MASTER

ORO 5126-42

UNITARITY OF THE BENCZE-REDISH-SLOAN EQUATIONS

Wayne Polyzou
Department of Physics and Astronomy
University of Maryland, College Park, Maryland 20742 U.S.A.

May 1978

U. of Md. TR #78-094
U. of Md. PP #78-207

UNIVERSITY OF MARYLAND
DEPARTMENT OF PHYSICS AND ASTRONOMY
COLLEGE PARK, MARYLAND

DISCLAIMER

ORO 5126-42
U. of Md. TR #78-094
U. of Md. PP #78-207

UNITARITY OF THE BENCZE-REDISH-SLOAN EQUATIONS

Wayne Polyzou
Department of Physics and Astronomy
University of Maryland, College Park, Maryland 20742 U.S.A.

May 1978

ABSTRACT

We show that the N-particle scattering equations of Bencze, Redish and Sloan[1] satisfy a proper unitarity relation. Our proof uses a recently proposed form of these equations given by Benoist-Gueutal, L'Huillier, Redish and Tandy.

KEYWORD ABSTRACT

NUCLEAR REACTIONS N-body problem, unitarity, connected kernel equations.

1. INTRODUCTION

Unitarity of the scattering matrix is a fundamental property that should be satisfied in any quantum mechanical scattering theory. In the past few years there have been many reformulations of the N-body Schrödinger equation as an integral equation whose kernel becomes compact after a finite number of iterations.[1] The advantage of compact kernel equations is that they are both theoretically well understood, and easy to treat numerically. One of the difficulties with these formulations is that the manipulations making the kernel compact often mix the dynamics in an unphysical way. In any application where one approximates these operators, it is important to understand the physics behind them to make a justifiable approximation. One tool in understanding the physics contained in these operators is unitarity. Through unitarity we can identify parts of the kernel responsible for the production of flux in the different scattering channels.

Because these theories are all equivalent to Schrödinger theory, it would be surprising if any of them failed to satisfy unitarity. Perhaps for this reason, there has been little interest in giving explicit proofs of unitarity. We argue that the interest in an explicit proof of unitarity is in understanding the physics contained in the N-body operators, rather than in giving a proof of a property that is implicitly built into the operators. In this paper we give an explicit proof of the unitarity of the Bencze-Redish-Sloan[2] (BRS) equations. Because our interest is in using the unitarity proof to understand the physical content of our operators, we concern ourselves primarily with the algebraic aspects of the proof, rather than the more difficult mathematical aspects. We prove the unitarity of an equation derived by Benoist-Gueutal, L'Huillier, Redish and Tandy (BLRT) that can be shown to reduce to the BRS equation[3] when we restrict considerations to two-body forces. In the next section we introduce our notation and introduce the BLRT formalism.

2. NOTATION

Consider N non-relativistic particles interacting via pair potentials.[4] We let lower case latin letters, {a,b,c,...} index partitions of N-particles into $n_a, \ldots (2 \leq n_a \leq N)$ clusters. We use the notation $a \subseteq b$ to indicate that the partition b can be obtained from a by joining some (possibly 0) of the clusters of a. The total Hamiltonian of our system is $H = K + V$, where K is the N-particle kinetic energy operator, and V is the sum of all interparticle potentials. For every partition a, V admits the decomposition $V = V_a + V^a$ where V_a is the sum of all interactions internal to the clusters of a, and V^a is the sum of all interactions externel to the clusters of a. In addition we define V_b^a which is the sum of all interactions both internal to the clusters of b and external to those of a. We define partition Hamiltonians $H_a = K + V_a$. The scattering states of the full H asymptotically behave like those eigenstates of H_a that have the particles of each cluster in a bound state. We denote these eigenstates of H_a by $|\phi_a(\alpha_a)\rangle$, where α denotes the internal quantum numbers of the bound clusters. The set of all scattering channels is denoted by A. We let $z = E + i\varepsilon$ and define the usual transition operators

$$T_{+}^{ab}(z) \equiv V^{a} + V^{a}G(z)V^{b} = V^{a}G(z)G_{b}^{-1}(z) \tag{1}$$

$$T_{-}^{ab}(z) \equiv T_{+}^{ab}(z) + V^{b} - V^{a} \tag{2}$$

where

$$G(z) = (z-H)^{-1};\; G_{a}(z) = (z-H_{a})^{-1} \tag{3}$$

$$G(z) - G_{a}(z) = G(z)V^{a}G_{a}(z) = G_{a}(z)V^{a}G(z). \tag{4}$$

We let $C_a = (-)^{n_a}(n_a - 1)!$ and define $\tilde{V}_a^c = C_a V_a^c$. BLRT show that $\tilde{V}_c^a$ defines a channel coupling scheme by the sum rule

$$\sum_{\{c|n_c \geq 2\}} \tilde{V}_c^b \equiv \sum_c{}' \tilde{V}_c^b = V^b. \tag{5}$$

Using (5) in (1) with (4) they write

$$T_{+}^{ab}(z) = \sum_c{}' \tilde{V}_c^a G_c(z) G_b^{-1}(z) + \sum_c{}' \tilde{V}_c^a G_c(z) T_{+}^{cb}(z). \tag{6}$$

This can be reduced to the pair

$$T_{+}^{ab}(z) = \sum_c{}' \tilde{T}^{ac}(z) G_c(z) G_b^{-1}(z) \tag{7}$$

$$\tilde{T}^{ab}(z) = \tilde{V}_b^a + \sum_c{}' \tilde{V}_c^a G_c(z) \tilde{T}^{cb}(z) = \tilde{V}_c^a + \sum_c{}' \tilde{T}^{ac}(z) G_c(z) \tilde{V}_b^c. \tag{8}$$

Equation (8) is the BRS equation in BLRT form, while (7) relates the full transition operator to the BRS operator. BLRT then proceed to show

$$\sum_c{}' \tilde{V}_c^a G_c(z) \tilde{V}_b^c \text{ is connected,} \tag{9}$$

$$\sum_c{}' \tilde{V}_c^a G_c(z) \tilde{V}_b^c = \sum_{\{c|n_c=2\}} K_{BR}^{ac}(z) \tilde{V}_b^c \tag{10}$$

where $K_{BR}^{ac}(z)$ is the kernel originally derived by BR.

3. UNITARITY

Unitarity can be expressed in many ways. For scattering theory, the optical theorem is one of the most natural expressions of unitarity. Its content is simple; flux is conserved and all of the scattered flux comes out in one of the open scattering channels. The most familiar expressions involve matrix elements of transition operators and have the form

$$\mathrm{Im}\langle\phi_a(\alpha_a)|T_+^{aa}(E+io^+)|\phi_a(\alpha_a)\rangle = -\frac{1}{2}|\vec{j}_{inc}|\sum_{\beta_b\epsilon A}\sigma_{\alpha_a\to\beta_b}(E) \qquad (11)$$

where $\vec{j}_{inc}$ is the current of the incident beam and $\sigma_{\alpha_a\to\beta_b}(E)$ is the partial cross section for scattering from the initial channel α_a to a final channel β_b. Expression (11) has an operator analogue which in the center of momentum frame has the structure

$$\lim_{\epsilon\to 0}\{T_+^{ab}(E+i\epsilon) - T_-^{ab}(E-i\epsilon)\} = \sum_{\gamma_c\epsilon A} T_+^{ac}(E+io^+)D_c(\gamma_c;E)T_-^{cb}(E-io^+) \qquad (12)$$

where

$$D_c(\gamma_c;E) = -2\pi i\int d\underset{\sim}{P}(c)|\phi_c(\gamma_c);\underset{\sim}{P}(c)\rangle\delta(E-E(\gamma_c,\underset{\sim}{P}(c)))\langle\phi_c(\gamma_c);\underset{\sim}{P}(c)| \qquad (13)$$

and $\underset{\sim}{P}(c)$ is the relative momenta of the clusters of c. We will prove expression (12) follows from (7), (8) assuming we know how to solve all of our subsystem (2...N-1 body) problems. In what follows it is convenient to introduce the notation

$$A(z) = A(E+i\epsilon)$$

$$A(E) = \lim_{\epsilon\to 0} A(E\pm i\epsilon)$$

$$\Delta_\epsilon A(E) \equiv A(z) - A(z^*)$$

$$\Delta A(E) \equiv \lim_{\epsilon\to 0}\Delta_\epsilon A(E). \qquad (14)$$

4. PROOF OF UNITARITY

In this section we show that the solutions of the BR equations satisfy (12). In order to isolate the algebraic aspects of this proof, we list those results that are not purely algebraic in nature, but are nevertheless needed in the proof. We then discuss these results in the next section. The basic technical results that we need are:

TR1: $$\lim_{\varepsilon\to 0} \sum_c{}' \tilde{T}^{ac}(z)\Delta_\varepsilon G_c(E)T_-^{cb}(z^*) = \sum_c{}' \tilde{T}^{ac}(E^+)\Delta G_c(E)T_-^{cb}(E^-). \tag{15}$$

TR2: $$\lim_{\varepsilon\to 0} \Delta_\varepsilon G_c(E) = \sum_{\{\gamma_d \in A \mid d\subseteq c\}} (1+G_c(E^+)V_c^d)D_d(\gamma_d,E)(1+V_c^dG_c(E^-)). \tag{16}$$

TR3: $$(1+V_c^aG_c(z))T_-^{cb}(z) = T_-^{ab}(z). \tag{17}$$

TR4: $$\tilde{T}^{ab}(z)G_b(z)G_c^{-1}(z)D_c(\gamma_c,E) \underset{\varepsilon\to 0}{\longrightarrow} 0 \text{ if } b \not\subseteq c. \tag{18}$$

Armed with these technical results we begin our proof of unitarity. The first step is to use (1), (5) and the subsystem resolvent relation

$$G_a(z) - G_b(z) = G_a(z)(V^b - V^a)G_b(z) \tag{19}$$

to rewrite our transition operators as

$$T_+^{ab}(z) = V^a + \sum_c{}' \tilde{T}^{ac}(z)G_c(z)V^b \tag{20}$$

$$T_-^{ab}(z) = V^b + \sum_c{}' \tilde{T}^{ac}(z)G_c(z)V^b. \tag{21}$$

Subtracting $T_+^{ab}(z^*)$ from $T_+^{ab}(z)$ one finds after some algebra

$$\Delta_\varepsilon T_+^{ab}(E) = \sum_{c,d}{}' \{\tilde{V}_c^a(\delta_{cd}\Delta_\varepsilon G_d(E) + G_c(z)\tilde{T}^{cd}(z)\Delta_\varepsilon G_d(E) + G_c(z)\Delta_\varepsilon\tilde{T}^{cd}(E)G_d(z^*)$$
$$+ \Delta_\varepsilon G_c(E)\tilde{T}^{cd}(z^*)G_d(z^*))V^b\}. \tag{22}$$

From (8) it follows that

$$\Delta_\varepsilon \tilde{T}^{cd}(E) = \sum_e{}' \tilde{T}^{ce}(z)\Delta_\varepsilon G_e(E)\tilde{T}^{ed}(z^*). \tag{23}$$

Using (23) in (22) after some algebra gives

$$\Delta_\varepsilon T_+^{ab}(E) = \sum_c{}' \tilde{T}^{ac}(z)\Delta_\varepsilon G_c(E)T_-^{cb}(z^*). \tag{24}$$

Taking the limit as $\varepsilon\to 0$ using TR1 and TR2 gives

$$\Delta T_+^{ab}(E) = \sum_c{}' \sum_{\substack{\{\gamma_d | d\subseteq c\} \\ \gamma_d \varepsilon A}} \tilde{T}^{ac}(E^+)(1+G_c(E^+)V_c^d)\, D(\gamma_c,E)(1+V_c^d\, G_c(E^-))T_-^{cb}(E^-). \tag{25}$$

Using TR3 and (19) we obtain

$$\Delta T_+^{ab}(E) = \sum_c{}' \sum_{\{\gamma_d\, \varepsilon A | d\subseteq c\}} \tilde{T}^{ac}(E^+)G_c(E^+)G_d^{-1}(E)D_d(\gamma_d,E)T_-^{db}(E^-)$$

$$= \sum_{\gamma_d \varepsilon A} \sum_{c\supseteq d}{}' \tilde{T}^{ac}(E^+)G_c(E^+)G_d^{-1}(E)D_d(\gamma_d,E)T_-^{db}(E^-). \tag{26}$$

From TR4 we see the sum over c can be extended over $c \not\supseteq d$ to give

$$\Delta T_+^{ab}(E) = \sum_{\gamma_c \varepsilon A} \{\sum_d{}' \tilde{T}^{ad}(E^+)G_d(E^+)G_c^{-1}(E)\}D_c(\gamma_c,E)T_-^{cb}(E^-). \tag{27}$$

Using (7) we obtain

$$\Delta T_+^{ab}(E) = \sum_{\gamma_c \varepsilon A} T_+^{ac}(E^+)D_c(\gamma_c,E)T_-^{cb}(E^-) \tag{28}$$

which agrees with the desired result (12).

5. TECHNICAL RESULTS

In arriving at this result we utilized four technical results that were non-algebraic in character. We give a brief discussion of each of these results in this section.

The first of these results is TR1. It says that we can bring the limit $\varepsilon\to 0$ inside the T operators in expression (24). The type of problem that can arise here is when a cut of $\tilde{T}^{ac}(z)$ and a cut of $T_-^{cb}(z^*)$ are separated by δ-functions in all of their continuum variables. This will cause additional singularities through the mechanism

$$\int dp \frac{2i\varepsilon}{(p^2+E)^2+\varepsilon^2} \to -2i\varepsilon \int \delta(E+p^2)\,dP.$$

We are protected from this situation because the singular parts of $\tilde{T}^{ac}(z)$ and $T_{-}^{cb}(z^*)$ are separated by the connected operator $\sum_c{}' \tilde{V}_c^a \Delta_\varepsilon G_e(z) \tilde{V}_e^c$, having no non-trival translational symmetries. That this is connected follows from BLRT. A slightly more refined argument shows that cuts of the pairs $\tilde{T}^{ac}(z)$, $\Delta_\varepsilon G_c(E)$ and $\Delta_\varepsilon G_c(E)$, $T_{-}^{cb}(z^*)$ are never separated by δ-functions in <u>all</u> of their continuum variables; i.e., every denominator in the T's has at least one integration performed. This prevents hidden singularities from arising in their products as $\varepsilon \to 0$.

The second technical result follows directly from the spectral resolution of the partition resolvents

$$G_c(z) = \sum_{\{\gamma_d \varepsilon A \mid d \subseteq c\}} \int d\underset{\sim}{P}(d) \,|\phi_c^+(\gamma_d)\underset{\sim}{P}(d)\rangle \frac{1}{E - E(\gamma_d, \underset{\sim}{P}(d)) + i\varepsilon} \langle\phi_c^+(\gamma_d)\underset{\sim}{P}(d)| \tag{29}$$

if we note

$$|\phi_c^+(\gamma_d)\underset{\sim}{P}(d)\rangle = (1 + G_c(E^+(\gamma_d, \underset{\sim}{P}(d))V_c^d)\,|\phi_d(\gamma_d)\underset{\sim}{P}(d)\rangle. \tag{30}$$

The third technical result is obvious from the solved form of the operator $(T_{-}^{ab}(z) = G_a^{-1}(z)G(z)V^b)$. Because our solutions are known to be meromorphic in the cut plane,[5] and because the kernel is contractive outside of a sufficiently wide strip about the real axis, it is sufficient to show $T_{-}^{ab}(z)$ and $G_a^{-1}(z)G_c(z)T_{-}^{cb}(z)$ have identical perturbation series to show that they agree on their domains of analyticity. The fact that the perturbation series (in $\tilde{V}_b^a$) are equal can be shown explicitly by observing

$$T_{-}^{ab}(z) = V^b + \sum_c{}' \tilde{V}_c^a G_c(z) V^b + \sum_{c,d} \tilde{V}_c^a G_c(z) \tilde{V}_d^c G_d(z) V^b + \ldots$$

and

$$G_a^{-1}(z)G_c(z) = 1 + \sum_d (\tilde{V}_d^a - \tilde{V}_d^c) G_d(z) + \sum_{d_1, d_2} (\tilde{V}_{d_1}^a - \tilde{V}_{d_1}^c) G_{d_1}(z) (\tilde{V}_{d_2}^{d_1} - \tilde{V}_{d_2}^c) G_{d_2}(z) + \ldots . \tag{32}$$

We note (31) follows directly from (8) and (21), while (32) (which is also convergent in a sufficiently wide strip about the real axis) follows from (5) and (19). Multiplying (31) and (32) together, grouping terms of a single power of $\tilde{V}^a_b$, one obtains the perturbation expansion for $T^{ab}_-(z)$. A slight generalization of this argument allows one to get z down to the scattering cut provided we suitably restrict the domains of our operators.

The last technical result follows if we note

$$\tilde{T}^{ad}(z)G_d(z)G_c^{-1}(z)D_c(\gamma_c,E) = i\varepsilon\, \tilde{T}^{ad}(z)G_d(z)D_c(\gamma_c,E). \qquad (33)$$

This will vanish as $\varepsilon\to 0$ unless we can factor a $G_c(z)$ out of the right-hand side of $\tilde{T}^{ad}(z)G_d(z)$. It is easy to show one must have $c \subseteq d$ in order to do this.

6. CONCLUSIONS

We have shown that the BRS equations satisfy a proper unitarity relation. This proof relied only on properties of the dynamic equations. This should be a useful consideration in understanding approximations made with these equations.

The author would like to thank E. F. Redish for both suggesting this problem and for useful suggestions in its preparation. This work was supported by the U. S. Department of Energy.

REFERENCES

1. S. Weinberg, Phys. Rev. 133, B232 (1964); O. A. Yakubovskii, Sov. J. Nucl. Phys. 5, 937 (1967); Gy. Bencze, Nucl. Phys. A210, 568 (1973); E. F. Redish, Nucl. Phys. A225, 16 (1974); D. J. Kouri and F. S. Levin, Nucl. Phys. A253, 395 (1975); W. Tobocman, Phys. Rev. C9, 2966 (1974).
2. Gy. Bencze, Nucl. Phys. A210, 568 (1973); E. F. Redish, Nucl. Phys. A225, 16 (1974).
3. P. Benoist-Gueutal, M. L'Huillier, E. F. Redish and P. C. Tandy, to be published; K. L. Kowalski, Phys. Rev. C16, 7 (1977).
4. The assumption of pair potentials is not needed in the proof of unitarity, only in showing the equivalence between the BLRT equations and the BRS equations.
5. M. Reed and B. Simon, Functional Analysis (Academic Press, 1972), Ch. VI.14-15; S. Steinberg, Arch. Rat. Mech. Anal. 31, 372 (1968).