1^{\,2} + \omega^2(t^*))^{5/4}} \sin Q_1 \tag{3.53a}$$

$$\frac{dP_1}{dt} = -\,\varepsilon \, \frac{\omega^2(t^*)}{2\,(P_1^{\,2} + \omega^2(t^*))^{1/4}} \cos Q_1 \tag{3.53b}$$

These functions are a modification of the equations derived from the spin-roll resonance model originally proposed by Kevorkian in [13] and [20] and discussed by Kath in [10] and [11]. The primary changes consist of a reduction of order of the original system, and an exponentially decreasing function for ω so that motion is now outward in the potential well.

For the calculation of the adiabatic invariant, we first examine Problem I for the initial conditions $k = 0.5$ and $q = 0.0$ at $t^* = 0.0$ and for the value $\hat{\varepsilon} = 0.1$. We find the numerical solutions for k and q, and in Figure 3.2a we graph p calculated from (3.39) and the adiabatic invariant $\mathcal{A}$ from (3.49), using these numerical solutions.

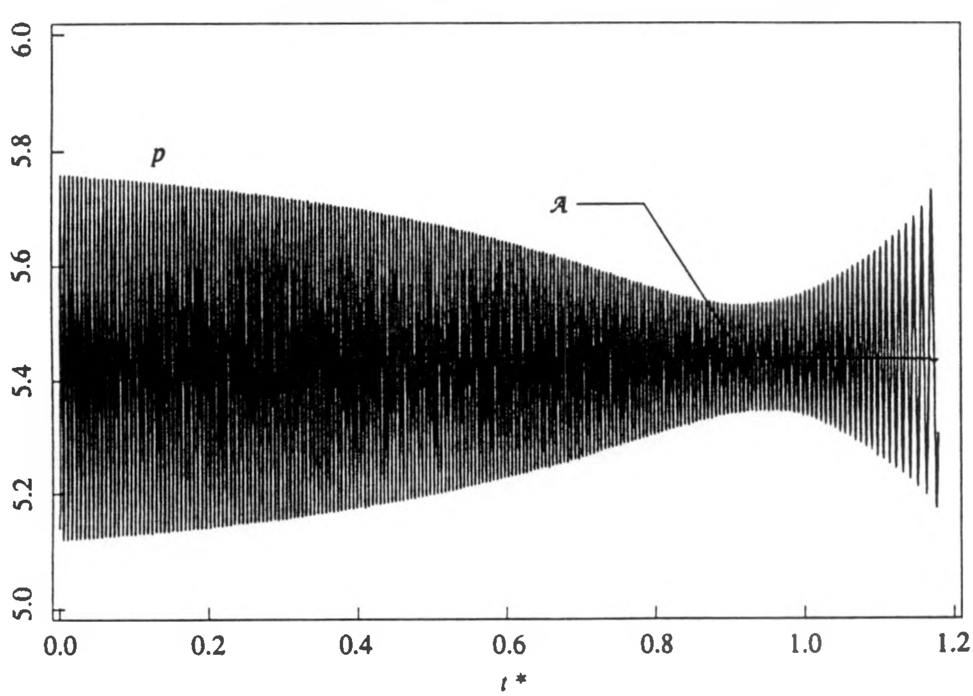

Fig. 3.2a

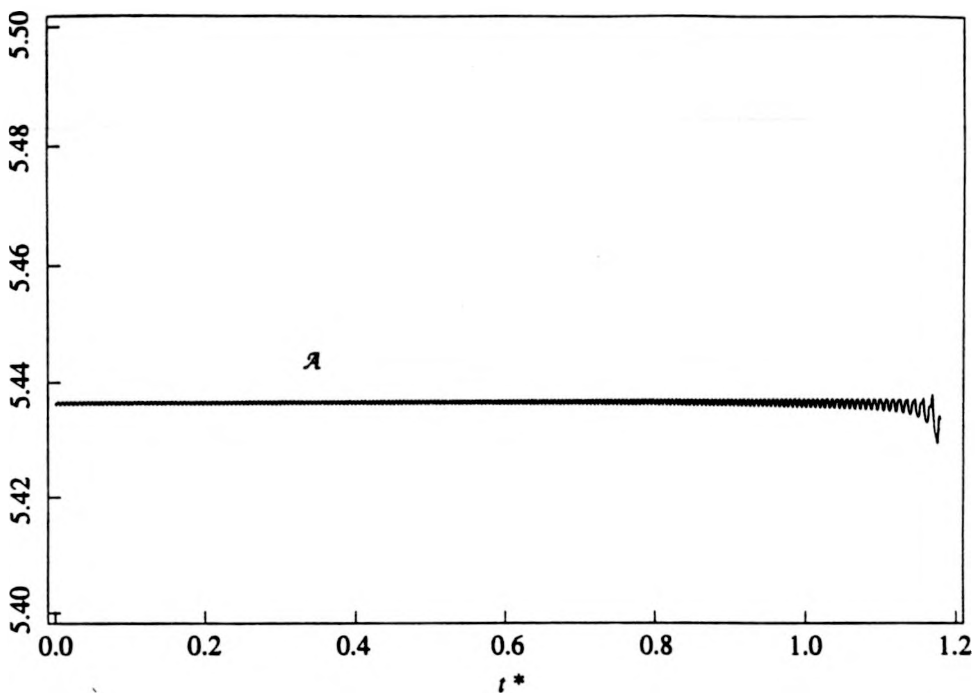

Fig. 3.2b

Figure 3.2 Adiabatic Invariant – Problem I

As expected, the function $\mathcal{A}$ proves remarkably constant, particularly in view of the large value of $\hat{\varepsilon}$. In Figure 3.2b we show an enlarged version of 3.2a and see the oscillatory behavior of $\mathcal{A}$ associated with the $O(\hat{\varepsilon}^4)$ oscillatory Hamiltonian not eliminated by the near-identity transformation.

We can verify that this error in the constancy is indeed $O(\hat{\varepsilon}^4)$ by choosing several different values of $\hat{\varepsilon}$, finding the magnitude of the oscillations numerically, then using a least squares log-log fit to calculate the order. Using $\hat{\varepsilon} = 0.1$, 0.05, 0.02, 0.01, and 0.005 and the fitting equation

$$|max\ \mathcal{A} - min\ \mathcal{A}| = C\,\hat{\varepsilon}^K \tag{3.54}$$

we find that the least squares fit gives $K = 4.012$ (with $C = 5.603$). This result is shown in Figure 3.3.

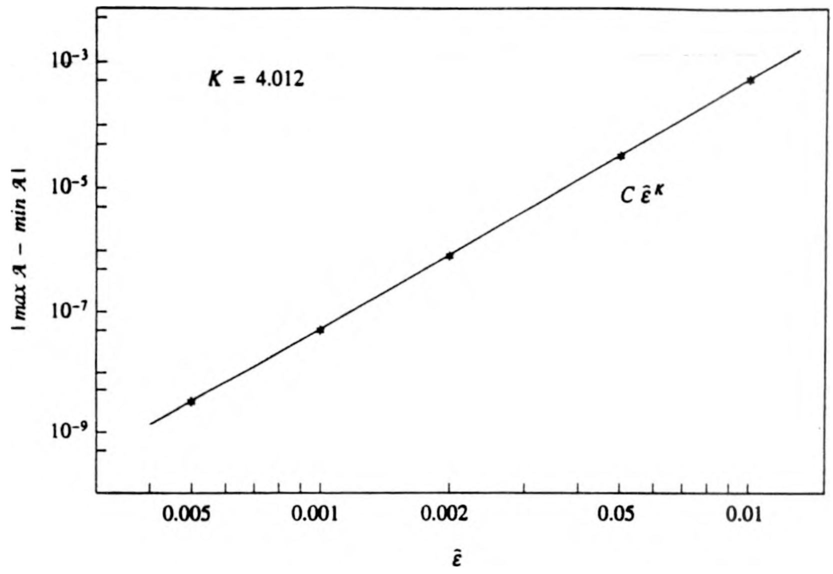

Figure 3.3 Order of the Oscillatory Error in the Adiabatic Invariant

There is a second contribution to the $O(\hat{\varepsilon}^4)$ error associated with the adiabatic invariant; this is due to a non-zero slowly varying average value for $\mathcal{A}$. When equation (3.46b) is integrated to find P (the adiabatic invariant) as a function of time, the $O(\hat{\varepsilon}^4)$ slowly varying oscillatory integrand gives rise to the solution,

$$P = \text{constant} + \hat{\varepsilon}^4 \left[\underset{\wedge}{P}_4(\hat{t}, t^*) + \underline{P}_4(t^*) \right] \tag{3.55}$$

This effect is seen more clearly in the integration of Problem II. Using initial conditions $k = 0.5$ and $q = 0.0$ at $t^* = 0.0$ for the value $\hat{\varepsilon} = 0.1$, we integrate equations (3.53) and use (3.49) to calculate the adiabatic invariant with the result shown in Figure 3.4.

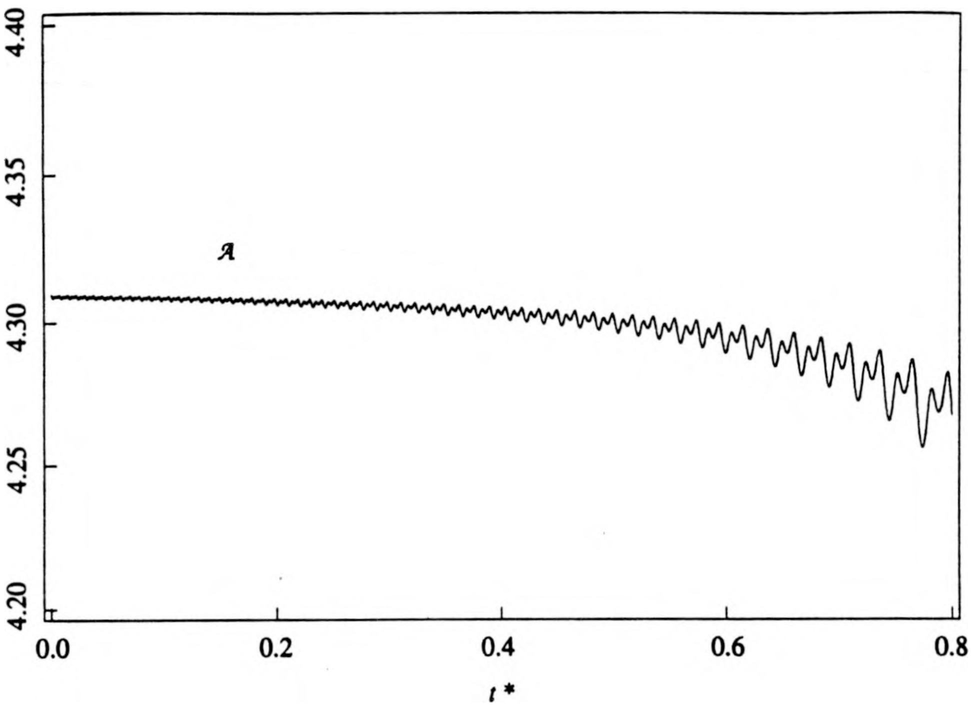

Figure 3.4 Averaged Drift of the Adiabatic Invariant – Problem II

The drift in the average value is small but definite. We can again show this effect to be $O(\hat{\varepsilon}^4)$ by integrating the equations for several different values of $\hat{\varepsilon}$ up to a fixed value of t^*, then subtracting the final average value of $\mathcal{A}$ from the initial average value in the following expression

$$\left|\mathcal{A}_{avg}(final) - \mathcal{A}_{avg}(initial)\right| = D\,\hat{\varepsilon}^M \tag{3.56}$$

for $\hat{\varepsilon} = 0.1,\ 0.05$, and 0.02, we integrate the modified spin-roll system to $t^* = 0.5$ and find $M = 4.050$ (with $D = 110.51$). This fit is shown in Figure 3.5.

Having numerically verified in the manner described above that the two sources of non-constancy in the adiabatic invariant are $O(\hat{\varepsilon}^4)$, we may reasonably conclude that the asymptotic formula (3.49) derived for $\mathcal{A}$ to $O(\hat{\varepsilon}^3)$ is indeed correct.

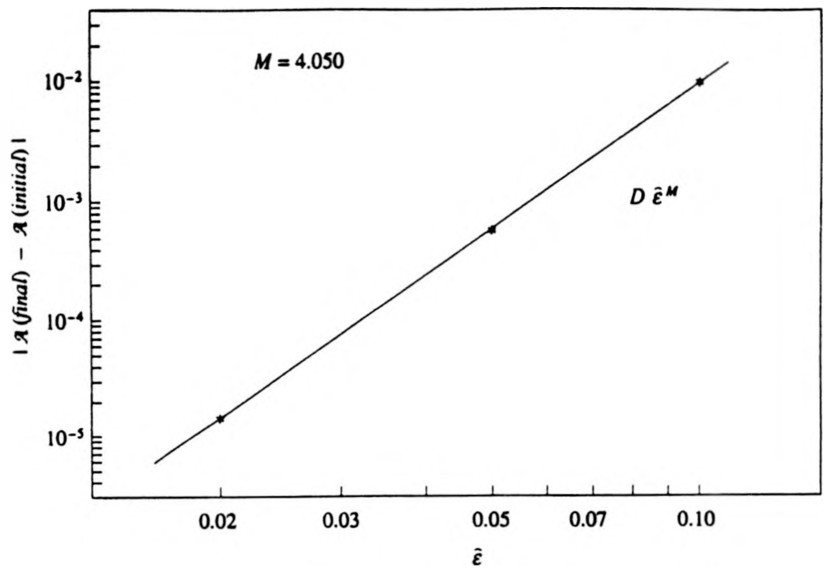

Figure 3.5 Order of the Averaged Error in the Adiabatic Invariant

In the verification of the average drift of the adiabatic invariant, we have not used smaller values of $\hat{\varepsilon}$ as numerical integration becomes increasingly difficult. This is one of the primary reasons for solving the problem asymptotically. For example, to solve the problem in the range $t^* \in [0.0, 1.0]$ for a value of $\hat{\varepsilon} = 1/1000$ requires an integration to a time of one billion on the fast time scale $\hat{t}$. Using FORTRAN subroutine DDRIV with a relative error tolerance of 1.0×10^{-14}, we find a numerical integration of Problem I for $\hat{\varepsilon} = 1/10$ takes only $2\,\frac{1}{2}$ minutes on a VAXserver 3500; correspondingly, the $\hat{\varepsilon} = 1/1000$ case would take roughly 1,700 days. Even if time were not a factor, it is very difficult to numerically integrate rapidly oscillating functions accurately over long times.

The asymptotic solution, on the other hand, requires only a numerical quadrature for Q from (3.47) which is a quadrature in t^*, and therefore takes the same time independent of the value of $\hat{\varepsilon}$ (only depending on the numerical accuracy required for a valid solution). Moreover, the asymptotic solutions become more accurate as

$\hat{\varepsilon} \to 0$, so that the very good results obtained for these larger values can be expected to get even better.

Since we know P to be a constant to $O(\hat{\varepsilon}^3)$ and we have the time dependence of Q specified by (3.47), we can express the variables Q^* and P^* as functions of time, hence the original Q_1 and P_1 as well. First we invert (3.45) to find $\bar{k}$ as a function of the adiabatic invariant and t^*. Defining

$$f(k) = E_c(k) - (1-k^2)\,K(k) \tag{3.57}$$

which is a known invertible function, then

$$\bar{k}\,(t^*) = f^{-1}\left(\frac{\mathcal{A}}{16}\sqrt{\frac{\sigma_P(t^*)}{A(t^*)}}\right) \tag{3.58}$$

can be easily computed, where the value of $\mathcal{A}$ is found from (3.49) and initial conditions. Because Q^* and P^* depend explicitly on k rather than on p, we express the final solution in terms of the averaged variables $\bar{k}$ and Q, of which we know the time dependence from (3.58) and (3.47). The asymptotic solutions for $Q^*(t^*)$ and $P^*(t^*)$ are given to $O(\hat{\varepsilon}^2)$ by

$$\begin{aligned}
Q^*(t^*) &= 2\sin^{-1}\left(\bar{k}\,\mathrm{sn}(4\,K(\bar{k})\,Q,\bar{k})\right) \\
&+ \hat{\varepsilon}\,\frac{(2\,\beta_{11} + \beta_{12})}{A}\,\bar{k}\,\mathrm{cn}(4\,K(\bar{k})\,Q,\bar{k})\,\sin^{-1}\left(\bar{k}\,\mathrm{sn}(4\,K(\bar{k})\,Q,\bar{k})\right) \\
&+ \hat{\varepsilon}^2\left[\frac{\beta_{21}}{2\,A}\,\frac{\bar{k}^2\,I_{SC2}\,\mathrm{cn} - \bar{k}\,\mathrm{cn}^2 - \mathrm{sn}\,\mathrm{dn}\,\sin^{-1}\left(\bar{k}\,\mathrm{sn}\right)}{\bar{k}\,(1-\bar{k}^2)}\right. \\
&\qquad + \frac{\left\{4\,A\,(3\,\beta_{22} + 2\,\beta_{23}) - (\beta_{11}+\beta_{12})(12\,\beta_{11} + 5\,\beta_{12})\right\}}{24\,A^2} \\
&\qquad \times \frac{E_c(\bar{k})\,(1-2\,\bar{k}^2) - K(\bar{k})\,(1-6\,\bar{k}^2)\,(1-\bar{k}^2)}{K(\bar{k})\,\bar{k}\,(1-\bar{k}^2)}\,(E_{osc}\,\mathrm{cn} - \mathrm{sn}\,\mathrm{dn}) \\
&\qquad + \frac{8\,A\,(4\,\beta_{22} + 3\,\beta_{23}) - 32\,\beta_{11}^2 - 48\,\beta_{11}\,\beta_{12} - 15\,\beta_{12}^2}{16\,A^2}\,\bar{k}\,\mathrm{sn}\,\mathrm{dn}
\end{aligned} \tag{3.59a}$$

$$+\frac{(2\,\beta_{12}+\beta_{12})^2}{4\,A^2}\,\bar{k}\,\sin^{-1}(\bar{k}\,\mathrm{sn})\,(2\,\bar{k}\,\mathrm{cn}^2-\mathrm{sn}\,\mathrm{dn}\,\sin^{-1}(\bar{k}\,\mathrm{sn}))\,]$$

and

$$P^*(t^*) = 2\,\bar{k}\sqrt{\frac{A(t^*)}{\sigma_P(t^*)}}\;\mathrm{cn}\,(4\,K(\bar{k})\,Q,\bar{k}\,)$$

$$-\,\hat{\varepsilon}\,[\,\frac{(2\,\bar{k}^2-1)\,\beta_{11}+\bar{k}^2\,\beta_{12}\,\mathrm{cn}^2+2\,\bar{k}\,(2\,\beta_{11}+\beta_{12})\,\mathrm{sn}\,\mathrm{dn}\,\sin^{-1}(\bar{k}\,\mathrm{sn})}{2\sqrt{A\,\sigma_P}}\,]$$

$$-\,\hat{\varepsilon}^2\,[\,\frac{\beta_{21}}{2\sqrt{A\,\sigma_P}\;\bar{k}\,(1-\bar{k}^2)}\,(\,\mathrm{cn}\,\sin^{-1}(\bar{k}\,\mathrm{sn})\,(1+\bar{k}^2\,\mathrm{cn}^2-2\,\bar{k}^2)$$

$$+\,\bar{k}\,\mathrm{sn}\,\mathrm{dn}\,(\,\bar{k}\,I_{SC2}-\mathrm{cn}\,))$$

$$+\,\frac{(1+2\,\bar{k}^2\,\mathrm{cn}^2-2\,\bar{k}^2)}{16\sqrt{A^3}\,\sigma_P}\,\bar{k}\,\mathrm{cn}\,\{4\,(2\,\beta_{11}+\beta_{12})^2\,[\sin^{-1}(\bar{k}\,\mathrm{sn})]^2 \tag{3.59b}$$

$$+\,8\,A\,(\beta_{22}+\beta_{23})-(2\,\beta_{11}+\beta_{12})(4\,\beta_{11}+5\,\beta_{12})\}$$

$$+\,\frac{\{4\,A\,(3\,\beta_{22}+2\,\beta_{23})-(\beta_{11}+\beta_{12})(12\,\beta_{11}+5\,\beta_{12})\}}{24\sqrt{A^3}\,\sigma_P\;K(\bar{k})\,\bar{k}\,(1-\bar{k}^2)}$$

$$\times\,\{\,(E_c(\bar{k})\,(1-2\,\bar{k}^2)-K(\bar{k})\,(1-\bar{k}^2))\,\mathrm{cn}\,(1+\bar{k}^2\,\mathrm{cn}^2-2\,\bar{k}^2)$$

$$+\,(E_c(\bar{k})\,(1-2\,\bar{k}^2)-K(\bar{k})\,(1-\bar{k}^2)\,(1-6\,\bar{k}^2))\;E_{osc}\,\mathrm{sn}\,\mathrm{dn}\,\}$$

$$+\,\frac{\bar{k}^2\,\beta_{11}\,(2\,\beta_{11}+\beta_{12})}{\sqrt{A^3}\,\sigma_P}\,\mathrm{sn}\,\mathrm{cn}\,\mathrm{dn}\,\sin^{-1}(\bar{k}\,\mathrm{sn})\,]$$

where we define

$$I_{SC2}(4K(\bar{k})Q,\bar{k}) = \int_0^{4K(\bar{k})Q}\mathrm{cn}^2(u,\bar{k})\;\sin^{-1}(\bar{k}\,\mathrm{sn}(u,\bar{k}))\,du \tag{3.60}$$

and where all elliptic functions in (3.59) are evaluated at the arguments $(4K(\bar{k})Q,\bar{k})$. The known functional dependence of $\bar{k}(t^*)$ and $Q(t^*)$ on time from (3.58) and (3.47) is used in the expansions (3.59). Although these expansions are relatively large, they still can be evaluated simply since SMP can automatically generate FORTRAN code for these expressions.

The equations (3.59) also represent a direct transformation of the Hamiltonian (3.36) from the variables (P^*, Q^*) to the final averaged variables (P, Q) encompassing both the action-angle transformation and the near-identity transformation. These solutions will be valid to $O(\hat{\varepsilon})$ for t^* of $O(1)$ if the $\hat{\varepsilon}^4 \underline{H}_4$ term is included in the integration for $Q(t^*)$. For $O(1)$ accuracy it is necessary only to integrate the $\underline{H}_0$ and $\underline{H}_2$ terms as $\underline{H}_1 = \underline{H}_3 = 0$. The solution for the original problem (3.34) is simply given by

$$Q_1 = Q^*(t^*) - \frac{\pi}{2} \tag{3.61a}$$

$$P_1 = P_c(t^*) + \hat{\varepsilon} P^*(t^*) \tag{3.61b}$$

where the expansions (3.59) are used for $Q^*(t^*)$ and $P^*(t^*)$.

The accuracy of these asymptotic solutions can be found by comparing an exact numerical solution of the original equations (3.34) for Q_1 and P_1 with the asymptotic solution given by (3.61) and (3.59). Using (3.51) for Problem I and given the same initial conditions as for the calculation of the adiabatic invariant, we find both the asymptotic and the exact numerical solution for $t^* \in [0.0, 1.2]$. In finding the asymptotic solution, we have not included an integration of the $\partial \underline{H}_4/\partial P$ term in the calculation of $Q(t^*)$ from (3.47). In Figures 3.6a and 3.6b we show both asymptotic and numerical solutions for Q_1 and P_1 in the range $t^* \in [1.1, 1.2]$, the last part of the solution region including escape from sustained resonance. In 3.6b we include a plot of the resonant momentum $P_c(t^*)$. As seen, the agreement is almost identical to escape, except for a very small phase error presumably due to not including the $\partial \underline{H}_4/\partial P$ term in the Q integration.

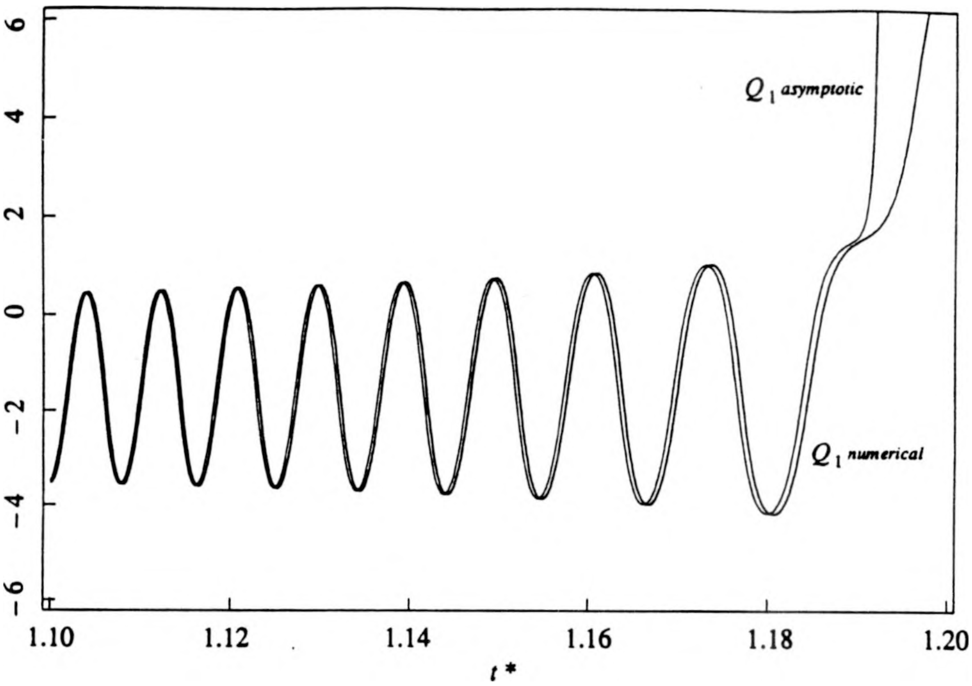

Fig. 3.6a

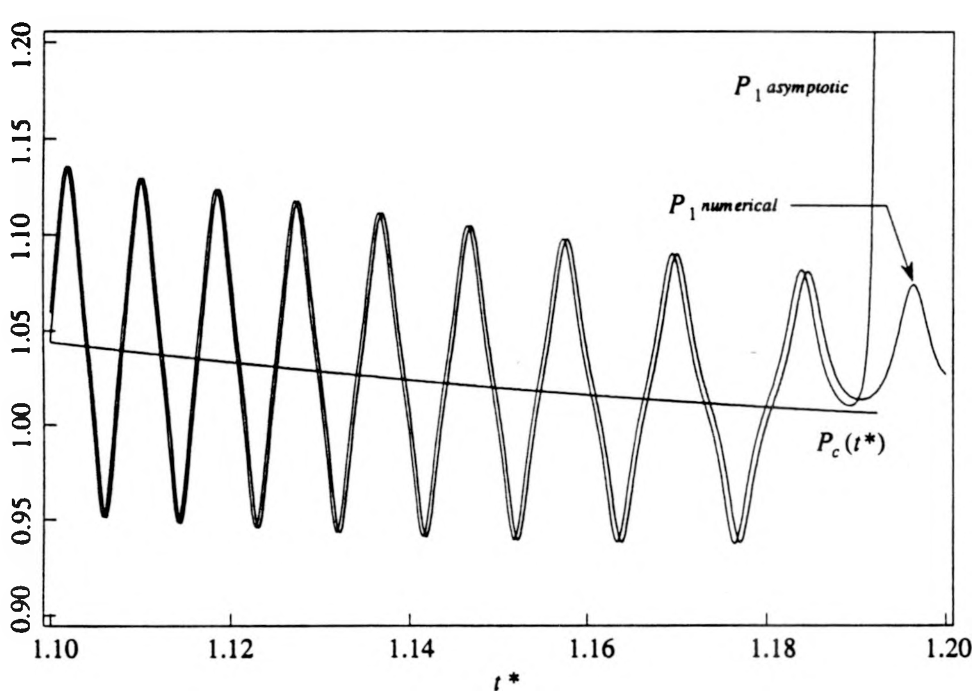

Fig. 3.6b

Figure 3.6 Asymptotic Solution for Problem I

When the same comparison is performed for Problem II with the same initial conditions as for Figure 3.4, the results for keeping track of the final phase are less satisfactory than for Problem I; but the agreement between the numerical and asymptotic results is still very good. In Figures 3.7a and 3.7b, we compare Q_1 and P_1 found asymptotically with the numerical result in the range $t^* \in [0.0, 1.0]$. In 3.7b we include a plot of the resonant momentum $P_c(t^*)$.

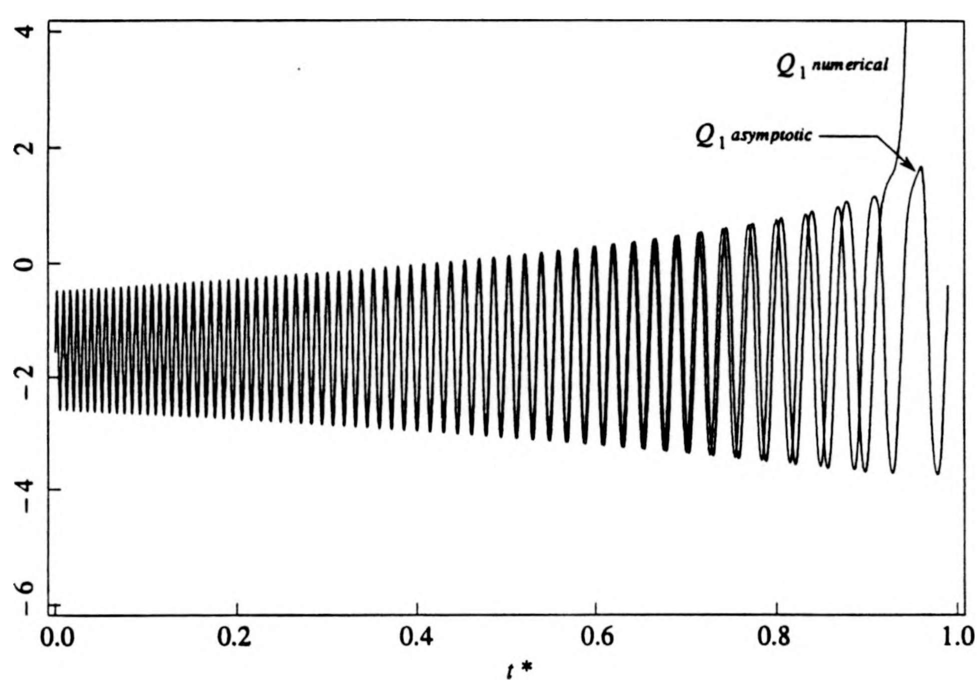

Fig. 3.7a

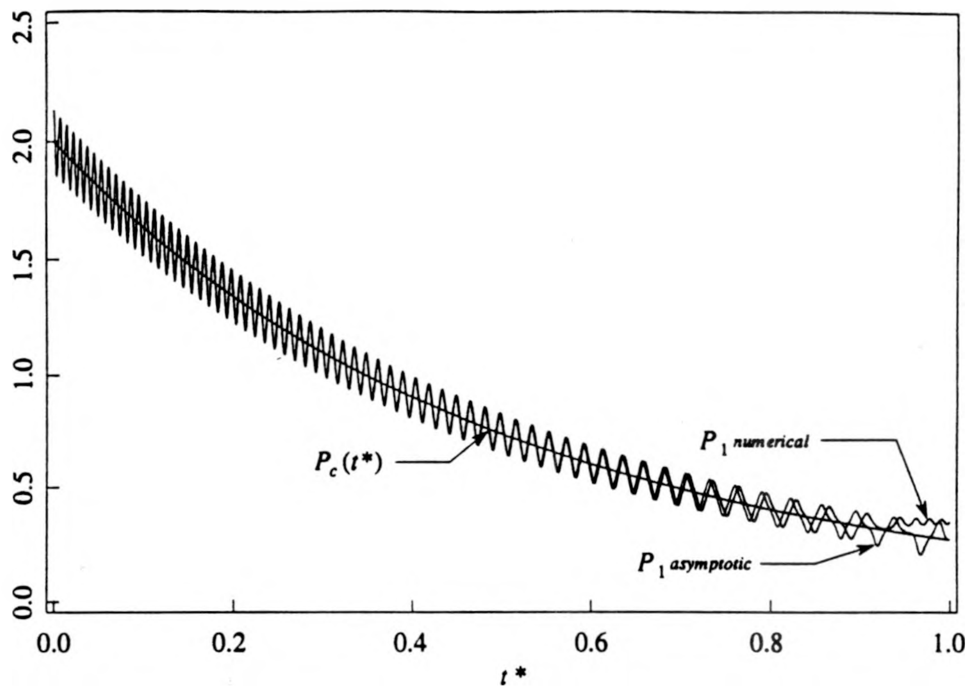

Fig. 3.7b

Figure 3.7 Asymptotic Solution for Problem II

The difference in the phase can be accounted for by including the $O(\hat{\varepsilon}^4)$ term in the integration for Q in (3.47), which in the case of Problem II contains terms which grow exponentially due to the choice for $\omega(t*)$. This correction is seen in Figures 3.8a and 3.8b where we have included this term in the integration. As the asymptotic solution becomes nearly identical to the numerical solution up until the point where the motion ceases to be periodic, we only show the final few oscillations for comparison of the two solutions to see the improvement over Figures 3.7.

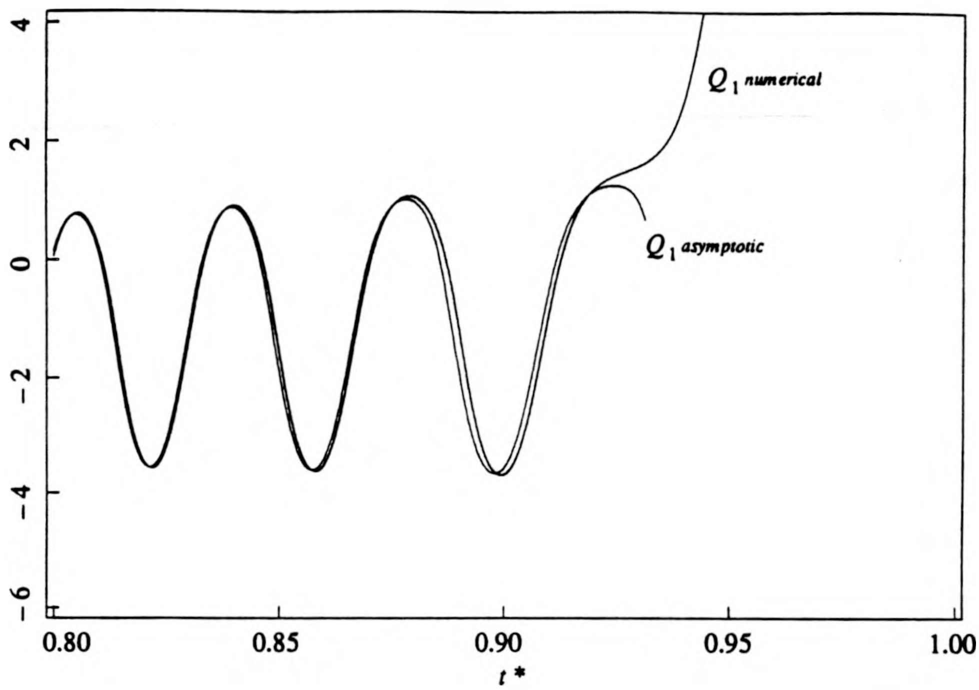

Fig. 3.8a

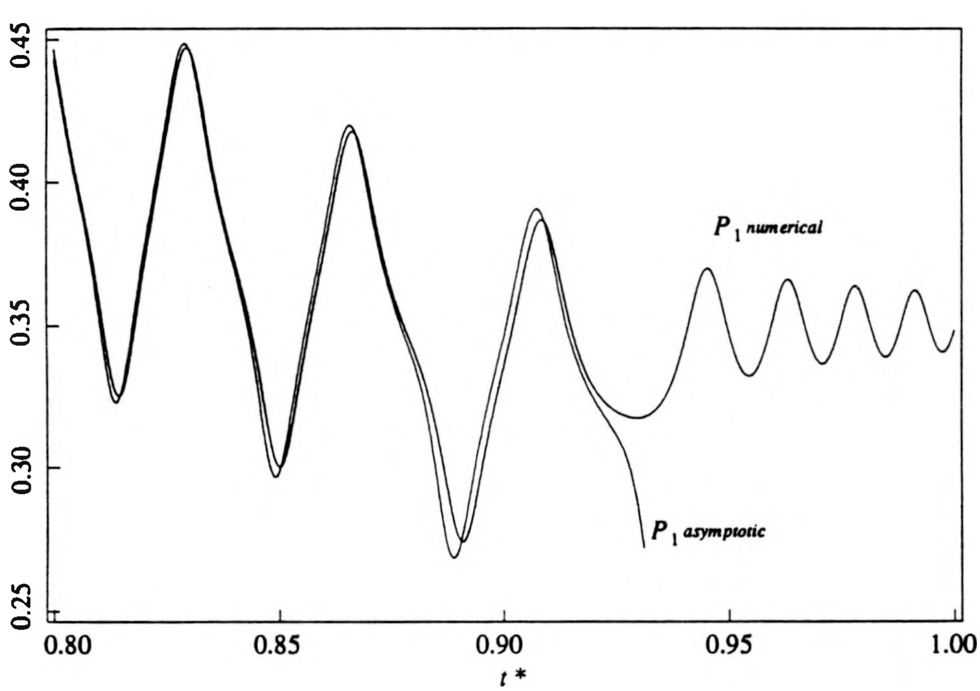

Fig. 3.8b

Figure 3.8 Asymptotic Solution with Corrected Phase – Problem II

As seen, the main error in the asymptotic expansions arises due to error in the phase of the oscillations. The amplitude, however, is accurate to the order of the expansion. We can use this fact to quickly calculate a very accurate predicted time of escape for a system initially caught in sustained resonance. Given a certain set of initial conditions within the potential well, we want to know at what time, if any, the motion will escape.

From (3.59a) for Q^*, the magnitude of the oscillations is governed to $O(1)$ by the value of $\bar{k}$. If $\bar{k}$ is an increasing function of t^* then Q^* will have oscillatory motion slowly moving outward in the potential well (as in Figure 3.1c). From (3.58) for $\bar{k}(t^*)$, since $\mathcal{A}$ is a constant and f^{-1} a monotonically increasing function, this condition simply translates to

$$\frac{d}{dt^*}\left(\frac{\sigma_P(t^*)}{A(t^*)}\right) > 0 \tag{3.62}$$

for motion outward in the potential well. The opposite situation occurs for $\bar{k}$ decreasing.

When $\bar{k}$ is increasing, we identify the time of escape at the point when Q^* first exceeds the value at one of the saddle points $\pm\pi$. We use this definition of escape rather than an energy condition since cases exist where the energy is decreasing, yet the motion is outward from the center of the potential well. This is seen in Figure 3.9a for Problem II for initial conditions $k = 0.7$ and $q = 0.0$ at $t^* = 0.0$ and for the value $\hat{\varepsilon} = 0.1$, where we have plotted the energy calculated from the Hamiltonian and Q_1 (scaled) on a common axis. At the point of escape the energy is clearly decreasing and has peaked much earlier. In Figure 3.9b we show the slowly varying "phase-plane" that is typical for decreasing energy escape. The overall motion is outward in the potential well but inward in $\dot{Q}_1$, which measures the kinetic energy.

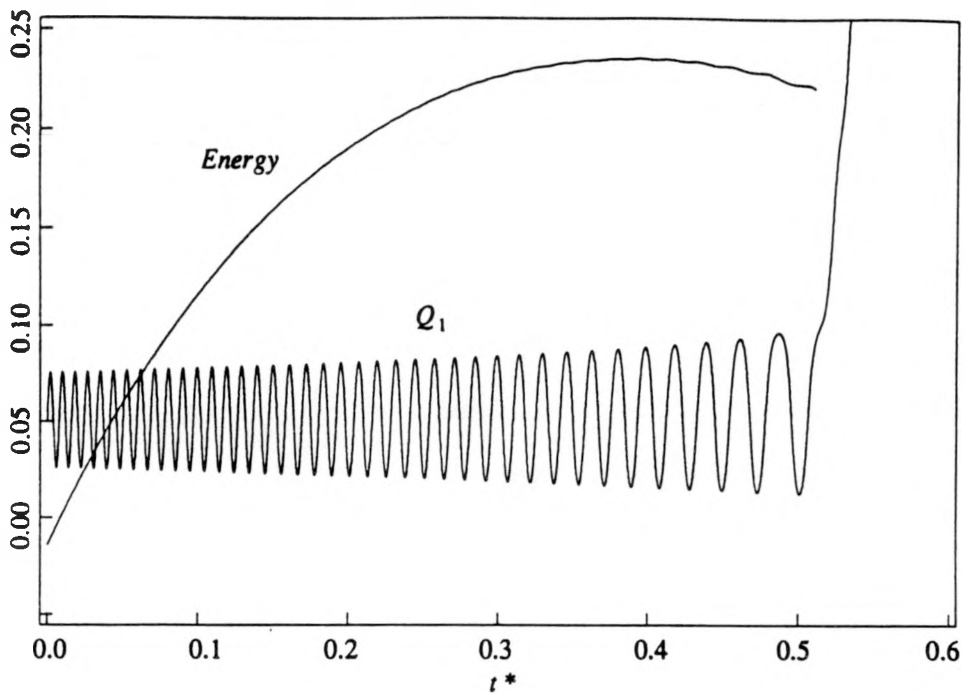

Fig. 3.9a

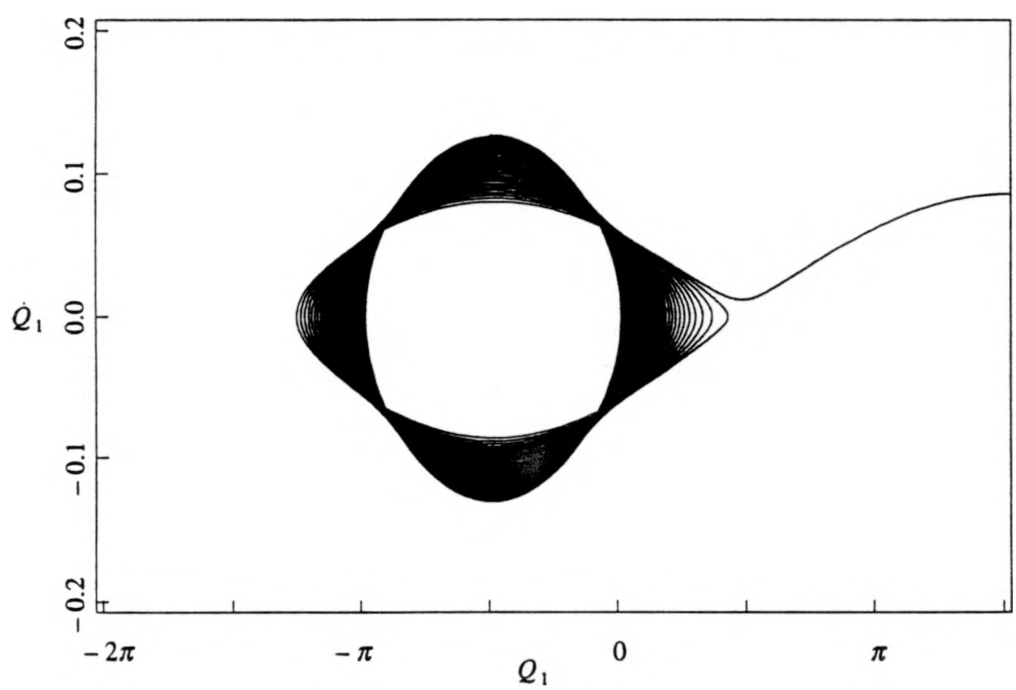

Fig. 3.9b

Figure 3.9 Escape from Potential Well

In order to predict escape, we asymptotically expand the quantity

$$\sin\left(Q^*/_2\right) \tag{3.63}$$

using (3.59a) for Q^*. When this quantity approaches ± 1, then $Q^* \to \pm\pi$, which corresponds to the saddle points in the phase plane. We find an expression for the envelope of the oscillations of (3.63), given by

$$\begin{aligned} k_{CR}^{\pm}(t^*; \hat{\varepsilon}) &= \pm \bar{k}(t^*) \\ &\pm \hat{\varepsilon}^2 \left\{ \frac{(2\bar{k}^2 - 1)\,(E_c(\bar{k}) - (1-\bar{k}^2)\,K(\bar{k}))}{48\,A^2\,\bar{k}\,K(\bar{k})} \right. \\ &\qquad \times \left\{4\,A\,(3\,\beta_{22} + 2\,\beta_{23}) - (\beta_{11} + \beta_{12})(12\,\beta_{11} + 5\,\beta_{12})\right\} \\ &\qquad \left. + \frac{\bar{k}\,(1-\bar{k}^2)\,(8\,A\,\beta_{23} - 8\,\beta_{11}\,\beta_{12} - 5\,\beta_{12}^2)}{96\,A^2} \right\} \\ &- \hat{\varepsilon}^2\,\frac{\beta_{21}\sin^{-1}(\bar{k})}{4\,A\,\bar{k}} \end{aligned} \tag{3.64}$$

where $\bar{k}(t^*)$ is given by (3.58). If k_{CR} never reaches either of the values ± 1, then escape will not occur; however, should the values reach either ± 1, the motion will cease to be oscillatory and Q^* will then increase or decrease monotonically.

Choosing the maximum of the absolute value of (3.64) for the two possibilities gives the algebraic condition

$$k_{CR}(t_{CR}^*; \hat{\varepsilon}) = 1 \tag{3.65}$$

for the escape time t_{CR}^*, which can be quickly solved.

Using the test cases for Problem I (3.51) and Problem II (3.53) problems, both of which have motion away from the center of the potential well, we check how accurate condition (3.65) is for predicting escape time. Using the same value of $\hat{\varepsilon} = 0.1$ and the same initial conditions at $t^* = 0.0$, $k = 0.5$ and $q = 0.0$, we plot the maximum value of $k_{CR}(t^*)$ from (3.64) and the value for Q_1 from a numerical integration of the original problem on a common axis. When k_{CR} crosses 1 we expect the motion of Q_1 to cease to be oscillatory. In Figure 3.10a we have done this for Problem I and have included the value of $\bar{k}$ as well. Q_1 is scaled and plotted to show when escape actually does occur. In Figure 3.10b we have enlarged the region where escape occurs to see how very accurate this calculation is.

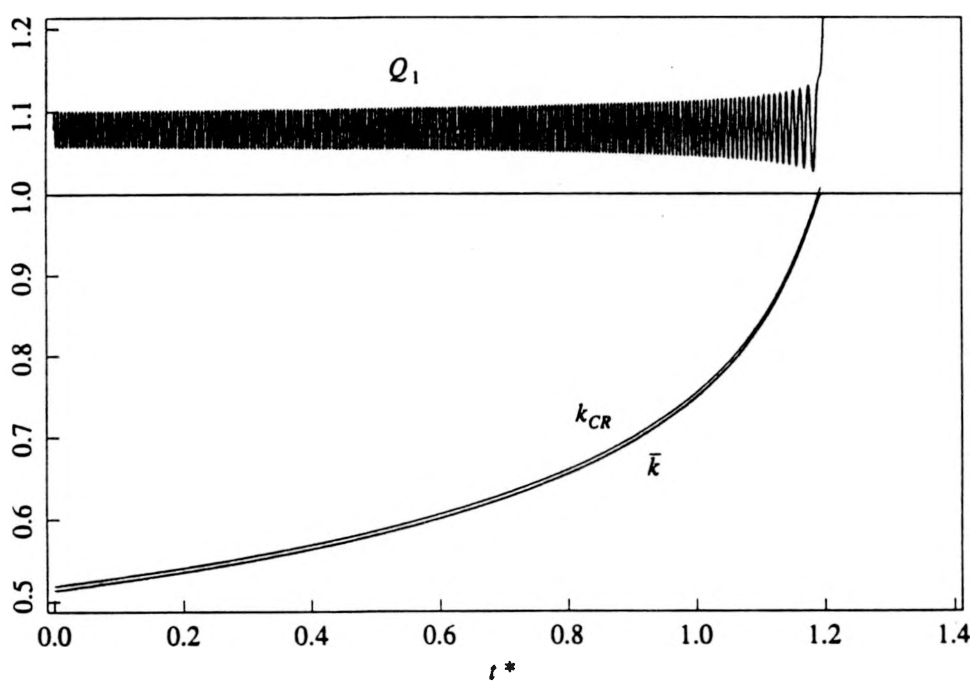

Fig. 3.10a

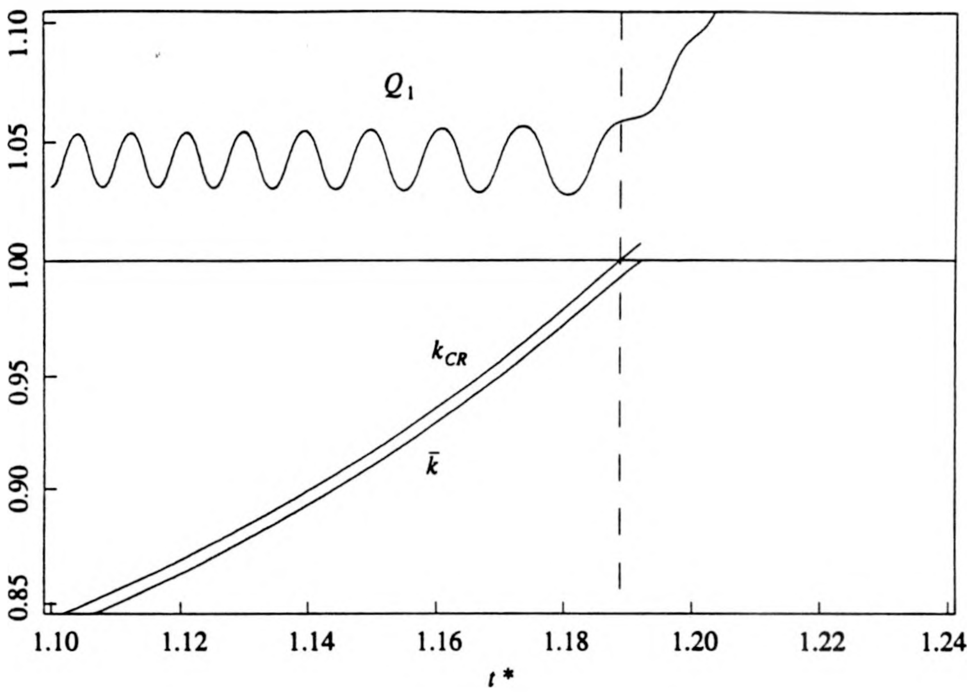

Fig. 3.10b

Figure 3.10 Escape Condition for Problem I

Figure 3.11 is the identical diagram for Problem II. Again, we find that the prediction from (3.65) is remarkably accurate. Other tests have been performed and the same excellent agreement is found to well within one oscillation of Q_1. In fact, if a more accurate prediction of escape is needed, the asymptotic solutions (3.59) can be used to find the exact phase on the last oscillation.

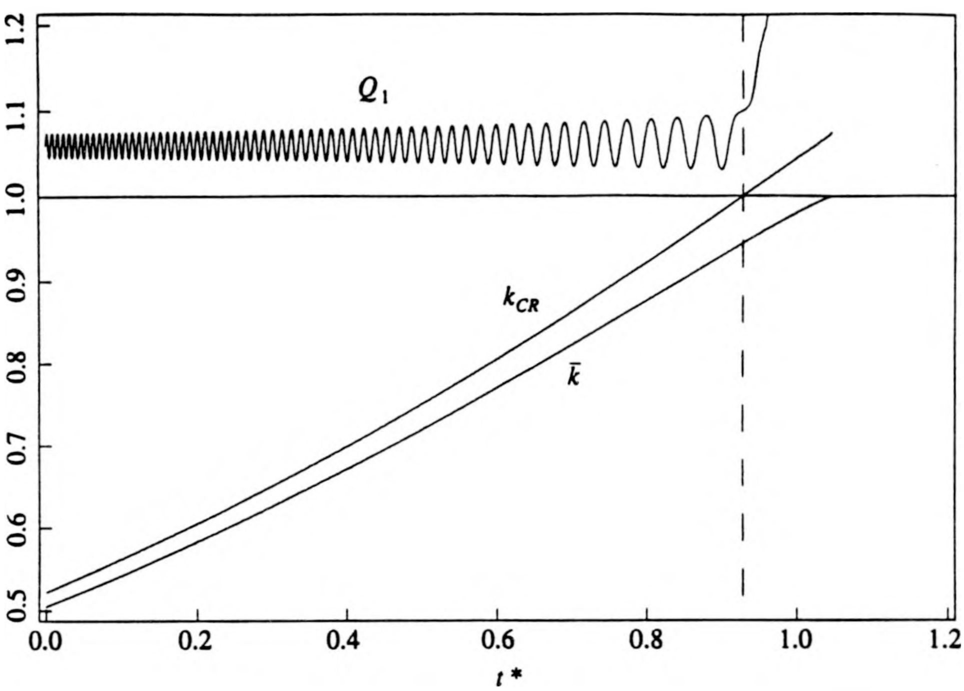

Figure 3.11 Escape Condition for Problem II

The model problem considered in this section and given by the Hamiltonian (3.34) contains only a single harmonic mode and no averaged term to $O(\varepsilon)$. Also, the $O(\varepsilon^2)$ Hamiltonian has been assumed independent of Q_1. We can generalize this problem to include averaged terms of $O(\varepsilon)$ and a general $O(\varepsilon^2)$ Hamiltonian

$$H(P_1, Q_1, t^*; \varepsilon) = H_0(P_1, t^*) + \varepsilon\,[A(P_1, t^*)\sin Q_1 + C(P_1, t^*)] \\ + \varepsilon^2 H_2(P_1, Q_1, t^*) + O(\varepsilon^3) \qquad (3.66)$$

and still find a solution nearly identical to the one in this section. The transformation removing the resonance and the action-angle transformation remain the same. Only the near-identity generating functions are modified slightly, changing the final asymptotic expansion.

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