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APPLICATION OF THE FINITE ELEMENT METHOD TO THE NONLINEAR INVERSE HEAT CONDUCTION PROBLEM USING BECK'S SECOND METHOD*

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ABSTRACT

The calculation of the surface temperature and surface heat flux from a measured temperature history at an interior point of a body is identified in the literature as the inverse heat conduction problem. This paper presents, to the author's knowledge, the first application of a solution technique for the inverse problem that utilizes a finite element heat conduction model and Beck's nonlinear estimation procedure. The technique is applicable to the one-dimensional nonlinear model with temperature-dependent thermophysical properties. The formulation is applied first to a numerical example with a known solution. The example treated is that of a periodic heat flux imposed on the surface of a rod. The computed surface heat flux is compared with the imposed heat flux to evaluate the performance of the technique in solving the inverse problem. Finally, the technique is applied to an experimentally determined temperature transient taken from an interior point of an electrically-heated composite rod. The results are compared with those obtained by applying a finite difference inverse technique to the same data.

NOMENCLATURE

Symbol	Definition
a	Radius of cylindrical rod
[B]	Matrix, as defined in equation (10)
[C]	Heat capacity matrix for assembly of elements
C_i	Thermal expansion coefficients in gap width model, equation (33)
c	Specific heat
e	Index of elements
E	Number of elements in assembly
$\{\bar{F}\}$	Vector for assembly of elements, equation (11)
$\{\bar{\bar{F}}\}$	Vector for assembly of elements, equation (12)
h	Convective heat transfer coefficient
h^r	Radiative heat transfer coefficient, equation (5)
J	Number of time steps in analysis interval; J - 1 equals number of "future" temperatures
[K]	Thermal conductivity matrix for assembly of elements
k	Thermal conductivity
L	Distance of temperature probe from surface of body
M	Number of nodes
N_I	Interpolation function
{N}	Vector of interpolation functions
$\underline{n}$	Unit outward normal to boundary surface
Q	Internal heat generation rate, per unit volume
q	Imposed surface heat flux
q^c	Surface heat flux due to convection
q^r	Surface heat flux due to radiation

Symbol	Definition
r	Radial coordinates, one-dimensional model
r_I	Radial coordinates of node point I
r^p	Radial coordinate of temperature probe location
Δr_{gap}	Width of sheath gap in heater rod model, equation (33)
r_{15}	Inner radius of sheath gap interface, equation (33)
r_{16}	Outer radius of sheath gap interface, equation (33)
$[S]$	Matrix, as defined in equation (16)
T	Temperature
T_I	Value of temperature at Ith node
$\{T\}$	Temperature vector
T^p	Measured temperature at internal point r^p of body
T^{ac}	Temperature at which no convection occurs
T^{ar}	Temperature at which no radiation occurs
T^w	Temperature at wall
T_{av}	Computed temperature at thermocouple probe position r^p averaged over analysis interval, equation (32)
T^p_{av}	Input temperature at thermocouple probe position r^p averaged over analysis interval, equation (32)
T_{15}	Computed temperature at inner surface of sheath gap interface in heater rod model, equation (33)
T_{16}	Computed temperature at outer surface of sheath gap interface in heater rod model, equation (33)
t	Time
TOL1	Convergence tolerance, equation (18)
TOL2	Tolerance for temperature error, equation (31)

Symbol	Definition
W	Summed weights function, equation (32)
w_j	Weighting functions, equation (28)
$\underline{x}$	General spatial coordinates

Greek Symbols	Definition
α	Thermal diffusivity
ρ	Density
Ω	General spatial domain
Ω^e	Element domain
Γ_1	Boundary on which condition (2) is prescribed
Γ_2	Boundary on which condition (3) is prescribed
Γ_2^e	Element external boundary on which condition (3) is prescribed
∇	Gradient operator
σ	Stefan constant
ε	Emissivity
η	One-dimensional coordinate of parent element
β	Surface heat flux parameter, equation (27)
Δ	Incremental change in kernel
τ	Dimensionless time
Σ	Summation symbol

Subscripts	
I	Index of nodes
i	Index of time steps in solution, where i is a non-negative integer
j	Index of time steps in analysis interval, $1 \leq j \leq J$
$(i)\Delta t$	Time $t = (i)\Delta t$ at which kernel is evaluated

Superscripts

$[\ ]^T$	Transpose of matrix
$\{\ \}^T$	Row vector
(P)	Iteration number at which kernel is evaluated

Other Symbols

$[\ ]$	Matrix
$\{\ \}$	Column vector
$\|\|\ \|\|$	Euclidean norm
$\int$	Integral sign

INTRODUCTION

In transient heat conduction analysis, a class of problems can be identified where the temperature history is known at some interior point in the body and the transient surface temperature and surface heat flux are to be determined. This class is generally referred to in the literature as the inverse problem, in contrast with the usual direct formulation where the interior temperature history is determined from specified boundary conditions. Typically, the inverse formulation arises in experimental studies where direct measurement of surface conditions is not feasible, such as convective heat transfer in rocket nozzles and quenching of solids in a fluid. One application examined in this paper deals with two-phase flow over the surface of an electrically-heated rod that contains thermocouple probes in the interior.

Various solution methods have been applied to the inverse problem over the past two decades, including integral equation solutions, series solutions, transform solutions, and function minimization techniques. In one of the earliest papers, Stolz (1)[1] obtained a linear solution by numerical inversion of the integral solution of the direct problem. His solution was found to be unstable for small time steps. Using an integral approach similar to that of Stolz, Beck (2) utilized a least squares technique to generate solutions for a much smaller time step. In a recent paper, Arledge, et al (3) also use an integral solution procedure which is valid for constant thermal properties. Burggraf (4) devised a series solution to the linear inverse problem which is exact only for

[1]Underlined numbers in parentheses identify References in the last section of the paper.

continuous input data. Makhin and Shmukin (5), Kover'yanov (6), Plummer, et al (7), and Mehta (8) also utilize a series solution. Sparrow, et al (9) and Imber and Khan (10) apply the transform method to the linear problem. Imber (11) has developed a transform solution to the inverse problem that is applicable to two-dimensional bodies of arbitrary shape when input data are known at suitable interior locations. In all of these papers, linearity of the model is essential to the formulation.

Several references consider the nonlinear problem of analyzing a composite body with temperature-dependent thermal properties. Ott and Hedrick (12) have developed a one-dimensional, implicit finite difference formulation and have applied it to an electrically-heated composite rod with temperature-dependent geometry and material properties. Beck has examined the nonlinear problem using a finite difference method (13) that builds on the ideas in Reference (2) and, more recently in Reference (14), has incorporated his function minimization technique into the framework of nonlinear estimation. Important to Beck's "second" method (14) is the observation that the temperature response at an interior location is delayed and damped with respect to changes in surface conditions. Beck, therefore, determines the surface heat flux in a given time step with a procedure that utilizes interior temperature data at "future" times. The surface heat flux is assumed to be a constant or low order polynomial over an analysis interval that consists of several time steps in the discretized data. The coefficients that describe the heat flux are adjusted iteratively to achieve the closest agreement in a least squares sense with the input "future" temperatures over the analysis interval. Less flexible versions of Beck's technique were considered earlier by Frank (15) and Davies (16). Muzzy, et al (17) have adapted Beck's second method, with some modifications, into an explicit finite difference scheme for one-dimensional composite bodies with temperature-dependent material properties.

The finite difference method has been the predominant numerical technique for solution of the direct problem of heat conduction, and is applied in the nonlinear inverse formulations of References (12), (13), (14), and (17). In recent years, the finite element method (18) has become well established as another numerical technique for heat conduction analysis. The finite element approach has demonstrated great versatility in modeling homogeneous or composite bodies with temperature-dependent material properties and complex geometries and boundary conditions. (See, for example, References (19), (20), (21).) In addition to these benefits, the technique shows considerable promise for the solution of coupled heat conduction and thermal stress problems (22), (23).

In a recent paper, Hore, et al (24) present what is evidently the first application of the finite element method to the inverse problem. They develop a procedure for determining the surface heat flux at one boundary of a one-dimensional linear system from a known temperature history at an interior point. An iterative technique is used to determine incremental changes in surface heat flux until the error in the computed temperature at the interior point is within a prescribed tolerance. In their analysis, the surface flux is evaluated using contemporary input temperatures only, i.e., no "future" temperatures are utilized to determine surface flux as in Beck's second method.

In Reference (24), the finite element formulation of the inverse problem is applied to two numerical examples with known solutions. The first example treated is that of a constant heat flux imposed on the surface, while the second considers a periodic square wave heat flux. For both cases, the predicted temperature at the interior node closely followed the input temperature values. However, the two examples exhibited numerical instabilities for the heat flux calculations, in the form of oscillations that progressively diverged in time from the known heat flux solution. Hore, et al identify some other difficulties with their inverse solution technique, including that of using temperature data measured at points far removed from the surface to solve for the surface heat flux. A potential difficulty with their technique is that it does not minimize the effect of experimental errors incurred in the temperature measurements at interior points. When these data are not smooth, oscillations in the calculated values of the heat flux can result. Hore, et al speculate that the nonlinear estimation techniques of Beck, along with the use of "future" temperatures, could possibly alleviate some of these difficulties.

The inverse solution technique described in the following sections represents, to the author's knowledge, the first application of Beck's nonlinear estimation procedure in a computational scheme based on a finite element model of the direct problem. Discussion of the finite element formulation and Beck's procedure is followed by application of this inverse technique to two problems. First, a numerical example with a known solution is treated to evaluate the performance of the technique in solving the inverse problem. Finally, the technique is applied to an experimentally determined temperature transient taken from an interior point of an electrically-heated composite rod. The finite element computations are compared with those obtained by applying a finite difference inverse technique to the same data.

FINITE ELEMENT FORMULATION OF THE DIRECT PROBLEM

The conduction of heat in the region Ω is governed by the quasi-linear parabolic equation

$$\nabla \cdot (k\nabla T) + Q = \rho c \frac{\partial T}{\partial t} \tag{1}$$

subject to the boundary conditions

$$T = T^{w} \text{ on } \Gamma_1 \tag{2}$$

and

$$k\nabla T \cdot \underline{n} + q + q^{r} + q^{c} = 0 \text{ on } \Gamma_2 \tag{3}$$

The heat flow rates per unit area on convection and radiation boundaries are written

$$q^{c} = h(T - T^{ac}) \; , \; q^{r} = h^{r}(T - T^{ar}) \quad , \tag{4}$$

where h^{r} is defined by

$$h^{r} = \varepsilon\sigma(T^2 + T^{ar^2}) \; (T + T^{ar}) \tag{5}$$

In general, k, c, and h^{r} are temperature and spatially dependent, while Q, q, and h are time and spatially dependent.

Let the region Ω be idealized by a system of finite elements and let the unknown temperature T be approximated throughout the solution domain at any time t by

$$T(\underline{x},t) = \sum_{I=1}^{M} N_I(\underline{x}) \; T_I(t) = \{N\}^T \; \{T\} \tag{6}$$

Here the N_I are the interpolation functions defined piecewise element by element and the T_I or $\{T\}$ are the nodal temperatures. The governing equations of the discretized system can be derived by minimizing a functional or by using Galerkin's method (18). In the Galerkin formulation employed here, the problem is recast in a weighted integral form using the interpolating functions N_I as the weighting functions:

$$\int_{\Omega^e} \{N\}[\nabla \cdot (k\nabla(\{N\}^T \{T\})) + Q - \rho c \frac{\partial}{\partial t} (\{N\}^T \{T\})] \, d\Omega$$

$$- \oint_{\Gamma_2^e} \{N\}[k\nabla(\{N\}^T \{T\}) \cdot \underline{n} + q \tag{7}$$

$$+ h(\{N\}^T \{T\} - T^{ac}) + h^r(\{N\}^T \{T\} - T^{ar})] \, d\Gamma = 0$$

Only a single finite element is considered in the integral (7), as the governing equations of the complete system of elements are obtained by assembling the individual finite element matrices. The surface integral over Γ_2^e refers only to those elements with external boundaries on which condition (3) is given.

Green's first identity is applied to the first volume integral of equation (7) so that the second derivatives do not impose unnecessary continuity conditions between elements. When use is made of the boundary conditions (2) and (3), the integral formulation (7) leads to a set of transient ordinary differential equations for the assemblage of finite elements:

$$[C] \frac{\partial\{T\}}{\partial t} + [K] \{T\} + \{\bar{F}\} + \{\bar{\bar{F}}\} = 0 \tag{8}$$

The components in equation (8) are defined by:

$$[C] = \sum_{e=1}^{E} \int_{\Omega^e} \rho c\{N\} \{N\}^T \, d\Omega \quad , \tag{9}$$

$$[K] = \sum_{e=1}^{E} \int_{\Omega^e} k[B]\ [B]^T\ d\Omega$$

$$+ \sum_{e=1}^{E} \oint_{\Gamma_2^e} (h + h^r)\ \{N\}\ \{N\}^T\ d\Gamma\ , \quad [B] = \nabla\{N\} \quad , \qquad (10)$$

$$\bar{F} = - \sum_{e=1}^{E} \int_{\Omega^e} \{N\}\ Q d\Omega$$

$$+ \sum_{e=1}^{E} \oint_{\Gamma_2^e} \{N\}\ (q - h\ T^{ac})\ d\Gamma \quad , \qquad (11)$$

$$\bar{\bar{F}} = - \sum_{e=1}^{E} \oint_{\Gamma_2^e} \{N\}\ h^r\ T^{ar}\ d\Gamma \quad , \qquad (12)$$

where the summations are taken over the individual finite element contributions. These integrals are evaluated numerically using Gauss-Legendre quadrature in the applications to be presented later.

The system of nonlinear equations (8) through (12) which defines the discretized problem can be solved using many different types of integration schemes. The implicit one-step Euler backward difference method is employed in this analysis. The time derivative of the temperature is approximated by

$$\frac{\partial\{T\}}{\partial t} = \frac{\{T\}_{(i+1)\Delta t} - \{T\}_{(i)\Delta t}}{\Delta t} \qquad (13)$$

where $\{T\}_{(i)\Delta t}$ is assumed known at time $(i)\Delta t$. In the nonlinear analysis, $\{T\}_{(i+1)\Delta t}$ is calculated using a computational scheme that iterates on the out-of-balance heat flow rate for a given time step. At time $(i+1)\Delta t$, the initial approximation of the node point temperatures is calculated by

$$(\frac{1}{\Delta t}[C]_{(i)\Delta t} + [K]_{(i)\Delta t})\{T\}^{(0)}_{(i+1)\Delta t} = \frac{1}{\Delta t}[C]_{(i)\Delta t}\{T\}_{(i)\Delta t} - \{\overline{F}\}_{(i+1)\Delta t} - \{\overline{\overline{F}}\}_{(i)\Delta t} \quad (14)$$

As demonstrated in Reference (25), the $(P)^{th}$ correction $\{\Delta T\}^{(P)}$ to the temperature vector $\{T\}_{(i+1)\Delta t}$ is given by

$$[S]^{(P-1)}_{(i+1)\Delta t}\{\Delta T\}^{(P)} = -\Big[[S]^{(P-1)}_{(i+1)\Delta t}\{T\}^{(P-1)}_{(i+1)\Delta t} - \frac{1}{\Delta t}[C]^{(P-1)}_{(i+1)\Delta t}\{T\}_{(i)\Delta t} + \{\overline{F}\}_{(i+1)\Delta t} + \{\overline{\overline{F}}\}^{(P-1)}_{(i+1)\Delta t}\Big] \quad (15)$$

where

$$[S]^{(P-1)}_{(i+1)\Delta t} = \frac{1}{\Delta t}[C]^{(P-1)}_{(i+1)\Delta t} + [K]^{(P-1)}_{(i+1)\Delta t} \quad (16)$$

is evaluated using temperatures $\{T\}^{(P-1)}_{(i+1)\Delta t}$.

In each iteration, a new temperature vector is computed according to

$$\{T\}^{(P)}_{(i+1)\Delta t} = \{T\}^{(P-1)}_{(i+1)\Delta t} + \{\Delta T\}^{(P)} \quad (17)$$

The iteration continues until convergence is obtained according to the criterion

$$||\{\Delta T\}^{(P)}|| \;/\; ||\{T\}^{(P)}_{(i+1)\Delta t}|| < \text{TOL1} \quad , \quad (18)$$

where TOL1 represents an adjustable tolerance.

The procedure represented by equations (14) through (18) is repeated in each time step of the calculation.

In this application of the finite element method to the inverse problem, the analysis is limited to a one-dimensional model expressed in cylindrical coordinates. The temperatures are assumed to be spatially dependent only upon the radial coordinate r, and an isoparametric (18) discretization is employed,

$$r = \sum_{I=1}^{M} N_I r_I \tag{19}$$

so that r is interpolated using the same functions N_I as those used for T in equation (6). Both linear and quadratic interpolation functions are used in the application to be presented later. These functions are defined for the element natural coordinate system depicted in Figure 1 as follows:

Linear:

$$N_1 = -\frac{1}{2}(\eta - 1) \qquad N_2 = \frac{1}{2}(\eta + 1) \tag{20}$$

Quadratic:

$$N_1 = \frac{1}{2}(\eta^2 - \eta) \quad , \quad N_2 = \frac{1}{2}(\eta^2 + \eta)$$

$$N_3 = 1 - \eta^2 \tag{21}$$

For the quadratic element, the center node (Figure 1) can be reduced out on the element level using static condensation procedures (26).

FORMULATION OF THE INVERSE PROBLEM

For the purposes of this study, the one-dimensional problem of a cylindrical body with flux boundary conditions at the surface is considered as depicted in Figure 2. The conditions

$$\frac{\partial T}{\partial r} = 0 \quad \text{at} \quad r = 0 \tag{23}$$

$$T(r^p,t) = T^p(t) \quad \text{at} \quad r = r^p < a \tag{23}$$

are prescribed, while the surface heat flux

$$-\,k\,\frac{\partial T}{\partial r} = q(a,t) \quad \text{at} \quad r = a \tag{24}$$

is unknown.

For convenience, a solid cylinder is assumed, but a hollow cylinder with any known boundary condition at the inner surface could be used. The material properties k and c are known functions of temperature T and spatial variable r. The problem is to determine q(a,t) and the spatial temperature distribution T(r,t), $0 \leq r \leq a$, when the temperature history $T(r^p,t) = T^p(t)$ is known at an interior point $r^p < a$.

The method developed by Beck (14), with certain modifications suggested by Muzzy, et al (17), is used in the solution of the nonlinear inverse problem presented here. Beck's technique focuses on the observation that the temperature response at an interior location is delayed and damped with respect to changes at the surface of the body, as verified by Burggraf's exact linear solution (4). To effectively deal with this observation, Beck determines the surface heat flux q(a,t) at time t using interior temperatures T^p measured at times greater than t. A common difficulty with other numerical inverse procedures (Reference (1), for example) is the occurrence of violent oscillations or instabilities in the calculated heat flux when the time steps are reduced to sufficiently small values. Beck's approach permits the use of small time steps for improved accuracy in the heat flux calculations without encountering these instabilities. His method also tends to reduce oscillations in the computed surface flux due to experimental errors incurred in measurement of the interior temperatures T^p.

In the application of Beck's method, the surface heat flux is represented by a vector of elements $(q_0, q_1, \ldots, q_n)$ such that in a given time step Δt, $q(a,t)$ is represented by

$$q(a,t) = q_{(i)\Delta t} \qquad (i-1)\Delta t < t \leq (i)\Delta t \quad , \quad i \geq 1 \tag{25}$$

For a given $i \geq 1$, it is assumed that $q_{1\Delta t}, q_{2\Delta t}, \ldots, q_{(i)\Delta t}$ are known. To determine $q_{(i+1)\Delta t}$, an analysis interval consisting of $J \geq 1$ time steps is selected, as depicted in Figure 3. In the next step of the calculation, q is estimated over the analysis interval $(i)\Delta t < t \leq (i+J)\Delta t$ using relations that take the trend of q into account.[2] For the first time step in the analysis interval,

$$q_{(i+1)\Delta t} = q_{(i)\Delta t} + (q_{(i)\Delta t} - q_{(i-1)\Delta t}) \tag{26}$$

and for the "future" time steps

$$q_{(i+j)\Delta t} = q_{(i+j-1)\Delta t} + \beta(q_{(i+j-1)\Delta t} - q_{(i+j-2)\Delta t}) \tag{27}$$

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[2] In his paper, Beck examines both constant and linearly varying heat flux estimates over the analysis interval.

for $2 \leq j \leq J$, where $0 \leq \beta \leq 1$ is an adjustable parameter.[3] Then the boundary value problem (equations (1) through (5)), cast in the discretized finite element formulation (equations (8) through (12)), is solved over the analysis interval $(i)\Delta t < t \leq (i+J)\Delta t$, using conditions (26) and (27).

The objective of the method is to select $q_{(i+1)\Delta t}$ to achieve the closest agreement in a least squares sense between the computed and input temperatures at r^p over the analysis interval. This is accomplished by minimizing the weighted sum of squares function

$$f(q) = \sum_{j=1}^{J} w_j (T_{(i+j)\Delta t} - T^p_{(i+j)\Delta t})^2 \tag{28}$$

with respect to the parameter $q_{(i+1)\Delta t}$. In equation (28), the weights are defined by $w_j = j^2$ and $T_{(i+j)\Delta t}$, $T^p_{(i+j)\Delta t}$ are the computed and input temperatures at the interior point r^p. The minimization is done using an iterative procedure that involves direct sampling of the function (28) and adjustment of $q_{(i+1)\Delta t}$ in each iteration.[4]

The solution value of $q_{(i+1)\Delta t}$ is taken as the accepted value of $q(a,t)$ over the single time step Δt only. The analysis interval is shifted by one time step and the process is repeated. For the special case $J = 1$, no future temperatures are used and least squares minimization is not required.

For convective boundary conditions, a heat transfer coefficient can be computed in each time step from the expression

$$h_{(i)\Delta t} = \frac{q_{(i)\Delta t}}{(T^w_{(i)\Delta t} - T^{ac}_{(i)\Delta t})} \tag{29}$$

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[3] q_0 is determined from conditions at the initial time.

[4] In his formulation, Beck uses an analytical scheme to minimize the summed square function f.

As an alternate formulation, the iteration scheme outlined above can be performed on the heat transfer coefficient $h_{(i)\Delta t}$ and the surface heat flux then computed from

$$q_{(i)\Delta t} = h_{(i)\Delta t}(T^{w}_{(i)\Delta t} - T^{ac}_{(i)\Delta t}) \tag{30}$$

The latter scheme is employed by Muzzy, et al (17) in a finite difference application of Beck's method.

Some of the modifications to Beck's method suggested in Reference (17) have been implemented in the procedure presented here. First of all, a weighted least squares criterion is used in the function (28). The weights reflect that the temperature difference at time $(i+j)\Delta t$ has more influence on $q_{(i+1)\Delta t}$ for increasing j over the analysis interval consisting of an appropriate number of time steps. Beck's formulation is obtained by defining $w_j \equiv 1$ for all j.

Secondly, before minimization of the summed square function (28) proceeds for a given analysis interval, $q_{(i+1)\Delta t}$ is adjusted iteratively to satisfy the requirement

$$|T_{av} - T^{p}_{av}| < TOL2 \tag{31}$$

for some prescribed TOL2 > 0, where T_{av} is given by

$$T_{av} = \frac{1}{W} \sum_{j=1}^{J} w_j \, T_{(i+j)\Delta t} \quad , \quad W = \sum_{j=1}^{J} w_j \tag{32}$$

and T^{p}_{av} is similarly defined. The resultant estimate for $q_{(i+1)\Delta t}$ is then refined in the minimization procedure for the function (28). This ensures that the input and computed temperatures at r^{p} agree closely in an averaged sense before minimization of the summed square function (28) is carried out. Otherwise, it may be possible for the algorithm to calculate $q_{(i+1)\Delta t}$ from a relative minimum rather than from the true minimum of the function (28) on the analysis interval. This problem is discussed in more detail in Reference (17).

A crucial factor in Beck's formulation is the relationship between the magnitude of the time step Δt and the required number of time steps J in the analysis interval, given a temperature probe located a distance L from the heated surface. Beck (14) explores this relationship by studying sensitivity coefficients that define the temperature change at an interior point due to a unit step in surface heat flux. He examines a one-dimensional model with the temperature probe fixed at distance $\frac{L}{a} = 1$ from the surface. Using the criteria derived from the sensitivity coefficients for this model, Beck recommends values of J that are appropriate for given values of the dimensionless time step $\Delta\tau = \frac{\alpha \Delta t}{a^2}$. The value of J is increased as the magnitude of $\Delta\tau$ is reduced, roughly preserving the length $J \cdot \Delta\tau$ of the analysis interval. Muzzy, et al (17) also study this relationship in applying Beck's formulation. Some additional results are presented in the numerical applications in this report. For a detailed discussion of this topic, the reader is referred to Beck's paper.

NUMERICAL APPLICATIONS

The inverse formulation developed in the preceding sections is applied here to a composite rod containing an electric heating element and thermocouple sensors. This heater rod represents one member of a 49-rod array that is designed for test purposes to simulate a nuclear fuel bundle. The heater rod bundle is positioned in a thermal-hydraulics test loop that is used to study hypothetical loss-of-coolant accidents in pressurized-water nuclear reactors (27).[5]

A heater rod cross section[6] and the corresponding one-dimensional finite element discretization used in the inverse analysis are depicted in Figures 4 and 5. The electric heater rods are from 548.64 to 640.08 cm (18 to 21 ft) in length, 1.077 cm (0.424 in.) in diameter, and have dual-sheath design. The outer sheath is 0.025 cm thick (0.010 in.) stainless steel; the inner sheath is 0.076 cm thick (0.030 in.) stainless steel and is grooved to accept the 0.051 cm (0.020 in.) chromel vs. alumel thermocouples. The next inner layer is boron nitride (BN), which electrically insulates the heating element from the stainless steel sheaths. In the section of the rod from which the cross section of Figure 4 is extracted, the heater element consists of an Inconel 600 tube.[7] The core of the heater element is filled with magnesium oxide (MgO), which is both a filler and insulator between the heating element and the central rod thermocouple sheaths.

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[5]This test facility is operated by the Oak Ridge National Laboratory (ORNL) Pressurized-Water Reactor Blowdown Heat Transfer Separate-Effects Program, which is part of the overall light-water reactor safety research program of the Nuclear Regulatory Commission.

[6]The heater rod cross section selected for the test models is that one identified in Reference (12) for LEVEL G (ZONE I).

[7]As described in Reference (12), the heater element configuration and heater output vary over the length of the rod.

The transient response of the heater rod is modeled as a coupled heat conduction and mechanical deformation problem due to the presence of a small air gap between the stainless steel sheaths that varies in width with temperature. The fabrication process that reduces the heater rod to its final diameter often creates an imperfect fit between the inner and outer sheaths at the thermocouple locations and produces a gap between the thermocouple junction and the outer sheath. The thermocouple is welded to the inner sheath, causing the gap between the junction and outer sheath to grow with increasing fluid temperature and to close with increasing heater power. Correspondingly, the change in the gap width alters the temperature profile in the cross section.

A one-dimensional model developed in Reference (12) is used to model the mechanical response of the gap:

$$\Delta r_{gap} = \Delta r_{gap_0} + r_{16}(\mathrm{EXP}[C_1(T_{16} - T_{16_0}) + \frac{C_2}{2}(T_{16}^2 - T_{16_0}^2) + \frac{C_3}{3}(T_{16}^3 - T_{16_0}^3)] - 1) - r_{15}(\mathrm{EXP}[C_1(T_{15} - T_{15_0}) + \frac{C_2}{2}(T_{15}^2 - T_{15_0}^2) + \frac{C_3}{3}(T_{15}^3 - T_{15_0}^3)] - 1) \quad (33)$$

In equation (33), the quantities Δr_{gap_0}, T_{16_0}, T_{15_0} are the bias gap and bias nodal temperatures determined in an initial steady-state configuration. The expansion coefficients $C_i (i=1,3)$ are determined in situ as part of a rod calibration procedure (28) for each test. In calculating the thermo-mechanical response of the heater rod model, the gap width Δr_{gap} and the appropriate geometric variables of the finite element model (equations (8) through (12)) are adjusted in each iteration of the solution process described in equations (14) through (18).

The thermophysical properties of thermal conductivity k and specific heat c are dependent upon temperature and the spatial coordinate. Except for the thermal conductivities of MgO and BN, these properties are determined for each material as a function of temperature from an optimum polynomial fit to available data, as given in Reference (12). The thermal conductivities for the MgO and BN depend on packing density and must be determined in situ as part of the rod calibration procedure (28) prior to each test.

The first numerical example[8] was selected to evaluate the performance of the technique in solving the inverse problem for the finite element model of Figure 5. The periodic surface heat flux depicted in Figure 6 was used as boundary condition input for a direct solution. This boundary condition is included because the ramp in heat flux is typical of surface transients in the test loop and because the finite element formulation used by Hore, et al (24) demonstrated divergence in the surface heat flux for a similar periodic problem. The temperature transient of Figure 6 was calculated at the thermocouple node 14 of the discrete model using a heat generation rate fixed at $Q = 9.19 \times 10^3$ watts/cm^3. With the temperature transient of Figure 6 serving as input, the corresponding inverse analysis was performed in an attempt to recreate the periodic surface heat flux boundary condition. Computed results were obtained using no future temperatures, one, and two

[8]The finite element inverse calculations described in this section were performed using TOL1=.001, equation (18); TOL2=1.0, equation (31); β=0.5, equation (27); Δt=0.05 seconds, which is equal to the data acquisition interval for the thermocouple sensors in the heater rod. For each analysis, the iterative procedure for minimizing the summed square function (equation (28)) was terminated when the uncertainty in the value of $q_{(i+1)\Delta t}$ was less than 1%.

future temperatures (corresponding to J = 1, 2, and 3) in the inverse solution. In Figures 7 through 9, the surface heat flux calculated for each J value is compared with the input boundary condition of the direct problem. The calculated and input thermocouple temperatures at node 14 are also compared for each case in these figures; however, the error in temperatures (TOL2 = 1.0) is not discernible on the scale of these plots. All three inverse solutions follow the input surface flux of the direct problem. The solutions using future temperatures reduce oscillations in the computed surface heat flux, but tend to "round off" rapid changes as J is increased. For the finite element model of Figure 5 and a selected time step of Δt = .05 seconds, the use of one future temperature appears optimal for reducing oscillations.

In the second numerical example, the inverse formulation is applied to an actual thermocouple transient taken from a representative test of the ORNL thermal-hydraulics facility.[9] The heater power input to the rod during this transient is depicted in Figure 10. Figures 11 through 13 depict the results of the finite element inverse analysis for the first 10 seconds of the test, the significant period of the transient. Included are plots of surface heat flux and surface temperatures for solutions using no future temperatures, one, and two future temperatures. Figure 10 also compares the thermocouple temperatures computed in the inverse solution for J = 2 with the data thermocouple temperatures; as in the first test case, the error in temperatures is not discernible on the scale of these plots. Results from the first test problem (Figures 6 through 9) suggest that

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[9]The thermocouple transient used in this analysis was recorded at thermocouple TE-325BG of rod bundle 1 in blowdown test 105 (29). The position of this thermocouple in bundle 1 and a complete description of the rod geometry are given in Reference (12).

the use of one future temperature in the inverse analysis is adequate to remove some of the "roughness" from the computed results without severe rounding of rapid changes in surface heat flux.

The measured thermocouple transient (Figure 10) examined in this study was also analyzed in Reference (29) using a one-dimensional finite difference inverse formulation developed by Ott and Hedrick (12). Results of this analysis, depicted in Figures 11 through 13, indicate good agreement between the finite element and the finite difference inverse techniques for the rod configuration of Figure 4. Agreement between the two techniques is equally good for other thermocouple transients that were analyzed and presented in Reference (25).

The inverse formulation described here has been used in Reference (30) to analyze the quenching of 304 stainless steel thick cylinders in liquid nitrogen. Analyses were performed for values of time up to 600 seconds on numerical test cases with known solutions and on experimental data collected at interior thermocouple sensors. None of the inverse analyses performed thus far on the quenched cylinders or on the electrically-heated rods have demonstrated any instability in the surface heat flux calculations. This contrasts sharply with the finite element formulation of Reference (24) in which the accuracy of the heat flux predictions progressively deteriorated with time for both example problems presented.

SUMMARY AND CONCLUDING REMARKS

In this paper, a formulation of the nonlinear inverse heat conduction problem has been presented that is applicable to composite bodies with temperature-dependent thermophysical properties. This formulation is based on a finite element model of the direct problem and on Beck's nonlinear estimation procedure. Applications of the inverse technique to an electrically-heated composite rod were examined in this study. In the first example, a periodic heat flux was imposed on the surface of the rod. The inverse calculations followed the input surface heat flux of the direct problem, with the use of one "future" temperature optimal for reducing oscillations without severe "rounding" of rapid changes in the computed flux. Finally, the technique was applied to an actual thermocouple transient recorded at an interior thermocouple sensor in the rod. For this transient, the results from the finite element inverse analysis were found to be in good agreement with those obtained from a finite difference inverse technique. None of the analyses performed thus far using the inverse technique developed here have demonstrated numerical instabilities in the heat flux calculations such as those encountered in the finite element formulation of Reference (24).

The results presented here clearly demonstrate that the inverse formulation based on the finite element technique and Beck's second method is capable of successfully treating experimental data. Consideration of "future" temperatures in calculating surface heat flux permits the use of small dimensionless time steps while avoiding severe oscillations or numerical instabilities in the computed results. This technique also reduces oscillations in the calculated heat flux that are due to experimental errors incurred in temperature measurements.

Studies are under way to extend the formulation presented here to treat the coupled inverse heat conduction-thermal deformation problem in two and three dimensions. While both the finite element and the finite difference methods have been applied successfully in one-dimensional inverse analyses, it is the author's opinion that the finite element technique offers advantages in modeling complex geometries and boundary conditions in a multidimensional system. The compatibility between the finite element heat conduction model and the well-known finite element displacement formulation used in analysis of the mechanical problem is particularly advantageous in these studies.

ACKNOWLEDGMENTS

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LIST OF FIGURES

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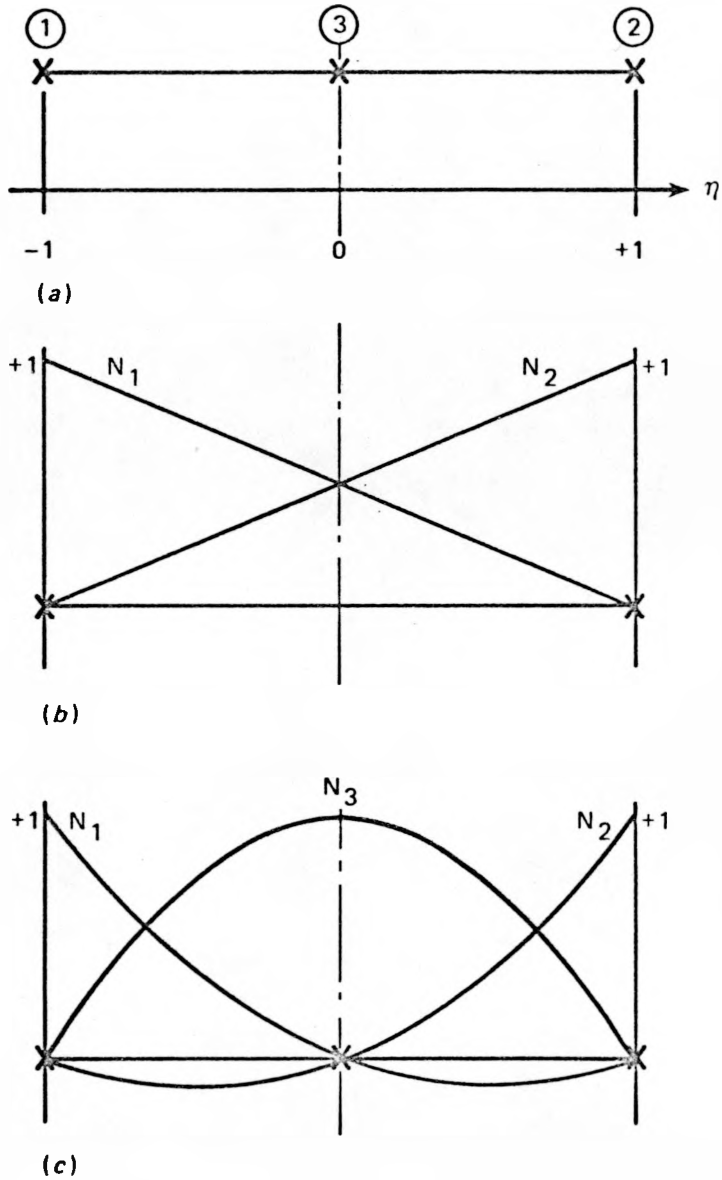

FIGURE 1

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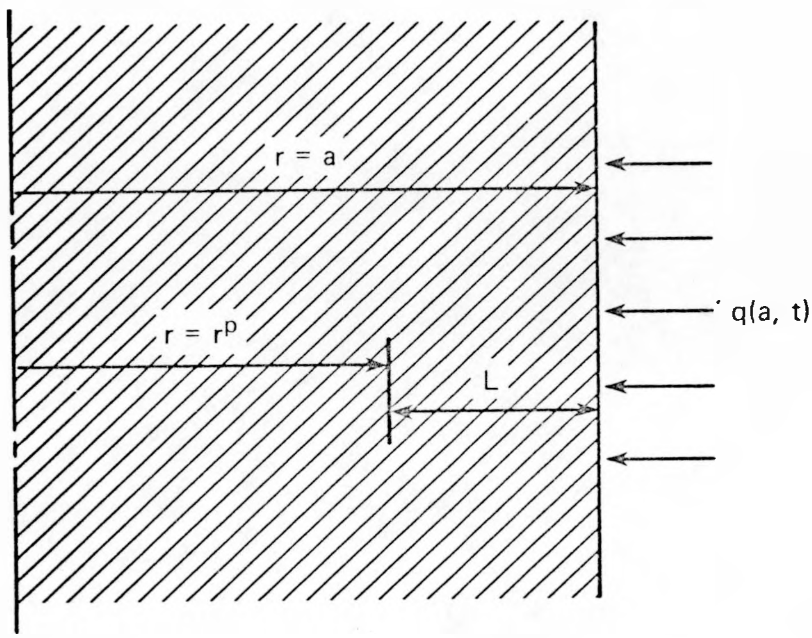

FIGURE 2

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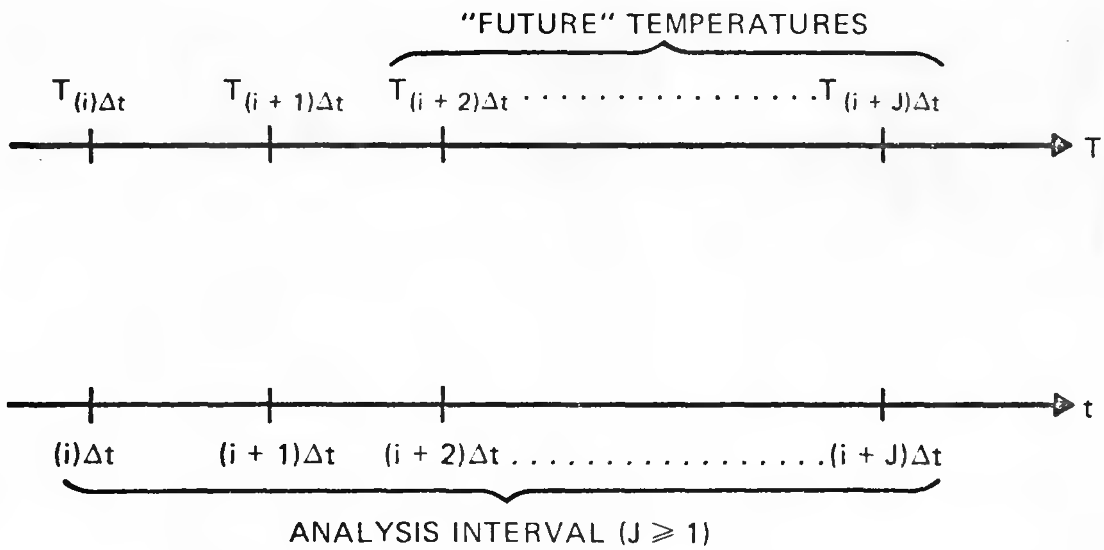

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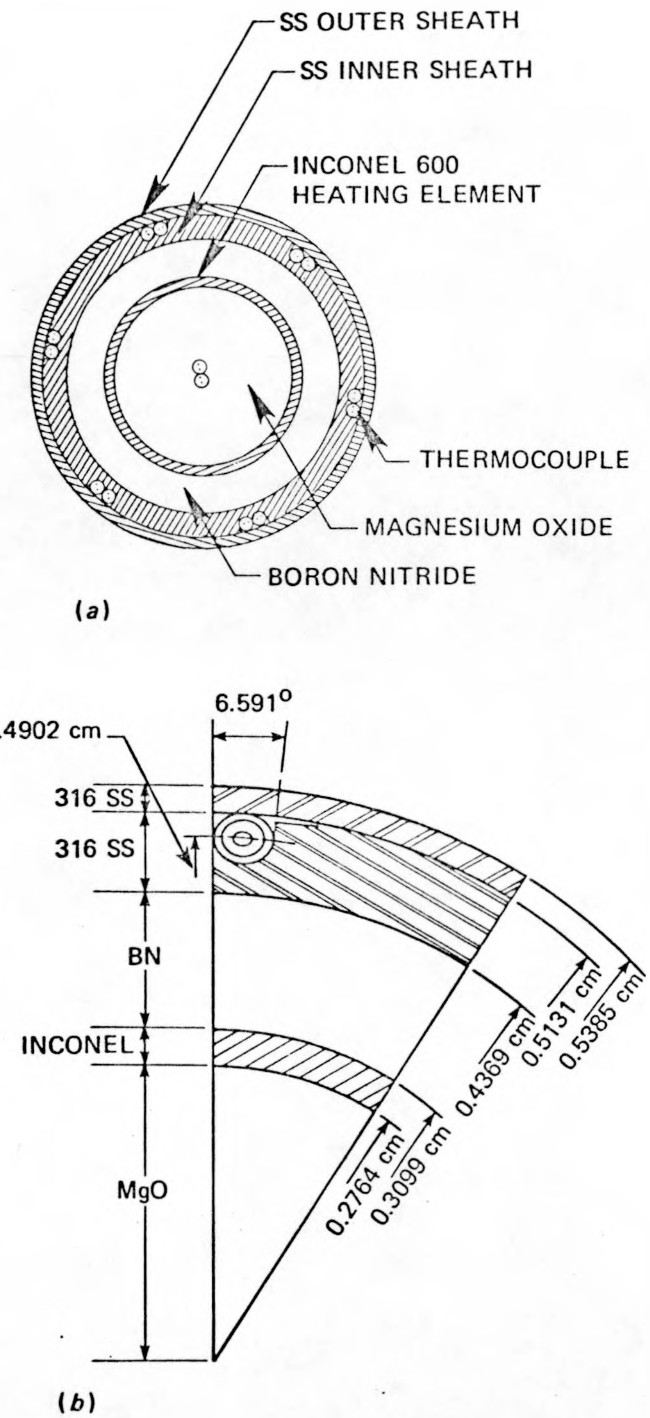

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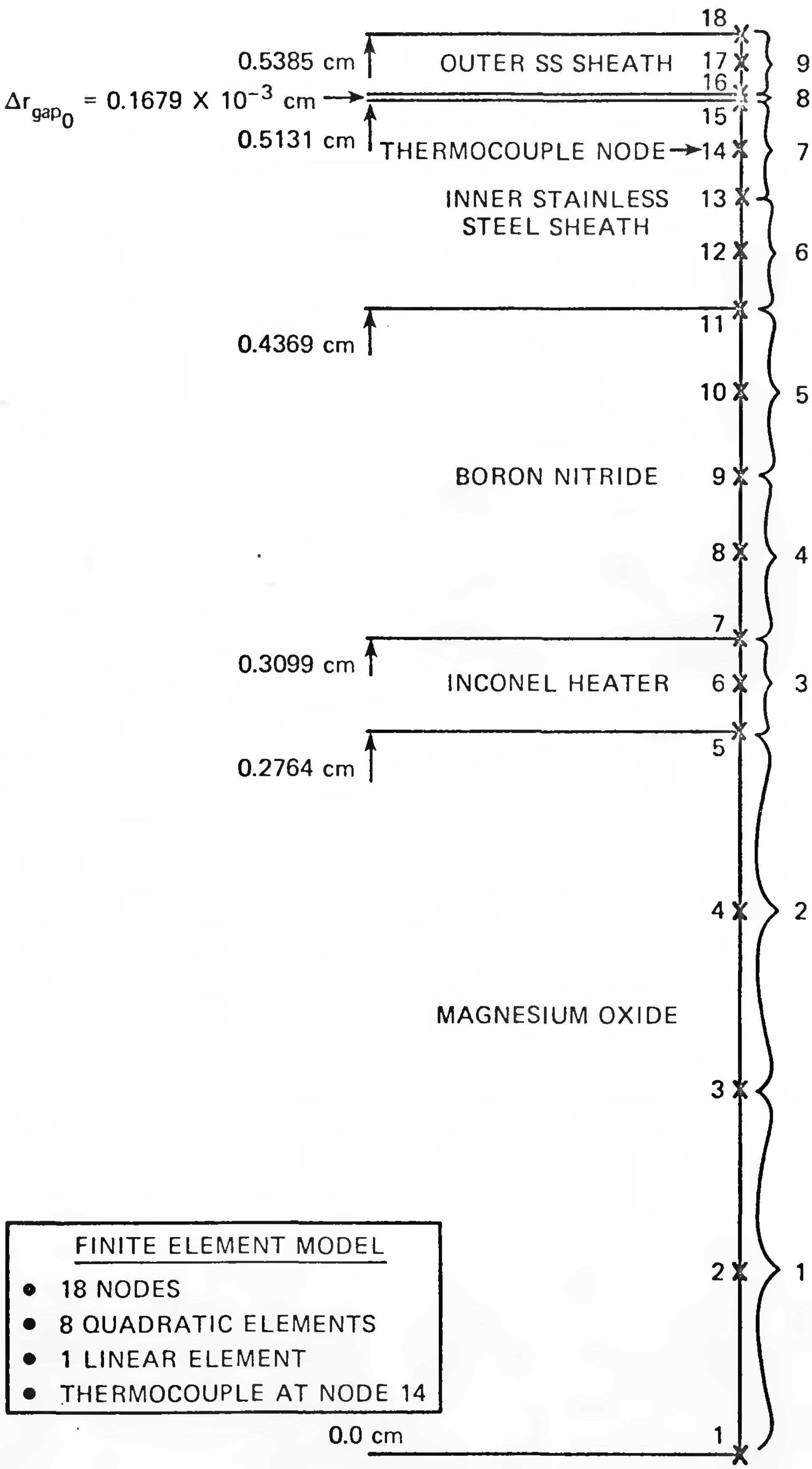

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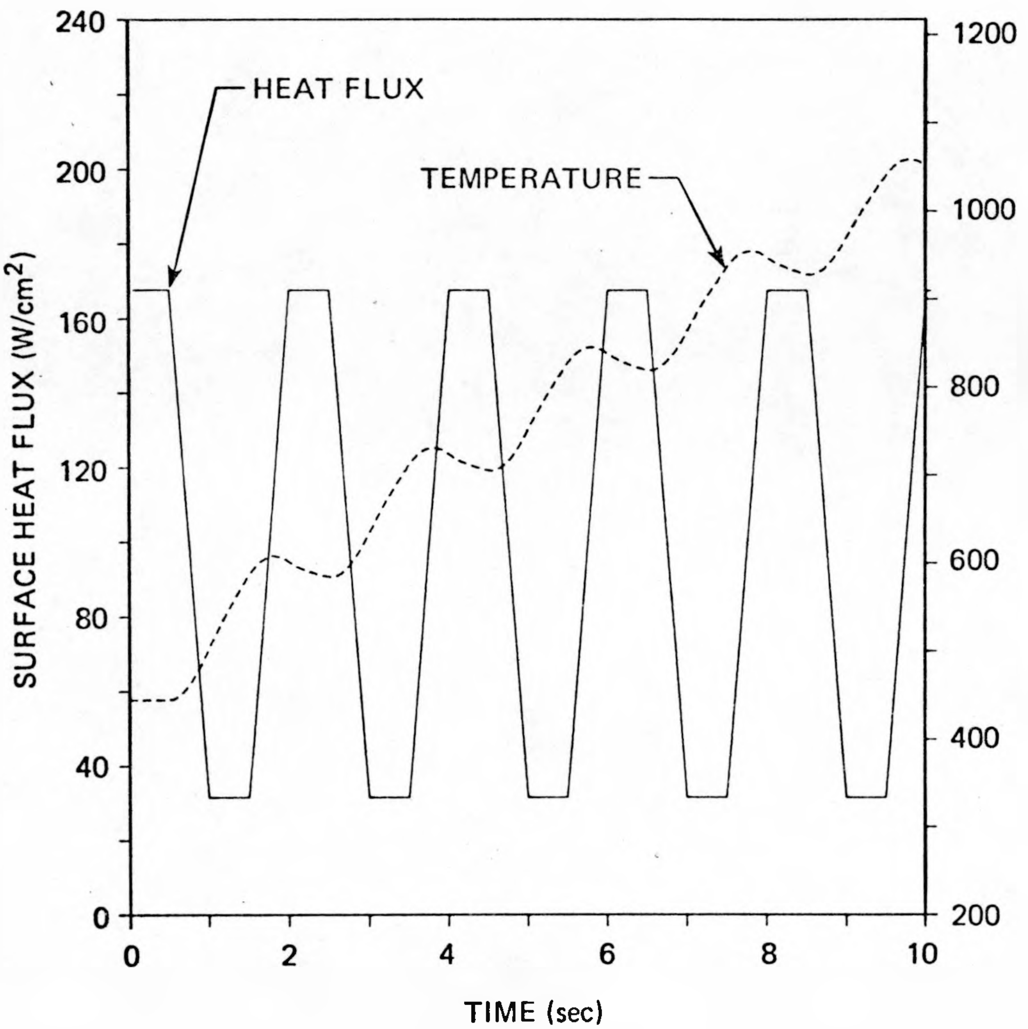

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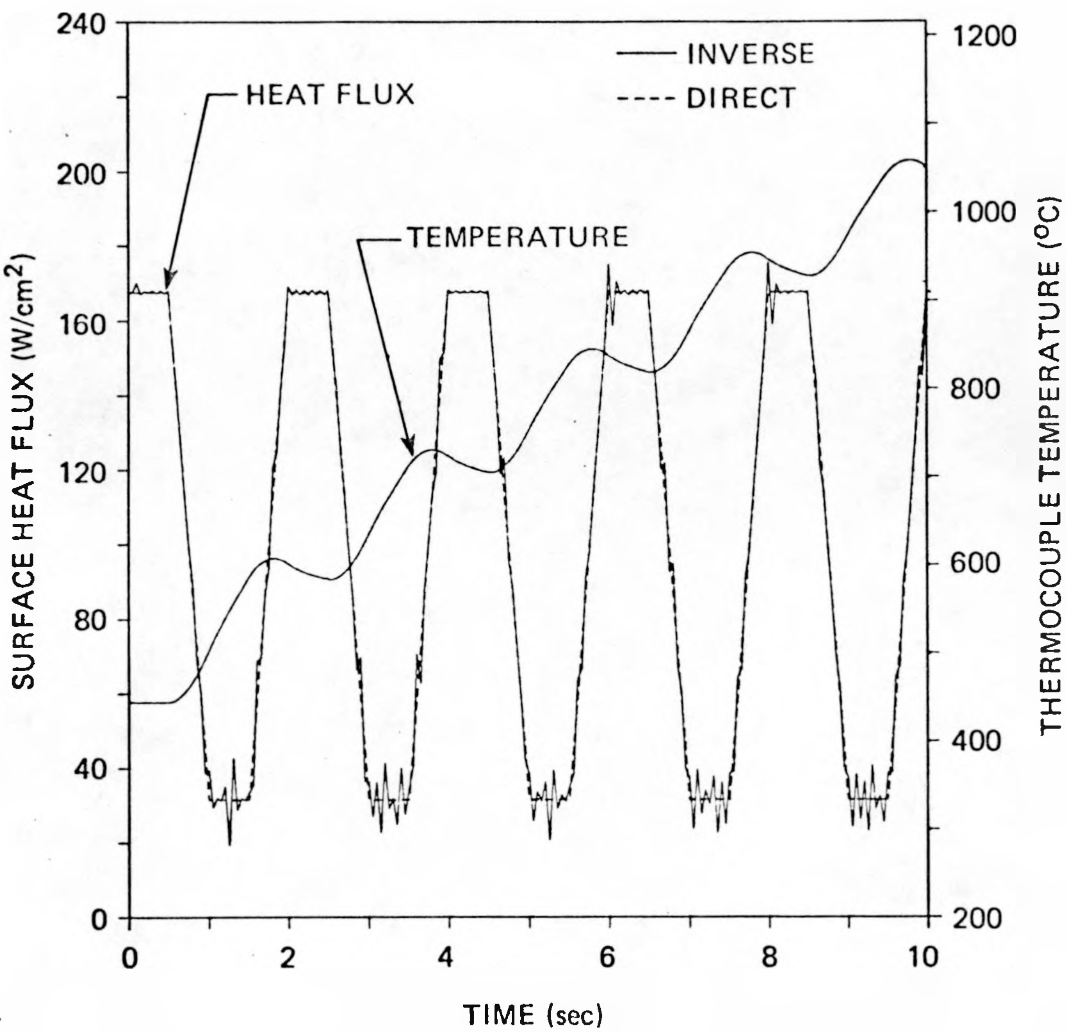

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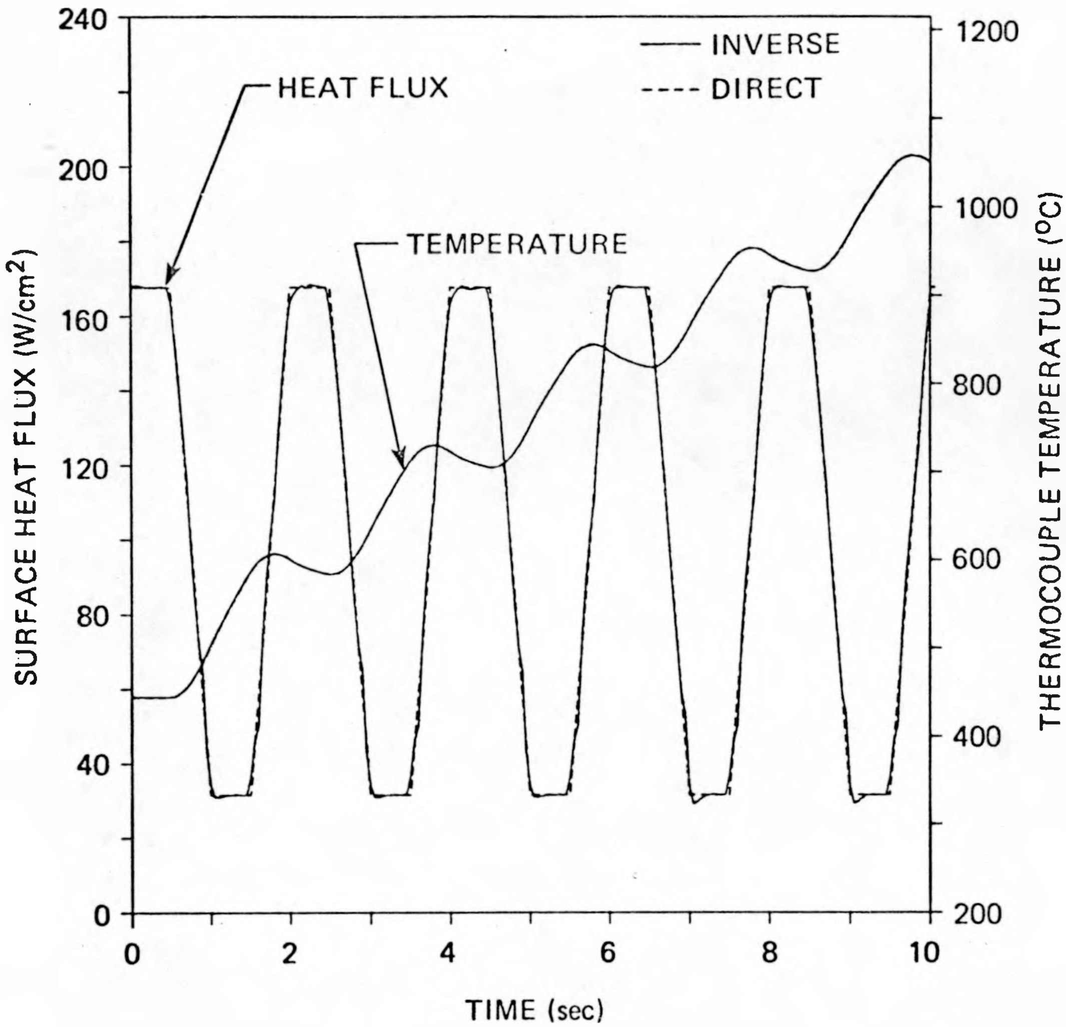

FIGURE 8

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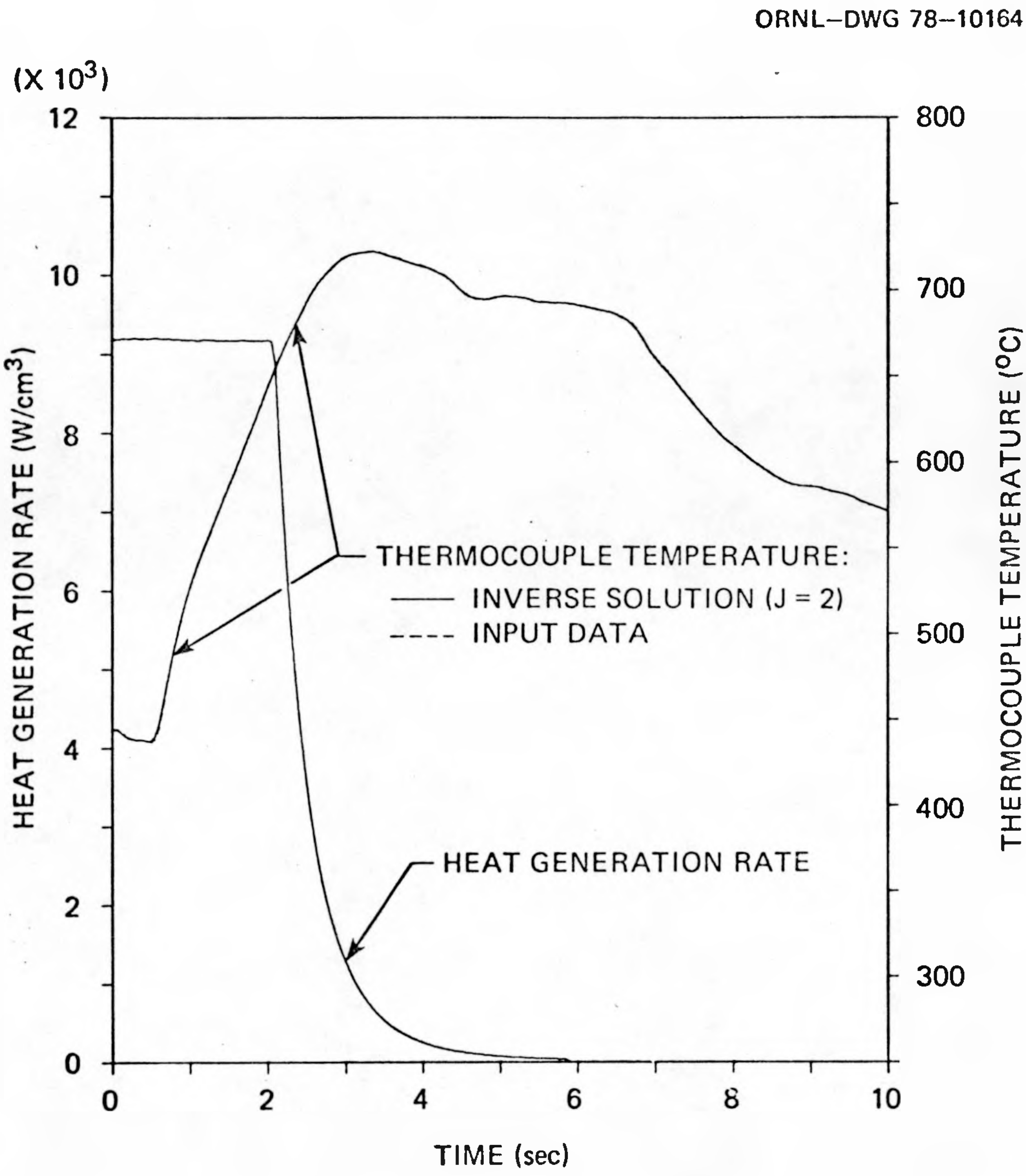
ORNL–DWG 78–10164
(X 10^3)
HEAT GENERATION RATE (W/cm^3)
THERMOCOUPLE TEMPERATURE (°C)
THERMOCOUPLE TEMPERATURE:
—— INVERSE SOLUTION (J = 2)
---- INPUT DATA
HEAT GENERATION RATE
TIME (sec)
0
2
4
6
8
10
12
300
400
500
600
700
800

FIGURE 10

ORNL–DWG 78–10165

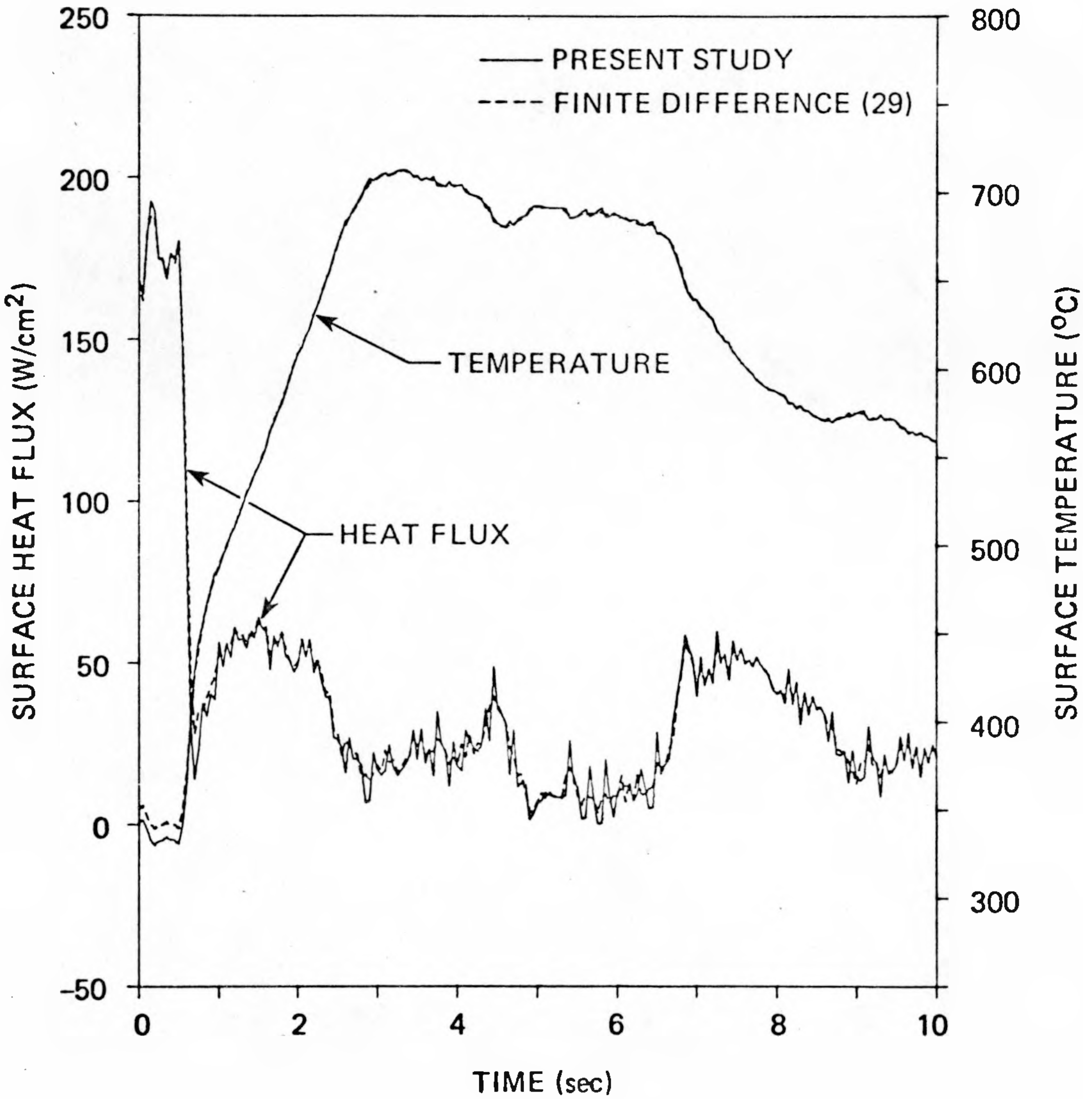

FIGURE 11

ORNL–DWG 78–10166

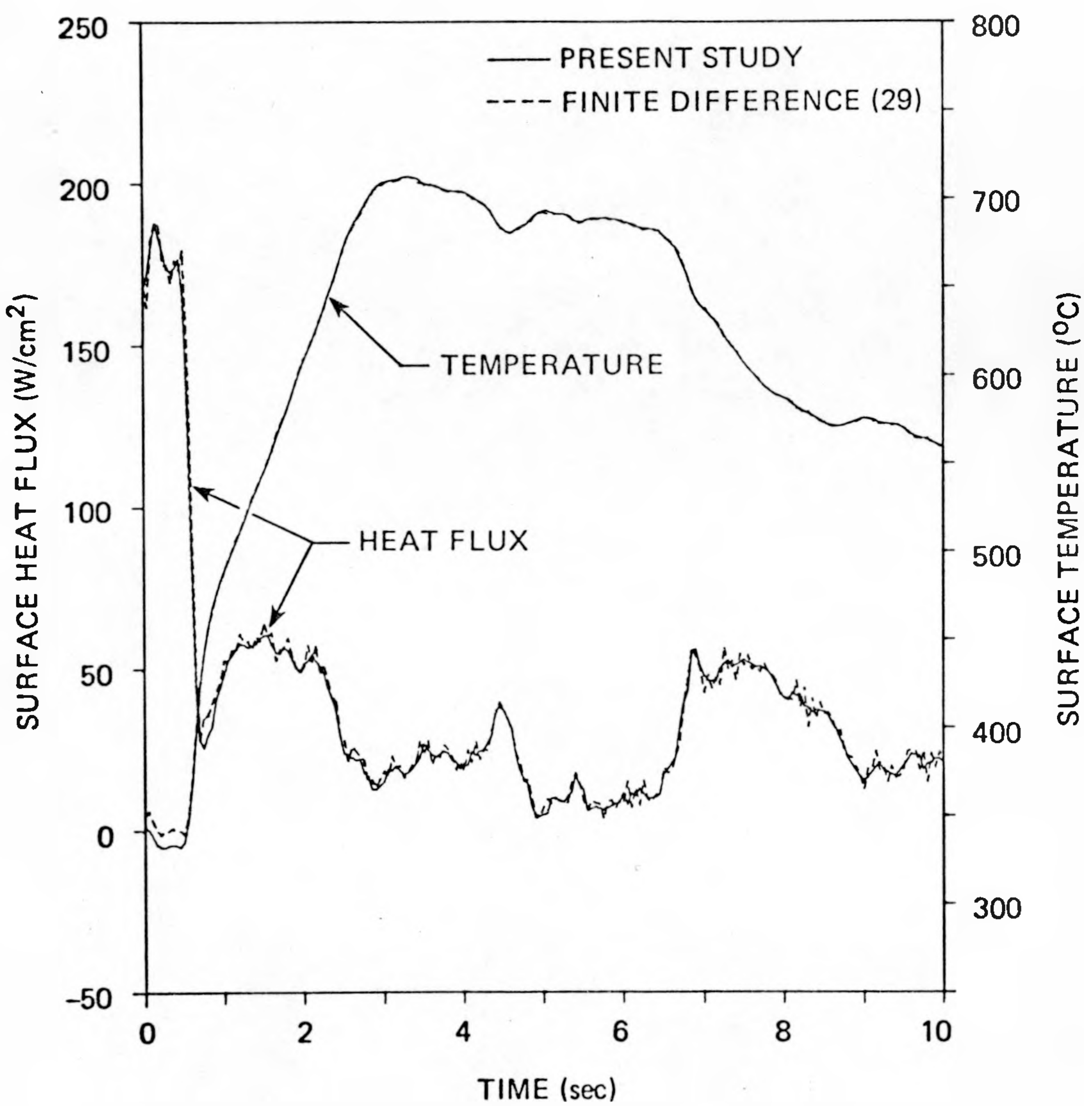

FIGURE 12

ORNL–DWG 78–10167

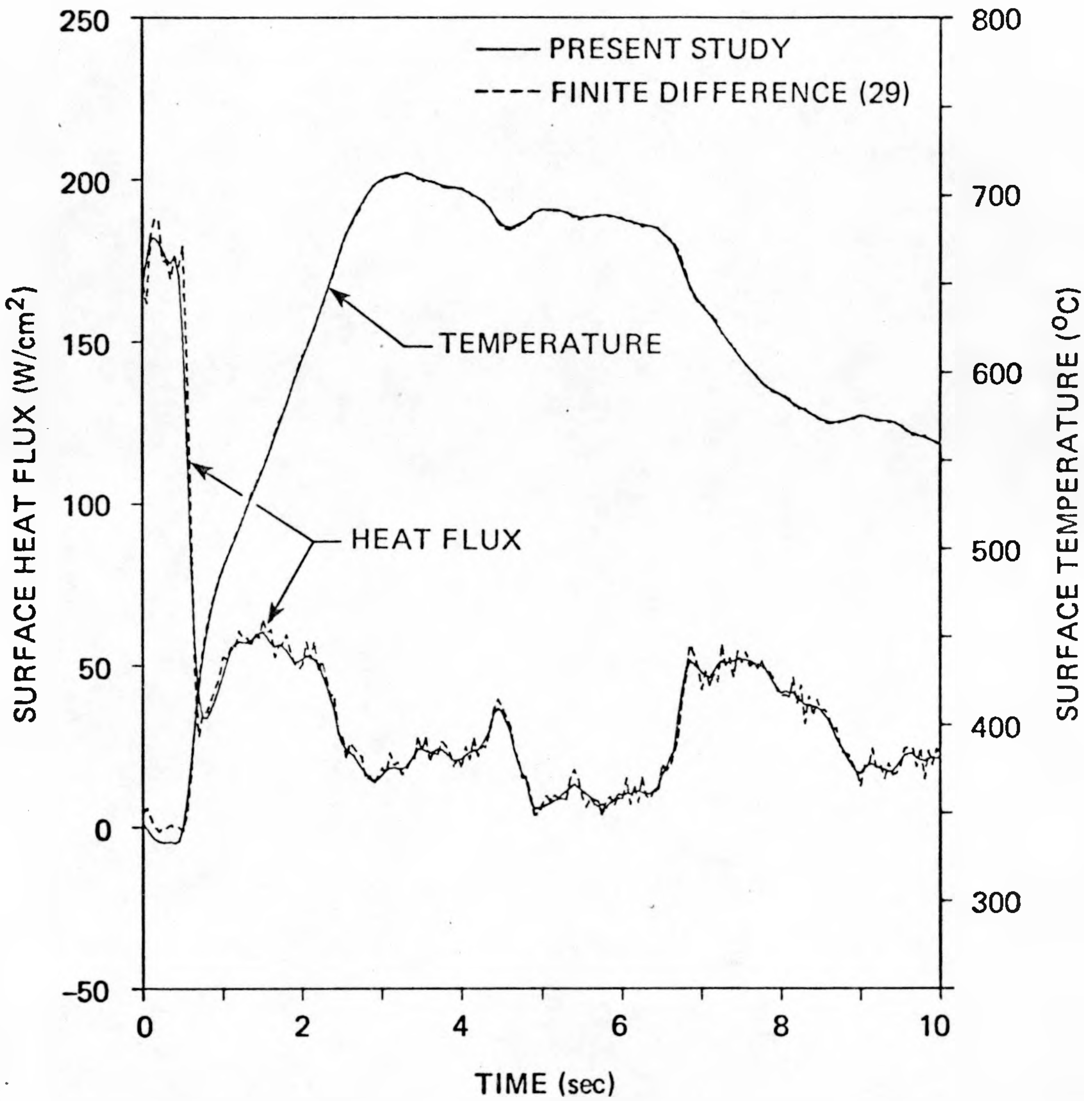

FIGURE 13