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SPECIAL FUNCTIONS ASSOCIATED WITH SU(3) WIGNER-CLEBSCH-GORDAN COEFFICIENTS

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ABSTRACT

The Wigner-Clebsch-Gordan (WCG) coefficients of the unitary groups are a rich source of multivariable special functions. The general algebraic setting of these coefficients is reviewed and several special functions associated with the SU(3) WCG coefficients defined and their properties presented.

1. Introduction

The relation between group representations of symmetry groups and special functions has been well-known since the classic lectures of Wigner[1] in 1955, Talman's[2] monograph (based on Wigner's lectures), and Vilenkin's[3] more extensive monograph. The application of Lie algebraic methods has also led to a uniform approach to many special functions as developed, for example, by Miller.[4] Not as well-known are the relations between Wigner-Clebsch-Gordan (WCG) coefficients of symmetry groups and special functions, although they are equally rich in structure. Examples are the classic relations of the WCG-coefficients of the quantal rotation group $\lambda SU(2)$, and the associated Racah coefficients to terminating ${}_3F_2$ and ${}_4F_3$ hypergeometric series. Only recently has this relation been put into the perspective of a general theory of orthogonal polynomials by Wilson[5] and Askey.[6]

It is this second type of relationship, between WCG-coefficients and special functions, that is the subject of this paper. Indeed, we shall develop it in detail only in the context of the WCG-coefficients of the group SU(3) of 3×3 unitary unimodular matrices. We shall show that already in this case one is led to the discovery of new classes of special functions of intrinsic interest in their own right, and going far beyond what one might expect from such a specialized problem. One can only speculate what a complete theory of SU(n) and other symmetry groups will unveil, as already indicated by the extensive work of Milne[7] and Gustafson.[8]

Much of what is presented here is already appeared in the literature[9-20] but is somewhat scattered. The goal here is to present a somewhat more organized viewpoint of this subject, showing how the original problem of calculating WCG-coefficients has led naturally to new special functions and the development of their properties.

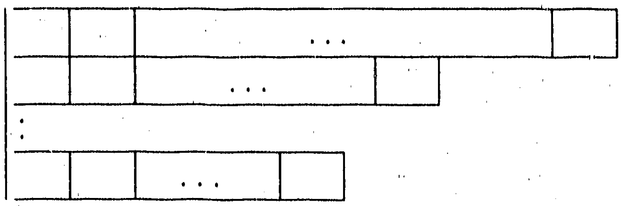
It is important to place the subject in its appropriate general framework. It is no accident that the subject is rich in its detailed structure, since it is rooted in combinatorics and invariant theory through Young-Weyl standard tableaux, in analysis through representation theory, and in algebra through the multiplication properties of basis functions and operators.

Let us outline the contents of this paper. In *Section 2*, the general mathematical setting of the subject is reviewed. In *Section 3*, we motivate and discuss how a certain class of U(3) invariant polynomials with remarkable symmetries and structural zeros enter into the problem of WCG-coefficients. Indeed, it was in proving these properties that all the subsequent discoveries of other special functions were made. In the sections following, we discuss the relationship of these various special functions to the original polynomials and summarize their important properties. These include: *Section 4*. Symmetric Generalized Hypergeometric Functions and Coefficients. *Section 5*. Two New Classes of Symmetric Basis Functions.

2. Review of Basic Concepts

2.1. Standard Tableaux and Gel'fand Patterns. Young-Weyl standard tableaux and Gel'fand-Zetlin-Weyl patterns are two distinct, but equivalent, methods of codifying information relevant to the representations of the general linear group $GL(n,\mathbb{C})$ and its subgroups. (For brevity, we call these patterns simply Gel'fand patterns). We shall require these entities both in the special context of &(3) and in general form. We present briefly their definitions.

A Young *frame* Y_λ *of shape* $\lambda=[\lambda_1\lambda_2\ldots\lambda_n]$, where the λ_i are nonnegative integers satisfying $\lambda_1\geq\lambda_2\geq\ldots\geq\lambda_n$, is a diagram consisting of λ_1 boxes (nodes) in row 1, λ_2 boxes in row 2, ..., λ_n boxes in row n, arranged as illustrated below:



λ_1
λ_2
$\vdots$
λ_n (2.1)

A Young-Weyl *tableau* is a Young frame in which the boxes have been "filled in" with integers selected from 1, 2, ..., n. The tableau is standard if the sequence of integers appearing in each row of Y_λ is *nondecreasing* as read from left to right and the sequence of integers appearing in each column is *strictly increasing* as read from top to bottom. The *weight* or *content* of a Young-Weyl tableau Y_λ is defined to be the row vector $\alpha=(\alpha_1, \alpha_2, \ldots, \alpha_n)$, where α_k equals the number of times integer k appears in the array. If $\lambda_1+\lambda_2+\ldots+\lambda_n=N$, then also $\alpha_1+\alpha_2+\ldots+\alpha_n=N$. We shall call λ a *partition of N* into n parts or, more often, a partition when N is unspecified. We always count the zeros in determining the parts of a partition.

A Gel'fand pattern is a triangular array of n rows of integers, there being one entry in the first (bottom) row, two entries in the second row ..., and n entries in the nth row. The entries in each row 1, 2, ..., $n-1$ are arranged so as to fall between the entries in the row above, as displayed in

$$(m)=\begin{pmatrix} m_{1n} & & m_{2n} & \cdots & m_{nn} \\ & & \vdots & & \\ & m_{13} & m_{23} & m_{33} & \\ & & m_{12} \quad m_{22} & & \\ & & m_{11} & & \end{pmatrix}. \tag{2.2}$$

The integral entries m_{ij}, $i\leq j=1, 2, \ldots, n$, in this array are required to satisfy the following rules:

(*i*) $$m_{1n}\geq m_{2n}\geq\ldots\geq m_{nn} \;. \tag{1.3a}$$

(*ii*) For each specified partition $[m_{1n}\ldots m_{nn}]$, the entries in the remaining rows $j=n-1, n-2, \ldots, 1$ may be any integers that satisfy the "betweenness conditions":

$$m_{1j+1}\geq m_{1j}\geq m_{2j+1}\geq m_{2j}\geq m_{3j+1}\geq m_{3j}\geq\ldots\geq m_{j-1j}\geq m_{jj}\geq m_{j+1j+1}. \tag{1.3b}$$

We denote by G_λ the set of all Gel'fand patterns corresponding to the partition $\lambda=[m](\lambda_i=m_{in})$. There is a natural one-to-one correspondence between the set G_λ of Gel'fand patterns and the set W_λ of Young-Weyl standard tableaux.

The mapping between Gel'fand patterns and standard tableaux is described as follows: [The shape of the frame is $\lambda=[m_{1n}m_{2n}\ldots m_{nn}]$ and row j of the frame as read from left to right, has m_{jj} js, $m_{jj+1}-m_{jj}$ $(j+1)$s, ..., $m_{jn}-m_{jn-1}$ ns for j=1,2,..., n-1, and row n has m_{nn} ns.

Using this rule, we see that the set of patterns G_λ is mapped to the set of tableaux W_λ. Conversely, from each standard tableau $T\in W_\lambda$ we construct in an obvious way the pattern in the set G_λ.

The weight or content of a Gel'fand pattern (m) is the row vector $w=(w_1, w_2, \ldots, w_n)$, where w_j is defined to be the sum of the entries in row j Of(m) minus the sum of the entries in row $j-1(w_1=m_{11})$:

$$w_j=\sum_{i=1}^{j} m_{ij}-\sum_{i=1}^{j-1} m_{ij-1}. \tag{2.3}$$

This definition of weight coincides with that of a standard tableau.

The constraint in a standard tableau that each row (column) should comprise a set of nondecreasing (strictly increasing) nonnegative integers is realized in a Gel'fand pattern by the "geometric" rule that the integers (m_{ij}) satisfy the betweenness conditions.

The significance of the integer $m_{ij}-m_{ij-1}$ in terms of the corresponding standard tableau is

$$m_{ij}-m_{ij-1} = \text{number of times integer } j \text{ appears in row } i. \tag{2.4a}$$

We define $m_{jj-1}=0, j=1,\ 2,\ \ldots,\ n$. Similarly, m_{ij} is the sum of entries in row j of the corressponding standard tableau given by

$$m_{ij} = (\text{number if } i\text{s}) + \cdots + (\text{number of } j\text{s}). \tag{2.4b}$$

The number of standard tableaux in the set W_λ (number of patterns in the set G_λ) is given by the Weyl dimension formula:

$$\dim \lambda = \prod_{i<j}^{n} {}_1 (\lambda_i - \lambda_j + j - i)/1!2! \ldots (n-1)! . \qquad (2.5)$$

For subsequent results, we require the *measure* M_λ of Y_λ defined by

$$M_\lambda = (\dim \lambda)^{-1} \prod_{s=1}^{n} (\lambda_i + n - i)!/(i-1)! . \qquad (2.10)$$

2.2 **Algebra of the General Linear Group.** So-called integral representations of the general linear group $GL(n,\mathbb{C})$ are irreducible under restriction to the unitary subgroup $U(n) \subset GL(n,\mathbb{C})$. It is convenient to formulate results for the unitary group in this general framework of $GL(n,\mathbb{C})$.

The integral representations of $GL(n,\mathbb{C})$ are enumerated by partitions having n parts; that is, by the elements of the set $P_n = \{[\lambda_1 \lambda_2 \cdots \lambda_n] \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0;\ \lambda_i \in \mathbf{N}\}$. The finite subset of P_n such that $\lambda_1 + \lambda_2 + \cdots + \lambda_n = N$ (partitions of N into n parts) is denoted P_n^N, and the set of all partitions by P; that is $P = \bigcup_{n \geq 1} P_n$. Thus, for each $\lambda \in P_n$, there corresponds a matrix $D^\lambda(Z)$ such that the correspondence $Z \longrightarrow D^\lambda(Z)$, each $Z \in GL(n,\mathbb{C})$, is an irreducible matrix representation fo $GL(n,\mathbb{C})$. The dimension of D^λ is dim λ as given by Eq. (2.7).

The matrices $D^\lambda(Z)$ are well-studied objects, and their explicit form has been given by many authors, based on many methods of analysis (see, for example, Grabmeier and Kerber[V]). The form favored by physicists is one in which these matrices are unitary under the group-subgroup restriction $GL(n,\mathbb{C}) \downarrow U(n)$. It is also customary to label the rows and columns of the matrix $D^\lambda(Z)$ by Gel'fand patterns in the following way: Let $\lambda = [\lambda_1 \lambda_2 \cdots \lambda_n] = [m_{1n} m_{2n} \cdots m_{nn}] \in P_n$ and define a double Gel'fand pattern by $\begin{pmatrix} m' \\ \lambda \\ m \end{pmatrix}$, e.g., for n=2, $\begin{pmatrix} & m'_{11} & \\ \lambda_1 & & \lambda_2 \\ & m_{11} & \end{pmatrix}$, where $\begin{pmatrix} \lambda \\ m \end{pmatrix}$ denotes the n-rowed triangular array (2.2) and $\begin{pmatrix} m' \\ \lambda \end{pmatrix}$ is a similar pattern, inverted above the first for convenience of display. Since the partition λ is shared, it is written only once.

The representation functions in the rows and columns of the matrix $D^\lambda(Z)$ are now denoted

$$D^{\lambda}_{mm'}(Z)=D\begin{pmatrix} m' \\ \lambda \\ m \end{pmatrix}(Z). \qquad (2.8)$$

One could also use double standard tableaux for this row-column enumeration. Important properties of the representation functions (2.8) include: (a) they are *homogeneous polynomials of total degree* $\lambda_1+\cdots+\lambda_n=N$; (b) they are a basis of the ring of all polynomials in any number of variables as λ runs over all $\lambda\in P$.

Implicit in our definition of the representation matrices (2.8) is the group-subgroup $GL(n,\mathbb{C})\downarrow GL(n-1,\mathbb{C})$ reduction given explicitly by the direct sum of matrices, $D^{\lambda}(Z)=\sum\oplus D^{\mu}(Z')$, where Z' is the $(n-1)\times(n-1)$ matrix obtained from Z by setting $z_{in}=z_{ni}=1$, $i=1, 2, \ldots, n$. The summation is over all partitions $\mu=[\mu_1\mu_2 \cdots \mu_{n-1}]\in P_{n-1}$ such that the betweenness conditions $\lambda_1\geq\mu_1\geq\lambda_2\geq\ldots\geq\mu_{n-1}\geq\lambda_n$ are satisfied.

The Kronecker product of two irreps D^{μ} and D^{ν} of $GL(n,\mathbb{C})$ is completely reducible into irreps of $GL(n,\mathbb{C})$ by the rule

$$D^{\mu}\times D^{\nu}=\sum_{\Delta\in W(\mu)} \oplus I(\mu\times\nu;\ \nu+\Delta)D^{\nu+\Delta}, \qquad (2.9)$$

where the summation is over all distinct weights Δ of irrep μ. The *intertwining numbers* $I(\mu\times\nu;\ \nu+\Delta)$ in this relation express the number of times irrep $\lambda=\nu+\Delta$ is contained in the direct product irrep $\mu\times\nu$. The are related to the Littlewood-Richardson numbers $g(\mu\nu\lambda)$ by

$$(\mu\times\nu;\lambda)=g(\mu\nu\lambda)=0,\ \text{unless}\ \lambda=\nu+\Delta\ \text{for some weight}\ \Delta\in W(\mu). \quad (2.23)$$

Indeed, the properties of these numbers when viewed as *functions over the set of all partitions* $\nu\in P_n$; that is, $I_{\mu,\Delta}:P_n \longrightarrow L_{\mu,\Delta}=\{0, 1,\cdots, K(\mu,\Delta)\}$, with values $I_{\mu,\Delta}(\nu)$ in the set $L_{\mu,\Delta}$, are crucial to the definition and construction of unit tensor (Wigner) operators in U(n) (see below).

There are two important algebras associated with $GL(n,\mathbb{C})$. The first is that of the homogeneous polynomials (2.8); the second that of U(n) unit tensor operators, as we now explain by giving the product law for basis elements of these algebras:

(*i*) Product law for representation functions:

$$D\begin{pmatrix} m'_1 \\ \mu' \\ m_1 \end{pmatrix} D\begin{pmatrix} m'_2 \\ \nu' \\ m_2 \end{pmatrix} = \sum_{\lambda} \begin{bmatrix} m' & m'_1 & m'_2 \\ \lambda & \mu & \nu \\ m & m_1 & m_2 \end{bmatrix} D\begin{pmatrix} m' \\ \lambda \\ m \end{pmatrix}. \tag{2.11a}$$

The bracket coefficient is expressed in terms of U(n) WCG coefficients by

$$\begin{bmatrix} m' & m'_1 & m'_2 \\ \lambda & \mu & \nu \\ m & m_1 & m_2 \end{bmatrix} = \sum_{\gamma} \left\langle \begin{matrix} \lambda \\ m \end{matrix} \middle| \left\langle \begin{matrix} \gamma \\ \mu \\ m_1 \end{matrix} \right\rangle \middle| \begin{matrix} \nu \\ m_2 \end{matrix} \right\rangle \left\langle \begin{matrix} \lambda \\ m' \end{matrix} \middle| \left\langle \begin{matrix} \gamma \\ \mu \\ m'_1 \end{matrix} \right\rangle \middle| \begin{matrix} \nu \\ m'_2 \end{matrix} \right\rangle. \tag{2.11b}$$

The bra-ket notation

$$\left\langle \begin{matrix} \lambda \\ m \end{matrix} \middle| \left\langle \begin{matrix} \gamma \\ \mu \\ m_1 \end{matrix} \right\rangle \middle| \begin{matrix} \nu \\ m_2 \end{matrix} \right\rangle \tag{2.11c}$$

for a WCG coefficient is further explained below.

(*ii*) Product law for Wigner operators:

$$\left\langle \begin{matrix} \gamma_1 \\ \mu \\ m_1 \end{matrix} \right\rangle \left\langle \begin{matrix} \gamma_2 \\ \mu \\ m_2 \end{matrix} \right\rangle = \sum_{\gamma} \begin{Bmatrix} \gamma & \gamma_1 & \gamma_2 \\ \lambda & \mu & \nu \\ m & m_1 & m_2 \end{Bmatrix} \left\langle \begin{matrix} \gamma \\ \mu \\ m \end{matrix} \right\rangle, \tag{2.12a}$$

The curly-bracket object denotes an *invariant operator* in U(n). It is expressed in terms of *Racah invariant operators* and Wigner coefficients by

$$\begin{Bmatrix} \gamma & \gamma_1 & \gamma_2 \\ \lambda & \mu & \nu \\ m & m_1 & m_2 \end{Bmatrix} = \sum_{\gamma_3} \left\langle \begin{matrix} \lambda \\ m \end{matrix} \middle| \left\langle \begin{matrix} \gamma_3 \\ \mu \\ m_1 \end{matrix} \right\rangle \middle| \begin{matrix} \nu \\ m_2 \end{matrix} \right\rangle \left\{ \begin{pmatrix} \lambda \\ \gamma \end{pmatrix} \begin{bmatrix} \gamma_3 \\ \mu \\ \gamma_1 \end{bmatrix} \begin{pmatrix} \nu \\ \gamma_2 \end{pmatrix} \right\} \tag{2.12b}$$

Let us explain the notations in these product laws.

The patterns $\begin{pmatrix} \lambda \\ m \end{pmatrix}, \begin{pmatrix} \mu \\ m_1 \end{pmatrix}, \begin{pmatrix} \nu \\ m_2 \end{pmatrix}$ are all Gel'fand patterns in which the labels in rows n, n-1, . . . , 1 have the significance of group-subgroup reductions for the chain $U(n) \supset U(n-1) \supset \ldots \supset U(1)$, in accordance with the Weyl rule. The n-rowed pattern ..., which inverted in the notation for a Wigner operator has no such group-subgroup significance, although, by definition, its entries γ_{ij}

run over all values satisfying the betweenness relations. The discovery that patterns *numerically* identical to Gel'fand patterns enumerate all Wigner coefficients for U(n) was one of the significant discoveries in the early 1960's (see Ref.). It takes into account beautifully the fact that the intertwining function $I_{\mu,\Delta}$ takes on only values in the set $L_{\mu,\Delta}$. This accounting is made through the weight $\Delta=(\Delta_1, \Delta_2, \ldots, \Delta_n)$ of an operator pattern, where each Δ_j is defined in terms of the entries γ_{ij} of the pattern exactly as in Eq. (2.5) for a Gel'fand pattern. A given weight $\Delta\in W(\mu)$ has a multiplicity $K(\mu,\Delta)$, and there are exactly this number of distinct operator patterns γ having this weight. Thus, the Wigner coefficient (2.11c) is, first of all, equal to zero, unless $\lambda=\nu+\Delta$, where $\Delta\in W(\mu)$ [the property of the intertwining number in Eq. (2.10)]; secondly, there are exactly $K(\mu,\Delta)$ operator patterns γ providing us with $K(\mu,\Delta)$ *sets* of orthogonal Wigner coefficients (this orthogonality is expressed in terms of summations over the Gel'fand patterns m_1 and m_2). These sets of orthogonal coefficients then effect completely the reduction of the direct product $(D^{\mu}\times D^{\nu})\downarrow D^{\nu+\Delta}$, $\Delta\in W(\mu)$, in the region of maximum multiplicity; that is, for all μ, ν, and $\lambda=\nu+\Delta$ such that $I(\mu\times\nu,\ \nu+\Delta)=K(\nu,\Delta)$.

Operator patterns must have still further structure. This is because the intertwining number can assume any value in the set $L_{\mu,\Delta}$ for certain ν. This means then that certain *whole sets* of Wigner coefficients must vanish. This property is best expressed through the notion of a Wigner operator and its characteristic null space. In its very conception, a U(n) Wigner operator is to have certain mapping properties when acting in the Hilbert space over which it is defined. We take this Hilbert space H to be a direct sum $H=\sum_{\lambda\in P_n}\oplus H_\lambda$, where H_λ is the carrier space of irrep λ of U(n) [or $GL(n,\mathbb{C})$], each such irrep space occurs exactly once in the direct sum, and the sum is over all $\lambda\in P_n$.

An (abstract) Wigner operator, denoted $\langle\dot{\mu}\rangle$ below, is then a map $H \longrightarrow H$ with the following specific properties for each $\nu\in P_n$, where Δ denotes the weight of the operator pattern γ:

$$\left\langle\begin{matrix}\gamma\\ \mu\\ m\end{matrix}\right\rangle : H_\nu \longrightarrow 0 \text{ if } \nu+\Delta\notin\mu\times\nu; \tag{2.13a}$$

$$\left\langle \begin{matrix} \gamma \\ \mu \\ m \end{matrix} \right\rangle : H_\nu \longrightarrow 0 \text{ or } H_\nu \longrightarrow H_{\nu+\Delta} \text{ ; if } \nu+\Delta \in \mu\times\nu, \tag{2.13b}$$

Among the $K(\mu,\Delta)$ unit tensor operators in the set

$$\left\{ \left\langle \begin{matrix} \gamma \\ \mu \\ m \end{matrix} \right\rangle \middle| \Delta\begin{pmatrix} \mu \\ \gamma \end{pmatrix} = \Delta \in W(\mu) \right\} \tag{2.14}$$

exactly $K(\mu,\Delta)-I(\mu\times\nu;\ \nu+\Delta)$ of them annihilate the irrep space H_ν [have the first property (2.13b)], while the remaining $I(\mu\times\nu,\ \nu+\Delta)$ operators effect maps $H_\nu \longrightarrow H_{\nu+\Delta}$ [have the second property (2.13b] as given explicitly by the sets of orthogonal Wigner coefficients that effect the reduction $(D^\mu \times D^\nu)\downarrow D^{\nu+\Delta}$:

$$\left\langle \begin{matrix} \gamma \\ \mu \\ m_1 \end{matrix} \right\rangle \left| \begin{matrix} \nu \\ m_2 \end{matrix} \right\rangle = \sum_m \left\langle \begin{matrix} \nu+\Delta \\ m \end{matrix} \middle| \left\langle \begin{matrix} \gamma \\ \mu \\ m_1 \end{matrix} \right\rangle \middle| \begin{matrix} \nu \\ m_2 \end{matrix} \right\rangle \left| \begin{matrix} \nu+\Delta \\ m \end{matrix} \right\rangle \tag{2.15}$$

Here the set of vectors $\left\{ \left| \begin{matrix} \nu \\ m_2 \end{matrix} \right\rangle \middle| m_2 \text{ is a lexical Gel'fand pattern} \right\}$ is an orthonormal basis of H_ν. As ν runs over all $\nu \in P_n$, these vectors are to be an orthonormal basis of the (separable) Hilbert space H.

The curly bracket object in Eq. (2.12b) denotes a Racah invariant. It is particularly significant that Racah invariants are fully labelled by *operator patterns*. There are other important forms of Eqs. (2.11) and (2.12), derived from these relations by using the orthogonality relations for Wigner coefficients and Racah invariants (see Ref.). We refer to the literature for these properties. Let us note here, however, that under the unitary transformation $K:H \longrightarrow H$ given by

$$\mathcal{U} \left| \begin{matrix} \nu \\ m_1 \end{matrix} \right\rangle = \sum_{m'} D^\mu_{m_1' m_1}(U) \left| \begin{matrix} \nu \\ m_1' \end{matrix} \right\rangle, \tag{2.16a}$$

the unit tensor operators transform irreducibly according to

$$\mathcal{U} \left\langle \begin{matrix} \gamma \\ \mu \\ m \end{matrix} \right\rangle \mathcal{U}^{-1} = \sum_{m'} D^\mu_{m_1' m}(U) \left\langle \begin{matrix} \gamma \\ \mu \\ m' \end{matrix} \right\rangle. \tag{2.16b}$$

It is useful to remark that the existence of the algebraic structures, Eq. (2.11) for functions and Eq. (2.12) for operators, is assured, since $D^{\mu}\times D^{\nu}$ is completely reducible. The important question is whether or not their exists canonical or natural realizations of these algebras, free of arbitrary choices. The answer for U(2) and U(3) is that the algebra of Wigner operators is canonically determined by characteristic null space alone, and is implied definitively by the intertwining number function. This structure is made precise for U(2) in Ref. and for U(3) in numerous publications (see, for example, Refs.).

It is against the background of the very general algebraic setting put forth in this section that one must view the results on special functions to follow in Sections 4 and 5. For structures of such sweeping scope, one must expect equally exquisite mathematics to appear in its concrete implementation.

3. **The G^t_q Polynomials of SU(3).** We have surveyed in the preceding subsection the general theory of representations and unit tensor operators in U(n). All of these results apply directly to U(3) by specialization to n=3.. In the case of U(3), however, all representations and all unit tensor operators, that is, all WCG coefficients are canonically determined by the null space of each unit tensor operator which itself is implied by properties of the intertwining number function $I_{\mu,\Delta}$. Indeed, the specific set of conditions implied by null space is that a class of Wigner coefficients must be zero [see Ref.], which, in turn, leads to a unique (up to phase) determination of *all* coefficients. In implementing this, one is lead to the study of some remarkable polynomials, denoted G^t_q, describe in this section.

We need some symbol definitions for describing the polynomials G^t_q and their properties:

(i) The set of real numbers and the set of nonnegative integers are denoted by $\mathbb{R}$ and **N**, respectively.

(ii) The Möbius plane and the subset of (lattice) points of M with integral coordinates are demoted by **M** and **L**, respectively.

(iii) Integers such that $q\in\mathbf{N}$, $p=q$, $q+1$, ... are denoted by q and p.

(iv) The three-tuple $(\Delta_1,\Delta_2,\Delta_3)$ wuch that $\Delta_i\in\mathbf{N}$, $0\leq\Delta_i\leq p$, and $\Delta_1+\Delta_2+\Delta_3=p+q$ is denoted $\Delta=(\Delta_1,\Delta_2,\Delta_3)$.

(v) A point in **M**, which is sometimes restricted to **L**, is denoted $x=(x_1,x_2,x_3)$.

(vi) Pochhammer's notation for the rising for the rising factorial for $a \in \mathbf{N}$ with $(x)_0=1$ is $(x)_a=x(x+1)\cdots(x+a-1)$.

(vii) $\lambda=[\lambda_1\lambda_2\ldots\lambda_t]$ denotes an irrep label of U(t). The symbols μ, ν, ... denote irrep labels of the same type as λ.

(viii) $h(\lambda\mu\nu\rho)$ denotes the number of times irrep $[q-t+1, \ldots, q-t+1]$ ($q-t+1$ repeated t times) is contained in the direct product $\lambda\times\mu\times\nu\times\rho$, and is defined to be zero if $[q-t+1, \ldots, q-t+1]\notin\lambda\times\mu\times\nu\times\rho$.

(ix) The symbol A denotes the 3×3 array of variables defined by

$$A=A_t(\Delta;x)=(a_{ij})$$

$$=\begin{bmatrix} \Delta_1 - t + 1 & \Delta_2 - t + 1 + x_1 & \Delta_3 - t + 1 - x_1 \\ \Delta_2 - t + 1 & \Delta_3 - t + 1 + x_2 & \Delta_1 - t + 1 - x_2 \\ \Delta_3 - t + 1 & \Delta_1 - t + 1 + x_3 & \Delta_2 - t + 1 - x_3 \end{bmatrix} \quad (3.1)$$

(x) For t=0 and t=q+1, we define $G_q^0(\Delta;x)=G_q^{q+1}(\Delta;x)=1$.

(xi) The notation

$$\mathcal{G}_q^t(A)=G_q^t(\Delta;x) \quad (3.2)$$

is used to signify that $\mathcal{G}_q^t$ is a polynomial in the variables a_{ij} of the array A, hence, of the Δ_i and x_j.

We can now give explicitly the polynomials $G_q^t(\Delta;x)$, $q\in\mathbf{N}$, t=1, 2, ..., q, using the terms defined above, and forward referencing to Section 4 for the definitions of hypergeometric coefficients in Eqs. (4.10, (4.4b), and (4.6b):

$$G_q^t(\Delta;x)=\mathcal{G}_q^t(A)=\prod_{s=1}^{t}\frac{(q-s+1)!}{(s-1)!}\times\prod_{i=1}^{3}\prod_{s=1}^{t}$$

$$(-a_{i1}-s+1)_{q-t+1}$$

$$\times\sum_{\lambda\mu\nu\rho} h(\lambda\mu\nu\rho)\langle{}_1\mathcal{F}_0(K-l)\,|\,\rho\rangle$$

$$\times\langle{}_2\mathcal{F}_1(-a_{12},-a_{13},;a_{11}-\ell)\,|\,\lambda\rangle$$

$$\times\langle{}_2\mathcal{F}_1(-a_{22},-a_{23},;a_{21}-\ell)\,|\,\mu\rangle$$

$$\times\langle{}_2\mathcal{F}_1(-a_{32},-a_{33},;a_{31}-\ell)\,|\,\nu\rangle \qquad (3.3)$$

where we have defined $\ell=q-2t+1$, $K+\Delta_1+\Delta_2+\Delta_3-3t+3$.

Let us summarize the properties of the G_q^t polynomials:

(i) Total degree $2t(q-t+1)$ in x. By this we mean that $G_q^t(\Delta;x)$ is a sum of monomials of the form $x_1^\alpha x_2^\beta x_3^\gamma$, where α, β, γ are non-negative integers such that $\alpha+\beta+\gamma\leq 2\tau(\theta-t+1)$ and the sum is over all such monomials multiplied by real coefficients that are themselves functions of Δ_i. This polynomial property is placed in evidence when Eq. (3.3) is rewritten in terms of the quantities (see Ref.) ($k\in N$):

$$\tilde{F}_{k,\lambda}(x,y,z)=\prod_{s=1}^{t}(x+t-kk-s-1)_k\langle{}_2\mathcal{F}_1(-y,-z;\; x+t-k\,|\,\lambda\rangle. \qquad (3.4)$$

(ii) Determinantal symmetry. This symmetry refers to the invariance of $G_q^t(\Delta;x)$ under the transformation of the six variables $(\Delta_1,\Delta_2,\Delta_3,x_1,x_2,x_3)$ induced by row interchange, column interchange, and transposition of the 3×3 array A defined by Eq. (3.1). For example, under matrix transposition of A, that is, $A\longrightarrow\tilde{A}$, we have

$$(\Delta_1,\Delta_2,\Delta_3,x_1,x_2,x_3)\longrightarrow(\Delta_1,\Delta_2+x_1,\Delta_3-x_1,-x_1,-x_3,-x_2). \qquad (3.5)$$

(iii) Weight space $\mathbf{W}_q^t(\Delta)$ are in one-to-one correspondence with those of the weight space of irrep $[q-t,0,-t+1]$ of U(3). With each point $x\in\mathbf{W}_q^t(\Delta)$, we associate a multiplicity number $M_q^t(\Delta;x)$,

$$M_q^t(\Delta;x)=\min\Big((t,q-t+1,1+d_t(x)\Big), \tag{3.6}$$

where $d_t(x)$ is the "distance" from lattice point $x\in\mathbf{W}_q^t(\Delta)$ to the nearest boundary point as measured along the direction of a coordinate axis (one lattice spacing = one unit of distance, with $d_t = 0$ at the boundary). The multiplicity function $M_q^t(\Delta;x)$ assigns to each point $x\in\mathbf{W}_q^t(\Delta)$ exactly the value of the multiplicity of the weight $w=(w_1,w_2,w_3)$ of irrep $[q-t,0,-t+1]$, where w is related to the point $x\in\mathbf{W}_q^t(\Delta)$ by $x_1=\Delta_3-t+1-w_1, x_2=\Delta_2-\Delta_3+q-1-w_2, x_3=\Delta_2-t+1-w_3$. By the phrase "a polynomial has the weight space $\mathbf{W}_q^t(\Delta)$ of zeros," we mean that each $x\in\mathbf{W}_q^t(\Delta)$ is a zero of the polynomial with multiplicity $M_q^t(\Delta;x)$.

Property (i) is already evident from the definition (2.44) of G_q^t, as is the invariance of $G_q^t(\Delta;x)$ under the transformation of the variables $(\Delta;x)$ corresponding to the column interchanges in the array A. Accordingly, the proof of the determinantal symmetry stated in (ii) requires only that for the invariance under the transformation (3.5) corresponding to transposition of the array A. This transpositional symmetry is also the key to proving that $G_q^t(\Delta;x)$ possesses the zeros described in (iii), a result proved in Ref. . It was the search for a proof of transpositional symmetry that led to the discovery of many of the special functions discussed in Sections 3 and 4, although the discovery of the ${}_2\mathcal{F}_1$ generalized hypergeometric function and the associated Saalschütz identity came earlier in developing properties of the $G_q^1(t=1)$ polynomials. We outline next how generalized hypergeometric coefficients enter into the proof of transpositional symmetry.

$$G_q^t(\Delta;x)=G_q^t(A)=\prod_{s=1}^{t}\frac{(q-s+1)!}{(s-1)!}\prod_{i=1}^{3}\prod_{s=1}^{t}(-a_{i1}-s+1)_{q-t+1}\sum_{\lambda\mu\nu\rho}h(\lambda\mu\nu\rho)$$

$$\langle {}_1F_0(K-\ell)|\rho\rangle\ \langle {}_2F_1(-a_{12},-a_{13},;a_{11}-\ell|\lambda\rangle\ \langle {}_2F_1(-a_{22},\ -a_{23},;$$

$$a_{21}-\ell|\mu\rangle\ \langle {}_2F_1(-a_{32},-a_{33},;a_{31}-\ell|\nu\rangle, \tag{3.7}$$

where we have defined $\ell=q-2t+1$, K = magic square parameter of A.

We now use the known identity $h(\lambda\mu\nu\rho)=\sum_\kappa g(\mu\rho\kappa)g(\nu\lambda\bar{\kappa})$, and the Saalschütz identity (4.8) with $(a,b,c,)=(-a_{22},-a_{23},a_{21}-l)\nu\longrightarrow\rho$, $\lambda\longrightarrow\kappa$ to effect a transformation of (3.3). In this intermediate result, we rename dummy summation partitions μ to by ν and κ to be ν, and then use $g(\mu\lambda\bar{\nu}) = g(\mu\nu\bar{\lambda})$. This brings $\mathscr{G}_q^t$ to the new form:

$$\mathscr{G}_q^t(A)=\prod_{s=1}^{t}\frac{(q\ -s\ +1)!}{(s\ -1)!}\prod_{i=1}^{3}\prod_{s=1}^{t}(-a_{i1}-s+1)_{q-t+1}$$

$$\times\sum_{\lambda}\times\langle {}_2\mathscr{F}_1(-a_{12},-a_{13},;a_{11}-\ell)|\lambda\rangle$$

$$\times\sum_{\mu\nu}g(\mu\nu\bar{\lambda})\ \langle {}_2\mathscr{F}_1(-a_{32},-a_{33},;a_{31}-\ell)|\mu\rangle$$

$$\times\ \langle {}_2\mathscr{F}_1(a_{21}+a_{22}-\ell,a_{21}+a_{23}-\ell;a_{21}-\ell)|\nu\rangle \tag{3.7}$$

We continue with the transformation of $\mathscr{G}_q^t$ as given by Eq. (3.7) in the following manner: Define new variables a, b, c, d, e by $a=-a_{33}$, $b=-a_{32}$, $d=-a_{22}$, $e=-a_{23}$, $c=K-\ell$ and define the polynomial A_λ of these variables by

$$A_\lambda\begin{bmatrix}a,b,d,e\\ c\end{bmatrix}=\prod_{s=1}^{t}(a+b+c-s+1)_{\lambda_s}(d+e+c-s+1)_{\lambda_s}$$

$$\times\sum_{\mu\nu}g(\mu\nu\lambda)\langle {}_2\mathscr{F}_1(a,b;a+b+c)|\mu\rangle\ \langle {}_2\mathscr{F}_1(d+c,e+c;d+e+c)|\nu\rangle. \tag{3.8}$$

Combining definition (3.8) and Eq. (3.7) and carrying out some simplifying algebraic steps, we find the following expression for $\mathcal{G}_q^t$ in terms of the A_λ functions:

$$\mathcal{G}_q^t(A)=(-1)^{t(q-t+1)}\left[\prod_{s=1}^{t}\frac{(q-s+1)!}{(s-1)!}\right]$$

$$\times\sum_{\lambda}M^{-1}(\lambda)\left[\prod_{s=1}^{t}(-1)^{\lambda_s}(-a_{11}-s+1)_{q-t+1-\lambda_s}(-a_{12}-s+1)_{\lambda_s}\right.$$

$$\left.\times(-a_{13}-s+1)_{\lambda_s}(-a_{21}-s+1)_{\lambda_s}(-a_{31}-s+1)_{\lambda_s}\right]A_\lambda\begin{bmatrix}a,b,d,e\\ c\end{bmatrix}. \qquad (3.9)$$

The array A is given in terms of the variables a, b, c, d, e by

$a_{11}=-(c+\ell)-(a+b+d+e)$, $a_{12}=(c+\ell)+(b+d)$, $a_{13}=(c+\ell)+(a+e)$, $a_{21}=(c+\ell)+(d+e)$, $a_{31}=(c+\ell)+(a+b)$, where we recall that $\ell=q-2t-1$.

The summation in Eq. (3.9) is over all partitions λ such that $q-t+q\geq\lambda_1\geq\cdots\geq\lambda_t\geq 0$. The functions A_λ, given by Eq. 3.8) are defined for all partitions, hence, for the $\bar{\lambda}$ occurring in (3.9). We see from Eq. (2.54) that a sufficient condition for transpositional symmetry of the polynomial $\mathcal{G}_q^t$, that is, for $\mathcal{G}_q^t(A) = \mathcal{G}_q^t(\tilde{A})$ is

$$A_\lambda\begin{bmatrix}a,b,d,e\\ c\end{bmatrix}\ A_\lambda\begin{bmatrix}a,e,d,b\\ c\end{bmatrix}. \qquad (3.10)$$

The above transformations of the original $\mathcal{G}_q^t(\Delta;x)$ polynomial to the forms given by Eqs. (2.51) and (2.54), respectively, show that we can prove transpositional symmetry by proving (*i*) the generalized Saalschütz identity; (*ii*) the b and e interchange symmetry of the functions A_λ.

This task has led to the discovery of the special functions discussed in Sections 4 and 5.

4. Symmetric Generalized Hypergeometric Coefficients and Functions.

Two basic identities are required to prove the determinantal symmetry of the polynomials $G_q^t(A)$ defined and discussed in Section 2.4. These are the *generalized Saalschütz identity* [relation (4.8) below] and the *generalized Bailey identity of the second kind* [relation (4.16) below]. It was the need for proofs of these relations between *generalized hypergeometric coefficients* that led to the introduction of a class of *symmetric generalized hypergeometric functions* in an arbitrary number of variables (indeterminates) and the development of some of their properties. In this section, we summarize the results obtained thus far, pointing out that the general theory is still incomplete.

We shall encounter several types of generalized hypergeometric functions all belonging to same general ${}_p\mathscr{F}_q$ class, which we now define. Let $\mathbf{a}=(a_1,\ldots,a_p)$, $\mathbf{b}=(b_1,\ldots,b_q)$, and $\mathbf{z}=(z_1,\ldots,z_t)$ denote arbitrary complex *numerator* and *denominator.parameters*, p and q in number, respectively, and z a set of t indeterminates. We define *generalized hypergeometric coefficients* by

$$\langle {}_p\mathscr{F}_q(\mathbf{a};\mathbf{b})\,|\,\mu\rangle = M_\mu^{-1} \prod_{s=1}^{t} \left[\frac{\Pi_{i=1}^{p}(a_i - s + 1)_{\mu_s}}{\Pi_{j=1}^{q}(b_j - s + 1)_{\mu_s}} \right], \tag{4.1}$$

where $\mu=[\mu_1\ \mu_2\cdots\mu_t]$ is an arbitrary partition and M_μ is the measure factor defined by Eq. (2.6). These coefficients are now used to define a *generalized hypergeometric function* by the formal series

$${}_p\mathscr{F}_q(\mathbf{a};\mathbf{b};z) = \sum_\mu \langle {}_p\mathscr{F}_q(\mathbf{a};\mathbf{b})\,|\,\mu\rangle e_\mu(z), \tag{4.2}$$

where $e_\mu(z)$ denotes a Schur function. The Schur functions are themselves defined in terms of a standard Young-Weyl tableaux by the formula:

$$e_\mu(z) = \sum_\alpha z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_t^{\alpha_t}. \tag{4.3}$$

where $\alpha=(\alpha_1,\ldots,\alpha_t)$ is a weight of the partition μ. Equivalently, α is the content of a Young frame μ "filled in" with 1, 2,..., t according to the usual rules for a standard tableau. The summation in Eq. (4.3) is over all weights α of m, *including repetitions*.

Shukla[V] introduced the coefficients (4.1) as the natural generalization of the ${}_p\mathscr{F}_q$ functions defined earlier in Ref. and defined in Eq. (4.6) below. We will encounter several of these functions in the sequel.

One of the simplest functions we require in the class (4.1) is

${}_1\mathscr{F}_0$. This function already occurs in the work of Littlewood.[V] It has the explicit definition from Eqs. (4.1)–(4.3) given by

$$ {}_1\mathscr{F}_0(a;\ z)=\sum_{\mu}\langle {}_1\mathscr{F}_0(a)\,|\,\mu\rangle e_{\mu}(z)=\prod_{s=1}^{t}(1-z_s)^{-a} \tag{4.4a} $$

where the hypergeometric coefficient is given by

$$ \langle {}_1\mathscr{F}_0(a)\,|\,\mu\rangle=(\dim\,\mu)\prod_{s=1}^{t}\frac{(a-s+1)_{\mu}}{(t-s+1)_{\mu}}. \tag{4.4b} $$

This function then satisfies the *addition rule*

$$ {}_1\mathscr{F}_0(a;\ z)\,{}_1\mathscr{F}_0(b;\ z)={}_1\mathscr{F}_0(a+b;z). \tag{4.5} $$

The second type of generalized hypergeometric functions we require from the general class (4.2) which are the generalized Gauss series given by

$$ {}_2\mathscr{F}_1(a,b;c;z)=\sum_{\mu}\langle {}_2\mathscr{F}_1(a,b;c)\,|\,\mu\rangle e_{\mu}(z). \tag{4.6} $$

For $t=1$ definition (4.6) reduces to the classic Gauss series.

The main results proved in Refs.   are the following theorem and properties (*i*)–(*iv*):

THEOREM. *The generalized Gauss series obeys the Euler identity*:

$$ {}_2\mathscr{F}_1(a,\ b;\ c;\ z)\,{}_1\mathscr{F}_0(c-a-b;\ z)={}_2\mathscr{F}_1(c-a,\ c-b;\ c;\ z). \tag{4.7} $$

An immediate consequence of this theorem is

(*i*) *Generalized Saalschütz identity*:

$$ \sum_{\mu\nu} g(\mu\nu\lambda)\langle {}_2\mathscr{F}_1(a,b;c)\,|\,\mu\rangle\langle {}_1\mathscr{F}_0(c-a-b\,|\,\nu\rangle=\langle {}_2\mathscr{F}_1(c-a,c-b;c)\,|\,\lambda\rangle. \tag{4.8} $$

This relation is an easy consequence of the Euler identity and the multiplicative property of Schur functions,

$$ e_{\mu}(z)e_{\nu}(z)=\sum_{\lambda} g(\mu\nu\lambda)e_{\lambda}(z), \tag{4.9} $$

where $g(\mu\nu\lambda)$ denotes the Littlewood-Richardson numbers for $GL(t,\mathbb{C})$. For $t=1$, we have $g(\mu\nu\lambda)=\delta_{\mu+\nu,\lambda}$, and Eq. (4.8) reduces to the classic Saalschütz identity [ ].

(*ii*) *Generalized Bailey Identity of the First Kind*:

$$\sum_{\mu\nu} g(\mu\nu\lambda)\langle {}_2\mathscr{F}_1(c-a,c-b;c)|\mu\rangle\langle {}_2\mathscr{F}_1(c'-a',c'-b';c')|\nu\rangle$$
$$=\sum_{\mu\nu} g(\mu\nu\lambda)\langle {}_2\mathscr{F}_1(a,b;c)|\mu\langle {}_2\mathscr{F}_1(a',b';c')|\nu\rangle, \qquad (4.10)$$

where the parameters are to satisfy $c-a-b+c'-a'-b'=0$. Relation (4.10) is called a generalized Bailey identity because for $t=1$, we have

$$\sum_{\mu+\nu=\lambda} \langle {}_2\mathscr{F}_1(a, b; c)|\mu\rangle\langle {}_2\mathscr{F}_1(a',b';c')|\nu\rangle$$
$$=\frac{(a)_\lambda(b)_\lambda}{\lambda!(c)_\lambda}\, {}_4F_3\begin{bmatrix} a', b', 1-c-\lambda, -\lambda; \\ c', 1-a-\lambda, 1-b-\lambda \end{bmatrix}$$
$$=\frac{(a)_\lambda(b)_\lambda}{\lambda!(c')_\lambda}\, {}_4F_3\begin{bmatrix} a, b, 1-c'-\lambda, -\lambda; \\ c, 1-a'-\lambda, 1-b'-\lambda \end{bmatrix}, \qquad (4.11)$$

in which $c-a-b+c'-a'-b'=0$. The identity between the two ${}_4F_3$ hypergeometric series (of unit argument) is the reversal identity (reverse the order of terms in the finite series expression). Using (4.11) in the identity (4.10) for $t=1$ now gives Bailey's identity. Shukla[v] independently obtained relation (4.10).

(*iii*) *Generalized Addition Rule of Binomial Type*:

$$\sum_{\mu\nu} g(\mu\nu\lambda)\langle {}_1\mathscr{F}_0(x)|\mu\rangle\langle {}_1\mathscr{F}_0(y)|\nu\rangle=\langle {}_1\mathscr{F}_0(x+y)|\lambda\rangle. \qquad (4.12)$$

For $t=1$, this relation reduces to

$$\sum_{\mu+\nu=\lambda} \frac{(x)_\mu(y)_\nu}{\mu!\nu!}=\frac{(x+y)_\lambda}{\lambda!}, \qquad (4.13)$$

hence, the designation of (4.12) as a *generalized binomial identity*.

(*iv*) *Generalized Bailey Identity of the Second Kind*:

$$A_\lambda\begin{bmatrix}a,b,d,e\\ c\end{bmatrix} = \prod_{s=1}^{t} (a+b+c-s+1)_{\lambda_s}(d+e+c-s+1)_{\lambda_s}$$

$$\times\sum_{\mu\nu} g(\mu\nu\lambda)\langle {}_2\mathscr{F}_1(a,b;a+b+c)\,|\mu\rangle\langle {}_2\mathscr{F}_1(d+c,e+c;d+e+c)\,|\nu\rangle.$$

$$=\prod_{s=1}^{t}(a+e+c-s+1)_{\lambda_s}(b+d+c-s+1)_{\lambda_s}$$

$$\times\sum_{\mu\nu} g(\mu\nu\lambda)\langle {}_2\mathscr{F}_1(a,e;a+e+c)\,|\mu\rangle\langle {}_2\mathscr{F}_1(b+c,d+c;b+d+c)\,|\nu\rangle. \qquad (4.14)$$

For $t=1$, this relation reduces to

$$(c+d)_n(c+e)_n\ {}_4F_3\begin{bmatrix}a,b,1-c-e-e-n,\ -n;\\ a+b+c,1-c-d-n,1-c-e-n\end{bmatrix}$$

$$=(a+c+d)_n(a+c+e)_n\ {}_4F_3\begin{bmatrix}a,a+c,a+b+d+e+2c+n-1,-n\\ a+b+c,a+c+d,a+c+e\end{bmatrix}, \qquad (4.15)$$

which is again an expression of Bailey's identity. Since relation (4.14) is distinct from (4.10), it is called a *generalized Bailey identity of the second kind*.

Let us remark that the proof of relation (4.14) [given in Ref.] is equivalent to the proof of the b,e interchange symmetry of the A_λ coefficients defined by Eq. (3.8). The proof of the generalized Bailey identity of the second kind is quite difficult. This proof was achieved in Ref. by showing that the left-hand side of Eq. (4.14) could be expressed in terms of yet another generalization of hypergeometric functions going beyond definition (4.2). This generalization, in turn, led to the discovery of a new class of symmetric functions, as we describe in the next section. Since the present section is about hypergeometric functions and their generalizations, we will describe here this second generalization in terms of the new symmetric functions, leaving the definition of the latter for the next section. The symmetric functions in question are the $T_\mu(\alpha;z)$ defined in Eq. ( ), in which μ is an arbitrary partition $\mu=[\mu_1\mu_2\cdots\mu_t]$, α is an arbitrary parameter, and the z_i in $z=(z_1,z_2,\cdots,z_t)$ are indeterminates.

Let us refer back to the definition (4.2) of the hypergeometric functions ${}_p\mathscr{F}_q$. For the new definition, we retain exactly the hypergeometric coefficients as defined by Eq. (4.1). We now, however, replace the Schur functions by the new symmetric functions $\{T_\mu(\alpha;z)\}$. In this way, we define the formal series:

$$ {}_p\mathscr{F}^T_q(\mathbf{a};\mathbf{b};z)\sum_\mu \langle {}_p\mathscr{F}_q(\mathbf{a};\mathbf{b})\,|\mu\rangle T_\mu(\alpha;z). \tag{4.16}$$

[Clearly, one could equally well consider generalizations of Eq. (4.2) in which one makes the replacement $e_\mu(z) \longrightarrow S_\mu(z)$, where the $\{S_\mu\}$ are an arbitrary basis of the ring of symmetric functions.]

The quite remarkable result for the generalized Bailey identity (4.14) is that one can also write

$$A_\lambda\begin{pmatrix} a,b,d,e \\ c \end{pmatrix} = M_\lambda^{-1}\left[\prod_{s=1}^{t}\prod_{i=1}^{3}(b_i-s+1)\lambda,\right]{}_2\mathscr{F}^T_3(\mathbf{a};\mathbf{b};p), \tag{4.17}$$

where $\mathbf{a}=(a_1,a_2)$, $\mathbf{b}=(b_1,b_2,b_3)$ $a_1=a$, $a_2=a+c$, $b_1=a+b+c$, $b_2=a+c+d$, $b_3=a+c+e$; $z_s=p_s=\lambda_s+t-s$, $s=1, 2, \cdots, t$; $\alpha=2c+a+b+d+e-2t+1=b_1+b_2+b_3-a_1-a_2-2t+1$. The identity (4.17), of course, makes the b,e interchange symmetry evident and proves relation (4.14). We also note that the summation over μ in the definition of ${}_2\mathscr{F}^T_3(\boldsymbol{a};\boldsymbol{b};p)$ is finite because of property (5.11) of the symmetric functions $T_\mu(\alpha;z)$.

We conclude this section by noting the following class of summation formulas for relations (4.14) and 4.17):

$$A_\lambda\begin{pmatrix} 0,b,d,e \\ c \end{pmatrix} = \langle {}_3\mathscr{F}_0(b+c,d+c,e+c)\,|\lambda\rangle, \tag{4.18a}$$

$$A_\lambda\begin{pmatrix} \alpha,b,0,e \\ c \end{pmatrix} = \langle {}_3\mathscr{F}_0(a+c,b+c,e+c)\,|\lambda\rangle, \tag{4.18b}$$

$$A_\lambda\begin{pmatrix} a,b,d,e \\ -a \end{pmatrix} = \langle {}_3\mathscr{F}_0(b,d,e)\,|\lambda\rangle, \tag{4.18c}$$

$$A_\lambda\begin{pmatrix} a,b,d,e \\ -d \end{pmatrix} = \langle {}_3\mathscr{F}_0(a,b,e)\,|\lambda\rangle, \tag{4.18d}$$

$$\langle {}_3\mathscr{F}_0(a,b,c)\,|\,\lambda\rangle = M_\lambda^{-1}\prod_{s=1}^{t}(a-s+1)_{\lambda_s}(b-s+1)_{\lambda_s}(c-s+1)_{\lambda_s}. \tag{4.19}$$

5. Two New Classes of Symmetric Functions

5.1. The Symmetric Polynomials $t_\mu(z)$. The proof of the generalized Bailey identity of the second kind, (Eq. (4.16), led to the discovery of two new classes of symmetric functions. These will be described in this section.

It is instructive to contrast the new symmetric functions with the classic Schur functions. In the definition (4.3) of the Schur function $e_\lambda(z)$, we associate one and the same monomial to each standard tableau having the same weight. In this sense, the Schur functions do not take into account all the "information" carried by the set of standard tableaux. This property is to be contrasted with the symmetric functions defined below, where we associate a distinct polynomial to each tableau and then sum over all tableaux.

We find it convenient to define the new symmetric functions in terms of Gel'fand patterns, although, since these are one-to-one with standard tableaux, the definition could equally well be given in terms of the latter. We remark that the symmetric functions t_λ defined below in this paper are *not* homogeneous polynomials. [This contrasts with the classic symmetric polynomials, which are homogeneous (see Macdonald).] Because of this inhomogeneity, their properties depend on the number of variables. For this reason, and because it is natural when using Gel'fand patterns to keep the zero parts of a partition, we will always employ an explicit notation in which symmetric functions labeled by partitions have a number of parts equal to the number of variables. For example, the symbol $t_{[21]}(z_1,z_2,z_3)$ is not defined.

We now define the class of symmetric polynomials mentioned earlier. Let $\lambda \in P_n$ be a partition. With each pattern $(m)\in G_\lambda$, where $[m_{1n}m_{2n}\ldots m_{nn}]=\lambda$, we associate the following polynomial $t_{(m)}(z)$ in the variables $z = (z_1,z_2,\ldots,z_n)$ (indeterminates):

$$t_{(m)}(z)=\prod_{j=1}^{n}\prod_{i=1}^{j}(z_j-m_{ij}-j+i+1)_{m_{ij}-m_{ij-1}}. \tag{5.1}$$

The dependence of the polynomial $t_{(m)}(z)$ on the variable z_j is given fully in terms of the entries in row j and row $j-1$ of the pattern (m)

as depicted by

$$\begin{matrix} m_{1j} & & m_{2j} & & m_{j-1j} & & m_{jj} \\ & m_{1,j-1} & & m_{2,j-1} & & m_{j-1,j-1} & & 0 \end{matrix} \qquad (5.2)$$

where we have adjoined $m_{jj-1}=0$ to row $j-1$. Thus, the z_j factors in the defining expression are

$$(z_j-m_{1j}-j+2)_{m_{1j}-m_{1,j-1}}(z_j-m_{jj}+1)_{m_{jj}}, \qquad (5.3)$$

The significance of the differences $m_{ij}-m_{ij-1}$ and of the m_{ij} in terms of the standard tableau corresponding to the Gel'fand pattern (m) is noted in Eqs. (2.4). We now define the polynomial t_λ by

$$t_\lambda(z)=\sum_{(m)\in G_\lambda} t_{(m)}(z), \qquad (5.4)$$

where the summation is carried out over all Gel'fand patterns (m) having the specified nth row $\lambda=[m_{1n}m_{2n}\cdots m_{nn}]$; that is, over all patterns, dim λ in number, belonging to the set G_λ.

The polynomials $t_\lambda(z)$ defined by expressions (5.1) and (5.4) are not obviously symmetric in the variables z_i. It is nontrivial to prove this property. This result and a number of other principal properties of the $t_\lambda(z)$ are presented as theorems below, without proofs. All these results are proved in Ref. . These properties establish the polynomials $t_\lambda(z)$ as an important class of symmetric functions.

Theorem 5.1.1. *The polynomials* $t_\lambda(z)$ *are symmetric polynomials in the variables* $(z_1,z_2,\ldots,z_n)$ *for each partition* $\lambda=[\lambda_1\lambda_2\ldots\lambda_n]$.

Theorem 5.1.2. *The set of symmetric polynomials* $\{t_\lambda(z)|\lambda\in P_n\}$ *forms a Z-basis of the ring* Λ_n *of symmetric polynomials.*

Theorem 5.1.3. *The symmetric polynomials* $t_\lambda(z)$ *satisfy the following two relations:*

$$t_{[\lambda_1\lambda_2\ldots\lambda_n]}(z_1,z_2,\ldots,z_n)$$

$$=(z_1z_2\ldots z_n)t_{[\lambda_1-1\ldots\lambda_n-1]}(z_1-1,\ldots z_n-1,\lambda_n\geq 1. \qquad (5.5a)$$

$$t_{[\lambda_1\lambda_2\ldots\lambda_n]}(z_1,z_2,\ldots,z_{n-1},0)$$

$$=\delta_{\lambda_n,0}t_{[\lambda_1\lambda_2\ldots\lambda_{n-1}]}(z_1-1,z_2-1,\ldots,z_{n-1}-1). \qquad (5.5b)$$

Theorem 5.1.4. *Let* $z=(m_1,m_2,\ldots,m_n)$, *where* $m_i=\mu_i+n-i$ *with* $\mu=[\mu_1\mu_2\ldots\mu_n]\in P_n$. *Then*

$$t_\lambda(m)=0, \text{ unless } \lambda_i\leq\mu_i, \ i=1, 2, \ldots, n. \qquad (5.6)$$

Theorem 5.1.5. *Let* D(z) *denote the Vandermonde determinant*

$$D(z)=\prod_{1\leq i<j\leq n}(z_i-z_j). \qquad (5.7a)$$

Then $t_\lambda(z)$ *may be written as*

$$t_\lambda(z)=\frac{1}{D(z)}\sum_\alpha K(\lambda,\alpha)D(z-\alpha)\prod_{i=1}^{n}(z_i-\alpha_i+1)_{\alpha_i}, \qquad (5.7b)$$

where the summation is over all distinct weights α *of the set of standard tableaux of shape* λ.

Theorem 5.1.6. *Let* x *be an arbitrary parameter. Then the following expansion is valid:*

$$\left[\prod_{i=1}^{n}(x+1-z_i)_k=\sum_\lambda\langle{}_1\mathscr{F}_0(-k)|\lambda\rangle\prod_{i=1}^{n}(x-n+i+1)_{k-\lambda_i}\right]t_\lambda(z), \qquad (5.8a)$$

where $\langle{}_1\mathscr{F}_0(-k)|\lambda\rangle$ *denotes the hypergeometric coefficient defined by*

$$\langle {}_1\mathcal{F}_0(-k)\,|\,\lambda\rangle = M_\lambda^{-1}\prod_{i=1}^{n}(-k-i+1)_{\lambda_i},$$

$$= \begin{cases} (-1)^{\lambda_1+\cdots+\lambda_n} dim[\lambda' 0^{k-\lambda_1}] \text{ for } \lambda_1 \le k, \\ 0 \text{ for } \lambda_1 > k, \end{cases} \qquad (5.8b')$$

where λ' *is the partition conjugate to* λ.

Remarks

(*i*) I.G. Macdonald has pointed out to us that $t_\lambda(z)$ is obtained from the Jacobi–Trudi determinantal form of the Schur functions $e_\lambda(z)$ by replacing all the ordinary powers x^a by falling factorials $[x]_a = x(x-1)\cdots(x-a+1)$. If it were possible to prove directly that the $t_\lambda(z)$ thus obtained has the tableau expansion (5.4), the proof of Theorem 5.1.1 would be greatly simplified.

(*ii*) *Theorem* 5.1.6, which gives the expansion of the symmetric polynomial $\sum_{i=1}^{n}(x+1-z_i)_k$ of rising factorials in terms of the basis $t_\lambda(z)$, is the natural generalization of the expansion of this form to ordinary powers k in terms of the Schur function basis $e_\lambda(z)$ (see Ref.).

5.2. **The Symmetric Polynomials** $T_\lambda(\alpha;z)$. In this section, we define another class of symmetric polynomials, denoted $T_\lambda(\alpha;z)$ and depending on n variables $(z_1, z_2, \cdots, z_n)$ and an arbitrary parameter α. We will show that these symmetric polynomials are the natural basis for the expansion of the symmetric polynomial

$$Q_n(x, y; z) = \prod_{i=1}^{n} (x+1-z_i)_k (y+1-z_i)_k \qquad (5.9)$$

in terms of a basis. This polynomial is clearly invariant under the action of the direct product group $S_2 \times S_n$, where S_2 is the group of permutation of (x,y) and S_n that of $(z_1, z_2, \cdots, z_n)$. It is also invariant under the transformation A_j defined by $A_j : z_i \longrightarrow z_i$

$(i=1,2,\cdots,n; i\neq j)$, $z_j \longrightarrow -z_j-\alpha(i=j)$. Here j may be $1,2,\cdots, n$; that is, there are n such transformations in all. We denote by H_n the group generated by these n commuting involution maps.

Having made these brief motivational remarks, let us now proceed directly to the definition of the $T_\lambda(\alpha;z)$. With each Gel'fand pattern (m) having the nth row $\lambda=[\lambda_1\lambda_2\cdots\lambda_n]=[m_{1n}m_{2n}\cdots m_{nn}]$ we associate a polynomial $T_{(m)}$. This polynomial is defined on an arbitrary parameter α and n variables $z=(z_1,z_2,\cdots,z_n)$ by

$$T_{(m)}(\alpha;z)=t_{(m)}(z)t_{(m)}(-z-\alpha), \tag{5.10a}$$

where $\alpha=(\alpha, \alpha,\cdots,\alpha)$ in the right-hand side and

$$z+\alpha=(z_1+\alpha,\ z_2+\alpha,\cdots,\ z_n+\alpha). \tag{5.10b}$$

Here $t_{(m)}$ is defined by Eq. (5.1). We define T_λ by

$$T_\lambda(a;z)=\sum_{(m)\in G_\lambda} T_{(m)}(\alpha;z). \tag{5.10c}$$

Onc again, it is not obvious that $T_\lambda(a;z)$ are symmetric polynomials in the variables $(z_1,z_2,\cdots,z_n)$, since this symmetry is not valid term-wise; that is, is not a property of each $T_{(m)}(a;z)$. It is, however, obvious that each $T_{(m)}(a;z)$, hence, also $T_\lambda(a;z)$, is invariant under each transformation A_j defined above.

We next state the principal properties of the polynomials $T_\lambda(a;z)$:

Theorem 5.2.1. *The polynomials $T_\lambda(a;z)$ are symmetric polynomials in the variables $(z_1,z_2,\cdots,z_n)$ for each partition $\lambda=[\lambda_1\lambda_2\cdots\lambda_n]$ and each parameter α.*

Theorem 5.2.2. *The set of symmetric polynomials $\{T_\lambda(a;z)\,|\,\lambda\in P_n\}$ forms a Z-basis of the ring of polynomials invariant under the group $S_n\times H_n$.*

Theorem 5.2.3. *Let $z=m_1,m_2,\cdots,m_n)$, where $m_i=\mu_i+n-i$ with $\mu=[\mu_1\mu_2\cdots\mu_n]\in P_n$. Then*

$$T_\lambda(\alpha;m)=0, \quad \text{unless } \lambda_i\le\mu_i,\ i=1,2,\cdots,n. \tag{5.11}$$

Theorem 5.2.4. *Let x and y be arbitrary parameters. Then the following expansion is valid*:

$$\prod_{i=1}^{n}(x+1-z_i)_k(y+1-z_i)_k=\sum_\lambda\langle {}_1\mathcal{F}_0(-k|\lambda)\rangle$$

$$\times\prod_{i=1}^{n}(x-n+i+1)_{k-\lambda_i}(y-n+i+1)_{k-\lambda_i}T_\lambda(\alpha;\ z), \tag{5.18a}$$

where

$$\alpha=-x-y-k-1. \tag{5.12a}$$

Theorems 5.2.1–5.2.4 all have crucial roles in the proof that the function A_λ of the last section may be expressed in the form (4.17). (This is carried out in detail in Ref. .) This latter work thus completes, via the properties of the generalized hypergeometric coerfficients of Section 4 and the symmetric functions of this section, the proof that the original G_q^t polynomials are invariant under the determinantal symmetries of the array A.

REFERENCES

1. E.P. WIGNER, *Application of Group Theory to the Special Functions of Mathematical Physics*, (Unpublished lecture notes, Princeton University, Princeton, NJ, 1955).

2. J. Talman, *Special Functions: A Group Theoretical Approach*, (W.A. Benjamin, NY, 1968).

3. N. Vilenkin, *Special Functions and the Theory of Group Representation*, (Transl. from the Russian, Amer. Math. Soc. Transl., Vol. 22, Amer. Math. Soc., Providence, RI, 1968).

4. W. Miller, Jr., *Symmetry and Separation of Variables*, Encycl. of Math. and its Appl., Vol. 4, G.-C. Rota, Ed., (Addison-Wesley, Reading, MA, 1977).

5. J. Wilson, *SIAM J. Math. Anal.* **11** (1980) 670.

6. R. Askey and J.A. Wilson, *SIAM J. Math. Anal.* **10** (1979) 1008; *ibid* **13** (1982) 651.

7. W.J. Holman, III, L.C. Biedenharn, and J.D. Louck, *SIAM J. Math. Anal.* **7** (1976) 529.

8. S.C. Milne, *J. Math. Anal. Appl.* **122** (1987) 223, and references therein.

9. R.A. Gustafson, *SIAM J. Math. Anal.* **18** (1987) 1576, and references therein.

10. G.E. Baird and L.C. Biedenharn, *J. Math. Phys.* **4** (1963) 1449; *ibid* **5** (1965) 1730.

11. J.D. Louck, *Am. J. Phys.* **38** (1970) 3.

12. J.D. Louck and L.C. Biedenharn, *J. Math. Phys.* **11** (1970) 2368; *ibid* **14** (1973) 1336.

13. L.C. Biedenharn, J.D. Louck, E. Chacon, and M. Ciftan, *J. Math. Phys.* **13** (1972) 1957.

14. L.C. Biedenharn and J.D. Louck, *J. Math. Phys.* **13** (1972) 1985.

15. J.D. Louck, M.A. Lohe, and L.C. Biedenharn, *J. Math. Phys.* **16** (1975) 2408.

16. J.D. Louck and L.C. Biedenharn, *J. Math. Anal.* **59** (1977) 423.

17. D.P. Shukla, *Indian J. Pure Appl. Math* **12** (1981) 994.

18. L.C. Biedenharn and J.D. Louck, *Angular Momentum in Quantum*

Physics, Encycl. of Math. and its Appl., Vol. 8, G.-C. Rota, Ed., (Addison-Wesley, Reading, MA, 1977).

19. J.D. Louck and L.C. Biedenharn, *Some Properties of the Intertwining Number of the General Linear Group*, in Science and Computers, Advances in Supplementary Studies, G.-C. Rota, Ed., (Academic Press, NY, 1984).

20. K. Baclawski, *Adv. in Appl. Math*. **5** (1984) 416.

21. L.C. Biedenharn, R.S. Gustafson, M.A. Lohe, J.D. Louck, and S.C. Milne, *Special Functions and Group Theory in Theoretical Physics*, in Special Functions, Group Theoretical Aspects and Applications, R. ASkey, T.H. Koornwinder, and W. Schempp, Eds., (Reidel, NY, 1984) 129.

22. L.C. Biedenharn, M.A. Lohe, and J.D. Louck, *J. Math. Phys*. **26** (1985) 1458; *ibid* **29** (1988) 1106.

23. L.C. Biedenharn and J.D. Louck, *Adv. in Appl. Math*. **9** (1988) 447; *ibid* **10** (1989) 396.

24. L.C. Biedenharn and J.D. Louck, *Proc. Natl. Acad. Sci.* **USA 87** (1990) 1441.

25. L.C. Biedenharn, A.M. Bincer, M.A. Lohe, and J.D. Louck (to appear in *Adv. in Appl. Math.*).

26. J. Grabmeier and A. Kerber, *Acta Appl. Math* **8** (1987) 271.

27. D.E. Littlewood, *The Theory of Group Characters and Matrix Representations of Groups*, (2nd ed., Oxford Univ. Press, London, 1950).

28. W.N. Bailey, *Generalized Hypergeometric Series*, (Cambridge Univ. Press, Cambridge, 1935), p. 56

29. I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, (Clarendon Press, Oxford, 1979).

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