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SKYRMIONS AND VECTOR MESONS: A SYMMETRIC APPROACH†

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ABSTRACT

We propose an extension of the effective, low-energy chiral Lagrangian known as the Skyrme model, to one formulated by a non-linear sigma model generalized to include vector mesons in a symmetric way. The model is based on chiral SU(6) × SU(6) symmetry spontaneously broken to static SU(6). The ρ and other vector mesons are "dormant" Goldstone bosons since they are in the same SU(6) multiplet as the pion and other pseudoscalars. Hence the manifold of our generalized non-linear sigma model is the coset space (SU(6) × SU(6))/SU(6). Relatavistic effects, via a spin-dependent mass term, break the static SU(6) and give the vectors a mass. The model can then be fully relativistic and covariant. The lowest-lying Skyrmion in this model is the whole baryonic 56-plet, which splits into the octet and decuplet in the presence of relativistic SU(6)-breaking. Due to the built-in SU(6) and the presence of vector mesons, the model is expected to have better phenomenological results, as well as providing a conceptually more unified picture of mesons and baryons.

I. INTRODUCTION

Although one believes that quantum chromodynamics (QCD) is the theory of the strong interactions, because it cannot (yet) be solved exactly there is an important role, especially in low-energy physics, for phenomenological, effective Lagrangians. Indeed one hopes that eventually it will be possible to derive the correct, effective, low-

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energy Lagrangian from QCD in much the same way that the Landau-Ginsberg equations are derived from the BCS theory of superconductivity. Until then we must content ourselves with guessing at the best effective Lagrangian, but of course using both the guidance from the experimental facts and their summarization in phenomenology on the one hand and whatever hints we can gather from QCD itself on the other, to help us in making an intelligent guess. In this pursuit, interest has recently refocused on a model written down over twenty years ago by Skyrme[1]. It is the standard chiral SU(2) × SU(2) non-linear sigma model of Goldstone pions, with the addition of a quartic term to stablize the solitons, the classical solutions of the model which have come to be called Skyrmions.

In order to establish notation and conventions, but without giving many of the details[2], we may write the Lagrangian using the group currents

$$L_\mu(x) = U^{-1}\partial_\mu U \ , \tag{1}$$

as

$$\mathscr{L} = \frac{f_\pi^2}{4}\,\mathrm{Tr}(L_\mu L_\mu) + \frac{1}{8e^2}\,\mathrm{Tr}([L_\mu,L_\nu]) \tag{2}$$

where $f_\pi = 93$ MeV. U(x) is a field transforming as a non-linear realization of the chiral group (originally with Skyrme SU(2) × SU(2), but more recently SU(3) × SU(3)) with the constraint that

$$UU^\dagger = I, \tag{3}$$

that is, U is unitary. For the case of two flavours, for example, we may parametrize U(x) as a unit quaternionic field, $U(x) = \phi^0 + i\vec{\tau}\cdot\vec{\phi}$, with $\phi^2 = 1$ and $\vec{\phi}$ being the triplet of pion fields. U(x) here takes values on the non-linear manifold

$$M = \frac{SU(2) \times SU(2)}{SU(2)} \approx SO(4)/SO(3) \approx S^3,$$

the 3-sphere. The static soliton solution comes from Skyrme's hedgehog ansatz:

$$U(x) = \exp[i\ \vec{\tau}\cdot\hat{x}\ F(r)] \tag{4}$$

with the boundary conditions $F(0) = \pi$, $F(\infty) = 0$.

As Witten has shown[3], with the addition of another term to (2), the Wess-Zumino term[4] which describes the effects of non-Abelian anomalies, one can demonstrate that the Skyrmion is a fermion for the number of colours N_c odd, and in particular $N_c = 3$. This with previous arguments going back to Skyrme establishes that the Skyrmion is a baryon.

Studies[5] investigating the phenomenological consequences of the Skyrme model have found results generally within about 30% of experimental values. This suggests that something important is missing, and included in that something must surely be the vector mesons. There are a number of reasons for believing that vector mesons should be included in the model. To begin with one knows that ρ's and ω's, as well as ϕ's and K^*'s, are necessary to give a good description of low-energy phenomenology and nuclear physics. One can phrase this in another way by noting that since the Skyrme model is a non-renormalizable, effective model requiring a cut-off, and that this cut-off is on the order of a typical baryon mass, say 1-2 GeV, then one had better include the ρ's and other vector mesons which are below this cut-off if one hopes to describe the physics in this energy regime accurately.

Another argument for including vectors comes from QCD in the large N limit. It is known[6,7] that as $N \to \infty$, QCD reduces to a theory of fr mesons, with interactions appearing in $O(1/N)$. But it is not just spin-0 mesons which appear, but higher spins as well, all on the same footing. Furthermore, there are various arguments[7] to suggest that baryons appear as solitons in this large N effective field theory that would be obtained from QCD. Since this as yet not completely known, large N effective field theory is supposed to reduce at low energy to the Skyrme-type, non-linear sigma model, it behooves us again to include the vector mesons in our model.

Motivated by not totally dissimilar reasoning, a number of authors have already attempted to incorporate vector mesons in the chiral model. The Syracuse group[8] has followed the traditional method going back to Sakurai[9] of adding spin-1 mesons to the chiral Lagrangian by using "covariant" derivative couplings as if the spin-1 fields were gauge fields. However, mass terms for the vector and axial vectors are, as expected, also included. Although they were really not concerned with the Skyrme model as such in that they did not study the Skyrmion-baryons, they did include the Wess-Zumino term. In such a framework they have been able to obtain some nice results, especially concerning ω decays. However, there are some subtleties concerning the explicit chiral-symmetry breaking which results from their using Bardeen's form[10] of the non-Abelian anomaly (i.e. non-anomalous vector currents). We would like to argue that despite the successes of this traditional approach, there are conceptual difficulties with including vector mesons in this way for reasons, which will be discussed below, concerning static SU(6).

Other, less complete attempts to add vector mesons include the work of Adkins and Nappi[11], in which only the ω was coupled to the baryonic current. While such a treatment eliminated the need for a higher derivative term to stablize the Skyrmion, only slight improvement of the phenomenological results was found. Finally a group at Ohio State[12] has considered vectors, but only in final state interactions, in a model with the Wess-Zumino term (again not really the Skyrmion situation); so their treatment does not really include vector mesons in the effective Lagrangian.

What will be described here is an approach to the problem which is essentially different from those mentioned above; indeed one might say it is an orthogonal treatment. It is based on a very different picture of the vector mesons which was developed some time ago by the author and H. Pagels[13]. This description of the vector mesons keeps the π's (really, all the pseudoscalars) and the ρ's (all the vectors) as much as possible on the same footing. The motivation for this comes from static SU(6), the relevant highlights of which we will briefly review below in Section II. This will lead us to a consideration of the ρ-π puzzle and our solution in which the ρ (along with the other vector mesons in the nonet) emerges as a "dormant" Goldstone boson of spontaneously broken chiral $U(6) \times U(6)$. (The pion and its pseudoscalar octet partners are, of course, also Goldstone states; indeed they are the remaining states of the whole Goldstone supermultiplet.) The term dormant Goldstone boson is used since a spin-1 state can be a true

Goldstone state only in a nonrelativistic theory. Relativistic effects break the static SU(6) symmetry and give the ρ a mass. (Throughout this paper ρ will often, depending on the context, be used generically for the whole vector meson nonet, likewise for π, mutatis mutandis.)

This scenario is realized in a generalized linear sigma model, reviewed in Section II. Its consequences in a relativistic framework, i.e. in QCD or at least in its progenitor, the quark model, are also discussed in Section II, where it is shown that how the ρ remembers it is a dormant Goldstone boson is by being in a $(3,\bar{3}) + (\bar{3},3)$ representation of chiral SU(3) × SU(3), the same chiral representation content as the pion. This leads to an understanding of vector-meson dominance (VMD) as a consequence of spontaneous symmetry breaking, just as for PCAC.

Having reviewed this description of vector mesons, we are able to present the model in Section III, first pointing out that the Skyrme model is an ideal framework to employ this picture of ρ's since it already has a number of aspects of a static model. We then give our formulation first of the nonrelativistic, generalized non-linear sigma model on the coset space (SU(6) × SU(6))/SU(6). We discuss the existence of the static, Skyrme-type solutions and the Wess-Zumino-type term leading to the generalized Skyrmion being quantized as a fermion, and so being the whole baryonic 56-plet. We then consider the addition of a spin-dependent mass term which breaks SU(6) and enables us to formulate a fully relativistic, covariant model. Finally in Section IV we draw some conclusions, point out the considerable work remaining to be done on the model, and also discuss the relevance for this whole approach of a simplified, one-flavour version of the model.

II. SU(6) AND THE ρ–π PUZZLE

A. The Static SU(6) Group

It is well appreciated, though perhaps by now somewhat hazily remembered, that the symmetry which results from combining internal SU(3) flavour symmetry with SU(2) of spin, namely static SU(6) symmetry, leads to a rather successful description of many features of hadronic phenomenology. Work on this symmetry goes back to the mid-60's and is associated with Gürsey and Radicati, Pais, and Sakita[14]. As there exist a number of excellent reviews[15], we will be extremely brief here. SU(6) is an approximate, dynamic, non-relativistic (exact really only in the static limit) symmetry of the quark model and so of its field theoretic embodiment, QCD. It is very successful in providing: i) a classification of hadrons into static SU(6) × O(3) supermultiplets, and ii) relations among magnetic moments, masses, mixing angles, decay rates, etc. We might note that for simplicity one may limit the discussion to static SU(4) (originating with Wigner[15] in 1937) which mixes spin and isospin, but for studying phenomenological applications SU(6) leads to much better results. At any rate what these static symmetries do is to mix an internal symmetry with a spatial symmetry. (One should note that it is not just a direct-product group like SU(3) × SU(2) that we are concerned with, otherwise there would be no need for all the members of a supermultiplet to have the same parity.)

Because of this mixing of spatial and internal symmetries, it was quickly realized that SU(6) would run into troubles with relativity. An easy wasy to see the difficulty is to note that a Lorentz boost only affects the spin part (mixing spin and orbital angular momentum), while the internal group is Lorentz invariant. There were actually a number of no-go theorems proven which forbid a relativistic version of SU(6). (Of course, later it was shown how to have a relativistic theory which mixes spatial and internal symmetries, namely supersymmetry, which avoids the no-go theorems by being based on graded Lie algebras.) So one knows that SU(6) is an approximate, dynamical symmetry much like the approximate, dynamical spin SU(2) group used in Russell-Saunders coupling in atoms. Nevertheless, because SU(6) does appear as an approximate symmetry of nature, it is important to recover it in the static limit, i.e. in the rest frame of a particle, where the particle has no orbital angular momentum.

B. The ρ-π Puzzle

Presumably by now almost everyone (certainly everyone at this Workshop) believes that the pion (etc.) is a Nambu-Goldstone boson of spontaneously broken SU(3) × SU(3) chiral symmetry. But the ρ meson (and its partners) is in the same static SU(6)-quark model (QCD) supermultiplet as the pion: the 35-plet. The ρ-π puzzle is how to reconcile this; or how is PCAC compatible with the quark model? The answer which Pagels and I gave[13] is that the ρ is a "dormant" Goldstone boson. In the static, nonrelativistic limit it is a Goldstone state which then becomes massive in a relativistic theory which of necessity breaks SU(6).

To see the puzzle and our solution a bit more quantitatively, we note that under chiral SU(3) × SU(3) the pseudoscalars transform like $(3,\bar{3}) + (\bar{3},3)$, that is in terms of quark bilinears, like $\pi^a \sim \bar{q}i\gamma_5(\lambda^a/2)q$. However, conventional vectors, i.e. $\rho^a_\mu \sim \bar{q}\gamma_\mu(\lambda^a/2)q$, transform like (1,8) + (8,1). This is difficult to reconcile with static SU(6). Instead we assume that like the π's the ρ's also belong to a $(3,\bar{3}) + (\bar{3},3)$ representation of chiral SU(3) × SU(3). This then requires that the components of the vector-meson field operators be part of an antisymmetric tensor quark-bilinear operator, i.e. $\bar{q}\,\sigma_{\mu\nu}(\lambda^a/2)q \equiv t^a_{\mu\nu}$. Then the phenomenological vector-meson field is projected out by

$$\rho^a_\nu(x) = \frac{\partial^\mu t^a_{\mu\nu}(x)}{m_\rho Z_\rho^{1/2}} , \tag{5}$$

with the automatic consequence that

$$\partial^\nu \rho^a_\nu = 0 , \tag{6}$$

so that there are only three independent components as desired for a massive spin-1 field. (In the static ($\vec{k}=0$) limit, or perhaps better in the ρ's rest frame, $\rho^a_i \sim \bar{q}\,\sigma_{oi}(\lambda^a/2)q$.)

From the dual of the antisymmetric tensor field,

$${}^*t^a_{\mu\nu} = 1/2\ \varepsilon_{\mu\nu\lambda\delta}\ t^{a\lambda\delta} = \bar{q}\ i\gamma_5\ \sigma_{\mu\nu}\ (\lambda^a/2)q,$$

one projects out the phenomenological field of the chiral partner of the ρ meson in this picture, namely the B(1235), $J^{PC} = 1^{+-}$ axial vector meson, by

$$B^a_\nu = \frac{\partial_\mu \ {}^*t^a_{\mu\nu}}{m_B \ Z_B^{1/2}} \tag{7a}$$

with

$$\partial_\nu \ B^a_\nu = 0 \quad . \tag{7b}$$

This is a nice feature of our description of the spin-1 mesons since the B is a well-established resonance, while the chiral partner of the ρ in the traditional picture, namely the A_1 axial-vector meson, $J^{PC} = 1^{++}$, if one believes that it has finally been found, appears to have too high a mass to satisfy the Weinberg sum rule prediction. For details the reader is referred to our original papers[13]. Perhaps it should be remarked here that this chiral representation assignment of the spin-1 meson makes clear the role of the antisymmetric tensor bilinear operators, a subject which has been of interest of late.

C. Chiral U(6) x U(6)

In order to make the ensuing discussion more intelligible, it is probably useful to remind ourselves which algebra we are discussing. There are 144 Hermitian currents which can be formed from quark bilinears with 3 flavours:

$$V^a_\mu = \bar{q} \ \gamma_\mu \frac{\lambda^a}{2} q, \qquad A^a_\mu = \bar{q} \ \gamma_\mu q_5 \frac{\lambda^a}{2} q \ ,$$

$$S^a = \bar{q} \frac{\lambda^a}{2} q, \quad P^a = \bar{q} \ i\gamma_5 \frac{\lambda^a}{2} q, \quad T^a_{\mu\nu} = \bar{q} \ \sigma_{\mu\nu} \frac{\lambda^a}{2} q, \tag{8}$$

$$a = 0, \ \ldots \ 8 .$$

The charges $Q^A = \int d^3x \ J^A(x,t)$ defined from these currents, naively using the canonical commutation relations for the quark fields, close on the algebra U(12). Focusing just on V^a_μ and A^a_μ, we note that chiral chiral U(3) x U(3) is generated by the charges

$$Q^a = \int d^3x \ V^a_0 \ , \quad {}^5Q^a = \int d^3x \ A^a_0 \quad , \tag{9}$$

and

$$[Q^a, \ Q^b] = i \ f^{abc} \ Q^c \tag{10a}$$

$$[Q^a, {}^5Q^b] = i f^{abc}\, {}^5Q^c \tag{10b}$$

$$[{}^5Q^a, {}^5Q^b] = i f^{abc}\, Q^c \quad . \tag{10c}$$

If we now also consider the charges defined from the spatial components of V^a_μ and A^a_μ:

$$Q^a_i = \int d^3x\, V^a_i \ , \qquad {}^5Q^a_i = \int d^3x\, A^a_i \ , \tag{11}$$

then the charges (9) and (11) together close on the chiral U(6) × U(6) algebra of Feynman,Gell-Mann and Zweig[16]. This is given by relations (10) plus

$$[Q^a, Q^b_i] = i f^{abc}\, Q^c_i \tag{12a}$$

$$[{}^5Q^a, Q^b_i] = i f^{abc}\, {}^5Q^c_i \tag{12b}$$

$$[Q^a, {}^5Q^b_i] = i f^{abc}\, {}^5Q^c_i \tag{12c}$$

$$[{}^5Q^a, {}^5Q^b_i] = i f^{abc}\, Q^c_i \tag{12d}$$

$$[Q^a_i, Q^b_j] = i\, \delta_{ij} f^{abc}\, Q^c - i\, \varepsilon_{ijk} d^{abc}\, {}^5Q^c_k \tag{12e}$$

$$[{}^5Q^a_i, {}^5Q^b_j] = i\, \delta_{ij} f^{abc}\, Q^c - i\, \varepsilon_{ijk} d^{abc}\, {}^5Q^c_k \tag{12f}$$

$$[Q^a_i, {}^5Q^b_j] = i\, \delta_{ij} f^{abc}\, {}^5Q^c - i\, \varepsilon_{ijk} d^{abc}\, Q^c_k \ . \tag{12g}$$

The U(6) subalgebra generated by Q^a and ${}^5Q_i{}^a$ with commutation relations given by (10a), (12c), and (12f) corresponds to the static U(6) group.

D. The Generalized Linear Sigma Model

We have explicitly realized our description of the vector mesons as dormant Goldstone bosons by considering a generalization of the linear sigma model to the chiral U(6) × U(6) symmetry we have just reviewed. The reason for picking this group is that if one wants to extend the U(3) × U(3) σ model to include U(6), then the smallest group which contains these groups and closes is U(6) × U(6). Of course, due to the apparently insurmountable difficulties of achieving a relativistic SU(6) symmetry, the model at first is only a static model, but it still has Galilean invariance.

The following notation is useful. The charges which generate chiral U(6) ×U(6) obey the commutation relations

$$[{}_\alpha Q^A, {}_\alpha Q^B] = i F^{ABC} {}_\alpha Q^C,$$

$$[{}_\beta Q^A, {}_\alpha Q^B] = i F^{ABC} {}_\beta Q^C,$$

$$[{}_\beta Q^A, {}_\beta Q^B] = i F^{ABC} {}_\alpha Q^C , \qquad (13)$$

where A = a or ai, and a = 0, ...,8, i = 1,...,3. The generators Λ_A of static U(6) obey

$$\mathrm{Tr}\, \Lambda_A \Lambda_B = 2\, \delta_{AB} ,$$

$$[\Lambda_A, \Lambda_B] = 2\, i\, F_{ABC}\, \Lambda_C ,$$

$$\{\Lambda_A, \Lambda_B\} = 2\, D_{ABC}\, \Lambda_C,$$

$$F_{abc} = \left(\tfrac{1}{2}\right)^{1/2} f_{abc} , \quad D_{abc} = \left(\tfrac{1}{2}\right)^{1/2} d_{abc}$$

$$F_{ai,bj,ck} = \left(\tfrac{1}{2}\right)^{1/2} \varepsilon_{ijk}\, d_{abc} + \left(\tfrac{1}{2}\right)^{1/2} \delta_{ij}\, f_{abc}$$

$$D_{ai,bj,ck} = \left(\tfrac{1}{2}\right)^{1/2} \varepsilon_{ijk}\, f_{abc} + \left(\tfrac{1}{2}\right)^{1/2} \delta_{ij}\, d_{abc} , \qquad (14)$$

where f_{abc} and d_{abc} are the usual SU(3) coefficients, and $d_{oaa} = (2/3)^{1/2}$.

The mesons are classified into two 36-plets of static U(6). Using Cartesian coordinates we identify an odd-parity 36-plet,

$$M_A: \ M_a = \pi^a, \ M_{ai} = \rho_i^a , \qquad (15)$$

and an even-parity 36-plet,

$$N_A: \ N_a = \sigma^a, \ N_{ai} = B_i^a , \qquad (16)$$

in terms of the usual phenomenological meson fields. Together these multiplets transform as a $(6,\bar{6}) + (\bar{6},6)$ representation of chiral $U(6) \times U(6)$,

$$[{}_{\alpha}Q^{A}, M^{B}] = i\, F^{ABC}\, M^{C} ,$$

$$[{}_{\alpha}Q^{A}, N^{B}] = i\, F^{ABC}\, N^{C} ,$$

$$[{}_{\beta}Q^{A}, M^{B}] = i\, D^{ABC}\, N^{C} ,$$

$$[{}_{\beta}Q^{A}, N^{B}] = i\, D^{ABC}\, M^{C} . \tag{17}$$

We should emphasize that the $(6,\bar{6}) + (\bar{6},6)$ representation is forced upon us once we have the pseudoscalar mesons and vector mesons in the same multiplet and then demand that the pseudoscalars be in $(3,\bar{3}) + (\bar{3},3)$ under chiral $SU(3) \times SU(3)$, as they are in the usual formulation of the sigma model. The requirement of the $(6,\bar{6}) + (\bar{6},6)$ then fixes the charge conjugation property of the axial-vectors to be the same as the vectors, i.e. odd, while the σ^{a} and π^{a} are even as usual.

The interaction or potential part of the Lagrangian is

$$\begin{aligned} \mathscr{L}_{pot} = & -\frac{1}{2}\mu^{2}(M_{A}M_{A}+N_{B}N_{B}) - \lambda(M_{A}M_{A}+N_{B}N_{B})^{2} \\ & - \gamma\big[D_{ABC}D_{A'B'C}\,(M_{A}M_{B}M_{A'}M_{B'}+N_{A}N_{B}N_{A'}N_{B'} \\ & + 2\, M_{A}M_{B}N_{A'}N_{B'}) + 4\, F_{ABC}F_{A'B'C}M_{A}N_{B}M_{A'}N_{B'}\big] . \end{aligned} \tag{18}$$

This is the most general $U(6) \times U(6)$-invariant Lagrangian (restricted to polynomials of degree ≤ 4). We note that there is no trilinear term here, in contrast with the case for $SU(3) \times SU(3)$, so that the maximal group which leaves the Lagrangian invariant is $U(6) \times U(6)$ and not just $SU(6) \times SU(6)$. However, we know how to solve the $U_{A}(1)$ problem via its anomaly, and after $SU(6)$-breaking this can be implemented in the phenomenological Lagrangian in a by-now-standard fashion. This is the reason we are somewhat cavalier about distinguishing between $U(6)$ and $SU(6)$ throughout this paper.

As in the usual linear sigma model, if $\mu^{2} < 0$ then σ^{0} has a vacuum expectation value $\langle\sigma^0\rangle = a$, and the vacuum is just U(6)-invariant so the U(6) × U(6) symmetry is spontaneously broken. Standard analysis yields the following masses for the various mesons:

$$m^2_{\sigma^0{}'} = \mu^2 + 4a^2(3\lambda+\gamma),$$

$$m_B{}^2 = m_{\mathcal{A}}{}^2 = \mu^2 + 4a^2(\lambda+\gamma),$$

$$m_\pi{}^2 = m_\rho{}^2 = \mu^2 + 4a^2(\lambda+\gamma/3) = 0 \; . \tag{19}$$

So the whole odd-parity 36-plet has become massless; it is the Goldstone mode of the U(6) × U(6) symmetry broken to U(6).

The vector mesons cannot of course be true Goldstone bosons in a realistic, relativistic theory. Indeed there exists a rigorous proof against Goldstone bosons of spin ≥ 1 in a relativistic theory[17]. However, for a non-relativistic theory not only is there no theoretical problem, but spin-1 Goldstone states (spin waves) have actually been observed, for example, in certain He systems[18]. We would also point out that this picture of vector mesons as Goldstone bosons in the static limit has subsequently been confirmed in lattice calculations by two different methods[19]. Nevertheless, they are still only dormant Goldstone bosons, which are roused by relativistic effects to become massive states.

To see this explicitly, we note that once one demands relativistic invariance, because Lorentz boosts affect spin but not U-spin, U(6) symmetry is broken by kinetic energy terms, and this must give rise to spin-dependent mass terms. To include such an explicit U(6)-symmetry-breaking mass term it is helpful to use the tensor notation

$$\hat{M}^{\alpha}_{\beta} = i\,\delta^i_j\, P^A_B + (\vec{\sigma}\cdot\vec{\varepsilon})^i_j\, V^A_B,$$

$$\hat{N}^{\alpha}_{\beta} = i\,\delta^i_j\, S^A_B + (\vec{\sigma}\cdot\vec{\varepsilon})^i_j\, B^A_B \; , \tag{20}$$

where $P^A_B = (1/\sqrt{2})\,\lambda^A_{Ba}\pi^a$, $V^A_B = (1/\sqrt{2})\,\lambda^A_{Ba}\rho^a$, etc., and $\vec{\varepsilon}$ is the polarization vector. Combining these tensors into

$$T = \hat{N} + i\,\hat{M}$$

$$T^{\dagger} = \hat{N}^{\dagger} - i\,\hat{M}^{\dagger} \,, \tag{21}$$

we may write the mass operator as

$$-\frac{1}{2}\mu^2\ \mathrm{Tr}TT^{\dagger} - \frac{1}{2}\beta^2\ \mathrm{Tr}[\vec{\sigma}\ T][\vec{\sigma}\ T^{\dagger}]$$

$$= -\frac{1}{2}\mu^2(\pi^2+\rho^2+\sigma^2+B^2) - \frac{1}{2}\beta^2(\rho^2+B^2), \tag{22}$$

where [ ] indicates the spin trace. The spin-1 mesons are now split from their spin-0 multiplet partners, and the mass formulae are

$$m_B^2 = \mu^2 + \beta^2 + 4a^2(\lambda+\gamma),$$

$$m_{\hat{\sigma}}^2 = \mu^2 + 4a^2(\lambda+\gamma),$$

$$m_\rho^2 = \beta^2,$$

$$m_\pi^2 = 0 \ . \tag{23}$$

From (23) we obtain the relation

$$m_\rho^2 - m_\pi^2 = m_B^2 - m_\sigma^2, \tag{24}$$

which is in remarkably good agreement with the experimentally determined masses. For example, putting in the masses of the I = 1 members of the octets, we obtain

$$m_{\vec{\rho}}^2 - m_{\vec{\pi}}^2 = m_{\vec{B}}^2 - m_{\vec{\delta}}^2 \ , \tag{25}$$

$$0.547\ \mathrm{GeV}^2\ \text{vs.}\ 0.577\ \mathrm{GeV}^2 .$$

From this it appears that U(6)-symmetry breaking due to spin-dependent effects independent of SU(3) breaking is rather well borne out experimentally, with the parameter $\beta^2 \approx 0.6\ \mathrm{GeV}^2$.

E. Some Consequences of a Dormant Goldstone ρ

We would now like to mention some of the important consequences of this description of the vector mesons in the context of a relativistic theory, or, to put it in more anthropomorphic terms, how does the ρ remember its origin as a dormant Goldstone boson? The answer lies in its being in a $(3,\bar{3}) + (\bar{3},3)$ representation of chiral

SU(3) × SU(3). Perhaps this is as good a point as any to note that in terms of chiral representation content there are at least two possible pions, namely, the (1,8) + (8,1) representation in addition to the $(3,\bar{3}) + (\bar{3},3)$. The same is true for the vectors. So perhaps the physical states are mixtures. There is not much evidence of this for the pion, and to emphasize our viewpoint we will ignore it for the vectors as well. Certainly for the non-linear sigma model discussed below this seems to be appropriate.

The implication of putting the vector mesons in $(3,\bar{3}) + (\bar{3},3)$ is that vector-meson dominance is a consequence of spontaneous symmetry breaking. Just as the $\pi[(3,\bar{3}) + (\bar{3},3)]$ couples to the axial-vector current $[(1,8) + (8,1)]$ via a σ^0 going into the vacuum [Fig. 1(a)], so also the $\rho[(3,\bar{3}) + (\bar{3},3)]$ couples to the vector current $[(1,8) + (8,1)]$ by the same mechanism [Fig. 1(b)]. Vacuum symmetry breaking in these two instances corresponds to the nonvanishing vacuum values of

$$<[{}^5Q^a,\ \pi^b]> = -\ i\ \left(\tfrac{2}{3}\right)^{1/2}\ \delta^{ab}\ <\sigma^0>,$$

$$<[Q^a_i,\ t^b_{oj}]> = -\ i\ \left(\tfrac{2}{3}\right)^{1/2}\ \delta^{ab}\ \delta_{ij}<\sigma^0>. \qquad (26)$$

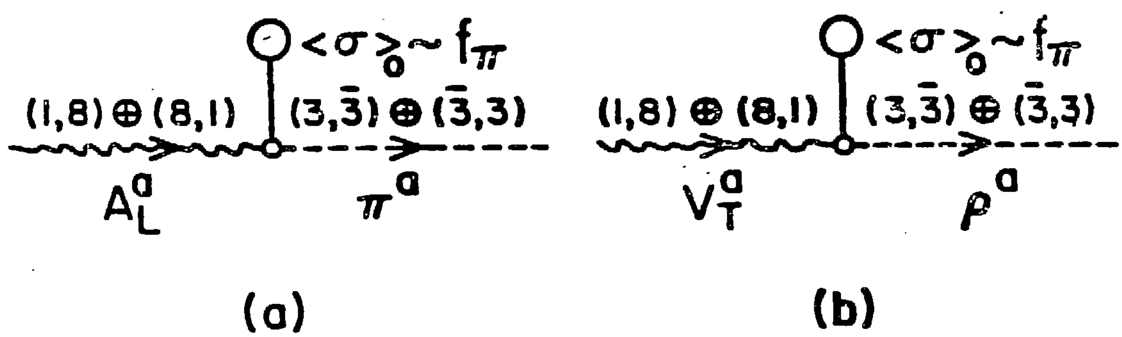

FIG. 1. Coupling of (a) axial-vector current to pion and (b) vector current to the ρ via spontaneous breaking of chiral symmetry.

In this description of VMD we can obtain an expression for the current-vector-meson coupling, $1/\gamma_\rho$, defined by

$$\langle 0 \left| V_\mu^a(0) \right| \rho^b(k,\varepsilon)\rangle = - i\, \varepsilon_\mu\, \delta^{ab} \frac{m_\rho^2}{\gamma_\rho} . \tag{27}$$

This is analogous to the expression for the pion-axial-vector-current coupling

$$\langle 0 \left| A_\mu^a(0) \right| \pi^b(k)\rangle = i\, k_\mu f_\pi \delta^{ab}. \tag{28}$$

Using the relationship $f_\pi = Z_\pi^{1/2}(2/3)^{1/2} \langle\sigma^0\rangle$, we obtain the relation between $1/\gamma_\rho$ and f_π to be

$$\frac{1}{\gamma_\rho} = \left(\frac{f_\pi}{m_\rho}\right)\left(\frac{Z_\pi}{Z_\rho}\right)^{1/2} . \tag{29}$$

In the SU(6) limit, $Z_\pi = Z_\rho$, while with $(Z_\pi/Z_\rho)^{1/2} \simeq 1.5$ one obtains the observed rate for $\rho \to e^+ + e^-$. Our result (29) is similar to the KSFR relation[20], but in our derivation we have an explanation for a relationship between $1/\gamma_\rho$ and f_π, namely, both are consequences of the same process (Fig. 1). As discussed in Ref. 13, there has been no really adequate derivation of the KSFR relation.

There are many other consequences of our description of the vector mesons, including the usual universality $\gamma_\rho = g_{\rho\pi\pi}$, soft ρ decoupling theorems, and tensor-field identities. We also examined a number of decay processes, as well as photon-ρ interactions. Referring the interested reader again to Ref. 13 for the details, we may summarize the situation by saying that not only is our picture consistent with the experimental behavior of vector mesons, but it also provides a unified framework for understanding the phenomenology of both pseudoscalar and vector mesons.

III. THE MODEL

A. Static Aspects of the Skyrme Model

With this review of the dormant Goldstone picture of the ρ's completed, and with the reader (one hopes) at least partially favourably disposed to allow its credibility, we are just about ready to discuss our generalized Skyrme model. But first it is worth pointing out that the usual Skyrme model already has some important static aspects to it, which make it an essentially ideal place to use this formulation of vector mesons.

If we first look at the Skyrme ansatz, Eq. (4), we notice the typical feature of such a soliton, namely that it mixes isospin and spatial indices. Furthermore, upon quantization the spin and isospin of the Skyrmion are intimately linked. Hence the usual Skrymion already has the signal aspect of static SU(6).

Perhaps this is not too surprising in light of our second point, which is that the Skyrmion, in so far as it can be considered the baryon of large-N QCD, is a static object. This is because in the large N limit the baryon mass is expected[7] to be proportional to N, i.e. to have the typical soliton behaviour with the mass proportional to the inverse coupling = 1/(1/N).

But there is an even deeper sense in which the usual Skyrme model is already a static model. As a number of authors[21] have recently pointed out, the Skyrme model, the static nonrelativistic quark model, and the static strong-coupling model are all related in that they share the same symmetry group, namely static SU(6) (or SU(4) in the original 2-flavour case), at least in the $N_c \to \infty$ limit. As Bardakci[21] in particular has shown, however, there are problems in next-to-leading order in 1/N, but these can be fixed at the cost of adding yet higher derivative (6th order) terms to satisfy the static SU(4) algebra. It seems to us that with all these strong clues, it is better to include static SU(4) or SU(6) from the start in the model. Of course, for a realistic, relativistic treatment of the meson sector, we will eventually want to break the SU(6).

B. The Nonrelativistic Model

In order to investigate most of the properties of the generalized Skyrmion which have been of principal interest for the usual Skyrmion, that is, the static properties of baryons, the following nonrelativistic model should be adequate. It is convenient to use for the parametrization of the meson field

$$\mathcal{U}(x) = \exp\big((i/f_\pi)\,\Lambda_A\, M_A(x)\big) \tag{30}$$

with the notation of (14) and (15) for Λ_A and M_A. A proto-Lagrangian may then be written as

$$\mathcal{L} = \frac{f_\pi^2}{4}\,\mathrm{Tr}\left[\partial_\mu \mathcal{U}\,\partial_\mu \mathcal{U}^\dagger\right] \ . \tag{31}$$

Again there is the all-important constraint $\mathcal{U}^\dagger\mathcal{U} = 1$, which would be expressed in a vector version of the model as $(\pi^a)^2 + (\rho_i^{\ a})^2 + \sigma^2 = 1$. This is what provides the non-linearity of the model and hence not only the interactions but also the non-trivial topology. So the manifold of the model is the coset space (SU(6) x (SU(6))/SU(6) (or (SU(4) x SU(4))/SU(4) without strangeness). The reader should be warned that despite the appearance of (31), the Lagrangian is not covariant, due to the spatial indices hiding on the $\rho_i^{\ a}$'s in M_A. Eq. (31) is only to be understood as a nonrelativistic model. But this is perfectly fine for the purposes of finding and studying <u>static</u> finite energy solutions, since being time-independent, they only involve ∂_i. So now, exactly as in the case for SU(2) or SU(3), the demand for finite energy is met by requiring

$$\mathcal{U}(x) \xrightarrow[|x| \to \infty]{} I \tag{32}$$

at all times t, compactifying $R^3 \to S^3$. $\mathcal{U}(x)$ thus describes mappings $S^3 \to SU(6)$ and these are described by the third homotopy group $\pi_3(SU(6))$. One of the crucial pieces of evidence that gives us confidence that we cannot be far wrong in our approach is that

$$\pi_3(SU(6)) \approx Z, \tag{33}$$

so that the space is indeed topologically non-trivial and splits up into an infinite set of topologically disconnected components. Actually (33) is a consequence of the Bott periodicity theorem[22)], which tells us in this case that $\pi_3(SU(n)) \approx Z$ for all $n \geq 2$.

The next question concerns the quartic-derivative, Skyrme term. Unlike the situation where one adds vector mesons on a different footing from the pseudoscalars and they hence can provide stabilization of the Skyrmion, in our case it is easy to show that the Lagrangian (31) as it stands does not avoid the Derrick, et al.

theorem. So the price of including vectors in this symmetric way is that we must keep the Skyrme term, at least in the unbroken version of the model. Hence the static Lagrangian so far, using the group currents (1) generalized to $\mathcal{U}(30)$, is

$$\mathcal{L} = -\frac{f_\pi^2}{4}\,\mathrm{Tr}(L_i L_i) + \frac{1}{8e^2}\,\mathrm{Tr}([L_i,L_j]^2). \tag{34}$$

There is one more term we would like our Lagrangian to have, that is the Wess-Zumino term. This term is important, as Witten[3] has demonstrated, in establishing the Skyrmion as a fermion, in reducing the parity of the model to that of QCD, and of course for its original purpose, that of incorporating the effects of the anomalies of current algebra. Since we certainly would want to retain these features in our generalized model, again it is fortunate that the same topological arguments that applied in the SU(3) case also apply in the case of SU(6). To begin with, the chiral SU(6) x SU(6) current algebra certainly is anomalous, having the anomalies from the SU(3) x SU(3) subalgebra, as well as those associated with the remaining currents. Nevertheless, at first sight it might appear that a Wess-Zumino term for a nonrelativistic chiral SU(6) x SU(6) model would not be easily admissible. That this turns out not to be the case can be seen from the remark[23] that the Wess-Zumino term is really most appropriately written[8] using differential forms and from this formulation one sees that it is independent of the metric. Related to this is the observation that the topological derivations of the Wess-Zumino term, as is usual with topological discussions in general, are in the context of <u>Euclidean</u> space-time, which is compactified to a sphere besides. The fact that the nonrelativistic model retains only Galilean invariance takes on much reduced significance in this context. (Please also note some remarks in the following subsection.)

So following, for example, Witten's derivation[3,24], we need to know what $\pi_4(SU(6))$ is since $\mathcal{U}$ is a mapping of the four-dimensional sphere M into the SU(6) manifold. Again the Bott periodicity theorem ensures that $\pi_4(SU(6)) = 0$, just as for SU(3). So the four sphere in SU(6) defined by $\mathcal{U}(x)$ is the boundary of a five-dimensional disc Q.

Then because Q is not unique, in an argument analogous to magnetic charge quantization, one has the requirement that on $Q + Q' = S^5$, a closed five-dimensional sphere,

$$\int_{S^5} \omega_{ijk\ell m} d\Sigma^{ijk\ell m} = 2\pi \cdot \text{integer} \tag{35}$$

for any S^5 in the SU(6) manifold. So one needs to know $\pi_5(SU(6))$, and once more the Bott periodicity theorem ensures that $\pi_5(SU((n)) = Z$, for $n \geq 3$, and so in particular for our case of SU(6). This then tells us that every S^5 in SU(6) is topologically a multiple of a basic five sphere $S_0{}^5$ which we use to normalize ω. We can make this more explicit by defining the one-form related to the group current (1),

$$\beta \equiv \mathcal{U}^{-1}(\partial_\mu \mathcal{U})\ dx^\mu \equiv \mathcal{U}^{-1} d\mathcal{U}. \tag{36}$$

The normalization condition is then

$$C \int_{S_0{}^5} \mathrm{Tr}(\beta^5) = 2\pi\ . \tag{37}$$

The proper normalization of C Tr(B^5) fortunately has already been carried out by Bott and Seely[25)] who proved that for C Tr(β^{2N-1}),

$$C = 2\pi\left(\frac{i}{2\pi}\right)^N \frac{(N-1)!}{(2N-1)!}\ . \tag{38}$$

The result is seen to depend on the dimension of the sphere and not on the group. We thus obtain the same normalization for SU(6) as for SU(3), and the Wess-Zumino term in our model can be written

$$\Gamma_{WZ} = \frac{-\ i\ n}{240\ \pi^2} \int_Q \mathrm{Tr}(\beta^5)\ . \tag{39}$$

The first term in the expansion in terms of the meson fields $M = \lambda_A M_A$ so that $\beta = (i/f_\pi)\ dM + \ldots$, is

$$\Gamma_{WZ} = \frac{n}{240\ \pi^2 f_\pi{}^5} \int_Q \mathrm{Tr}(d\ M)^5 + \ldots$$

$$= \frac{n}{240\ \pi^2 f_\pi{}^5} \int d^4x\ \varepsilon^{\mu\nu\alpha\beta}\ \mathrm{Tr}(M\partial_\mu M\partial_\nu M\partial_\alpha M\partial_\beta M) + \ldots\ . \tag{40}$$

As for the identification of n with the number of colours N_c, this requires a comparison with the QCD triangle diagram prediction for

$\pi^0 \to 2\gamma$ after gauging (39) with electromagnetism. This is then more properly done in the relativistic model after SU(6) breaking[26]. Assuming no unexpected complications, the result should be that n indeed equals N_c and so for $N_c = 3$ the generalized Skyrmion, following the same argument[3,24] as for SU(3), is indeed a fermion, and so a baryon.

As for the actual Skrymion solution, there are, not surprisingly considering the larger nature of the group, a number of options, some of which will presumably lead to exotic new states[26]. We mention here only the simplest option which corresponds to the lowest-lying state, namely embedding the Skyrme ansatz (4) in an otherwise diagonal SU(6) matrix, in a generalization of what has already been done[3,27] for SU(3). Upon the standard quantization of the collective coordinates, we should obtain that this Skyrmion is the whole baryonic 56-plet[26]. The first excitation of this object should then be the whole baryonic 70-plet. In addition, there will also be an embedding of an SO(3) ansatz, which corresponds to the SU(6) generalization of the dibaryon[28] of the SU(3) model.

C. Relativistic Version

As we have already noted a number of times, in the end we want a relativistic theory. As in the linear model we may add a spin-dependent mass term which will provide explicit SU(6)-breaking. Although such a term is presumed to be the effect of relativistic kinetic energy terms in the Lagrangian (namely Eq. (31) now viewed as valid in any Lorentz frame), an actual derivation seems to be excluded by the dynamical nature of approximate SU(6) symmetry. At any rate the actual spin-dependent mass term may be written in very much the same way as Eq. (22), with only notational differences to conform to the non-linear model. So the mass term is

$$\frac{m_V^2 f_\pi^2}{4} \mathrm{Tr}[\vec{\sigma}\mathcal{U}][\sigma\mathcal{U}^\dagger] \quad . \tag{41}$$

The parametrization

$$\hat{\mathcal{U}}(x) = \exp\{(i/f_\pi)\hat{M}(x)\}$$

$$\hat{M} = i\ I\ P + (\vec{\sigma}\cdot\vec{\varepsilon})V$$

$$P = \lambda^a \nu^a, \quad V = \lambda^a \rho^a \tag{42}$$

should make it apparent that upon expansion of $\hat{\mathcal{U}}$, to lowest order (41) implies $(m_V{}^2/2)V^2$. This will also result in the splitting of the baryonic 56 into the octet and decuplet.

Having thus implemented explicit SU(6)-symmetry breaking, we may formulate the model covariantly. Some subtleties still remain, so we will here only briefly sketch one approach. The vector meson fields can now be made four-vectors, $\rho_\mu^a(x)$. They are massive Proca fields, but instead of the usual Proca Lagrangian we are free to use

$$\mathcal{L}_P = -\frac{1}{2}(\partial_\mu \rho_\nu^a)^2 + \frac{1}{2} m^2 (\rho_\nu^a)^2 \tag{43}$$

with the condition $\partial_\mu \rho^{a\mu} = 0$. This is fortunate since an $F_{\mu\nu}$-type term would not naturally arise in the formalism using the group-element field $\mathcal{U}(x)$.

As long as we are able to remain in Euclidean space-time, then it appears rather straightforward to write down a field $\mathcal{U}(x)$. There are not the usual problems with unitarity here, with which attempts at relativistic extensions of SU(6) were plagued[15]. This is because a partial way out of these problems, which arise from the noncompact nature of the groups considered (due to the noncompactness of the Lorentz group) is to use the Weyl unitary trick, which in these cases essentially takes one to Euclidean space. So, for example, we may use matrices $\mathcal{M}$ based on the U(6,6) formulation[15] of Bég and Pais, but again rotated via the Weyl unitary trick to Euclidean space. In this formulation $\mathcal{U}(q) = \exp((i/f_\pi)\lambda\mathcal{M})$ with

$$\mathcal{M}^a_b(q) = -(i/m)(\sigma_{\mu\nu} q_\mu \varepsilon_\nu(q))^\sigma_\rho V^A_B - i(\gamma_5)^\sigma_\rho P^A_B, \tag{44}$$

where $\vec{\varepsilon}(q) = \vec{\varepsilon} + [q(\vec{q}\cdot\vec{\varepsilon})/m(q_0+m)]$ and $\varepsilon_0(q) = \vec{q}\cdot\vec{\varepsilon}/m$. In the static limit $\mathcal{M} = \hat{M}$ the static SU(6) matrix. One can repeat the analysis of the preceding subsection concerning both the non-trivial topology and the Wess-Zumino term, but now for this U(12) group, with the results essentially unchanged because of the Bott periodicity theorem.

Eventually we must return to Minkowski space-time. But since we have already broken the SU(6) symmetry, there should not be any difficulty especially if, as will generally be the case, we work with the expansion of $\mathcal{U}$ in terms of meson fields and with these explicitly separated into the pseudoscalar and vector pieces.

IV. CONCLUDING REMARKS

The generalized Skyrme model we have presented has a number of features which seem attractive. By including the vector mesons in a manner symmetrical to the pions, the model is a more realistic description of low-energy physics. Furthermore it goes some way towards a realization of the large N effective theory of QCD which a semiclassical Lagrangian like the Skyrme model is supposed to simulate.

By incorporating approximate, static SU(6) symmetry, the phenomenological successes of that symmetry should be manifest in the predictions of the model. Particularly noteworthy in this regard is the famous SU(6) result for $(\mu_P/\mu_N) = -3/2$. This is especially significant because the usual Skyrme models have had troubles[29)] with the baryon magnetic moments. In general it is to be hoped that the model discussed here will improve on the 30% accuracy of the usual Skyrme models.

Of course being a new model, much of the work we have heard about at this Workshop which has been done on the SU(2) and SU(3) Skyrme models has to be repeated on this model before its true worth can be determined. Unfortunately the analysis is rather more complicated since a larger group is involved. Nevertheless, it is clear that conceptually the model has some advantages. It gives a more unified picture in which the π's and ρ's are collective excitations and the baryons emerge as solitons. A rather rich spectrum also results from this model. The danger is that it is perhaps too rich.

A final remark, due to Witten. If we simplify to a world of just one flavour, then the lowest-lying meson in large-N QCD must be a vector meson, the equivalent roughly of the ω. In the same theory the lowest-lying baryon would be the equivalent of the Δ^{++} and would emerge as a soliton constructed from the vector-meson field. Its spin would go like N/2. This not only lends additional credence to the model we have developed, but also offers a simplified testing ground to analyze and verify its implications. The one-flavour model also of course offers the opportunity to realize explicitly this prediction from arguments based on large-N QCD. Work is currently in progress along these lines.

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