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A FINITE ELEMENT FORMULATION OF THE TWO-DIMENSIONAL NONLINEAR INVERSE HEAT CONDUCTION PROBLEM*

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ABSTRACT

The calculation of the surface temperature and surface heat flux from measured temperature transients at one or more interior points of a body is identified in the literature as the inverse heat conduction problem. Heretofore, analytical and computational methods of treating this problem have been limited to one-dimensional nonlinear or two-dimensional linear material models. This paper presents, to the authors' knowledge, the first inverse solution technique applicable to the two-dimensional nonlinear model with temperature-dependent thermophysical properties. This technique, representing an extension of the one-dimensional formulation previously developed by one of the authors, utilizes a finite element heat conduction model and a generalization of Beck's one-dimensional nonlinear estimation procedure. The formulation is applied to the cross section of a composite cylinder with temperature-dependent material properties. Results are presented to demonstrate that the inverse formulation is capable of successfully treating experimental data. An important feature of the method is that small time steps are permitted while avoiding severe oscillations or numerical instabilities due to experimental errors in measured data.

NOMENCLATURE

- a = Radius of cylindrical rod
- $[A]$ = Matrix, as defined in equation (33)
- $[B]$ = Matrix, equation (10)
- $[C]$ = Heat capacity matrix for assembly of elements
- $\{D\}$ = Vector, equation (34)
- c = Specific heat
- e = Index of elements
- E = Number of elements in assembly
- $\{\bar{F}\}$ = Vector for assembly of elements, equation (11)
- $\{\bar{\bar{F}}\}$ = Vector for assembly of elements, equation (12)
- h = Convective heat transfer coefficient
- h^r = Radiative heat transfer coefficient, equation (5)
- J = Number of time steps in analysis interval; $J - 1$ equals number of "future" temperatures
- $[K]$ = Thermal conductivity matrix for assembly of elements
- k = Thermal conductivity
- L = Number of interior temperature probes; also, number of nodes in surface heat flux interpolation
- M = Number of nodes in temperature interpolation
- N_I = Interpolation function for temperature
- $\{N\}$ = Vector of interpolation functions for temperature
- $\underline{n}$ = Unit outward normal to boundary surface
- Q = Internal heat generation rate, per unit volume
- q = Imposed surface heat flux
- q_ℓ = Value of surface heat flux at ℓ^{th} node, equation (22)
- $\{q\}$ = Surface heat flux vector of dimension L, equation (22)
- q^c = Surface heat flux due to convection
- q^r = Surface heat flux due to radiation
- R_ℓ = Interpolation function for surface heat flux
- $\{R\}$ = Vector of interpolation functions for surface heat flux
- r = Radial coordinate
- r^p = Radial coordinate of temperature probe location
- $[S]$ = Matrix, equation (17)
- T = Temperature
- T_I = Value of temperature at I^{th} node
- $\{T\}$ = Temperature vector
- T_ℓ^p = Measured temperature at internal point (r^p, θ_ℓ)
- T^{ac} = Temperature at which no convection occurs
- T^{ar} = Temperature at which no radiation occurs
- T^w = Temperature at wall
- t = Time
- TOL1 = Convergence tolerance for temperature vector, equation (19)
- TOL2 = Convergence tolerance for surface heat flux, equation (35)
- w_j = Weighting functions, equation (27)
- $\underline{x}$ = General spatial coordinates

<u>Greek Symbols</u>

- α = Thermal diffusivity
- ρ = Density
- Ω = General spatial domain
- Ω^e = Element domain
- Γ_1 = Boundary on which condition (2) is prescribed

Γ_2 = Boundary on which condition (3) is prescribed

Γ_2^e = Element external boundary on which condition (3) is prescribed

∇ = Gradient operator

σ = Stefan-Boltzmann constant

ε = Emissivity

β = Surface heat flux parameter, equation (25)

Δ = Incremental change in kernel

θ = Angular coordinate

θ_ℓ = Angular coordinate of ℓ^{th} temperature probe and related surface heat flux node

ϕ = Sensitivity coefficient, equation (30)

λ = Perturbation factor for q_ℓ, equation (30)

τ = Dimensionless time

$\sum$ = Summation symbol

Subscripts

I = Index of nodes

i = Index of time steps in solution, where i is a nonnegative integer

j = Index of time steps in analysis interval, $1 \leq j \leq J$

k,ℓ,m = Index of nodes for temperature probes and surface heat flux nodes

$(i)\Delta t$ or $;(i)\Delta t$ = Time $t = (i)\Delta t$ at which kernel is evaluated

Superscripts

$[\ ]^T$ = Transpose of matrix

$\{\ \}$ = Row vector

(P),(H) = Iteration number at which kernel is evaluated

Other Symbols

$[\ ]$ = Matrix

$\{\ \}$ = Column vector

$||\ ||$ = Euclidean norm

$\int$ = Integral sign

INTRODUCTION

In heat transfer studies, a class of problems can be identified where the surface temperature and surface heat flux are determined from the temperature history measured at a set of discrete points in the interior of the body. Generally, this class is referred to in the literature as the inverse problem, in contrast with the usual direct formulation where the interior temperature history is determined from specified initial and boundary conditions. Typically, the inverse formulation arises in experimental studies where direct measurement of surface conditions is not feasible, such as convective heat transfer in rocket nozzles or quenching processes for materials. An application presented in this paper treats an electrically heated composite rod with two-phase flow boundary conditions. Temperature transients recorded by thermocouple probes in the rod are used to investigate the time-history of surface conditions. Because these probes are positioned in the interior of the rod to avoid disturbing surface conditions and the flow adjacent to the surface, an inverse problem must be solved.

Various methods that have been applied to the inverse problem include integral equation solutions, series solutions, transform solutions, and function minimization techniques. Extensive bibliographies that survey these methods are readily available in the literature (see, for example, (1) and (2)); the limited number of references mentioned here deal with materially nonlinear or multidimensional inverse formulations. Heretofore, analytical and computational methods for treating the nonlinear inverse problem of temperature-dependent thermophysical properties have been restricted to one-dimensional models. Beck (2,3) has developed a nonlinear formulation based on a finite difference heat conduction model and nonlinear estimation procedures. Muzzy et al. (4) and Bass (5) have applied Beck's method, with some modifications, to one-dimensional composite models with temperature-dependent material properties. Other nonlinear formulations include a finite difference technique developed by Ott and Hedrick (6) and a transform method by Imber (1). Apparently, the only two-dimensional inverse formulation appearing in the open literature is that of Imber (7,8). His transform technique is applicable to two-dimensional geometries of arbitrary shape, but assumes a linear material model with constant properties.

This paper presents, to the authors' knowledge, the first inverse solution technique applicable to the two-dimensional nonlinear model with temperature-dependent properties. This technique, representing an extension of the one-dimensional formulation previously developed by Bass (5), utilizes a finite element heat conduction model and a generalization of Beck's one-dimensional nonlinear estimation procedure. The computational technique assumes several thermocouple sensors judiciously positioned in the interior of the material body. In the formulation, the unknown surface heat flux is discretized on the boundary domain of the body using a prescribed set of nodal points and suitable interpolating functions. Because the temperature response at interior locations is delayed and damped with respect to changes in surface conditions, these nodal point values of surface heat flux are determined in a given time step with a procedure that utilizes interior temperatures at "future" times. Specifically, the nodal values of flux are assumed to be constant or to vary piecewise linearly over an analysis interval that consists of several time steps in the discretized data. The coefficients that describe the nodal values are adjusted iteratively to achieve the closest agreement in a least squares sense with the input "future" temperatures over the analysis interval. The discretized approximation of the surface heat flux thus determined provides a conventional boundary condition for the forward problem in the next time step. The inverse solution computed in this way represents a "best approximation" in the finite dimensional subspace of solutions defined by the surface heat flux interpolation. An important feature of the method is that small time steps are permitted while avoiding severe oscillations or numerical instabilities due to experimental errors in measured data.

The formulation is applied to the cross section of a composite cylinder with temperature-dependent material properties. To evaluate the performance of the technique in solving the inverse problem, a standard initial-boundary value solution, with a known surface heat flux, is used as input for the inverse calculation. The computed surface heat flux is compared with the (known) imposed heat flux for two different thermocouple configurations. Finally, the technique is applied to experimentally determined temperature transients recorded at interior points of an electrically heated cylinder used to simulate a nuclear fuel rod in reactor loss-of-coolant analyses.

FINITE ELEMENT FORMULATION OF THE DIRECT PROBLEM

The conduction of heat in the region Ω is governed by the quasilinear parabolic equation

$$\nabla \cdot (k\nabla T) + Q = \rho c \frac{\partial T}{\partial t} \tag{1}$$

subject to the boundary conditions

$$T = T^{w} \text{ on } \Gamma_1 \tag{2}$$

and

$$k\nabla T \cdot \underline{n} + q + q^{r} + q^{c} = 0 \text{ on } \Gamma_2 \quad . \tag{3}$$

The heat flow rates per unit area on convection and radiation boundaries are written

$$q^{c} = h(T - T^{ac}) \quad , \quad q^{r} = h^{r}(T - T^{ar}) \quad , \tag{4}$$

where h^{r} is defined by

$$h^{r} = \varepsilon\sigma(T^2 + T^{ar^2})\ (T + T^{ar}) \quad . \tag{5}$$

In general, k, c, h, and h^{r} are temperature and spatially dependent, while Q and q are time and spatially dependent.

Let the region Ω be partitioned by a system of finite elements and let the unknown temperature T be approximated throughout the solution domain at any time t by

$$T(\underline{x},t) = \sum_{I=1}^{M} N_I(\underline{x})\ T_I(t) = \{N\}^T \{T\} \quad . \tag{6}$$

Here the N_I are the interpolation functions defined piecewise element by element and the T_I or $\{T\}$ are the nodal temperatures. The governing equations of the discretized system can be derived by minimizing a functional or by using Galerkin's method (9). In the Galerkin formulation employed here, the problem is recast in a weighted integral form using the interpolating functions N_I as the weighting functions:

$$\int_{\Omega^e} \{N\}[\nabla \cdot (k\nabla(\{N\}^T \{T\})) + Q - \rho c \frac{\partial}{\partial t} (\{N\}^T \{T\})]\ d\Omega$$

$$- \oint_{\Gamma_2{}^e} \{N\}[k\nabla(\{N\}^T \{T\}) \cdot \underline{n} + q + h(\{N\}^T \{T\} - T^{ac})$$

$$+ h^{r}(\{N\}^T \{T\} - T^{ar})]\ d\Gamma = 0 \quad . \tag{7}$$

Only a single finite element is considered in the integral (7), as the governing equations of the complete system of elements are obtained by assembling the individual finite element matrices. The surface integral over $\Gamma_2{}^e$ refers only to those elements with external boundaries on which condition (3) is given.

Green's first identity is applied to the first volume integral of equation (7) so that the second derivatives do not impose unnecessary continuity conditions between elements. When use is made of the boundary conditions (2) and (3), the integral formulation (7) leads to a set of transient ordinary differential equations for the assemblage of finite elements:

$$[C] \frac{\partial\{T\}}{\partial t} + [K]\ \{T\} + \{\bar{F}\} + \{\bar{\bar{F}}\} = 0 \quad . \tag{8}$$

The components in equation (8) are defined by:

$$[C] = \sum_{e=1}^{E} \int_{\Omega^e} \rho c\{N\}\ \{N\}^T\ d\Omega \quad , \tag{9}$$

$$[K] = \sum_{e=1}^{E} \int_{\Omega^e} k[B]\ [B]^T\ d\Omega$$

$$+ \sum_{e=1}^{E} \oint_{\Gamma_2{}^e} (h + h^{r})\ \{N\}\ \{N\}^T\ d\Gamma \quad ,$$

$$[B] = \nabla\{N\} \quad , \tag{10}$$

$$\{\bar{F}\} = - \sum_{e=1}^{E} \int_{\Omega^e} \{N\}\ Q d\Omega + \sum_{e=1}^{E} \oint_{\Gamma_2{}^e} \{N\}\ q d\Gamma \quad , \tag{11}$$

$$\{\bar{\bar{F}}\} = - \sum_{e=1}^{E} \oint_{\Gamma_2{}^e} \{N\}\ (h^{r}\ T^{ar} + h\ T^{ac})\ d\Gamma \quad , \tag{12}$$

where the summations are taken over the individual finite element contributions. These integrals are evaluated numerically using Gauss-Legendre quadrature in the applications to be presented later.

The system of nonlinear equations (8) through (12) which defines the discretized problem can be solved using many different types of integration schemes. The implicit one-step Euler backward difference method is employed in this analysis. The time derivative of the temperature is approximated by

$$\frac{\partial\{T\}}{\partial t} \cong \frac{\{T\}_{(i+1)\Delta t} - \{T\}_{(i)\Delta t}}{\Delta t} = \frac{\{\Delta T\}}{\Delta t} \quad , \tag{13}$$

where $\{T\}_{(i)\Delta t}$ is assumed known at time $(i)\Delta t$. In the nonlinear analysis, $\{T\}_{(i+1)\Delta t}$ is calculated using a computational scheme that iterates on the out-of-balance heat flow rate for a given time step. At time $(i+1)\Delta t$, the initial approximation of the increment $\{\Delta T\}^{(0)}$ in nodal point temperatures is calculated by

$$(\frac{1}{\Delta t}\ [C]_{(i)\Delta t} + [K]_{(i)\Delta t})\ \{\Delta T\}^{(0)} =$$

$$- [K]_{(i)\Delta t}\ \{T\}_{(i)\Delta t}$$

$$- \{\bar{F}\}_{(i+1)\Delta t} - \{\bar{\bar{F}}\}_{(i)\Delta t} \quad . \tag{14}$$

In each iteration, a new temperature increment is computed from

$$\{\Delta T\}^{(P)} = \{\Delta T\}^{(P-1)} + \{\delta T\}^{(P)} \quad , \tag{15}$$

where $\{\delta T\}^{(P)}$ is the $(P)^{th}$ correction to the temperature increment $\{\Delta T\}$. The expression for computing the correction $\{\delta T\}^{(P)}$ is determined by substituting (15) into (13) and using (8) in the form

$$[S]^{(P-1)}_{(i+1)\Delta t} \{\delta T\}^{(P)} = - \Big[[K]^{(P-1)}_{(i+1)\Delta t} \{T\}^{(P-1)}_{(i+1)\Delta t} + \frac{1}{\Delta t} [C]^{(P-1)}_{(i+1)\Delta t} \{\Delta T\}^{(P-1)} + \{\overline{F}\}_{(i+1)\Delta t} + \{\overline{\overline{F}}\}^{(P-1)}_{(i+1)\Delta t} \Big] \quad , \tag{16}$$

where

$$[S]^{(P-1)}_{(i+1)\Delta t} = \frac{1}{\Delta t} [C]^{(P-1)}_{(i+1)\Delta t} + [K]^{(P-1)}_{(i+1)\Delta t} \tag{17}$$

is evaluated using temperatures

$$\{T\}^{(P-1)}_{(i+1)\Delta t} = \{T\}_{(i)\Delta t} + \{\Delta T\}^{(P-1)} \quad . \tag{18}$$

The iteration continues until convergence is obtained according to the criterion

$$||\{\delta T\}^{(P)}|| \ / \ ||\{T\}^{(P)}_{(i+1)\Delta t}|| < \text{TOL1} \quad , \tag{19}$$

where TOL1 represents an adjustable tolerance.

Equations (14) through (19) constitute the full Newton iterative solution of the governing system of equations (8). To avoid the undesirable computational expense of updating and factorizing the effective stiffnexx matrix $[S]^{(P-1)}_{(i+1)\Delta t}$ in each iteration, the applications presented in this paper make use of the modified Newton-Raphson scheme. In this method, a new tangent stiffness matrix $[S]_{(n)\Delta t}$ is computed periodically from one of the converged solutions at time $(n)\Delta t$, n=0,1, 2,... i, and used in place of $[S]^{(P-1)}_{(i+1)\Delta t}$ in equation (16). Because the matrix $[S]_{(n)\Delta t}$ is held fixed in a given time step, this modified method involves fewer stiffness reformations than full Newton iteration. The frequency of the stiffness updates can be adjusted according to the degree of nonlinearity in the computational model to avoid an excessive number of iterative corrections.

This application of the finite element method to the inverse heat conduction problem considers a two-dimensional model of the (r,θ) cross section of a circular cylinder. An isoparametric (10) discretization is employed, so that the spatial coordinates are interpolated using the same functions N_I as those used for T in equation (6). The N_I associated with the 4- to 8-noded two-dimensional isoparametric element are described in numerous references, including (10) and (11), and will not be given here.

FORMULATION OF THE INVERSE PROBLEM

In this study, the two-dimensional problem of a cylindrical body subjected to a planar surface heat flux $q(\theta,t)$ is considered as depicted in Figure 1. The conditions

$$T(r^p,\theta_\ell,t) = T^p_\ell(t) \qquad 0 \le t \le \hat{t} \quad , \quad \ell = 1,L \tag{20}$$

are prescribed at L equally spaced interior points along a contour of radius r^p near the surface, while the surface heat flux function

$$-k\left(\frac{\partial T}{\partial r}\right)_{r=a} = q(\theta,t) \tag{21}$$

is unknown. The problem is to determine $q(\theta,t)$ and the temperature distribution $T(r,\theta,t)$, $0 \le r \le a$, $0 \le \theta \le 2\pi$, on a specified time domain. Although a circular geometry is assumed here, the basic technique described below for treating the inverse problem is applicable to other geometric shapes with a multiple number of thermocouple sensors judiciously positioned near the surface of the body.

In his treatment of the linear inverse problem, Imber (7) indicates that a successful extrapolation procedure requires the temperature distribution to be known, a priori, throughout a closed region within the body. For a one-dimensional axisymmetric analysis of a cylinder such as that depicted in Figure 1, the temperature can be determined in the closed region $r \le r^p < a$ using data from a single thermocouple sensor positioned at radius r^p. The two-dimensional analog achieved by relaxing the condition of axisymmetry then presumes a time-history of temperature data recorded pointwise on a closed contour of radius r^p, i.e., a "line-source" of temperature data. Because such a volume of measured data would not be available in any realistic experimental program, the technique described below is based on a limited number of thermocouple sensors discretely positioned on a contour near the surface of the body. Numerical examples presented in the next section illustrate that the accuracy of the technique in approximating the flux boundary condition is improved as the number of temperature sensors per unit arc length on the contour is increased.

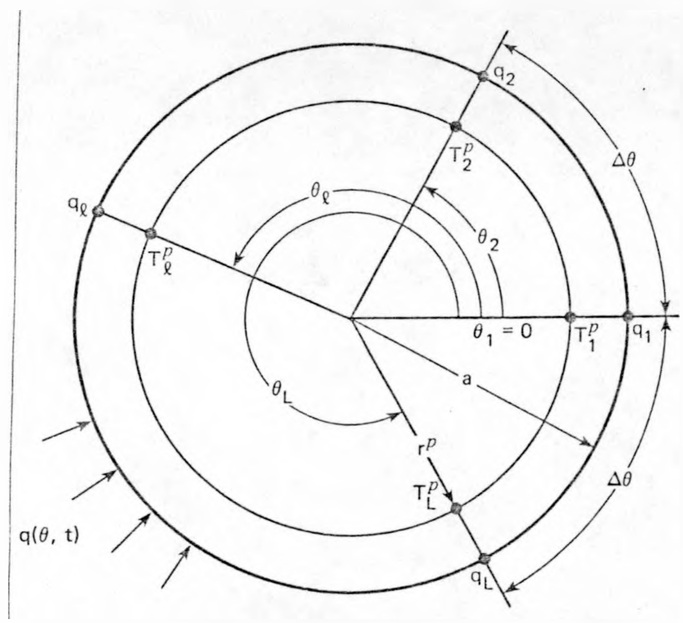

Fig. 1 Cross section of heated cylinder

The initial step in the development of the method is the discretization of the unknown surface heat flux on the boundary domain using a set of nodal values q_ℓ, $\ell = 1,L$, and suitable interpolating functions R_ℓ (to be specified later), as depicted in Figure 1. Thus, the approximation of the surface heat flux q is given by

$$q(\theta,t) = \sum_{\ell=1}^{L} R_\ell(\theta)\, q_\ell(t) = \{R\}^T \{q\} \quad . \tag{22}$$

One surface flux node is designated for each active thermocouple sensor and positioned at the minimum distance from the sensor node. Numerical tests have indicated that this geometric arrangement produces a stable, well conditioned system of equations for approximating the boundary heat flux function q.

In addition, the nodal values of surface heat flux will be temporally discretized such that in a given time step Δt, $q(\theta,t)$ is represented by

$$q(\theta,t) = \sum_{\ell=1}^{L} R_\ell(\theta)\, q_{\ell;(i)\Delta t} = \{R\}^T \{q\}_{(i)\Delta t}$$

$$(i-1)\Delta t < t \leq (i)\Delta t \qquad i \geq 1 \quad . \tag{23}$$

For a given $i \geq 1$, it is assumed that $\{q\}_{(1)\Delta t}$, $\{q\}_{(2)\Delta t}$, ..., $\{q\}_{(i)\Delta t}$ are known. To determine $\{q\}_{(i+1)\Delta t}$, an analysis interval of $J \geq 1$ time steps is selected, as depicted in Figure 2.[1] In the next step, $\{q\}$ is estimated over the analysis interval $(i)\Delta t < t \leq (i+J)\Delta t$ using relations that take the trend of q into account. For the first time step in the interval,

$$\{q\}_{(i+1)\Delta t} = \{q\}_{(i)\Delta t} + (\{q\}_{(i)\Delta t} - \{q\}_{(i-1)\Delta t}) \tag{24}$$

and for the "future" time steps

$$\{q\}_{(i+j)\Delta t} = \{q\}_{(i+j-1)\Delta t} + \beta(\{q\}_{(i+j-1)\Delta t} - \{q\}_{(i+j-2)\Delta t}) \tag{25}$$

for $2 \leq j \leq J$, where $0 \leq \beta \leq 1$ is an adjustable parameter.[2] Thus, the interpolated boundary conditions can be estimated for each time step in the analysis interval according to the relation

$$q(\theta)_{(i+j)\Delta t} = \{R\}^T \{q\}_{(i+j)\Delta t} \qquad 1 \leq j \leq J \quad . \tag{26}$$

Then the boundary value problem (equations (1) through (5)) cast in the discretized finite element formulation (equations (8) through (12)) is solved over the analysis interval $(i)\Delta t < t \leq (i+J)\Delta t$ using conditions (24) - (26) in the surface integral of equation (11).

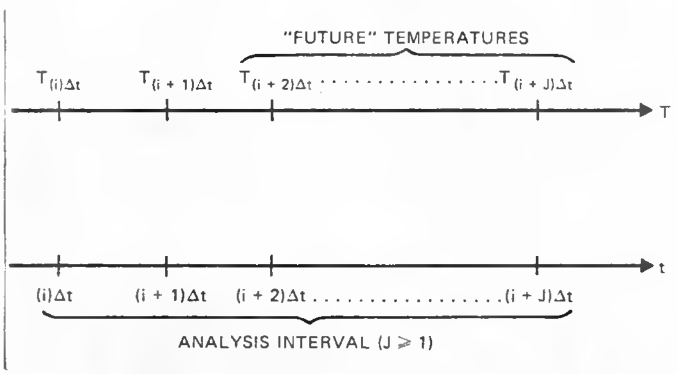

Fig. 2 Analysis interval for computing surface heat flux q

The objective of the method is to select $\{q\}_{(i+1)\Delta t}$ to achieve the closest agreement in a least squares sense between the computed and input thermocouple temperatures over the analysis interval. This is accomplished by minimizing the weighted sum of squares function

$$f(\{q\}_{(i+1)\Delta t}) = \sum_{j=1}^{J} w_j \sum_{\ell=1}^{L} (T_{\ell;(i+j)\Delta t} - T^p_{\ell;(i+j)\Delta t})^2$$

$$= \sum_{j=1}^{J} w_j \{T - T^p\}^T_{(i+j)\Delta t} \{T - T^p\}_{(i+j)\Delta t} \tag{27}$$

with respect to the L nodal parameters represented by the array $\{q\}_{(i+1)\Delta t}$. In equation (27), $\{T\}$ and $\{T^p\}$ are the computed and input temperatures at the interior thermocouple locations (r^p,θ_ℓ), $\ell = 1,L$. The weighting functions defined by $w_j = j^2$ were suggested by Muzzy et al. (4) in a one-dimensional finite difference application of Beck's method.[3]

The minimization procedure for the function f of (27) is based on an iterative technique that is a generalization of Beck's one-dimensional formulation. For the $(H)^{th}$ iterative correction $\{\Delta q\}^{(H)}$ to the minimizing nodal parameters $\{q\}_{(i+1)\Delta t}$, the elements of the temperature array $\{T\}^{(H)}_{(i+j)\Delta t}$ in (27) are approximated by a truncated Taylor series expansion

$$T^{(H)}_{\ell;(i+j)\Delta t} \simeq T^{(H-1)}_{\ell;(i+j)\Delta t} + \sum_{k=1}^{L} \frac{\partial T^{(H-1)}_{\ell;(i+j)\Delta t}}{\partial q^{(H)}_{k;(i+1)\Delta t}} \Delta q^{(H)}_k \quad ,$$

$$\ell = 1,L \tag{28}$$

where

$$\Delta q^{(H)}_k = q^{(H)}_{k;(i+1)\Delta t} - q^{(H-1)}_{k;(i+1)\Delta t} \qquad k = 1,L \quad . \tag{29}$$

If the heat conduction model is linear, this expression is exact and no iteration is required. The partial derivatives in (28), referred to as sensitivity coefficients, are approximated numerically according to the expression

$${}^j\phi^{(H-1)}_{\ell k} = \frac{\partial T^{(H-1)}_{\ell;(i+j)\Delta t}}{\partial q^{(H)}_{k;(i+1)\Delta t}} \qquad \begin{array}{l} \ell,k = 1,L \\ j = 1,J \end{array}$$

$$\approx \frac{T_{\ell;(i+j)\Delta t}\left(\{q^*\}^{(H-1)}_{(i+1)\Delta t}\right) - T_{\ell;(i+j)\Delta t}\left(\{q\}^{(H-1)}_{(i+1)\Delta t}\right)}{\lambda q^{(H-1)}_{k;(i+1)\Delta t}} \quad , \tag{30}$$

where $\{q^*\}$ is obtained from $\{q\}$ by perturbing the k^{th} component, i.e., $q^*_k = (1+\lambda)q_k$ and $q^*_m = q_m$, $m \neq k$. A value of $\lambda = 1 \times 10^{-3}$ is used in the present study.

In each iterative correction to $\{q\}_{(i+1)\Delta t}$, (J)(1+L) conventional solutions of the finite element heat conduction model (equations (8) - (12)) are required to compute the array (30) of sensitivity coefficients. With the ϕ's thus determined, the

[1] For elementary one-dimensional models with characteristic dimension a, Beck (2) recommends values of J that are appropriate for given values of the dimensionless time step $\Delta\tau = \alpha\Delta t/a^2$.

[2] $\{q\}_0$ is determined from conditions at the initial time.

[3] Beck's one-dimensional formulation uses $w_j = 1$ for all j.

extremizing condition

$$\frac{\partial f}{\partial \{q\}_{(i+1)\Delta t}} = \underline{0} \qquad (31)$$

is used to compute the incremental correction. When (28) is substituted into (27) and the differentiation (31) is performed, the $(H)^{th}$ correction $\{\Delta q\}^{(H)}$ is determined from the expression

$$[A]^{(H-1)} \{\Delta q\}^{(H)} = \{D\}^{(H-1)} \quad , \qquad (32)$$

where the components are given by

$$A_{\ell k}^{(H-1)} = \sum_{j=1}^{J} w_j \sum_{m=1}^{L} {}^j\phi_{m\ell}^{(H-1)} \, {}^j\phi_{mk}^{(H-1)} \quad , \qquad (33)$$

$$D_{\ell}^{(H-1)} = \sum_{j=1}^{J} w_j \sum_{k=1}^{L} (T^{p}_{k;(i+j)\Delta t} - T^{(H-1)}_{k;(i+j)\Delta t}) \, {}^j\phi_{k\ell}^{(H-1)} \quad . \qquad (34)$$

The correction (32) is then used to update the nodal array $\{q\}^{(H-1)}_{(i+1)\Delta t}$ of surface flux according to equation (29). Generally, the iteration is continued until convergence is achieved according to the criterion

$$||\{\Delta q\}^{(H)}|| \; / \; ||\{q\}^{(H)}_{(i+1)\Delta t}|| < TOL2 \qquad (35)$$

for some prescribed tolerance TOL2 > 0.

The discretized approximation of the surface heat flux

$$q(\theta)_{(i+1)\Delta t} = \{R\}^T \{q\}_{(i+1)\Delta t} \qquad (36)$$

thus determined provides a conventional boundary condition (for equation (11)) in the next single time step Δt only. The analysis interval is then shifted by one time step and the process is repeated.

In the numerical applications of this technique in the next section, the surface flux interpolating functions $\{R\}$ have the form

$$R_\ell(\theta) = \begin{cases} \cos^2[\frac{\pi}{2}\frac{(\theta - \theta_\ell)}{\Delta\theta}] & \theta_\ell - \Delta\theta \leq \theta \leq \theta_\ell + \Delta\theta \\ 0 & \text{otherwise} \end{cases} \qquad (37)$$

for $\ell = 1,L$. The functions (37) depicted in Figure 3 have the properties

$$R_\ell(\theta_k) = \begin{cases} 1 & \ell = k \\ 0 & \ell \neq k \end{cases} \qquad \ell,k = 1,L \qquad (38)$$

$$R_\ell(\theta) + R_{\ell+1}(\theta) = 1 \qquad \theta_\ell \leq \theta \leq \theta_{\ell+1}$$
$$(R_{L+1} \equiv R_1 \; , \; \theta_{L+1} \equiv \theta_1) \quad . \qquad (39)$$

It follows from equation (39) that the interpolation (22) can represent a uniform surface heat flux.

NUMERICAL APPLICATIONS

The two-dimensional inverse formulation developed in the preceding section is applied here to a composite rod containing an electric heating element and

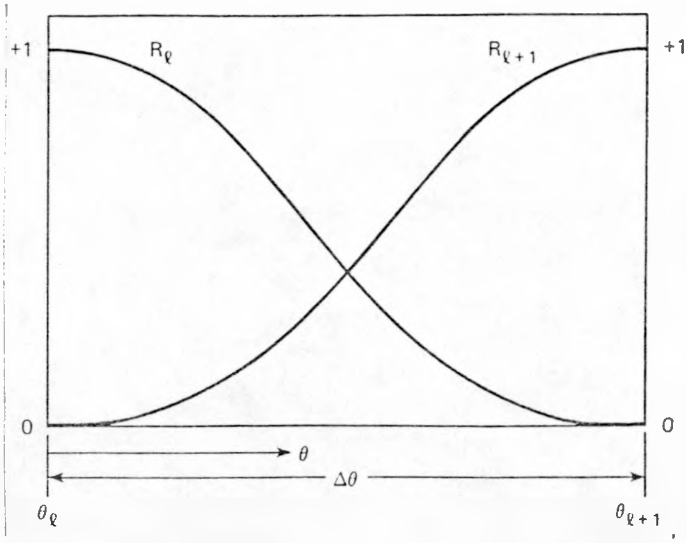

Fig. 3 Interpolating functions for surface heat flux

thermocouple sensors. This heater rod represents one member of a rod array that is designed for test purposes to simulate a nuclear fuel bundle. The heater rod bundle is positioned in a thermal-hydraulics test loop that is used to study hypothetical loss-of-coolant accidents in pressurized-water nuclear reactors (12).[4]

A heater rod cross section and the corresponding two-dimensional finite element discretization used in the inverse analysis are depicted in Figures 4 and 5. The rod has a nominal heated length of 366 cm (144 in.) and is constructed with a stainless steel outer sheath. Attached to the inner surface of this sheath at equal intervals are twelve chromel-alumel thermocouple assemblies, 0.05 cm (0.02 in.) in diameter. Four additional sensors are positioned in the center of the rod. Only four of the sixteen thermocouples actively record data in the cross section of Figure 4, namely the three boron nitride (BN)-filled thermocouples attached to the outer sheath and one of the center rod thermocouples; the junctions of the remaining thermocouples are positioned in different axial planes of the rod. Boron nitride is used as a filler and an insulator between the inconel heating element and the thermocouple assemblies. In the finite element model of the heater rod (Figure 5), each thermocouple at the outer sheath is modeled with two quadrilateral elements that are assigned the appropriate material properties of BN or MgO and the same total cross sectional area as the *in situ* circular sheaths. Those in the center of the rod are not used to drive the inverse computation and are not included in the finite element discretization.

The thermophysical properties of thermal conductivity k and specific heat c are temperature dependent for each material in the rod. Except for the thermal conductivities of MgO and BN, these properties are determined for each material as a function of temperature from an optimum polynomial fit to available data, as given in Reference (6). The thermal conductivities for the MgO and BN depend on packing density and must be determined *in situ* as part of the rod calibration procedure (13) prior to each test.

[4]This test facility is operated by the Oak Ridge National Laboratory (ORNL) Pressurized-Water Reactor Blowdown Heat Transfer Separate-Effects Program, which is part of the overall light-water reactor safety research program of the Nuclear Regulatory Commission.

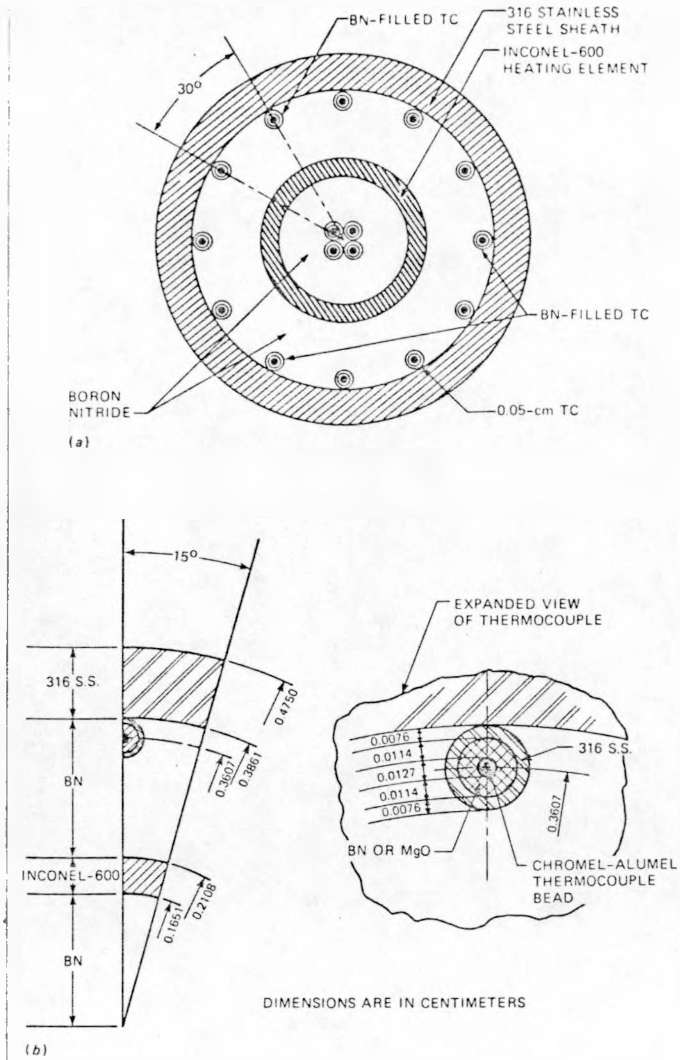

Fig. 4 Electrically heated rod with interior thermocouple sensors

(a) cross section
(b) dimensions

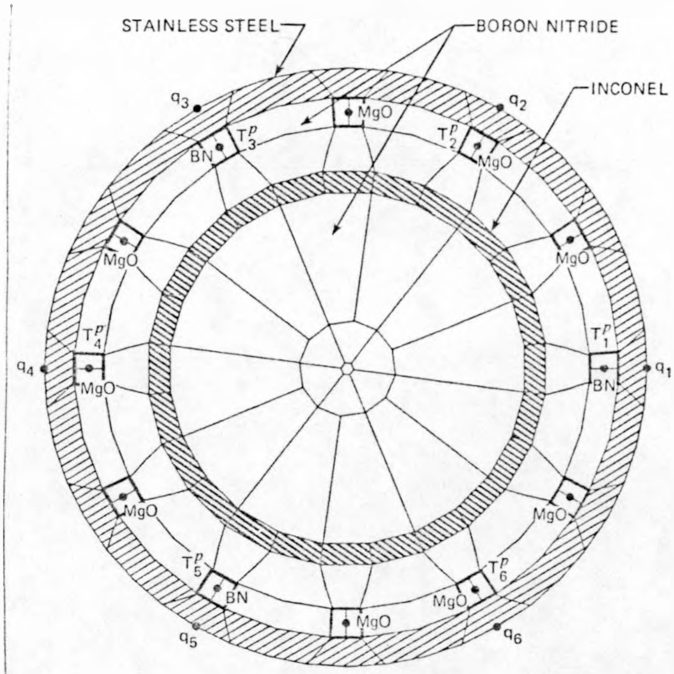

Fig. 5 Two-dimensional finite element model of heater rod cross section: 126 elements; 288 nodes

The first numerical example[5] was selected to evaluate the performance of the technique in solving the inverse problem for the finite element model of Figure 5. A standard initial-boundary value solution was obtained from the finite element formulation (8) - (12) using the prescribed surface heat flux function

$$q(\theta,t) = 47.31 + 126.2[\sin(\tfrac{1}{2}\{\theta - 2\pi t\})]^{2+12t} \ \mathrm{W/cm^2} , \quad 0 \le t \le 1.0 \qquad (40)$$

a constant heat generation rate $Q = 5274\ \mathrm{W/cm^3}$, a time step $\Delta t = .01$ s, and initial center rod temperature $T_{center} = 441.2$ C. From this direct solution, the temperature transients of Figure 6 were calculated

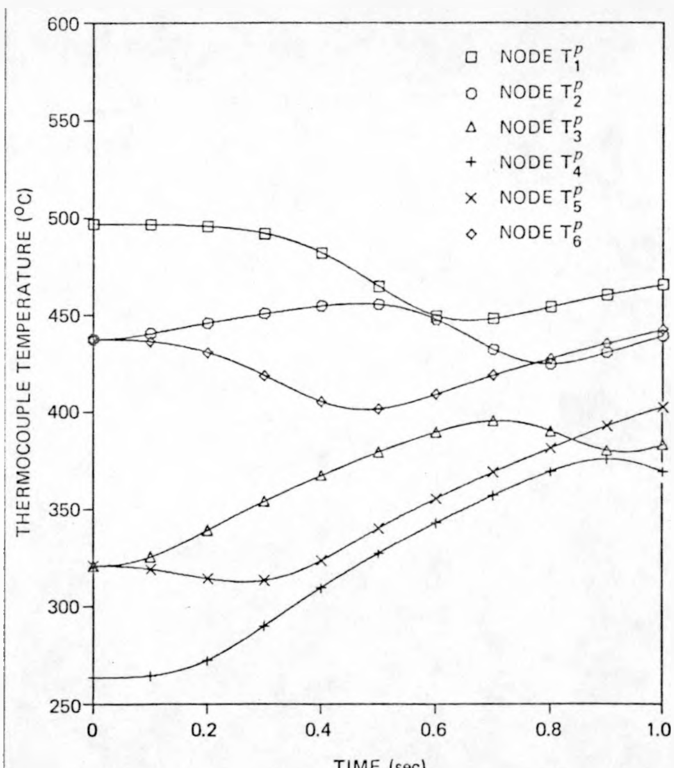

Fig. 6 Test case: calculated temperatures at thermocouple locations from direct solution

[5]The inverse calculations presented in this section were performed using TOL1 = .001, equation (19); $\beta = 0.5$, equation (25); TOL2 = 0.05, equation (35).

at the thermocouple locations 1 through 6 of the discrete model (Figure 5). With the thermocouple transients of Figure 6 serving as input, two different inverse analyses were performed in an attempt to reproduce the surface flux boundary condition (40). The first analysis utilized input data from only three of the thermocouples (L = 3), those numbered 1, 3, and 5 in Figure 5, while the second utilized data from six thermocouples (L = 6). The analysis interval consisted of only one time step (J = 1). Results from the two inverse analyses are compared with the known direct solution in Figures 7 and 8 at different times. Throughout the transient, the inverse analysis using six active thermocouples consistently produced a good approximation of both the surface flux function (40) and the surface temperatures. As illustrated in Figure 7, the solution using three active thermocouples was not as successful in approximating the surface variables at those times when the localized perturbation in the surface flux was not "near" an active sensor. This example demonstrates that, within practical limits, the prediction of surface conditions is improved as the number of thermocouple sensors per unit length of contour is increased.

In the second numerical example, the inverse formulation is applied to actual thermocouple transients taken from a representative test of ORNL's single-rod test apparatus.[6] The heater power input to the rod during the period of the transient considered here is essentially constant at Q = 5300 W/cm^3. Figure 9 illustrates the time-history of the thermocouple temperatures recorded by the active BN-filled sensors (1, 3, and 5 in Figure 5) and by the one active center thermocouple. The acquisition interval for these data

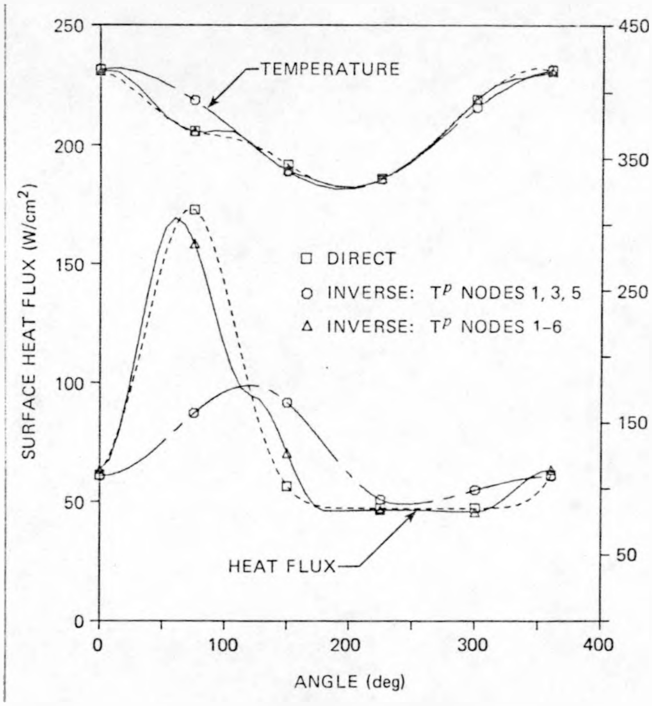

Fig. 7 Test case: comparison of direct solution with inverse solutions at time 0.7 s

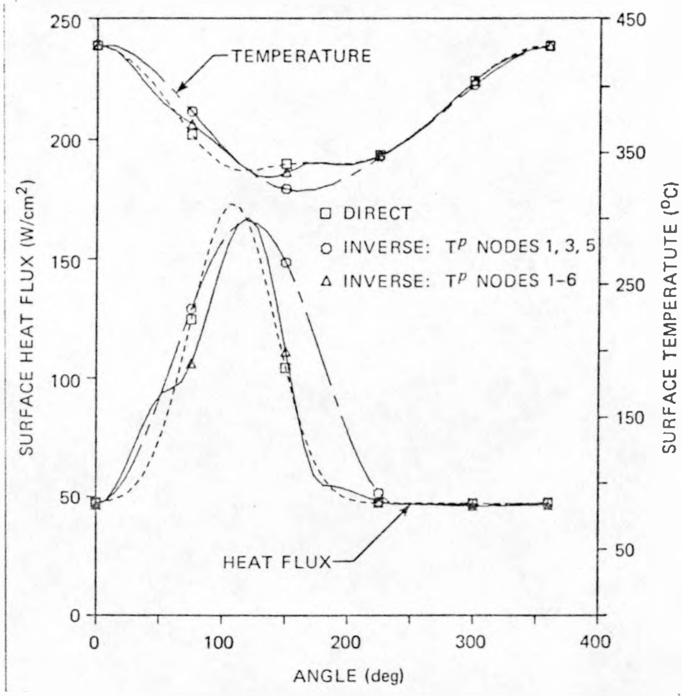

Fig. 8 Test case: comparison of direct solution with inverse solutions at time 0.8 s

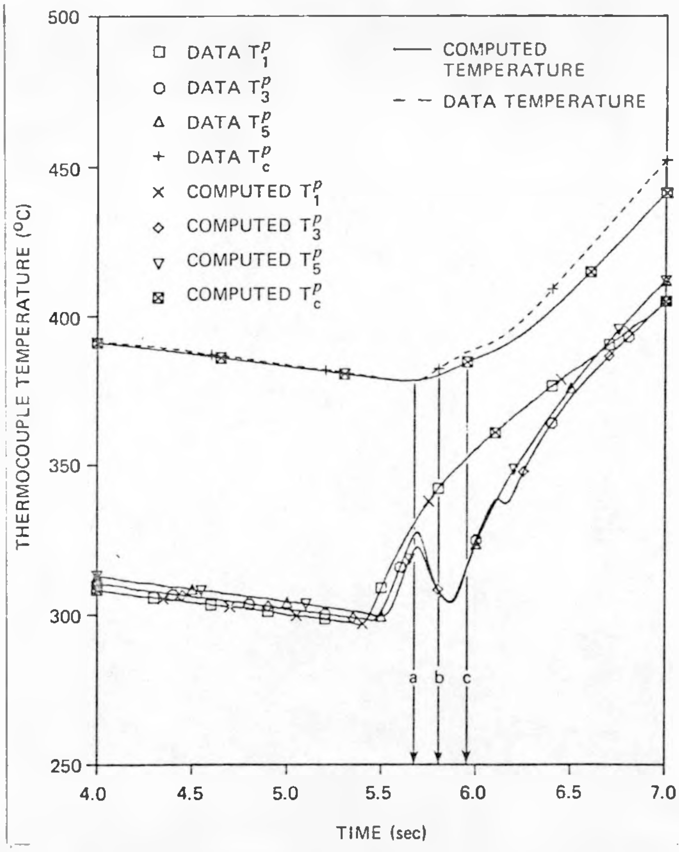

Fig. 9 Experimental case: comparison of inverse solution (Δt=0.01 s and J=2) with measured temperature data

[6] The single-rod test facility (12) at ORNL is used primarily to qualify heaters for the large rod bundle loop and to obtain blowdown heat transfer results for a single rod in an annular geometry.

is $\Delta t = 0.01$ s. In the same figure, the temperatures computed at the BN thermocouple locations in an inverse solution (for $\Delta t = 0.01$ s and J = 2) are compared with the input data; the error is not discernible on the scale of these plots. Because the center thermocouple data are not used in the inverse computation, comparison of these data with the computed center rod temperatures permits an evaluation of the rod finite element model. This comparison can be only approximate due to uncertainty in the precise orientation of the center thermocouple assembly and to the absence of appropriate material modeling of the assembly in the discretization. Agreement between the measured and computed values is generally good, although a slight divergence appears at time $t \approx 5.75$ s when gradients and time rates of temperature become pronounced in the center of the rod.

Figures 10 - 12 illustrate the computed time-history of surface conditions at node q_3 in Figure 5 for a time step $\Delta t = 0.01$ s and three different analysis intervals. The results for one time step in the analysis interval, J = 1 (Figure 10), indicate that the measured data of Figure 6 require the use of future temperatures to reduce oscillations in the computed values. The solution using two time steps, J = 2 (Figure 11), removes much of the "noise" from the flux time-history without severe rounding of rapid changes that begin at time $t \approx 5.5$ s. The results for J = 3 (Figure 12) lead to additional smoothing of the solution and illustrate the tendency to "round off" rapid changes as J is increased. For the finite element model of Figure 5 and a selected time step of $\Delta t = .01$ s, the use of one future temperature appears optimal for reducing oscillations. Figure 13 compares the surface conditions at time $t = 5.85$ s for the above three solutions.

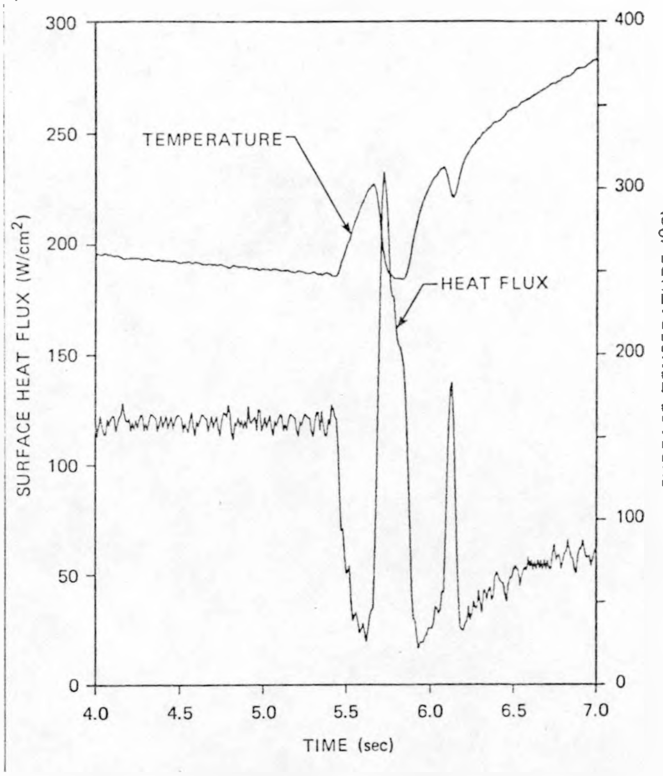

Fig. 11 Experimental case: inverse solution at surface node q_3 using two time steps in analysis interval (J=2)

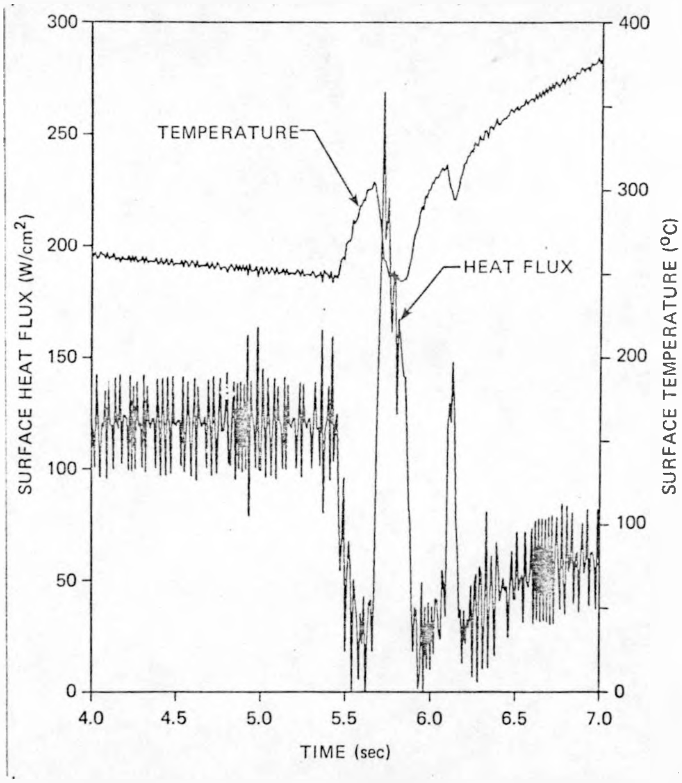

Fig. 10 Experimental case: inverse solution at surface node q_3 using one time step in analysis interval (J=1)

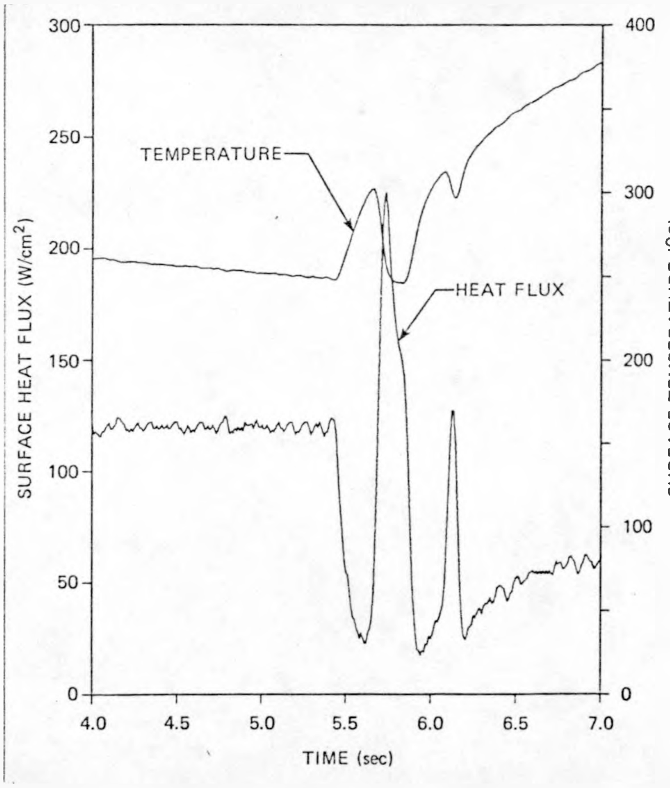

Fig. 12 Experimental case: inverse solution at surface node q_3 using three time steps in analysis interval (J=3)

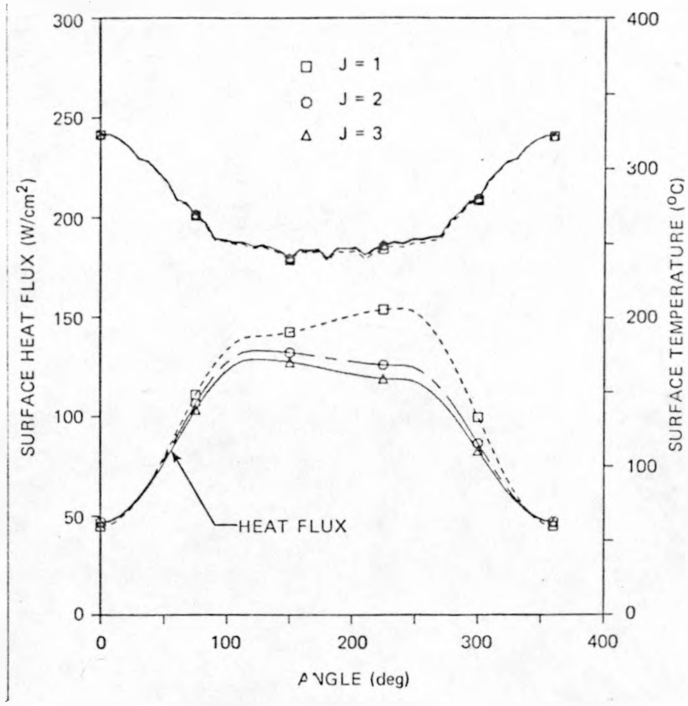

Fig. 13 Experimental case: surface conditions at time 5.85 s

In the experimental apparatus that produced the thermocouple transients of Figure 9, the heater rod surface is exposed to a transient two-phase flow that is primarily parallel to the rod axis. At point (a) in Figure 9 (time t = 5.65 s), the entire surface of the rod cross section has departed from nucleate boiling; at point (b) (time t = 5.80 s), part of the surface experiences a "rewet" with an accompanying drop in temperature; at (c) (time t = 5.95 s) the entire surface is in transition to film boiling. Contour plots in Figure 14 illustrate the change in temperature distribution for the cross section of the rod during this portion of the transient.

SUMMARY AND CONCLUDING REMARKS

This paper has presented a two-dimensional formulation of the inverse heat conduction problem that is applicable to composite bodies with temperature-dependent thermophysical properties. The formulation utilizes a finite element heat conduction model and a generalization of Beck's one-dimensional nonlinear estimation procedure. Applications of the inverse technique to an electrically heated composite rod were examined in the study. In the first example, a conventional initial-boundary value solution, with a known surface heat flux, was used as input for the inverse calculation. The computed surface heat flux was compared with the imposed heat flux for two different thermocouple configurations. These comparisons indicate that, within practical limits, the approximation of surface conditions is improved as the number of thermocouple sensors per unit length of contour is increased. Finally, the technique was applied to experimentally determined temperature transients recorded at thermocouple sensors in the interior of the rod. The results presented here demonstrate that the inverse formulation is capable of successfully treating experimental data. Consideration of future temperatures in calculating surface conditions permits the use of small time steps while avoiding severe oscillations or numerical instabilities due to errors in measured data.

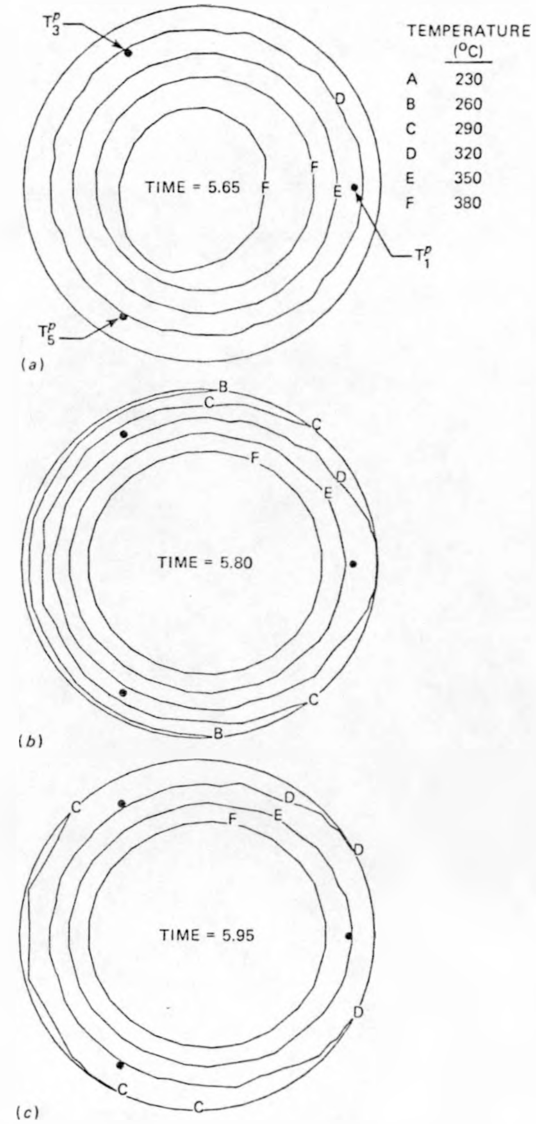

Fig. 14 Experimental case: temperature contours (deg C)

(a) time 5.65 s
(b) time 5.80 s
(c) time 5.95 s

ACKNOWLEDGMENTS

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