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THE PROPAGATION OF PLANE WAVES IN GRANULAR MEDIA†

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by

Jace W. Nunziato
Sandia Laboratories
Albuquerque, New Mexico 87185

Synopsis

In this paper, we discuss some recent theoretical results we have obtained on the propagation of plane longitudinal waves in granular materials. Using a one-dimensional formulation of the continuum theory of granular materials proposed by Goodman and Cowin [1], we model the longitudinal response of two classes of materials: materials with compressible granules, such as pressed powders and sands at high confining pressures, and flowing materials with incompressible granules, such as sands at low confining pressures. We then consider the speed and the amplitude of the various types of acceleration waves possible in each class. In materials with compressible granules, two waves are shown to exist, an elastic wave followed by a compaction wave. However, in flowing materials with incompressible granules, the only wave possible is a wave of dilatancy. In each case, we discuss the influence of the initial material non-uniformity on the wave behavior and cite some experimental observations which correlate with the results of the analysis.

1. Introduction

The subject of wave propagation in granular media has been one of continual interest over the years, especially in the fields of geophysics and civil engineering. However, the subject has recently gained even greater attention due to increasing demands for energy and the renewed general interest in applications involving pressed powders and porous materials. As a result, a considerable amount of research on wave propagation in these materials is currently in progress and many new and important results are being reported.

In this paper, I would like to describe some of the recent theoretical results we have obtained in this area and discuss how these results correlate with experimental observations. In particular, we will consider the propagation of plane longitudinal waves both in materials with compressible granules, such as pressed powders and sands at high confining pressures, and in materials with incompressible granules, such as sands at low confining pressures where flow is possible. Recognizing that longitudinal waves can be represented in terms of a one-dimensional motion, we employ a one-dimensional formulation of the continuum theory of granular materials proposed by Goodman and Cowin [1] as a model and then examine the various types of waves possible

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in each class of materials. In the case of materials with compressible granules, we show that in general there exists two possible longitudinal acceleration waves, the first wave being the well-known elastic (or compressional) wave and the second being the compaction wave. The effects of the initial porosity and material non-uniformity are considered in terms of the material's stress-strain curve and they are shown to have a significant influence on the attenuation or amplification of these waves. Furthermore, these effects determine the possible existence of shock waves in these materials. In general, we find the predictions of the theory to be consistent with experimental observations.

In the case of flowing granular materials with incompressible granules, the only wave possible is shown to be a longitudinal expansive wave at which there is a discontinuity in the gradient of porosity, and which propagates at a speed near the compaction wave speed calculated for materials with compressible granules. These waves are called waves of dilatancy and we note again that the wave's behavior is strongly influenced by the initial porosity and material non-uniformity. In particular, for uniformly distributed materials, we find that for most materials the wave will propagate faster and have a greater amplitude in looser-packed material. These results also appear to be supported by experimental observations.

2. Kinematics and Balance Laws of Granular Materials

Here we consider plane longitudinal motions of granular materials in which the interstices are void of material. Following the approach proposed by Goodman and Cowin [1], we assume that the material can be represented as a continuum and assign to the continuum the mathematical structure of a distributed body. In such a body, only one type of material point need be considered. Hence, for longitudinal motions, the body can be identified with an interval of the real line R in its reference configuration and each material point with its position X in R. The motion of the material is then described in terms of the displacement u of the material point X at time t:

$$u = \hat{u}(X,t) \quad . \tag{1}$$

For suitably smooth motions,

$$\epsilon = -\hat{u}_X(X,t) \quad , \quad v = \hat{u}_t(X,t) \quad , \quad L = u_{tX}(X,t) \quad , \tag{2}$$

define the strain ϵ, the particle velocity v, and velocity gradient L of X at time t. As a matter of convenience, we have taken the strain ϵ (and hence the stress σ) to be positive in compression.

An important consequence of the notion of a distributed body is that the bulk density ρ at any point X and time t has the decomposition

$$\rho = \nu\gamma \tag{3}$$

in terms of the density of the granules $\gamma = \hat{\gamma}(X,t)$ and the volume fraction of the granules $\nu = \hat{\nu}(X,t)$, $0 < \nu \leq 1$. The volume fraction ν is related to the porosity n by $\nu = 1 - n$ and is a measure of the dilatancy of the material which results from void compaction or distention.† It should be noted that (3) also holds in the reference configuration, i.e., $\rho_O = \nu_O\gamma_O$. In many materials, the initial distribution of granules may be non-uniform on a large scale and thus, ν_o may be a function of X.

Goodman and Cowin [1] have discussed in some detail the balance laws for granular materials and, in one dimension, the balance of mass and the balance of linear momentum become

†This notion of dilatancy is consistent with the definition put forth by Reynolds [2].

$$\frac{\rho_o}{\rho} = \frac{\nu_o\gamma_o}{\nu\gamma} = 1 - \epsilon \quad , \tag{4}$$

$$\rho_o\dot{v} = -\sigma_X + \rho_o b \quad , \tag{5}$$

respectively, where σ is the stress and b is the body force. Now, it should be evident from (4) that in order to calculate the density of the granules γ, both the strain ϵ and the volume fraction ν must be specified. Thus, ϵ and ν are kinematically independent variables and this fact necessitates the introduction of an additional force balance equation governing the void collapse. Goodman and Cowin [1] proposed such an equation based on self-equilibrated, spatially isotropic force systems and, in one dimension, it takes the form

$$\rho_o k\ddot{\nu} = h_X + \rho_o g \quad , \quad \dot{k} = 0 \quad , \tag{6}$$

where k is called the equilibrated inertia, h the equilibrated stress, and g the intrinsic equilibrated body force. This equation can also be motivated by a variational analysis outlined by Cowin and Goodman [3].

The quantities k, h, and g are difficult to interpret physically and this has provoked a considerable amount of discussion. However, comparison of (6) with the recent work of Carroll and Holt [4] on porous materials suggests that k is related to the initial surface area of the voids. It is also evident from (6) that the body force g provides the coupling between the total deformation of the material and changes in the void volume.† In granular media, this force would be associated with the Hertzian-type of contact forces acting on the granules and would also include the frictional effects associated with these forces. However, if the void distribution is non-uniform, the granules are non-spherical, or the granules are of different sizes, there will be other contact forces acting on the surface of the granules which will tend to change the packing or the fabric of the material. We associate the equilibrated stress h with these forces and suggest that it is this quantity which controls the dilatancy of the material.

3. Kinematics of Acceleration Waves

By a wave, we mean a propagating disturbance across which certain kinematical fields undergo jump discontinuities.‡ The intrinsic velocity U of such a discontinuity is defined by

$$U(t) = \frac{d}{dt} Y(t) > 0 \tag{7}$$

where Y(t) is the material point at which the front is to be found at time t. Thus, U expresses the rate of advance of the front with respect to the material in the reference configuration and Y(t) gives the material trajectory of the front. The jump $[\![f]\!]$ in a function f(X,t) across the wave front at time t is defined by

$$[\![f]\!](t) = f^-(t) - f^+(t) \tag{8}$$

where f^+ and f^- are the limiting values of f(X,t) immediately ahead of and behind the wave front.

†The fact that this coupling must be defined in terms of a constitutive equation was first suggested by Herrmann [5] in a study of porous materials.

‡Such waves are often called propagating singular surfaces and are discussed in more detail by Truesdell and Toupin [6].

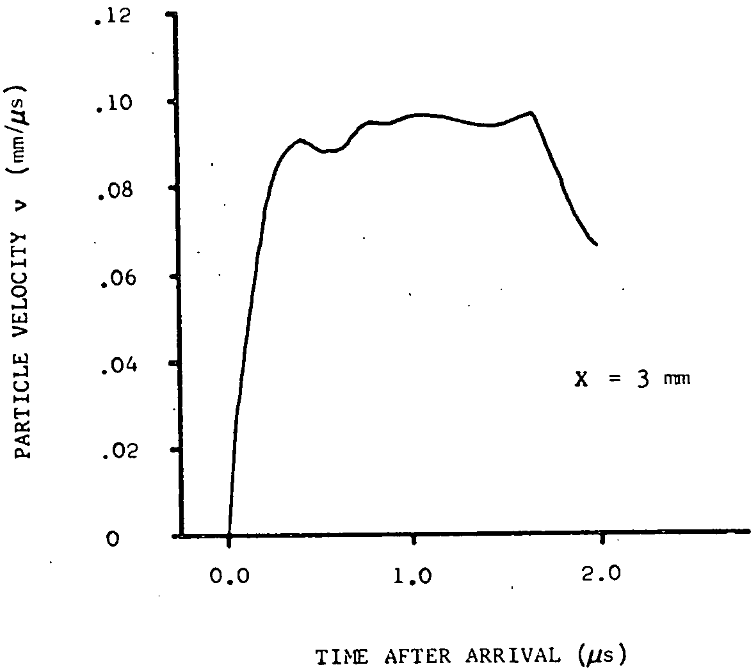

Figure 1. A compressive wave profile observed in a pressed granular material, PBX-9404 [8]. The discontinuity in acceleration occurs at the foot of the wave.

An acceleration wave is a wave across which the volume fraction ν, the strain ε, and the particle velocity v are continuous but their derivatives are not. Thus, at the wave

$$\llbracket \dot{v} \rrbracket = -U\llbracket L \rrbracket = -U^2\llbracket \varepsilon_X \rrbracket \quad , \qquad \llbracket \dot{\nu} \rrbracket = -U\llbracket \nu_X \rrbracket \quad . \tag{9}$$

Furthermore, the balance of linear momentum (5) (with $b \equiv 0$) implies that at $X = Y(t)$

$$\llbracket \sigma \rrbracket = 0 \quad , \quad \rho_o \llbracket \dot{v} \rrbracket = -\llbracket \sigma_X \rrbracket \quad , \tag{10}$$

and the balance of equilibrated force (6) implies that (cf. [7])

$$\rho_o k U \llbracket \dot{\nu} \rrbracket = -\llbracket h \rrbracket \quad , \qquad \rho_o k \llbracket \ddot{\nu} \rrbracket = \llbracket h_X \rrbracket + \rho_o \llbracket g \rrbracket \quad . \tag{11}$$

The jump in the particle acceleration $\llbracket \dot{v} \rrbracket(t)$ is called the amplitude $a(t)$ of the wave. For a compressive wave, $a(t) > 0$; while for an expansive wave, $a(t) < 0$. A compressive acceleration wave is shown in Figure 1.

4. Constitutive Assumptions

First, let us consider the one-dimensional counterpart of the general theory of granular materials studied by Goodman and Cowin [1]. Motivated by physical observations, they assumed that the material response was a function of the initial distribution of granules, the spatial and temporal changes in this distribution, the compressibility of the granules, and the fluid-like properties of the material. Hence, the stress σ, the equilibrated stress h, and the equilibrated body force g were defined by constitutive equations of the form

$$\sigma = \hat{\sigma}^*(\nu_o,\nu,\nu_X,\varepsilon) + \hat{\sigma}^+(\nu_o,\nu,\nu_X,\dot{\nu},\varepsilon,L) \quad , \tag{12}$$

$$h = \hat{h}(\nu_o,\nu,\nu_X,\varepsilon) \quad , \tag{13}$$

$$g = \hat{g}^*(\nu_o,\nu,\nu_X,\varepsilon) + \hat{g}^+(\nu_o,\nu,\nu_X,\dot{\nu},\varepsilon,L) \quad , \tag{14}$$

where the functions $\hat{\sigma}^*$, $\hat{h}$, and $\hat{g}^*$ are derivable from a stored energy function $\hat{e}$, i.e.,

$$\sigma^* = \hat{e}_\varepsilon \quad , \quad h = \hat{e}_{\nu_X} \quad , \quad g^* = -\hat{e}_\nu \quad , \tag{15}$$

and the functions σ^+ and g^+ satisfy the dissipation inequality

$$g^+\dot{\nu} + \sigma^+ L \le 0 \quad . \tag{16}$$

This inequality is a consequence of energy losses due to the local ($g^+\dot{\nu}$) and global ($\sigma^+ L$) effects of grain friction.

We should note that if the initial distribution of granules is uniform and, in a three-dimensional context, the material has a center of symmetry, then there are some additional restrictions on these constitutive equations. In particular, σ and g must be even functions of ν_X and h must be an odd function of ν_X.

This constitutive formulation encompasses the two special classes of materials which are of interest to us here.

Solid materials with compressible granules. Materials such as pressed powders and sands at high confining pressure do not, in general, exhibit fluid-like behavior. Thus, the constitutive equations (12)-(14) must be independent of the velocity gradient L. It follows then from (16) that (12)-(14) reduce to

$$\sigma = \hat{\sigma}^*(\nu_o,\nu,\nu_X,\epsilon) \quad , \quad h = \hat{h}(\nu_o,\nu,\nu_X,\epsilon) \quad , \tag{17}$$

$$g = \hat{g}^*(\nu_o,\nu,\nu_X,\epsilon) + \hat{g}^+(\nu_o,\nu,\nu_X,\dot{\nu},\epsilon) \quad , \tag{18}$$

where $\hat{\sigma}^*$, $\hat{h}$, and $\hat{g}^*$ are given by (15), and $g^+\dot{\nu} \le 0$.

Flowing materials with incompressible granules. On the other hand, sands at low confining pressure will exhibit fluid-like behavior and the granules will be essentially incompressible. In view of (4) and (2), this condition of incompressibility asserts that

$$\epsilon = 1 - \frac{\nu_o}{\nu} \quad , \quad \dot{\nu} = -\frac{\nu^2}{\nu_o} L \quad . \tag{19}$$

Thus, the constitutive equations (12)-(14) reduce to

$$\sigma = p + \tilde{\sigma}^+(\nu_o,\nu,\nu_X,L) \quad , \quad h = \tilde{h}(\nu_o,\nu,\nu_X) \quad , \tag{20}$$

$$g = \tilde{g}^*(\nu_o,\nu,\nu_X) + \tilde{g}^+(\nu_o,\nu,\nu_X,L) \quad , \tag{21}$$

where p is an indeterminate hydrostatic pressure, h and g^* are given by $(15)_{2,3}$, and $[\sigma^+ - \nu^2 g^+/\nu_o]L \le 0$.

5. Wave Propagation in Solid Materials with Compressible Granules

In order to properly interpret the behavior of longitudinal waves in granular materials with compressible granules, we need to initially consider some of the distinguishing features of the uniaxial stress-strain curves for these materials. It should be apparent from $(17)_1$ that, in fact, granular materials of this type have two stress-strain curves of interest. The curve defined by

$$\sigma = \hat{\sigma}^d(\epsilon,X) = \hat{\sigma}^*(\nu_o,\nu_o,(\nu_X)_o,\epsilon) \tag{22}$$

is called the dynamic stress-strain curve since it characterizes very rapid changes in stress with virtually no change in porosity. Thus, this curve is related to the stress-strain curve of the granules, properly corrected, however, for the initial porosity and its spatial variations. The other stress-strain curve of interest is the static curve defined by

$$\sigma = \hat{\sigma}^s(\epsilon,X) = \hat{\sigma}^*(\nu_o,\bar{\nu}(\epsilon,X),(\nu_X)_o,\epsilon) \tag{23}$$

where the function $\bar{\nu}(\varepsilon,X)$ represents the final equilibrium value of the volume fraction for a given value of strain and is the solution of the equation

$$\hat{g}^*(\nu_o,\nu,(\nu_X)_o,\varepsilon) = 0 \quad . \tag{24}$$

Clearly, this curve reflects the effects of void compaction. Hence, in any loading process, the loading path will lie somewhere between the two curves (22) and (23) and any hysteresis upon unloading will result from local frictional effects represented by the dependence of g on $\dot{\nu}$.†

In discussing wave behavior, it is the dynamic curve which is of importance. Experimental evidence on pressed granular materials [11] and sands at high confining pressures [12-14] indicate that the shape of this curve can vary greatly, depending upon the material, the porosity and the confinement. However, in general, the slope E,

$$E = \hat{\sigma}^d_\varepsilon = \hat{\sigma}^*_\varepsilon(\nu_o,\nu_o,(\nu_X)_o,\varepsilon) \quad , \tag{25}$$

is positive for all ε; while, the curvature $\tilde{E}$,

$$\tilde{E} = \hat{\sigma}^d_{\varepsilon\varepsilon} = \hat{\sigma}^*_{\varepsilon\varepsilon}(\nu_o,\nu_o(\nu_X)_o,\varepsilon) \quad , \tag{26}$$

may vary in sign with the value of ε. It is also important to note that the curve may depend on the position X due to the initial non-uniformity of the packing (i.e., dependence on $\nu_o(X)$). Consequently, the shape of the curve may vary from point to point in the material. This certainly can occur in large billets of pressed powders which were subjected initially to a non-uniform pressure [11] or in a column of sand where the porosity can depend on depth due to the overburden [12]. All of these factors can have an influence on wave behavior.

Speed and amplitude of acceleration waves. Having considered some of the properties of the dynamic stress-strain curve of granular materials, we now review some recent results on the behavior of acceleration waves obtained by Nunziato and Walsh [7].

First, we note that the continuity of the stress across an acceleration wave $(10)_1$ requires that the discontinuity in the volume fraction ν be second-order, i.e.,

$$[\dot{\nu}] = [\nu_X] = 0 \quad , \quad [\ddot{\nu}] = -U[\nu_{XX}] \neq 0 \quad . \tag{27}$$

Then, it can be shown that in general there exists two waves with non-zero amplitude a: the "fast" wave propagates at a speed

$$U_F^2 = \frac{1}{2}\left\{(c_1^2+c_2^2) + \sqrt{(c_1^2-c_2^2)^2 + 4\beta}\right\} \tag{28}$$

and the "slow" wave has the speed

$$U_S^2 = \frac{1}{2}\left\{(c_1^2+c_2^2) - \sqrt{(c_1^2-c_2^2)^2 + 4\beta}\right\} \quad , \tag{29}$$

where

$$c_1^2 = \frac{\hat{\sigma}^*_\varepsilon}{\rho_o} \quad , \quad c_2^2 = \frac{\hat{h}_{\nu_X}}{\rho_o k} \quad , \quad \beta = \frac{(\hat{\sigma}^*_{\nu_X})^2}{\rho_o^2 k} \quad . \tag{30}$$

With $k > 0$, U_F and U_S will be real and positive if $c_1^2 c_2^2 > \beta \geq 0$ and, furthermore,

†Hysteretic effects are an extremely important part of granular material response; see, for example, Stoll [9] and Krizek [10].

$$U_F \geq C_1 \quad , \quad U_S \leq C_2 \tag{31}$$

with the equalities holding when $\beta = 0$. For the "fast" wave propagating into unstrained material at rest, C_1 is the speed one would calculate from the initial slope of the dynamic stress-strain curve, E_o, and thus, (31) suggests that the "fast" wave is predominately associated with the elasticity of the material. The fact that U_F may exceed C_1 is a direct consequence of non-linear coupling effects (i.e., β) which result from the stress-strain curve changing with propagation distance. On the other hand, the "slow" wave always propagates into deforming material behind the "fast" wave and, since h plays an important role in void collapse through (6), $(30)_2$ and (31) suggest that this wave is associated with the compaction process in the material.

Nunziato and Walsh [7] also considered the amplitude behavior of the waves and showed that the growth and decay of the amplitude a(t) of each wave obeys a differential equation of the form

$$\frac{d}{dt} a(t) = -\mu(t)a + \varkappa(t)a^2 \quad . \tag{32}$$

For the "fast" wave, the coefficient $\mu(t)$ is a result of dispersive effects arising from the initially non-uniform distribution of granules. This coefficient is generally positive and tends to dampen the wave amplitude. The coefficient $\varkappa(t)$ depends on the curvature of the dynamic stress-strain curve, $\tilde{E}_o$, as well as several other parameters reflecting non-linear effects resulting from changes in the stress-strain curve with propagation distance. It is this coefficient which determines whether the wave simply attenuates or is amplified to form a shock wave (i.e., $a \to \infty$).† In particular, if

$$J(t) = \int_0^t \varkappa(s)ds < 0 \quad ,$$

then the amplitude a(t) of a compressive wave is bounded for all time t. However, if $J(t) > 0$, then the amplitude will grow and a shock wave will form in a finite time. For expansive waves, the conditions for growth and decay are reversed.

Nunziato, et al. [8] have used the results for the "fast" wave to predict the attenuation of acceleration waves observed in a pressed granular material, PBX-9404. This material is 98% dense and consists of a bimodal distribution of HMX granules with a small amount of binding compound. The waves were generated in a light-gas gun and the particle velocity histories observed in the material by laser interferometry (cf., Fig. 1). Knowing the properties of the granules, the initial density distribution, and how the wave speed U_F varied with propagation distance, Nunziato, et al. were able to construct a model for the material in the context of the present theory and calculate the amplitude as a function of propagation distance for two initial input amplitudes. The comparison of their analysis with the experimental measurements is fairly good. It is of interest to note that for this material, the shape of the dynamic stress-strain varies periodically with propagation distance and the curvature at low stresses is predominately negative.

Waves in uniformly distributed materials. In this case, ν_o is constant and the behavior of acceleration waves simplifies considerably. In particular, there are again two waves, but at the "fast" wave, the discontinuity in the volume fraction becomes third-order. Furthermore, the wave speed U_F and the amplitude a(t) reduce simply to‡

†Across a shock wave, the particle velocity v is discontinuous.

‡This speed corresponds to the ultrasonic speed of acoustic waves in the same material, cf. Nunziato and Walsh [15].

$$U_F^2 = C_1^2 = \frac{E_o}{\rho_o} \quad , \quad a(t) = a(0)\left\{1 - a(0)\,\frac{\tilde{E}_o t}{2E_oC_1}\right\}^{-1} .$$

Thus, the behavior of the "fast" wave is determined entirely by the initial slope and curvature of the dynamic stress-strain curve. In particular, $\tilde{E}_o < 0 \Rightarrow$ wave attenuation, and $\tilde{E}_o > 0 \Rightarrow$ wave growth and shock formation ($a \to \infty$) at

$$t_\infty = \frac{2E_oC_1}{a(0)\tilde{E}_o} .$$

Such shock formation has been observed in sands by Baker and Triandafilidis [11], who also correlated the wave's behavior with the shape of the dynamic stress-strain curve.

The subsequent behavior of a shock wave in uniformly distributed granular materials is determined by the shape of the dynamic curve and the nature of the boundary conditions. Nunziato and Walsh [16] have considered the growth and decay of shock waves in the context of the present theory and, in particular, have established the conditions for steady waves and for precursor decay. These conditions are consistent with the experimental observations of steady waves [11] and of precursor decay [8] in sand at high confining pressure.

6. Wave Propagation in Flowing Materials with Incompressible Granules

In this section, we wish to examine the properties of waves in granular materials where flow is possible. Experiments have shown that when these materials begin to flow through a constriction, a wave is initiated at the constriction and propagates outward into the material carrying an increase in porosity. This wave is called a wave of dilatancy. Cowin and Nunziato [17] have studied these waves in some detail using a singular surface analysis and a three-dimensional model of the type (20)-(21). Here we shall review their results in a one-dimensional context and assume that constitutive equations (20) have the specific form

$$\sigma = p + \beta + 2\alpha\nu_X^2 - (\lambda+2\mu)L \quad , \quad h = 2\alpha\nu_X \quad , \tag{33}$$

where α, β, λ, and μ are functions of the volume fraction ν and its reference value ν_o. It is of interest to note that for situations of limiting equilibrium, the three-dimensional counterpart of the stress-deformation relation (33) requires that the state of stress in the material satisfy exactly a generalized Mohr-Coulomb criterion in which $b = \alpha(\nu\alpha_\nu)^{-1}$ is the coefficient of friction. If, as experiment suggests, b is a function of ν only, then α depends on ν and ν_o according to

$$\alpha = \zeta \exp\left\{\int_{\nu_o}^{\nu} \frac{d\omega}{\omega \sin \varphi(\omega)}\right\} \tag{34}$$

where $\zeta > 0$ is a constant and $\varphi = \sin^{-1}b$ is called the angle of internal friction. We call α the modulus of dilatancy.

Using $(10)_1$ and $(11)_1$, it can be shown that in these materials, only one wave exists. Furthermore, this wave propagates with the speed

$$U^2 = \frac{2\alpha}{\rho_o k} \tag{35}$$

and the amplitude a is given by

$$a = -\frac{U^2}{\nu_o}[\nu_X] = -U^2\left\{\frac{\lambda + 2\mu}{\nu_o^{5/2}\sqrt{2\alpha\gamma k}} - \frac{2}{\nu_o}(\nu_X)_o\right\} \quad . \tag{36}$$

Since the speed of the wave is determined entirely by equation (6) which governs changes in porosity, the wave can clearly be identified as a wave of dilatancy.† The mechanical properties λ and μ only influence the amplitude of the wave. Notice that the wave may be either compressive of expansive depending upon the magnitude of the volume fraction gradient ahead of the wave. Of course, if the distribution of granules is initially uniform, then $(\nu_X)_o$ vanishes and (36) becomes

$$a = -\frac{U^2}{\nu_o}[\nu_X] = -\frac{U^2(\lambda+2\mu)}{\nu_o^{5/2}\sqrt{2\alpha\gamma k}} \quad . \tag{37}$$

Assuming $(\lambda+2\mu)$ positive, then the wave is always expansive and carries an increase in porosity ($[\nu_X] > 0$), as expected.

It is also interesting to note the dependence of the speed U and the amplitude a on the initial volume fraction ν_o. In particular, by (34), (35), and (37), it is evident that if the angle of internal friction φ is a function of ν only, and k, λ, and μ are constants, then the speed and the magnitude of the amplitude of a wave of dilatancy will be greater in loosely packed material than in densely packed material. The first of these conclusions is supported by some experimental observations reported by Cowin and Nunziato [17] on a dilatant wave in a column of sand.

Finally, we should note that Cowin and Nunziato [17] were able to establish several other results which can only be obtained in the three-dimensional theory. In particular, they found that there were no transverse waves possible and that the passage of the longitudinal wave resulted in a reduction in the transverse compressive stress. Both of these results also appear to be consistent with experimental observations.

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†In view of (33), the speed C_2, associated with the compaction process in granular solids with compressible granules, is precisely the speed of a wave of dilatancy.

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