DOE/ER/05223-31

THREE-BODY HELICITY FORMALISM APPLIED TO PION-DEUTERON SCATTERING*

MASTER

Humberto Garcilazo**

Physics Department, University of Texas A & M,

College Station, Texas 77843, U. S. A.

We apply Wick's three-body helicity formalism, to covariantly reduce the relativistic Faddeev equations that describe pion-deuteron elastic scattering, obtaining a set of equations that satisfy two and three-body unitarity and are Lorentz invariant with regard to both the space and the spin variables. We use all the pion-nucleon S and P-wave channels and the two nucleon-nucleon S-wave channels as input, to give a good description of the data for pion kinetic energies in the region from 142 MeV up to 512 MeV.

* Work supported in part by the United States Department of Energy under contract No. DE-AS05-76ER05223

** On leave from Escuela Superior de Física y Matemáticas, Instituto Politécnico Nacional, México 14 D. F., México

1. Introduction

The non-relativistic three-body problem can be solved exactly through the use of Faddeev equations.[1] In the relativistic case, the analog of Faddeev equations can be obtained, by restricting the intermediate states to contain allways three particles.[2-4] The equations that one obtains however, are harder to solve since one now has to deal with four-component variables instead of the usual three-component variables of the non-relativistic theory. These extra variables can be eliminated in a covariant way, by putting the particles on their mass shells and performing a dispersion integral in the total energy squared of the system, as shown by Blankenbecler and Sugar.[3] The Blankenbecler-Sugar prescription leads to equations which satisfy two and three-body unitarity, and which are amenable to numerical solution by the same methods used for the non-relativistic theory. One problem that remains however, is the reduction of the equations in terms of angular momentum states, in such a way that the resulting partial-wave equations remain relativistically invariant. This is particularly important when one deals with particles with spin, since a spin wave function in one reference frame, appears as a different combination of spinors in another reference frame.

As has been shown by Jacob and Wick,[5] the simplest way to deal with relativistic particles that have spin, is by means of helicity amplitudes, since the helicity quantum number is invariant under rotations and under Lorentz transformations along the direction of motion of the particle. Moreover, the helicity formalism has been extended by Wick to the case of three particles,[6] so that it provides a natural basis for the relativistic three-body problem.

We should point out that the helicity expansion is not the only way to perform a partial-wave decomposition of the relativistic Faddeev equations. In principle, if one knows the form of the vertices and the full structure of the two-body amplitudes in terms of Dirac γ matrices, one can perform first the Dirac algebra obtaining a set of spin quantum numbers, while the space variables can be projected out separately in terms of orbital angular momentum quantum numbers. The spin and orbital angular momentum quantum numbers can then be combined by means of Clebsch-Gordan coefficients to obtain amplitudes of total angular momentum. This procedure was used by Aaron, Amado and Young[7)] in their treatment of the coupled πN and $\pi\pi N$ systems using the pseudoscalar coupling for the $N \rightarrow \pi N$ vertex. The advantage of the helicity formalism however, is that one does not need to specify the form of the vertices, but one can obtain directly the three-body partial-wave equations in terms of the two-body partial-wave amplitudes. This is particularly useful if one does not know the form of the various vertices, such as in the pion-deuteron system, where in addition to the $N \rightarrow \pi N$ vertex, one would have to specify the $\Delta \rightarrow \pi N$ vertex, where Δ is any isobar with quantum numbers other than the P_{11}, as well as the $d \rightarrow NN$ vertex, where d is not only the deuteron channel but any nucleon-nucleon state. As far as we know, nobody has applied the Aaron, Amado and Young method to the pion-deuteron system. Instead, all previous treatments of pion-deuteron scattering[8-14)] have made the approximation of neglecting the small components of the Dirac spinors treating them as ordinary two-component spinors, so that standard angular momentum techniques can then be used to obtain the partial-wave equations.

Pion-deuteron elastic scattering has been studied considerably in recent years, since it offers the possibility of an exact treatment within the formalism of the relativistic three-body problem. The first serious attempt to describe this system including all the complications of spin and isospin variables, was that of Mandelzweig et al.[8] which included only the pion-nucleon P_{33} resonant channel. Subsequent works by Rinat and Thomas[9] and by Rivera and Garcilazo,[10] included both the pion-nucleon resonant channel and the S-wave nucleon-nucleon channels. Later studies by Myhrer and Thomas,[11] and by Rinat et al.,[12] have investigated the effect of other nucleon-nucleon channels, as well as the contribution of pion absorption and the exchange of rho mesons. Finally, in two recent works by Rinat et al.[13] and Giraud et al.,[14] the effect of the small pion-nucleon partial waves has been included by means of perturbation theory.

In this work, we will formulate the pion-deuteron problem using a set of linear equations similar to those used in previous works, but dealing with the spin variables in a relativistic way, by first of all doing the reduction from eight to six continuous variables, taking into account the fermion propagators for the nucleons, and secondly, by performing the partial-wave decomposition of the equations using Wick's three-body helicity states which are constructed by performing the Lorentz transformations of the spinors from the three-body center of mass frame to the two-body frames. As we will show in section three, the use of Fermion propagators for the nucleons in the Blankenbecler-Sugar reduction, leads in a natural way to the inclussion of only positive energy solutions of the Dirac equation for the nucleon spinors. Thus, the introduction of Fermion propagators is equivalent to the prescription of Aaron, Amado and Young,[7] who took a scalar propagator for the nucleon but introduced a complete set of Dirac spinors, of which they kept only the positive energy states.

Our theory differs from those of previous works,[12-14] not only in the relativistic treatment that we give to the spin, but also in the treatment of the small pion-nucleon partial waves, since we do not use perturbation theory, but include all the pion-nucleon channels exactly. While the use of perturbation theory may be justified in the resonance region where the pion-nucleon P_{33} channel is dominant, such an approach can not be expected to work at energies far from the 3,3 resonance such as the measurements of the Virginia group,[15] which cover the energy region from 230 to 512 MeV.

We do not include in our calculations the effect of true pion absorption, since this requires a theory of the coupled NN and πNN systems that has become available until very recently with the developement of the Avishai-Mizutani equations,[16,17] which have been shown rigorously to satisfy two and three-body unitarity. As has been pointed out before,[16] the only existing calculations in which the effect of true pion absorption has been included,[12,13] were based in a theory[18,19] for which no proof exists that three-body unitarity is satisfied.

In section two of this paper, we will review briefly the three-body helicity formalism of Wick, and in section three apply it to perform the partial-wave decomposition of the integral equations that describe pion-deuteron scattering. We will then use these partial-wave equations in section four to obtain our results, and finally give our conclusions in section five.

2. The helicity basis states

The three-body helicity states of Wick,[6] are of the form

$$|q_i p_i;\alpha_i\rangle = |q_i p_i;JMj_i m_i\lambda_i\lambda_j\lambda_k\rangle; \qquad i=1,2,3, \tag{1}$$

where p_i is the magnitude of the relative momentum between particles j and k measured in the center of mass frame frame of the pair, and q_i is the magnitude of the relative momentum between the pair j,k and particle i measured in the three-body center of mass frame. The discrete quantum numbers α_i are the helicities λ_j and λ_k, and the angular momentum j_i and its Z-component m_i, which are all measured in the two-body center of mass frame, and the helicity λ_i as well as the total angular momentum J and its Z-component M, which are measured in the three-body center of mass frame.

The states (1) are constructed as superpositions of helicity states in which the three helicities and the three momenta of the particles are measured in the three-body center of mass frame, as

$$|q_i p_i;JMj_i m_i\lambda_i\lambda_j\lambda_k\rangle = \eta_J\eta_{j_i} e^{-i\pi s_k}\int d\cos\theta'\, d\phi'\, d\cos\theta\, d\phi\, \mathcal{D}^{J^*}_{M,m_i-\lambda_i}(\phi,\theta,\phi')$$

$$\times d^{j_i}_{m_i,\lambda_j-\lambda_k}(\theta')\sum_{\nu_i\nu_j\nu_k}\delta_{\nu_i\lambda_i} d^{s_j}_{\nu_j\lambda_j}(\beta_j) d^{s_k}_{\nu_k\lambda_k}(\beta_k)$$

$$\times |\underset{\sim}{k}_1\underset{\sim}{k}_2\underset{\sim}{k}_3;\nu_1\nu_2\nu_3\rangle \,, \tag{2}$$

$$\eta_J = \left(\frac{2J+1}{4\pi}\right)^{\frac{1}{2}} \,, \tag{3}$$

where s_i is the spin of particle i, and $\underset{\sim}{k}_i$ and ν_i its momentum and helicity in the three-body center of mass frame, that is $\underset{\sim}{k}_1+\underset{\sim}{k}_2+\underset{\sim}{k}_3=0$. The angles θ' and ϕ' specify the direction of $\underset{\sim}{p}_i$ in the two-body center of mass frame, while the angles θ and ϕ specify the direction of $\underset{\sim}{q}_i$ in the three-body center of mass frame. The functions $d^{s_j}_{\nu_j\lambda_j}(\beta_j)$ and $d^{s_k}_{\nu_k\lambda_k}(\beta_k)$, are the matrix elements of the unitary transformation that transforms the helicity spinors from the three-body center of mass frame to the two -body frame.

The Jacobian transformation between the relative momenta $\underset{\sim}{p}_i, \underset{\sim}{q}_i$ and the momenta of the three particles $\underset{\sim}{k}_1, \underset{\sim}{k}_2, \underset{\sim}{k}_3$ in the three-body center of mass frame, is

$$\frac{d\underset{\sim}{k}_1}{2\omega_1(k_1)} \frac{d\underset{\sim}{k}_2}{2\omega_2(k_2)} \frac{d\underset{\sim}{k}_3}{2\omega_3(k_3)} \delta(\underset{\sim}{k}_1 + \underset{\sim}{k}_2 + \underset{\sim}{k}_3) = J_i(p_i q_i)\, d\underset{\sim}{q}_i d\underset{\sim}{p}_i , \tag{4}$$

with $\omega_i(k_i) = (k_i^2 + m_i^2)^{\frac{1}{2}}$, and

$$J_i(p_i q_i) = \frac{\omega(p_i)}{8W_i(p_i q_i)\omega_i(q_i)\omega_j(p_i)\omega_k(p_i)} , \tag{5}$$

$$\omega(p_i) = (m_j^2 + p_i^2)^{\frac{1}{2}} + (m_k^2 + p_i^2)^{\frac{1}{2}} , \tag{6}$$

$$W_i(p_i q_i) = \left[\omega^2(p_i) + q_i^2\right]^{\frac{1}{2}} , \tag{7}$$

$$W(p_i q_i) = W_i(p_i q_i) + (m_i^2 + q_i^2)^{\frac{1}{2}} . \tag{8}$$

From Eqs. (4) and (5), we see that if the one-particle states are normalized invariantly as

$$\langle \underset{\sim}{k}_i' \lambda_i' | \underset{\sim}{k}_i \lambda_i \rangle = 2\omega_i(k_i)\delta(\underset{\sim}{k}_i' - \underset{\sim}{k}_i)\delta_{\lambda_i'\lambda_i} , \tag{9}$$

then the partial-wave states (2), will be normalized as

$$\langle q_i' p_i' ; \alpha_i' | q_i p_i ; \alpha_i \rangle = J_i^{-1}(p_i q_i) \frac{1}{q_i^2} \delta(q_i' - q_i) \frac{1}{p_i^2} \delta(p_i' - p_i) \delta_{\alpha_i' \alpha_i} , \tag{10}$$

and will form a complete set such that

$$1 = \sum_{\alpha_i} \int q_i^2 dq_i p_i^2 dp_i \; J_i(p_i q_i) | q_i p_i ; \alpha_i \rangle \langle q_i p_i ; \alpha_i | . \tag{11}$$

A state of type i can be expressed in terms of states of type j, by means of Wick's recoupling coefficients,[6] which in our normalization take the form

$$\langle q_j p_j ; \alpha_j' | q_i p_i ; \alpha_i \rangle = \delta_{J',J}\delta_{M',M}\delta\left[W(p_j q_j) - W(p_i q_i)\right] H(1 - \cos^2\chi)$$

$$\times \frac{8W(p_i q_i)\omega(p_j)\omega(p_i)}{p_j q_j p_i q_i}\left[(2j_j + 1)(2j_i + 1)\right]^{\frac{1}{2}}$$

$$\times (-)^{s_j - \lambda_j' + s_k + \lambda_k'} d^{j_j}_{m_j, \lambda_k' - \lambda_i'}(\theta') d^{j_i}_{m_i, \lambda_j - \lambda_k}(\theta)$$

$$\times d^J_{m_j - \lambda_j', m_i - \lambda_i}(\chi) d^{s_k}_{\lambda_k', \lambda_k}(\beta_k) d^{s_j}_{\lambda_j', \lambda_j}(\beta_j) d^{s_i}_{\lambda_i', \lambda_i}(-\beta_i), \quad (12)$$

where H is the step function, and the arguments of the rotation matrices in Eq. (12), are the angles of the Wick triangle which are calculated by performing the various Lorentz transformations between the three-body center of mass frame, the center of mass frame of each pair, and the rest frames of the three particles, as shown in the appendix of Ref. 6.

Using the symmetry properties of the rotation matrices that appear in Eq. (12), it is easy to show that

$$\langle q_j p_j ; \alpha_j | q_i p_i ; \alpha_i \rangle = \langle q_j p_j ; -\alpha_j | q_i p_i ; -\alpha_i \rangle , \quad (13)$$

where we use $-\alpha_i$ to indicate that the magnetic quantum numbers are to be taken with opposite sign.

If t_i is the T-matrix of the pair j,k with particle i acting as spectator, the matrix elements of t_i between the states (2), are of the form

$$\langle q_i' p_i' ; \alpha_i' | t_i(W_0) | q_i p_i ; \alpha_i \rangle = \delta_{J'J}\delta_{M'M}\delta_{j_i' j_i}\delta_{m_i' m_i}\delta_{\lambda_i' \lambda_i} 2\omega_i(q_i)\frac{1}{q_i^2}\delta(q_i' - q_i)$$

$$\times \langle p_i' ; \lambda_j' \lambda_k' | t_i^{j_i}(\omega_0(q_i)) | p_i ; \lambda_j \lambda_k \rangle \ , \qquad (14)$$

where W_0 is the invariant mass of the three-body system, and $\omega_0(q_i)$ that of the two-body subsystem which is related to W_0 as

$$\omega_0(q_i) = \left[W_0^2 + m_i^2 - 2W_0(m_i^2 + q_i^2)^{\frac{1}{2}} \right]^{\frac{1}{2}} \ . \qquad (15)$$

A form for the two-body amplitudes which has been used extensively in three-body problems, is the one-term separable form in the LS representation

$$\langle p_i' | t_i^{L_i S_i j_i}(\omega_0) | p_i \rangle = g_i^{L_i S_i j_i}(p_i') \tau_i^{L_i S_i j_i}(\omega_0) g_i^{L_i S_i j_i}(p_i), \qquad (16)$$

which in the helicity basis becomes

$$\langle p_i' ; \lambda_j' \lambda_k' | t_i^{j_i}(\omega_0) | p_i ; \lambda_j \lambda_k \rangle = \sum_{L_i S_i} h_{\lambda_j' \lambda_k'}^{L_i S_i j_i}(p_i') \tau_i^{L_i S_i j_i}(\omega_0) h_{\lambda_j \lambda_k}^{L_i S_i j_i}(p_i) \ , \qquad (17)$$

with

$$h_{\lambda_j \lambda_k}^{L_i S_i j_i}(p_i) = \left(\frac{2L_i + 1}{2j_i + 1} \right)^{1/2} C_{0,\lambda_j - \lambda_k}^{L_i S_i j_i} C_{\lambda_j, -\lambda_k}^{s_j s_k S_i} g_i^{L_i S_i j_i}(p_i). \qquad (18)$$

Corresponding expressions analogous to Eqs. (16)-(18), can be written in the case of coupled orbital angular momentum states. Using Eqs. (16)-(18) into Eq. (14), it is easy to prove the symmetry relation

$$\langle q_i' p_i' ; \alpha_i' | t_i(W_0) | q_i p_i ; \alpha_i \rangle = \langle q_i' p_i' ; -\alpha_i' | t_i(W_0) | q_i p_i ; -\alpha_i \rangle \ . \qquad (19)$$

3. The Integral Equations for Pion-Deuteron Scattering

The amplitude for pion-deuteron elastic scattering, is given by

$$T_{\mu'\mu} = \langle \phi_{\mu'} | T_2 + T_3 | \phi_\mu \rangle \ , \tag{20}$$

where ϕ_μ and $\phi_{\mu'}$ are the initial and final wave functions of the system in which the pion and the deuteron are in a relative plane-wave state with the deuteron helicity being μ and μ' respectively, and T_i are the solutions of the Faddeev equations

$$T_i = (1 - \delta_{1i})t_i + \sum_{j\neq i} t_i G_0 T_j \ ; \qquad i,j = 1,2,3. \tag{21}$$

In Eqs. (21), G_0 is the propagator for three free particles, and we have taken the pion as particle 1, and the nucleons as particles 2 and 3. Eqs. (21), can be derived in the relativistic case, starting with the Bethe-Salpeter equation, and summing all the ladder diagrams in which only two particles interact, as shown by several authors.[2-4] These equations are of course integral equations depending on eight continuous variables, since the total four-momentum is conserved at every stage, that is, the kernel of Eqs. (21) is given by

$$t_i G_0 T_j = \frac{1}{(4\pi)^2 m_2 m_3} \int d^4k_1 d^4k_2 d^4k_3 \ t_i | k_1 k_2 k_3 \rangle \ \delta^4(k_1 + k_2 + k_3 - K)$$

$$\frac{1}{k_1^2 - m_1^2 + i\epsilon} \ \frac{\gamma \cdot k_2 + m_2}{k_2^2 - m_2^2 + i\epsilon} \ \frac{\gamma \cdot k_3 + m_3}{k_3^2 - m_3^2 + i\epsilon} \langle k_1 k_2 k_3 | T_j \ , \tag{22}$$

where we have introduced fermion propagators for particles 2 and 3 which are the nucleons. In order to eliminate two of the variables of integration in Eq. (22), while still mantaining two and three-body unitarity, we use the Blankenbecler-Sugar prescription in which we replace the denominators of the propagators of the three particles by their delta-function parts and perform a dispersion integral in the total energy squared of the system, that is

$$t_i G_0 T_j \rightarrow \frac{1}{4m_2m_3}\int \frac{dS'}{S'-S}\int d^4k_1 d^4k_2 d^4k_3 \; t_i|k_1k_2k_3\rangle\, \delta^4(k_1+k_2+k_3-K)$$

$$\times\, \delta^{(+)}(k_1^2-m_1^2)\,\delta^{(+)}(k_2^2-m_2^2)\,\delta^{(+)}(k_3^2-m_3^2)\,(\gamma\cdot k_2+m_2)\,(\gamma\cdot k_3+m_3)$$

$$\times\, \langle k_1k_2k_3|T_j \quad , \tag{23}$$

where $K=(\underset{\sim}{0},\sqrt{S'})$. The integration over the fourth components of the momenta in Eq. (23), can be carried out in a straightforward way, to get in the three-body center of mass frame

$$t_i G_0 T_j = \frac{1}{4m_2m_3}\int \frac{d\underset{\sim}{k}_1}{2\omega_1(\underset{\sim}{k}_1)}\,\frac{d\underset{\sim}{k}_2}{2\omega_2(\underset{\sim}{k}_2)}\,\frac{d\underset{\sim}{k}_3}{2\omega_3(\underset{\sim}{k}_3)}\;\delta(\underset{\sim}{k}_1+\underset{\sim}{k}_2+\underset{\sim}{k}_3)\; t_i|\underset{\sim}{k}_1\underset{\sim}{k}_2\underset{\sim}{k}_3\rangle$$

$$\times\, \frac{2(\omega_1+\omega_2+\omega_3)}{(\omega_1+\omega_2+\omega_3)^2 - S - i\varepsilon}\,\Lambda_+^{(2)}(\underset{\sim}{k}_2)\,\Lambda_+^{(3)}(\underset{\sim}{k}_3)\,\langle \underset{\sim}{k}_1\underset{\sim}{k}_2\underset{\sim}{k}_3|T_j \quad , \tag{24}$$

where $\omega_i=(\underset{\sim}{k}_i^2+m_i^2)^{\frac{1}{2}}$, and

$$\Lambda_+^{(i)}(\underset{\sim}{k}_i) = -\underset{\sim}{\gamma}\cdot\underset{\sim}{k}_i + \gamma_0\omega_i + m_i \;, \tag{25}$$

are projection operators for positive energy spinors, which obey the relation[20]

$$\Lambda_+^{(i)}(\underset{\sim}{k}_i) = 2m_i \sum_{\nu_i=1,2} u_{\nu_i}(\underset{\sim}{k}_i)\,\bar{u}_{\nu_i}(\underset{\sim}{k}_i) \;, \tag{26}$$

where the sum goes only over the two positive energy solutions of the Dirac equation. In the case when one chooses the axis of quantization along the direction of motion of the particles, one can use helicity spinors in Eq. (26), where the sum now goes over the two possible components of the helicity. Substituting Eqs. (24) and (26) into Eq. (21) and multiplying from the left and right by positive energy spinors, we get the covariantly reduced Faddeev equations

$$\bar{u}_{\nu_2'}(\underset{\sim}{k}_2')\bar{u}_{\nu_3'}(\underset{\sim}{k}_3')\langle \underset{\sim}{k}_1'\underset{\sim}{k}_2'\underset{\sim}{k}_3'|T_i|\underset{\sim}{k}_1''\underset{\sim}{k}_2''\underset{\sim}{k}_3''\rangle u_{\nu_2''}(\underset{\sim}{k}_2'')u_{\nu_3''}(\underset{\sim}{k}_3'') = (1-\delta_{1i})\bar{u}_{\nu_2'}(\underset{\sim}{k}_2')\bar{u}_{\nu_3'}(\underset{\sim}{k}_3')$$

$$\times \langle \underset{\sim}{k}_1'\underset{\sim}{k}_2'\underset{\sim}{k}_3'|t_i|\underset{\sim}{k}_1''\underset{\sim}{k}_2''\underset{\sim}{k}_3''\rangle u_{\nu_2''}(\underset{\sim}{k}_2'')u_{\nu_3''}(\underset{\sim}{k}_3'') + \sum_{j\neq i}\sum_{\nu_2\nu_3}\int \frac{d\underset{\sim}{k}_1}{2\omega_1(\underset{\sim}{k}_1)}\frac{d\underset{\sim}{k}_2}{2\omega_2(\underset{\sim}{k}_2)}\frac{d\underset{\sim}{k}_3}{2\omega_3(\underset{\sim}{k}_3)}$$

$$\times \delta(\underset{\sim}{k}_1+\underset{\sim}{k}_2+\underset{\sim}{k}_3)\ \bar{u}_{\nu_2'}(\underset{\sim}{k}_2')\bar{u}_{\nu_3'}(\underset{\sim}{k}_3')\langle \underset{\sim}{k}_1'\underset{\sim}{k}_2'\underset{\sim}{k}_3'|t_i|\underset{\sim}{k}_1\underset{\sim}{k}_2\underset{\sim}{k}_3\rangle u_{\nu_2}(\underset{\sim}{k}_2)u_{\nu_3}(\underset{\sim}{k}_3)$$

$$\times \frac{2(\omega_1+\omega_2+\omega_3)}{(\omega_1+\omega_2+\omega_3)^2 - S - i\varepsilon}\ \bar{u}_{\nu_2}(\underset{\sim}{k}_2)\bar{u}_{\nu_3}(\underset{\sim}{k}_3)\langle \underset{\sim}{k}_1\underset{\sim}{k}_2\underset{\sim}{k}_3|T_j|\underset{\sim}{k}_1''\underset{\sim}{k}_2''\underset{\sim}{k}_3''\rangle u_{\nu_2''}(\underset{\sim}{k}_2'')u_{\nu_3''}(\underset{\sim}{k}_3''). \tag{27}$$

If we introduce a dummy quantum number ν_1 to represent the helicity of the pion which is of course zero, we can write Eq. (27) as

$$\langle \underset{\sim}{k}_1'\underset{\sim}{k}_2'\underset{\sim}{k}_3';\nu_1'\nu_2'\nu_3'|T_i|\underset{\sim}{k}_1''\underset{\sim}{k}_2''\underset{\sim}{k}_3'';\nu_1''\nu_2''\nu_3''\rangle = (1-\delta_{1i})\langle \underset{\sim}{k}_1'\underset{\sim}{k}_2'\underset{\sim}{k}_3';\nu_1'\nu_2'\nu_3'|t_i|\underset{\sim}{k}_1''\underset{\sim}{k}_2''\underset{\sim}{k}_3'';\nu_1''\nu_2''\nu_3''\rangle$$

$$+ \sum_{j\neq i}\sum_{\nu_1\nu_2\nu_3}\int \frac{d\underset{\sim}{k}_1}{2\omega_1(\underset{\sim}{k}_1)}\frac{d\underset{\sim}{k}_2}{2\omega_2(\underset{\sim}{k}_2)}\frac{d\underset{\sim}{k}_3}{2\omega_3(\underset{\sim}{k}_3)}\delta(\underset{\sim}{k}_1+\underset{\sim}{k}_2+\underset{\sim}{k}_3)$$

$$\times \langle \underset{\sim}{k}_1'\underset{\sim}{k}_2'\underset{\sim}{k}_3';\nu_1'\nu_2'\nu_3'|t_i|\underset{\sim}{k}_1\underset{\sim}{k}_2\underset{\sim}{k}_3;\nu_1\nu_2\nu_3\rangle \frac{2(\omega_1+\omega_2+\omega_3)}{(\omega_1+\omega_2+\omega_3)^2 - S - i\varepsilon}$$

$$\times \langle \underset{\sim}{k}_1\underset{\sim}{k}_2\underset{\sim}{k}_3;\nu_1\nu_2\nu_3|T_j|\underset{\sim}{k}_1''\underset{\sim}{k}_2''\underset{\sim}{k}_3'';\nu_1''\nu_2''\nu_3''\rangle\ . \tag{28}$$

where the states $|\underset{\sim}{k}_1\underset{\sim}{k}_2\underset{\sim}{k}_3;\nu_1\nu_2\nu_3\rangle$, etc., are plane-wave states in which the three momenta and the three helicities are measured in the three-body center of mass frame.

Now that we have reduced the integral equations from eight to six continuous variables, the next step is to partial-wave decompose them such that Lorentz-invariance is mantained. There are two important effects that must be considered while performing the partial-wave decomposition. The first one, is the fact that the two-body operators t_i appear between plane-wave states which are defined in the three-body center of mass frame, so that we must expand these states in terms of the partial-wave states of the previous section, where the quantum numbers of the pair j,k are measured in the center of mass frame of the pair, so that the matrix elements of the operators t_i take the simple form exibited by Eq. (14). The second effect, is due to the fact that the amplitudes T_i are coupled to amplitudes T_j with $j \neq i$, so that after one has performed the partial-wave decomposition with the basis states of type i which are appropiate for the amplitudes T_i, one has to reexpand in terms of the states of type j which are appropiate for the amplitude T_j, and this can be achieved by means of Wick's three-body recoupling coefficients (12).

We can expand the plane-wave states in the three-body center of mass frame $|\underset{\sim}{k}_1\underset{\sim}{k}_2\underset{\sim}{k}_3;\nu_1\nu_2\nu_3\rangle$ that appear in Eq. (28), in terms of the basis states of the previous section, by using the inverse of the transformation (2), that is

$$|\underset{\sim}{k}_1\underset{\sim}{k}_2\underset{\sim}{k}_3;\nu_1\nu_2\nu_3\rangle = \eta_J\eta_{j_i} e^{i\pi s_k} \sum_{\substack{\lambda_i\lambda_j\lambda_k \\ j_i m_i JM}} \mathcal{D}^{J}_{M,m_i-\lambda_i}(\phi,\theta,\phi')\, d^{j_i}_{m_i,\lambda_j-\lambda_k}(\theta')$$

$$\times\, \delta_{\nu_i\lambda_i} d^{s_j}_{\nu_j\lambda_j}(\beta_j)\, d^{s_k}_{\nu_k\lambda_k}(\beta_k)\, |p_i q_i;JMj_i m_i\lambda_i\lambda_j\lambda_k\rangle \;, \qquad (29)$$

where according to the conventions of the previous section, the angles θ' and ϕ' specify the direction of the momentum $\underset{\sim}{p}_i$ in the two-body center of mass frame, and the angles θ and ϕ specify the direction of the momentum $\underset{\sim}{q}_i$ in the three-body center of mass frame. In order to carry out the partial-wave decomposition of Eq. (28), we first apply the Jacobian transformation (4) to change the element of integration from $d\underset{\sim}{k}_1 d\underset{\sim}{k}_2 d\underset{\sim}{k}_3$ to $d\underset{\sim}{q}_i d\underset{\sim}{p}_i$, and expand the two intermediate states by means of Eq. (29). Next, we use the completeness relation

$$\sum_{\nu_1\nu_2\nu_3}\delta_{\nu_i\lambda_i}\delta_{\nu_i\lambda_i'}d^{s_j}_{\nu_j\lambda_j}(\beta_j)d^{s_j}_{\nu_j\lambda_j'}(\beta_j)d^{s_k}_{\nu_k\lambda_k}(\beta_k)d^{s_k}_{\nu_k\lambda_k'}(\beta_k)=\delta_{\lambda_i\lambda_i'}\delta_{\lambda_j\lambda_j'}\delta_{\lambda_k\lambda_k'}\,, \tag{30}$$

as well as the orthogonality relations of the rotation matrices

$$\eta_J^2\eta_{j_i}^2\int d\phi'\; d\cos\theta'\; d\phi\; d\cos\theta\; \mathcal{D}^{J}_{M,m_i-\lambda_i}(\phi,\theta,\phi')\,\mathcal{D}^{J'*}_{M',m_i'-\lambda_i}(\phi,\theta,\phi')$$

$$\times d^{j_i}_{m_i,\lambda_j-\lambda_k}(\theta')d^{j_i'*}_{m_i',\lambda_j-\lambda_k}(\theta')=\delta_{JJ'}\delta_{MM'}\delta_{j_ij_i'}\delta_{m_im_i'}\quad, \tag{31}$$

and finally expand the external states on the right and left of Eq. (28) by means of Eq. (29), to get the partial-wave equations

$$\langle q_i'p_i';\alpha_i'|T_i|q_i''p_i'';\alpha_i''\rangle=(1-\delta_{1i})\langle q_i'p_i';\alpha_i'|t_i|q_i''p_i'';\alpha_i''\rangle+\sum_{j\neq i}\sum_{\alpha_i}\int q_i^2dq_ip_i^2dp_i$$

$$\times J_i(p_iq_i)\langle q_i'p_i';\alpha_i'|t_i|q_ip_i;\alpha_i\rangle\frac{2W(p_iq_i)}{W^2(p_iq_i)-S-i\varepsilon}$$

$$\times\langle q_ip_i;\alpha_i|T_j|q_i''p_i'';\alpha_i''\rangle\,. \tag{32}$$

Since Eqs (32) are integral equations only for the variables of the left-hand side, we can integrate over the variables of the right-hand side introducing the initial-state wave function of the system ϕ^J_μ with total angular momentum J and helicity of the deuteron μ, to get

$$\langle q_i'p_i';\alpha_i'|T_i|\phi^J_\mu\rangle=(1-\delta_{1i})\langle q_i'p_i';\alpha_i'|t_i|\phi^J_\mu\rangle+\sum_{j\neq i}\sum_{\alpha_i}\int q_i^2dq_ip_i^2dp_iJ_i(p_iq_i)$$

$$\times\langle q_i'p_i';\alpha_i'|t_i|q_ip_i;\alpha_i\rangle\frac{2W(p_iq_i)}{W^2(p_iq_i)-S-i\varepsilon}$$

$$\times\langle q_ip_i;\alpha_i|T_j|\phi^J_\mu\rangle\,. \tag{33}$$

Since in this last equation we have the amplitudes T_j as functions of the variables of type i, we now introduce a complete set of states of type j given by Eq. (11), to get finally

$$\langle q_i'p_i';\alpha_i'|T_i|\phi_\mu^J\rangle = (1-\delta_{1i})\langle q_i'p_i';\alpha_i'|t_i|\phi_\mu^J\rangle + \sum_{j\neq i}\sum_{\alpha_i}\sum_{\alpha_j}\int q_i^2dq_i p_i^2dp_i q_j^2dq_j$$

$$\times p_j^2dp_j J_i(p_iq_i)J_j(p_jq_j)\langle q_i'p_i';\alpha_i'|t_i|q_ip_i;\alpha_i\rangle$$

$$\times \frac{2W(p_iq_i)}{W^2(p_iq_i) - S - i\varepsilon}\langle q_ip_i;\alpha_i|q_jp_j;\alpha_j\rangle\langle q_jp_j;\alpha_j|T_j|\phi_\mu^J\rangle \, , \tag{34}$$

where $\langle q_ip_i;\alpha_i|q_jp_j;\alpha_j\rangle$ are Wick's three-body recoupling coefficients given by Eq. (12). Although there are four variables of integration in Eq. (34), we notice that Wick's recoupling coefficients (12) contain a delta function for the conservation of total energy, while the matrix elements of t_i given by Eq. (14), contain a delta function for the conservation of momentum of the spectator particle, so that Eqs. (34) are integral equations in only two continuous variables. If we assume in addition that the two-body amplitudes t_i are of separable form as given by Eq. (17), then Eqs. (34) reduce to integral equations in only one continuous variable. The equations can be further reduced, by taking advantage of the identity of the two nucleons and performing the anti-symmetrization of the equations[16,21] (including isospin), which reduces the number of coupled amplitudes to about one half.

One problem that still remains with Eq. (34), is the separation into states of definite parity which allows to decouple them even further. In order to do this, it is easy to show using the symmetry properties (13) and (19), that the solutions of Eq. (34) satisfy

$$\langle q_ip_i;\alpha_i|T_i|\phi_\mu^J\rangle = \langle q_ip_i;-\alpha_i|T_i|\phi_{-\mu}^J\rangle \, , \tag{35}$$

so that if we define the even and odd combinations of T_i

$$\langle q_ip_i;\alpha_i|T_i^{(\pm)}|\phi_\mu^J\rangle = \langle q_ip_i;\alpha_i|T_i|\phi_\mu^J\rangle \pm \langle q_ip_i;\alpha_i|T_i|\phi_{-\mu}^J\rangle \, , \tag{36}$$

they are also solutions of Eq. (34), and we see from Eq. (35) that they obey the parity relation

$$\langle q_i p_i ; -\alpha_i | T_i^{(\pm)} | \phi_\mu^J \rangle = \pm \langle q_i p_i ; \alpha_i | T_i^{(\pm)} | \phi_\mu^J \rangle , \qquad (37)$$

so that only about half of the amplitudes that enter into Eq. (34) will be coupled together, with the remaining amplitudes being given by Eq. (37).

The transition amplitude for going from a state in which the deuteron has helicity μ to a state in which it has helicity μ', is

$$T^J_{\mu'\mu} = \langle \phi^J_{\mu'} | T_2 + T_3 | \phi^J_\mu \rangle \quad , \qquad (38)$$

and from Eq. (35) we see that

$$T^J_{\mu'\mu} = T^J_{-\mu' -\mu} \; . \qquad (39)$$

Since in addition the Faddeev equations (21) are time-reversal invariant, the amplitudes (38) satisfy also

$$T^J_{\mu'\mu} = T^J_{\mu\mu'} \quad , \qquad (40)$$

so that there are only four independent pion-deuteron helicity amplitudes which can be chosen to be T^J_{00}, T^J_{10}, T^J_{11}, and T^J_{-11}.

4. Results

As an application of our formalism, we solved the integral equations (34) using separable T matrices as given by Eq. (17), with the pion-nucleon interaction being represented by the six S and P-wave channels, and the nucleon-nucleon interaction by the two S-wave channels.

In the case of the pion-nucleon T matrices, we took the form factors in Eq. (16), to depend only on the orbital angular momentum L_i, as

$$g_i^{L_i S_i j_i I_i}(p_i) = \frac{(p_i)^{L_i}}{1+(p_i/\alpha_{L_i})^2} \quad ; \qquad L_i = 0,1, \tag{41}$$

where I_i is the isospin of the pair. The energy dependence of the T matrices $\tau_i(\omega_0)$ in Eq. (16), is determined by the experimental phase shifts. We used the CERN phase shifts and inelaticities,[22] and extended these T matrices to the unphysical region $\omega_0 < m_1 + m_2$, by using the extrapolation formula

$$\tau_i^{L_i S_i j_i I_i}(\omega_0) = \left(\frac{\omega_0}{m_1+m_2}\right)^n \tau_i^{L_i S_i j_i I_i}(m_1+m_2) \ , \tag{42}$$

with $n = 2$. We checked that the results are rather insensitive to the extrapolation used in the unphysical region, by performing some calculations with $n = 1$ and $n = 3$, and seeing that the differential cross sections differed in all cases by less than half of a percent. We choosed the ranges α_{L_i} in Eq. (41), to be $\alpha_0 = 3.5\ \text{fm}^{-1}$ and $\alpha_1 = 1.8\ \text{fm}^{-1}$, which were obtained in Refs. 13 and 23 for the S_{11} and P_{33} channels respectively, by solving the Blankenbecler-Sugar equation with separable potentials. Our separable T matrices of course, do not require the use of separable potentials, since they are constructed directly from the experimental phase shifts, but we use these ranges so as to be able to compare our results in the resonance region with those of previous calculations. Since our goal is to cover an energy region which starts at 142 MeV and goes up to 512 MeV, it would be very difficult to find separable potentials that fit the phase shifts over such a wide range of energies.

In the case of the two S-wave nucleon-nucleon T matrices, we took the form factors in Eq. (16), as

$$g_i^{L_i S_i j_i I_i}(p_i) = \left[1 + (p_i/m_2)^2\right]^{\frac{1}{4}} \frac{1}{1 + (p_i/\alpha_{j_i})^2} \;; \quad j_i = 0,1, \qquad (43)$$

with the energy dependence $\Upsilon_i(\omega_0)$ in Eq. (16) determined by the phase shifts and mixing parameters.

We used the experimental phase shifts and mixing parameters of Ref. 24, and extended the T matrices to the unphysical region $\omega_0 < 2m_2$, by solving the Blankenbecler-Sugar equation using separable potentials with the form factors given by Eq. (43), and fitting the scattering lengths, efective ranges and deuteron pole. For equal mass particles, the form factors (43) transform the Blankenbecler-Sugar equation into the non-relativistic Lippmann-Schwinger equation with Yamaguchi potentials, so that we can use the potentials that were constructed in a previous work;[10] in particular, the ranges are $\alpha_0 = 1.153$ fm^{-1}, and $\alpha_1 = 1.449$ fm^{-1}. We should point out that if one tries to use separable potentials to represent the energy dependence of the T matrix in the physical region $\omega_0 > 2m_2$, one immediately runs into problems, since the experimental phase shifts change sign at a kinetic energy in the center of mass of about 150 MeV, while the phase shifts produced by separable potentials are positive for all energies.

We represented the initial and final states of the system ϕ^J_μ, using the deuteron wave function of Moravcsik[25] which has a D-state admixture of 6.7 %; and solved the integral equations along the real axis by means of Padé approximants,[26] for all partial waves with total angular momentum $J < 6$, and took the impulse approximation for the other partial waves up to $J = 14$.

We show in Fig. 1 our results for the total and integrated elastic cross sections, and compare them with the total cross-section data of Pedroni et al.[27] We see that the theory and experiment are in good agreement with each other except by the fact that they are shifted by approximately 10 MeV throughout the energy region considered. This energy shift is a very puzzling result and we have no explanation for it at the present time.

We show in Figs. 2-4, our results for the differential cross sections in the energy region between 142 and 512 MeV. As we can see, the agreement between theory and experiment is very good at 142 and 182 MeV, although it becomes somewhat worse as we move to 230 MeV, and then at 256 MeV there is a discrepancy by a factor of between 3 and 4 in the large-angle region, although in the forward direction the agreement is still very good up to $\theta \sim 70^\circ$. In the last three energies, the situation is somewhat better, since as we move to higher energies the discrepancy at large angles tends to dissappear and we are able to predict quite well the position of the minimum at 100 degrees.

Our results in Figs. 2 and 3 for the lowest four energies, are in good agreement with those of Giraud et al.[14] and of Rinat et al.[13] (without absorption), who use comparable input but treat the small partial waves in perturbation theory, so that this confirms the claim made in Refs. 13 and 14 that the use of perturbation theory is adequate in the resonance region. The effect of the small pion-nucleon partial waves as well as the relativistic treatment of the spin, become more important as we move to higher energies, such as those shown in Fig.

4. These results are the first full three-body calculations performed for these high energies, so that it is satisfying to see that the theory does such a good job.

5. Conclusions

We have incorporated Wick's three-body helicity formalism, to the integral equations that describe pion-deuteron elastic scattering, so as to obtain equations which are relativistically invariant with respect to both the space and the spin variables. Using the six S and P-wave pion-nucleon channels and the two nucleon-nucleon S-wave channels as input, we are able to give a good description of the data covering the energy region from 142 MeV to 512 MeV. Our results for the total cross sections, are in good agreement with the data of Pedroni et al.,[27] except that they are shifted by approximately 10 MeV throughout the energy range.

References

1. L. D. Faddeev, Mathematical aspects of the three-body problem in the quantum scattering theory (Davey, New York, 1965)
2. V. A. Alessandrini and R. L. Omnes, Phys. Rev. 139 (1965) B167
3. R. Blankenbecler and R. Sugar, Phys. Rev. 142 (1966) 1051
4. J. M. Namyslowsky, Nuovo Cim. 57A (1968) 355
5. M. Jacob and G. C. Wick, Ann. of Phys. 7 (1959) 404
6. G. C. Wick, Ann. of Phys. 18 (1962) 65
7. R. Aaron, R. D. Amado and J. E. Young, Phys. Rev. 174 (1968) 2022.
8. V. B. Mandelzweig, H. Garcilazo and J. M. Eisenberg, Nucl. Phys. A256 (1976) 461
9. A. S. Rinat and A. W. Thomas, Nucl. Phys. A282 (1977) 365
10. J. M. Rivera and H. Garcilazo, Nucl. Phys. A285 (1977) 505
11. F. Myhrer and A. W. Thomas, Nucl. Phys. A326 (1979) 497
12. A. S. Rinat, Y. Starkand, E. Hammel and A. W. Thomas, Phys. Lett. 80B (1979) 166
13. A. S. Rinat, E. Hammel, Y. Starkand and A. W. Thomas, Nucl. Phys. A329 (1979) 285
14. N. Giraud, C. Fayard and G. H. Lamot, Phys. Rev. C21 (1980) 1959
15. R. H. Cole, J. S. McCarthy, R. C. Minehart and E. A. Wadlinger, Phys. Rev. C17 (1978) 681
16. Y. Avishai and T. Mizutani, Nucl. Phys. A326 (1979) 352
17. Y. Avishai and T. Mizutani, Nucl. Phys. A338 (1980) 377
18. A. S. Rinat, Nucl. Phys. A287 (1977) 399
19. A. W. Thomas and A. S. Rinat, Phys. Rev. C20 (1979) 216
20. S. Gasiorowicz, Elementary particle physics (Wiley, New York, 1966)
21. I. R. Afnan and A. W. Thomas, Phys. Rev. C10 (1974) 109
22. D. J. Herndon, A. Barbaro-Galtieri and A. H. Rosenfeld, LRL report No. UCRL-20030 πN, 1970 (unpublished), CERN Kirsopp solution
23. R. M. Woloshyn, E. J. Moniz and R. Aaron, Phys. Rev. C13 (1976) 286

24. R. A. Arndt, R. H. Hackman and L. D. Roper, Phys. Rev. C15 (1977) 1002

25. M. J. Moravcsik, Nucl. Phys. 7 (1958) 113

26. G. A. Baker, Jr., and J. L. Gammel, ed., The Padé approximant in theoretical physics (Academic, New York, 1970)

27. E. Pedroni, K. Gabathuler, J. J. Domingo, W. Hirt, P. Schwaller, J. Arvieux, C. H. Q. Ingram, P. Gretillat, J. Piffaretti, N. W. Tanner and C. Wilkin, Nucl. Phys. A300 (1978) 321

28. J. H. Norem, Nucl. Phys. B33 (1971) 512

29. E. G. Pewitt, T. H. Fields, G. B. Yodh, J. G. Fetkovich and M. Derrick, Phys. Rev. 131 (1963) 1826

30. K. Gabathuler, C. R. Cox, J. J. Domingo, J. Rohlin, N. W. Tanner and C. Wilkin, Nucl. Phys. B55 (1973) 397

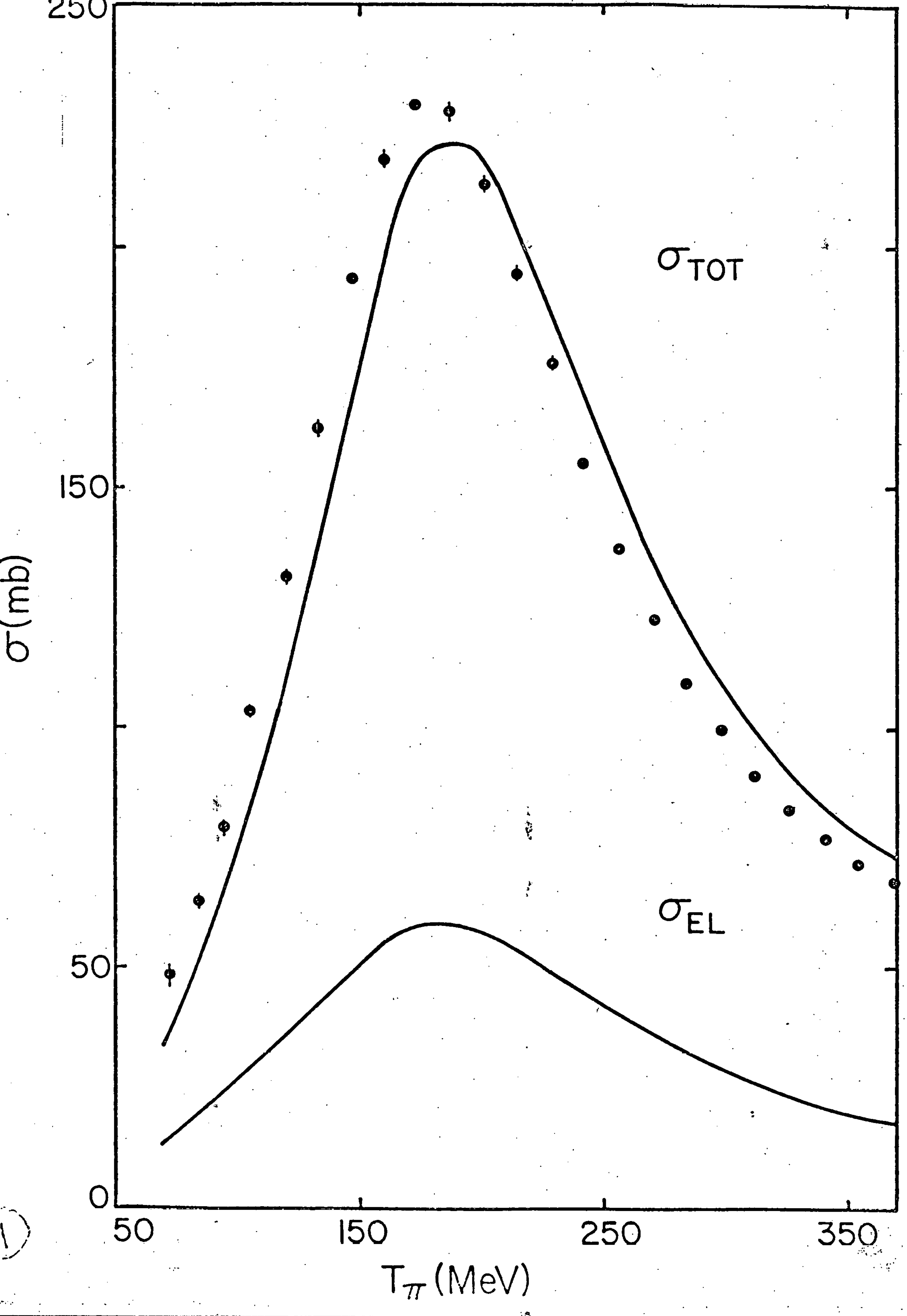
Figure Captions

Fig. 1. Total and integrated elastic cross sections as a function of the laboratory kinetic energy of the pion. The experimental points are from Ref. 27.

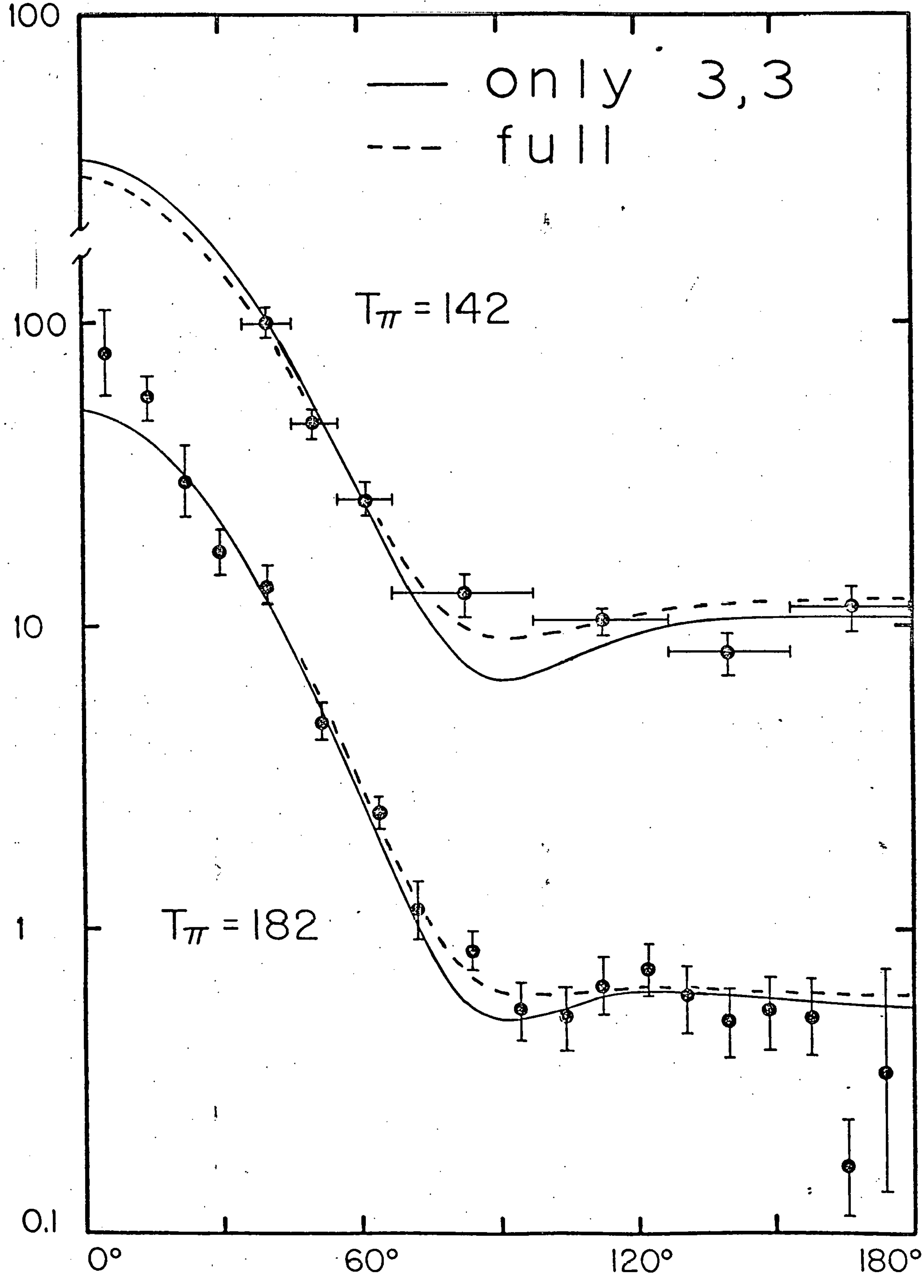
Fig. 2. Differential cross sections in the center of mass system, for pion laboratory kinetic energies of 142 and 182 MeV. The dashed lines are the result of the full calculation and the solid lines the result considering only the P_{33} channel for the pion-nucleon interaction. The experimental data are from Refs. 28 and 29.

Fig. 3. Differential cross sections in the center of mass system, for pion laboratory kinetic energies of 230 and 256 MeV. The dashed lines are the result of the full calculation and the solid lines the result considering only the P_{33} channel for the pion-nucleon interaction. The experimental data are from Refs. 15 and 30.

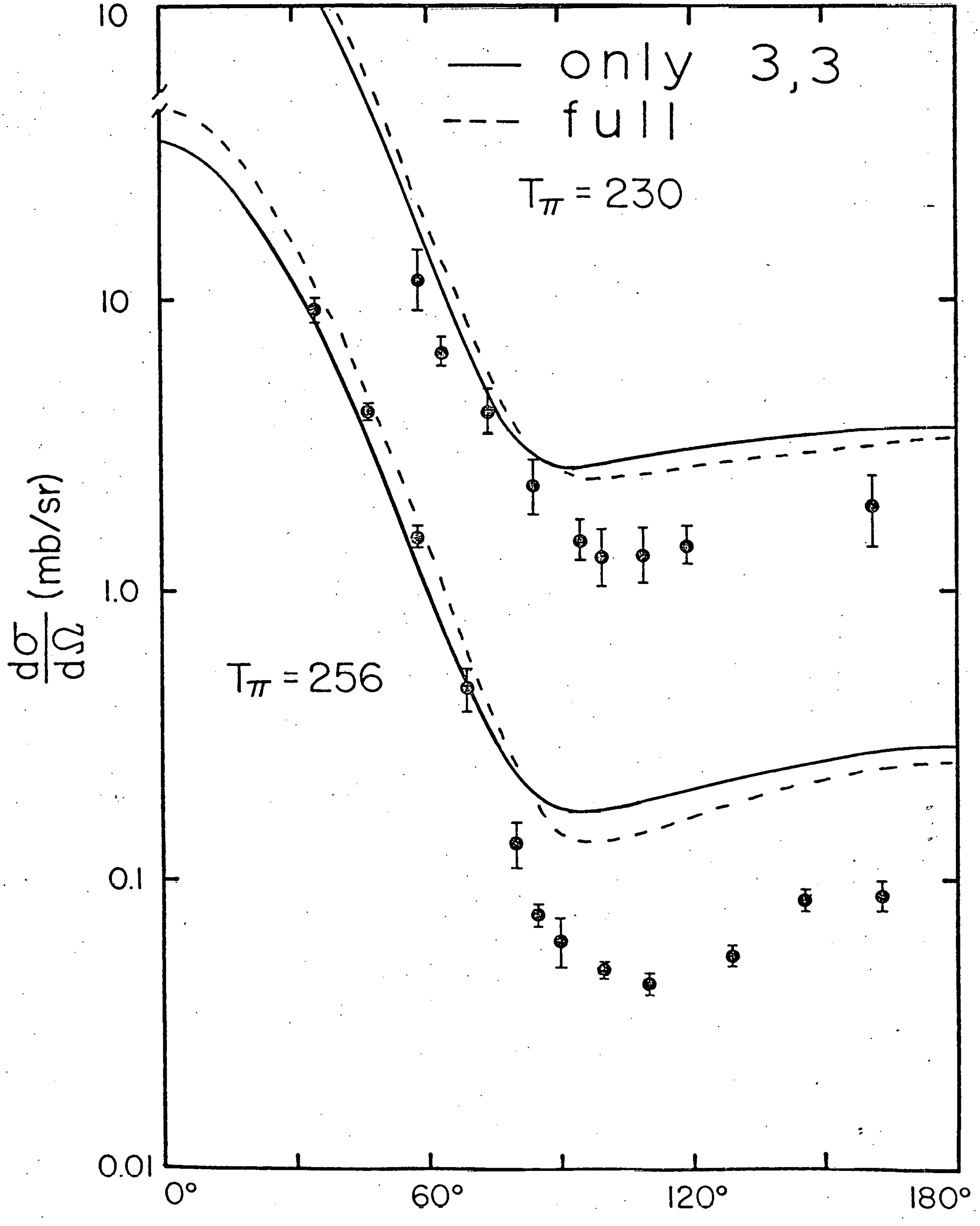
Fig. 4. Differential cross sections in the center of mass system, for pion laboratory kinetic energies of 323, 417 and 512 MeV. The dashed lines are the result of the full calculation and the solid lines the result considering only the P_{33} channel for the pion-nucleon interaction. The experimental data are from Ref. 15.



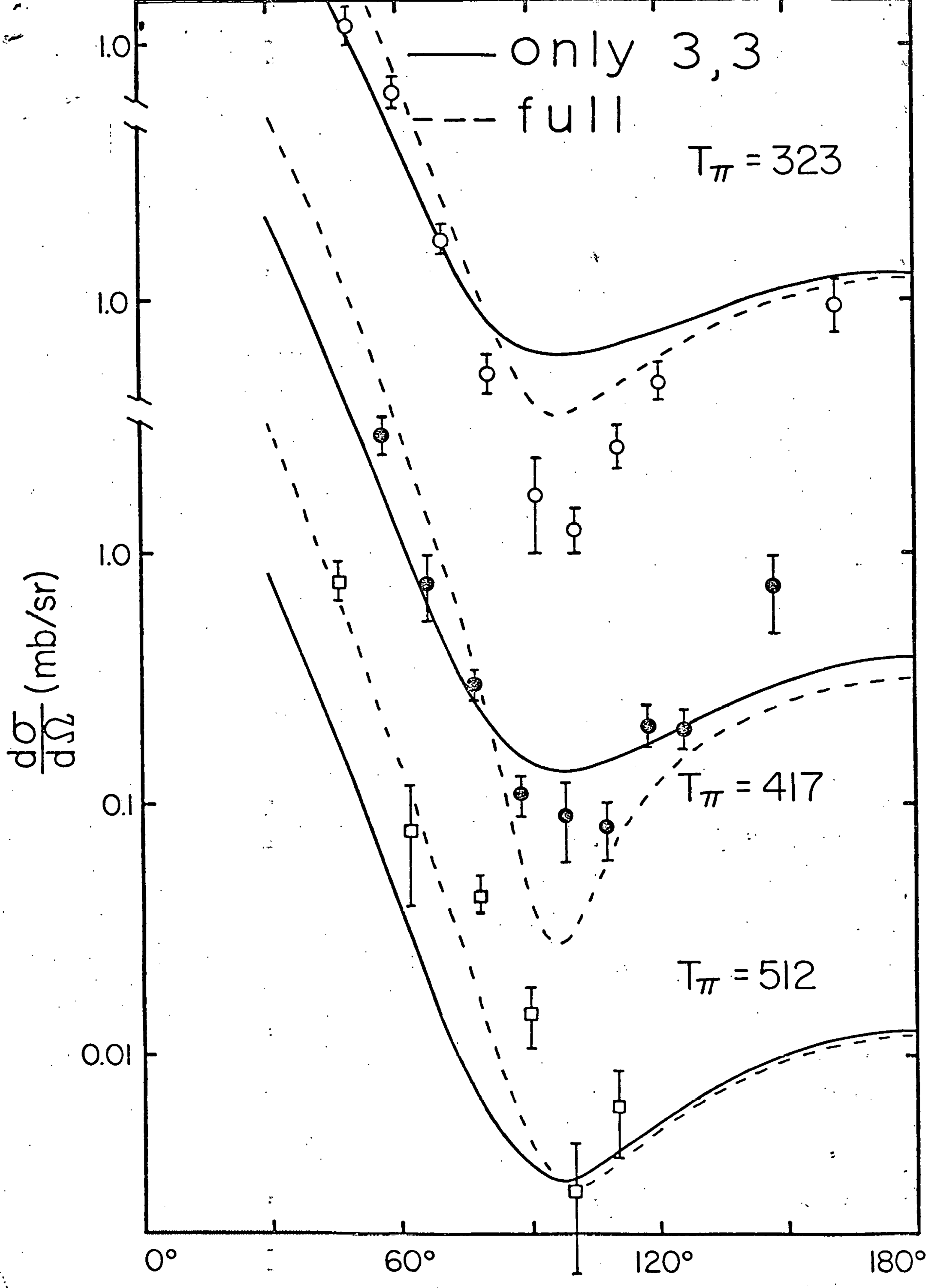
1



only 3,3
full
$T_\pi = 142$
$T_\pi = 182$
100
100
10
1
0.1
$\frac{d\sigma}{d\Omega}$ (mb/sr)
0°
60°
120°
180°



10
10
1.0
0.1
0.01
0°
60°
120°
180°
$\frac{d\sigma}{d\Omega}$ (mb/sr)
only 3,3
full
T_π = 230
T_π = 256


only 3,3
full
T_π = 323
T_π = 417
T_π = 512
1.0
1.0
1.0
0.1
0.01
$\frac{d\sigma}{d\Omega}$ (mb/sr)
0°
60°
120°
180°

11