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On the Geometry of Inhomogeneous Quantum Groups

Paolo Aschieri

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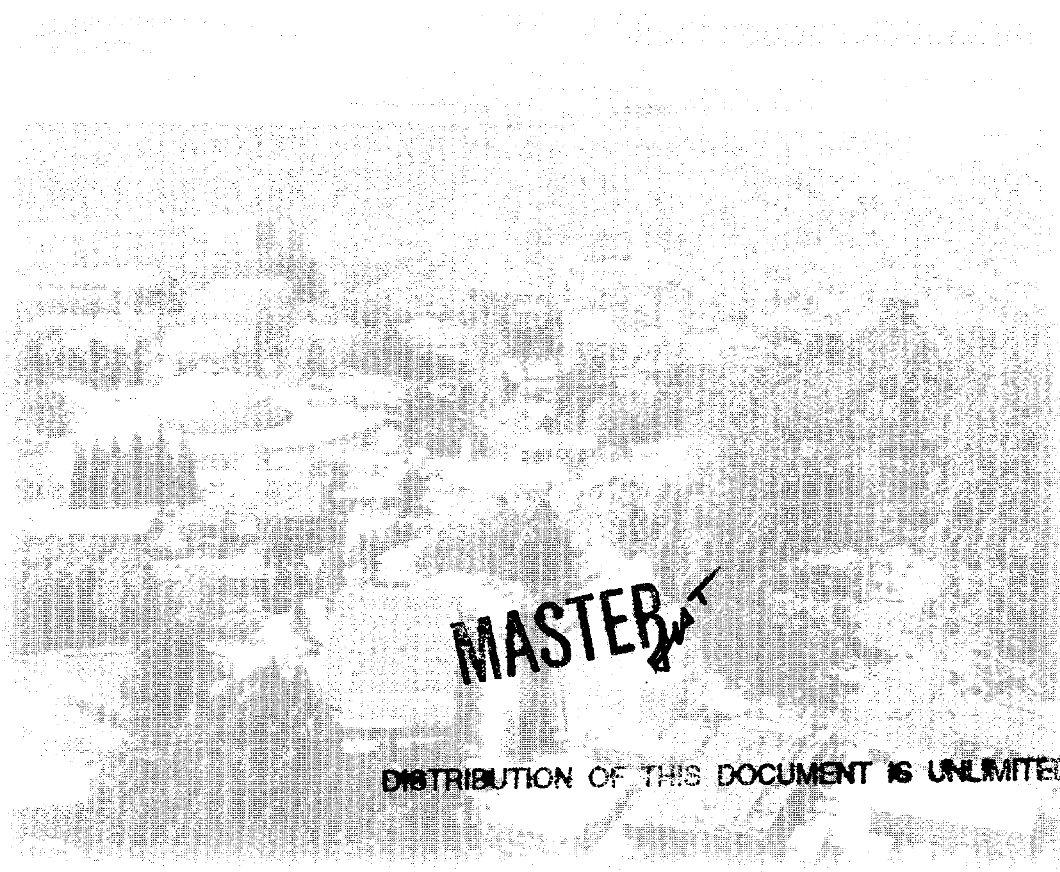
Ph.D. Thesis



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SCUOLA NORMALE SUPERIORE DI PISA

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TESI DI PERFEZIONAMENTO

On the Geometry of Inhomogeneous Quantum Groups

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Abstract

We give a pedagogical introduction to the differential calculus on quantum groups by stressing at all stages the connection with the classical case. We further analize the relation between differential calculus and quantum Lie algebra of left (right) invariant vectorfields. Equivalent definitions of bicovariant differential calculus are studied and their geometrical interpretation is explained. From these data we construct and analize the space of vectorfields, and naturally introduce a contraction operator and a Lie derivative, their properties are discussed.

After a review of the geometry of the multiparametric deformation of the linear group $GL_{q,r}(N)$ and its real forms, we then construct the multiparametric linear inhomogeneous quantum group $IGL_{q,r}(N)$ as a projection from $GL_{q,r}(N+1)$, or equivalently, as a quotient of $GL_{q,r}(N+1)$ with respect to a suitable Hopf algebra ideal. The semidirect product structure of $IGL_{q,r}(N)$ given by the $GL_{q,r}(N)$ quantum subgroup times translations is analized. A bicovariant differential calculus on $IGL_{q,r}(N)$ is explicitly obtained as a projection from the one on $GL_{q,r}(N+1)$. The universal enveloping algebra of $IGL_{q,r}(N)$ and its R-matrix formulation are constructed along the same lines. This quotient procedure unifies in a single structure the quantum plane coordinates and the q-group matrix elements $T^a{}_b$, and allows to deduce without effort the differential calculus on the q-plane $IGL_{q,r}(N)/GL_{q,r}(N)$. The general theory is illustrated on the example of $IGL_{q,r}(2)$.

We proceed similarly in the orthogonal and symplectic case. The inhomogeneous multiparametric q-groups of the B_n, C_n, D_n series are found by means of a projection from $B_{n+1}, C_{n+1}, D_{n+1}$. A matrix formulation is given in terms of the R-matrix of $B_{n+1}, C_{n+1}, D_{n+1}$, and real forms are discussed: in particular we obtain the q-groups $ISO_{q,r}(n+1, n-1)$, including the quantum Poincaré group. The universal enveloping algebras of the multiparametric $B_{n+1}, C_{n+1}, D_{n+1}$ q-groups are studied, they include as Hopf subalgebras the universal enveloping algebras of the inhomogeneous B_n, C_n, D_n. Bicovariant calculi on the minimal multiparametric deformations (twists) of these inhomogeneous groups are similarly found by means of a projection from the bicovariant calculus on B_{n+1}, C_{n+1}, D_{n+1}. In particular we obtain the bicovariant calculus on a dilatation-free minimal deformation of the Poincaré group $ISO_q(3,1)$. Then we construct differential calculi on multiparametric quantum orthogonal planes in any dimension N. These calculi are bicovariant under the action of the full inhomogeneous multiparametric quantum group $ISO_{q,r}(N)$, and do contain dilatations. We find a canonical group-geometric procedure to restrict these calculi on the q-plane and expressed them in terms of coordinates x^a, differentials dx^a and partial derivatives ∂_a without the need of dilatations, thus generalizing known results to the multiparametric case.Real forms are studied and in particular we obtain the quantum Minkowski space $ISO_{q,r}(3,1)/SO_{q,r}(3,1)$.The conjugated partial derivatives ∂_a^* can be expressed as linear combinations of the ∂_a. This allows a deformation of the phase-space with hermitian operators x^a and p_a.

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Contents

Introduction

Quantum Groups are particular deformations of Lie Groups. They are algebraic structures G_q depending on one (or more) continuous parameter q. When $q = 1$ we have a standard Lie group.

A method to modify a group could be to continuously deform the structure constants of the relative Lie algebra. This kind of deformations is uninteresting because it can be shown that for a semisimple Lie group, with a suitable redefinition of the generators it is always possible to recover the undeformed structure constants. At most, for special values of the deformation parameter, we get a contraction of the original group. (For example the Galilei group is a contraction of the Lorentz group, and the Poincaré group is a contraction of the de Sitter group).

Quantum deformations, on the contrary, allow to obtain new structures not isomorphic to the preceding ones. For example the quantum $SU(2)$ group is given by

$$[J^+, J^-] = \frac{q^{2J^0} - q^{-2J^0}}{q - q^{-1}} \quad , \quad [J^0, J^\pm] = \pm J^\pm \qquad (0.0.1)$$

Here we clearly see that the commutator depends continuously on q, moreover we loose the concept of structure constant (in sections 2.1 and 2.3, we will see a generalization of this concept). We however still have a rich structure (Hopf algebra structure), as rich as the $SU(2)$ one [16], [17]. In the $q \to 1$ limit we recover the $SU(2)$ algebra $[J^+, J^-] = 2J^0$, $[J^0, J^\pm] = \pm J^\pm$. Relations (0.0.1) define the universal enveloping algebra of $SU(2)$, i.e. the space given by all (formal) polynomials $\sum \lambda_{mnp}(J^0)^m(J^+)^n(J^-)^p$ where the ordering is always possible thanks to (0.0.1).

We know that a new physical theory can always be seen as a generalization of the previous existing one, in the sense that this is recovered as a limiting case in which some parameters of the new theory become negligible, e.g. special relativity where the Galilei symmetry group is replaced by the Lorentz one. Here the deformation parameter is c, and for $c \to \infty$ we recover Galiley relativity. The study of continuous deformation of Lie groups is therefore physically interesting, the symmetry group underlying many (particle) physics theories being Lie Groups.

The deformations we will deal with are in the context of noncommutative geometry. Both the group coordinates and the space coordinates on which the group acts, consist of non-commuting elements. It is of interest to apply these rich mathematical structures to the study of spacetime physics at Planck scale. Indeed at this scale due to the gravitational forces, Gedanken experiments show that it is not possible to probe spacetime structure[1], the description of spacetime as a smooth manifold becomes just a mathematical assumption. We can relax it and conceive a more general noncommutative structure. This is intresting because in a noncommutative spacetime uncertainty relations and discretization naturally arise. In this way one could incorporate the impossibility of an operational definition of spacetime structure beyond the Planck scale (due to gravity) in the noncommutative geometric structure of spacetime itself. A dynamic feature of spacetime would be incorporated at a kinematical level. This could be a fertile setting for the study of quantum gravity and field theory at Planck scale.

The easiest example of noncommuting space is given by the quantum plane, [48, 49, 50] where the coordinates x and y satisfy $xy = qyx$ where q is a complex parameter. The q-plane and similar higher dimensional q-spaces admit a differential structure where the derivative operators are finite difference operators. This resembles lattice like structures. At the phase-space level the quantum plane relations imply expressions of the type $x\partial_x - q\partial_x x = 1$ that lead to selfadjoint position and momentum operators with discrete heigenvalues: we obtain again a lattice structure [9]. In this context q may play the role of short distance regularization parameter preserving the q-symmetries, see also [10]. In the particular deformations where q is a dimensionful parameter, one can also try to relate this parameter to the Planck length [11].

Notice that a minimal uncertainty relation in position measurement is also in agreement with string theory models [2]. Moreover, non-perturbative attempts to describe string theories have shown that a noncommutative structure of spacetime emerges [3], noncommutative geometry can be the correct geometrical framework for the description of such theories [4].

In our study of inhomogeneous q-groups we will consider examples of noncommuting quantum planes (in 2 and higher dimension) that have quantum symmetry groups. This is not the only approach to a possible model of noncommutative spacetime. For example one can consider a noncommutative Minkowski spacetime which is covariant under the classical Poincaré group, see [5], and [6] where also first attempts to construct quantum field theories on noncommutative spaces have shown that in some cases these theories are equivalent to a non local quantum field theory on ordinary space. We also mention a related approach [8] that, in the spirit

[1]For example, in relativistic quantum mechanics the position of a particle can be detected with a precision at most of the order of its Compton wave length $\lambda_C = \hbar/mc$. Probing spacetime at infinitesimal distances implies an extremely heavy particle that in turn curves spacetime itself. When λ_C is of the order of the Planck length, the spacetime curvature radius due to the particle has the same order of magnitude and the attempt to measure spacetime structure beyond Planck scale fails.

of [7], uses noncommutative geometric methods to study a Kaluza-Klein gravity theory with a discrete two point internal space.

In this thesis we generalize to Hopf algebras basic structures of differential geometry on commutative manifolds, we then examine examples of inhomogeneous quantum groups and consider their differential calculus.

In the first chapters we introduce the concept of quantum group, and, stressing at all stages the connection with the classical case ($q \to 1$ limit), we develop the differential calculus on quantum groups, first studied in the seminal paper by Woronowicz [21]. We then examine in detail the quantum Lie algebra of left-invariant vectorfields (see Section 2.3). All the properties of the differential calculus can be derived from intuitive properties of the q-Lie algebra, in this way we emphasize the space of vectorfields that is more fundamental, for physical applications, than the space of 1-forms. For example, this q-Lie algebras are a good starting point for the formulation of gauge theories based on deformed Lie groups [12]. Next, a Lie derivative and a contraction operator (inner derivative) are found and, for left-invariant vectorfields, we prove the Cartan identity $\ell_{\vec{t}} = i_{\vec{t}}d + di_{\vec{t}}$ [27, 26, 34, 37]. These are basic tools in differential geometry, and are of interest in a geometric formulation of Einstein gravity. For example the invariance of Einstein action under diffeomorphisms can be expressed by $\ell_{\vec{t}} \int \mathcal{L} d^4x = 0$, this relation leads to the covariant conservation of the matter energy-momentum tensor (if torsion vanishes). Also, in the soft group manifold approach to gravity theories [13], the Cartan-Maurer equation, the Lie derivative and the contraction operator for the q-Poincaré group are fundamental to formulate a geometric definition of curvature, covariant derivative and Lorentz gauge transformation. There the curved general relativity space time (with the Lorentz gauge group) is obtained as the coset space Poincaré/Lorentz where the rigid Poincaré group structure has been softened allowing for a curvature two form term in the Cartan-Maurer equations. For a first example of this construction in the case of a minimal q-deformation (twist) of the Poincaré group see [14].

The second part of this thesis deals with the specific study of deformations of the inhomogeneous general linear group $GL(N)$ and of the inhomogeneous orthogonal and symplectyc groups. Contrary to the case of semisimple Lie groups [where there is a canonical Poisson (symplectic) structure that can be quantized to give the quantum group] there is not a canonical deformation procedure for these groups; providing examples of inhomogeneous deformation is therefore per se interesting. The $GL(N)$ and $SL(N)$ cases are easier to consider than the orthogonal and symplectic ones and the basic structures of inhomogeneous quantum groups are first found in this cases and later shown for the B_n, C_n, D_n series as well. There are many studies on inhomogeneous quantum groups [65, 57, 58, 59] and in particular on deformed Poincaré groups [66]. The analysis we present [60], [67] is based on a quotient procedure, we find deformations of the inhomogeneous version of the A_n, B_n, C_n, D_n groups via a quotient from the q-deformed homogeneous $A_{n+1}, B_{n+1}, C_{n+1}, D_{n+1}$ groups. An R-matrix formulation is thus provided. The universal enveloping algebra of these groups is also studied as well as their semidirect product structure of their homogeneous subgroups times translations. Then we apply the general theory of the differential calculus on q-groups to these specific cases. Differential calculi on these inhomogeneous q-groups are found using again the quotient structure with respect to $A_{n+1}, B_{n+1}, C_{n+1}, D_{n+1}$.

In particular we obtain a deformation of the Poincaré group, of its q-Lie algebra and differential calculus. It turns out that the differential calculus on inhomogeneous orthogonal and symplectic quantum groups, contrary to the linear case, cannot be constructed for general values of the deformation parameters. It exists only for minimal deformations (twists), these are the same noncommutative deformations that do not require the presence of a dilatation in the fully q-deformed inhomogeneous structure. Our analysis includes the first example of bicovariant differential calculus on quantum Poincaré group with deformed Lorentz subgroup. For bicovariant calculi on other deformations of the Poincaré group see [70]. The study of the differential calculi on orthogonal groups discussed in Chapter 4, leads to a good candidate for the differential calculus on the q-orthogonal planes and in particular on the q-Minkowski plane, with no restriction on the deformation parameters. Using the powerful result: q-Minkowski = q-Poincaré/q-Lorentz, we derive canonically the q-Minkowski geometry from the q-Poincaré geometry [55]. In this way we rederive the known results of [48, 53, 54] using the broader setting of the differential calculus on quantum $ISO(N)$. A detailed analysis of the reality condition on the quantum Minkowski plane is possible since on the quantum $ISO(3,1)$ differential calculus there is a canonical definition of $*$-conjugation. This operation is linear and one can canonically obtain real coordinates and momenta and a q-version of the Heisenberg x, p commutation relations. An interesting issue, in the spirit of [9] is then the analysis of the representations in Hilbert space of this algebra in order to study the admissible (discrete) values of momentum and position of particle states.

Chapter 1

Quantum Groups

1.1 Hopf structures in ordinary Lie groups and Lie algebras

Let us begin by considering $Fun(G)$, the set of differentiable functions from a Lie group G into the complex numbers $\mathbf{C}$. $Fun(G)$ is an algebra with the usual pointwise sum and product $(f+h)(g) = f(g)+h(g)$, $(f \cdot h) = f(g)h(g)$, $(\lambda f)(g) = \lambda f(g)$, for $f, h \in Fun(G)$, $g \in G$, $\lambda \in \mathbf{C}$. The unit of this algebra is I, defined by $I(g) = 1$, $\forall g \in G$.

Using the group structure of G, we can introduce on $Fun(G)$ three other linear mappings, the coproduct Δ, the counit ε, and the coinverse (or antipode) κ:

$$\Delta(f)(g,g') \equiv f(gg'), \quad \Delta : Fun(G) \to Fun(G) \otimes Fun(G) \tag{1.1.1}$$
$$\varepsilon(f) \equiv f(1_G), \quad \varepsilon : Fun(G) \to \mathbf{C} \tag{1.1.2}$$
$$(\kappa f)(g) \equiv f(g^{-1}), \quad \kappa : Fun(G) \to Fun(G) \tag{1.1.3}$$

where 1_G is the unit of G. It is not difficult to verify the following properties of the co-structures:

$$(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta \quad \text{(coassociativity of } \Delta) \tag{1.1.4}$$
$$(id \otimes \varepsilon)\Delta(a) = (\varepsilon \otimes id)\Delta(a) = a \tag{1.1.5}$$
$$m(\kappa \otimes id)\Delta(a) = m(id \otimes \kappa)\Delta(a) = \varepsilon(a)I \tag{1.1.6}$$

and

$$\Delta(ab) = \Delta(a)\Delta(b), \quad \Delta(I) = I \otimes I \tag{1.1.7}$$
$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b), \quad \varepsilon(I) = 1 \tag{1.1.8}$$
$$\kappa(ab) = \kappa(b)\kappa(a), \quad \kappa(I) = I \tag{1.1.9}$$

where $a, b \in A = Fun(G)$ and m is the multiplication map $m(a \otimes b) \equiv ab$. The product in $\Delta(a)\Delta(b)$ is the product in $A \otimes A$: $(a \otimes b)(c \otimes d) = ab \otimes cd$.

In general a coproduct can be expanded on $A \otimes A$ as:

$$\Delta(a) = \sum_i a_1^i \otimes a_2^i \equiv a_1 \otimes a_2, \tag{1.1.10}$$

where $a_1^i, a_2^i \in A$ and $a_1 \otimes a_2$ is a shorthand notation we will often use in the sequel. For example for $A = Fun(G)$ we have:

$$\Delta(f)(g, g') = (f_1 \otimes f_2)(g, g') = f_1(g) f_2(g') = f(gg'). \tag{1.1.11}$$

Using (1.1.11), the proof of (1.1.4)-(1.1.6) is immediate. We will also use the following notation: $\Delta^2(a) \equiv (\Delta \otimes id)\Delta(a) = (id \otimes \Delta)\Delta(a) = a_1 \otimes a_2 \otimes a_3$, more in general $\Delta^n(a) = a_1 \otimes a_2 \otimes, \ldots a_{n+1}$.

An algebra A endowed with the homomorphisms $\Delta : A \to A \otimes A$ and $\varepsilon : A \to \mathbf{C}$, and the antimorphism $\kappa : A \to A$ satisfying the properties (1.1.4)-(1.1.9) is a *Hopf algebra*. Thus $Fun(G)$ is a Hopf algebra.[1] Note that the properties (1.1.4)-(1.1.9) imply the relations:

$$\Delta(\kappa(a)) = \kappa(a_2) \otimes \kappa(a_1) \quad , \qquad \kappa(a)_1 \otimes \kappa(a)_2 \, , \ldots \kappa(a)_n = \kappa(a_n) \otimes \kappa(a_{n-1}) \, , \ldots \kappa(a_1) \tag{1.1.12}$$

$$\varepsilon(\kappa(a)) = \varepsilon(a). \tag{1.1.13}$$

Consider now the algebra A of polynomials in the matrix elements $T^a{}_b$ of the fundamental representation of G, i.e. the algebra A generated by the $T^a{}_b$.

It is clear that $A \subset Fun(G)$, since $T^a{}_b(g)$ are functions on G. In fact A is dense in $Fun(G)$ [the reason is that the matrix elements of all *finite* irreducible dimensional representations of G can be constructed out of appropriate products of $T^a{}_b(g)$; these products span a dense subset in $Fun(G)$ because the matrix elements of all irreducible representations of G form a complete basis of $Fun(G)$], therefore, a suitable completion $\hat{A}$ of A is $Fun(G)$: $\hat{A} = Fun(G)$. In the following we will drop the hat and we will not be concerned about these topological aspects. The group manifold G can be completely characterized by $Fun(G)$, the co-structures on $Fun(G)$ carrying the information about the group structure of G. Thus a classical Lie group can be "defined" as the algebra A generated by the (commuting) matrix elements $T^a{}_b$ of the fundamental representation of G, seen as functions on G. This definition admits noncommutative generalizations, i.e. the quantum groups discussed in the next section.

Using the elements $T^a{}_b$ we can write an explicit formula for the expansion (1.1.10) or (1.1.11): indeed (1.1.1) becomes

$$\Delta(T^a{}_b)(g, g') = T^a{}_b(gg') = T^a{}_c(g) T^c{}_b(g'), \tag{1.1.14}$$

[1]To be precise, $Fun(G)$ is a Hopf algebra when $Fun(G \times G)$ can be identified with $Fun(G) \otimes Fun(G)$, since only then can one define a coproduct as in (1.1.1).

since T is a matrix representation of G. Therefore:

$$\Delta(T^a{}_b) = T^a{}_c \otimes T^c{}_b. \tag{1.1.15}$$

Moreover, using (1.1.2) and (1.1.3), one finds:

$$\varepsilon(T^a{}_b) = \delta^a_b \tag{1.1.16}$$
$$\kappa(T^a{}_b) = (T^{-1})^a{}_b. \tag{1.1.17}$$

Thus the algebra $A = Fun(G)$ of polynomials in the elements $T^a{}_b$ is a Hopf algebra with co-structures defined by (1.1.15)-(1.1.17) and (1.1.7)-(1.1.9).

Another example of Hopf algebra is given by any ordinary Lie algebra g, or more precisely by the universal enveloping algebra $U(g)$ of a Lie algebra g, i.e. (by the Poincaré-Birkhoff-Witt theorem) the algebra, with unit I, of polynomials in the generators χ_i modulo the commutation relations

$$[\chi_i, \chi_j] = C_{ij}{}^k \chi_k. \tag{1.1.18}$$

Here we define the co-structures as:

$$\Delta'(\chi_i) = \chi_i \otimes I + I \otimes \chi_i \quad \Delta'(I) = I \otimes I \tag{1.1.19}$$
$$\varepsilon'(\chi_i) = 0 \quad \varepsilon'(I) = 1 \tag{1.1.20}$$
$$\kappa'(\chi_i) = -\chi_i \quad \kappa'(I) = I \tag{1.1.21}$$

The reader can check that (1.1.4)-(1.1.6) are satisfied.

1.2 Quantum groups. The example of $GL_q(2)$

Quantum groups can be introduced as noncommutative deformations of the algebra $A = Fun(G)$ of the previous section [more precisely as noncommutative Hopf algebras obtained by continuous deformations of the Hopf algebra $A = Fun(G)$]. The term quantum stems for the fact that they are obtained quantizing a Poisson (symplectic) structure of the algebra $Fun(G)$ [16]. Here, following [19] (see also [20]), we will consider quantum groups defined as the associative algebras A freely generated by non-commuting matrix entries $T^a{}_b$ satisfying the relation

$$R^{ab}{}_{ef} T^e{}_c T^f{}_d = T^b{}_f T^a{}_e R^{ef}{}_{cd} \tag{1.2.1}$$

and some other conditions depending on which classical group we are deforming (see later). The matrix R controls the non-commutativity of the $T^a{}_b$, and its elements depend continuously on a (in general complex) parameter q, or even a set of parameters. For $q \to 1$, the so-called "classical limit", we have

$$R^{ab}{}_{cd} \xrightarrow{q\to 1} \delta^a_c \delta^b_d, \tag{1.2.2}$$

i.e. the matrix entries $T^a{}_b$ commute for $q = 1$, and one recovers the ordinary $Fun(G)$.

The associativity of A leads to a consistency condition on the R matrix, the quantum Yang–Baxter equation:

$$R^{a_1b_1}{}_{a_2b_2}R^{a_2c_1}{}_{a_3c_2}R^{b_2c_2}{}_{b_3c_3} = R^{b_1c_1}{}_{b_2c_2}R^{a_1c_2}{}_{a_2c_3}R^{a_2b_2}{}_{a_3b_3}. \tag{1.2.3}$$

For simplicity we rewrite the "RTT" equation (1.2.1) and the quantum Yang–Baxter equation as

$$R_{12}T_1T_2 = T_2T_1R_{12} \tag{1.2.4}$$

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{1.2.5}$$

where the subscripts 1, 2 and 3 refer to different couples of indices. Thus T_1 indicates the matrix $T^a{}_b$, T_1T_1 indicates $T^a{}_cT^c{}_b$, $R_{12}T_2$ indicates $R^{ab}{}_{cd}T^d{}_e$ and so on, repeated subscripts meaning matrix multiplication. The quantum Yang–Baxter equation (1.2.5) is a condition sufficient for the consistency of the RTT equation (1.2.4). Indeed the product of three distinct elements $T^a{}_b$, $T^c{}_d$ and $T^e{}_f$, indicated by $T_1T_2T_3$, can be reordered as $T_3T_2T_1$ via two differents paths:

$$T_1T_2T_3 \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} T_1T_3T_2 \rightarrow T_3T_1T_2 \\ \\ T_2T_1T_3 \rightarrow T_2T_3T_1 \end{array} \begin{array}{c} \searrow \\ \nearrow \end{array} T_3T_2T_1 \tag{1.2.6}$$

by repeated use of the RTT equation. The relation (1.2.5) ensures that the two paths lead to the same result.

The algebra A ("the quantum group") is a noncommutative Hopf algebra whose co-structures are the same of those defined for the commutative Hopf algebra $Fun(G)$ of the previous section, eqs. (1.1.15)-(1.1.17), (1.1.7)-(1.1.9).

Let us give the example of $SL_q(2) \equiv Fun_q(SL(2))$, the algebra freely generated by the elements α, β, γ and δ of the 2×2 matrix

$$T^a{}_b = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{1.2.7}$$

satisfying the commutations

$$\begin{array}{c} \alpha\beta = q\beta\alpha, \quad \alpha\gamma = q\gamma\alpha, \quad \beta\delta = q\delta\beta, \quad \gamma\delta = q\delta\gamma \\ \beta\gamma = \gamma\beta, \quad \alpha\delta - \delta\alpha = (q - q^{-1})\beta\gamma, \qquad q \in \mathbf{C} \end{array} \tag{1.2.8}$$

and

$$\det_q T \equiv \alpha\delta - q\beta\gamma = I. \tag{1.2.9}$$

The commutations (1.2.8) can be obtained from (1.2.1) via the R matrix

$$R^{ab}{}_{cd} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \tag{1.2.10}$$

where the rows and columns are numbered in the order 11, 12, 21, 22.

It is easy to verify that the "quantum determinant" defined in (1.2.9) commutes with α, β, γ and δ, so that the requirement $\det_q T = I$ is consistent. The matrix inverse of $T^a{}_b$ is

$$(T^{-1})^a{}_b = (\det_q T)^{-1} \begin{pmatrix} \delta & -q^{-1}\beta \\ -q\gamma & \alpha \end{pmatrix} \tag{1.2.11}$$

The coproduct, counit and coinverse of α, β, γ and δ are determined via formulas (1.1.15)-(1.1.17) to be:

$$\Delta(\alpha) = \alpha \otimes \alpha + \beta \otimes \gamma, \quad \Delta(\beta) = \alpha \otimes \beta + \beta \otimes \delta$$
$$\Delta(\gamma) = \gamma \otimes \alpha + \delta \otimes \gamma, \quad \Delta(\delta) = \gamma \otimes \beta + \delta \otimes \delta \tag{1.2.12}$$
$$\varepsilon(\alpha) = \varepsilon(\delta) = 1, \quad \varepsilon(\beta) = \varepsilon(\gamma) = 0 \tag{1.2.13}$$
$$\kappa(\alpha) = \delta, \;\; \kappa(\beta) = -q^{-1}\beta, \;\; \kappa(\gamma) = -q\gamma, \;\; \kappa(\delta) = \alpha. \tag{1.2.14}$$

Note 1.2.1 In general $\kappa^2 \neq 1$, as can be seen from (1.2.14). The following useful relation holds [19]:

$$\kappa^2(T^a{}_b) = D^a{}_c T^c{}_d (D^{-1})^d{}_b = d^a d_b^{-1} T^a{}_b, \tag{1.2.15}$$

where D is a diagonal matrix, $D^a{}_b = d^a \delta^a_b$, given by $d^a = q^{2a-1}$ for the q-groups A_{n-1} and $GL_q(n)$.

Note 1.2.2 The commutations (1.2.8) are compatible with the coproduct Δ, in the sense that $\Delta(\alpha\beta) = q\Delta(\beta\alpha)$ and so on. In general we must have

$$\Delta(R_{12}T_1T_2) = \Delta(T_2T_1R_{12}), \tag{1.2.16}$$

which is easily verified using $\Delta(R_{12}T_1T_2) = R_{12}\Delta(T_1)\Delta(T_2)$ and $\Delta(T_1) = T_1 \otimes T_1$. This is equivalent to proving that the matrix elements of the matrix product T_1T_1', where T' is a matrix [satisfying (1.2.1)] whose elements *commute* with those of $T^a{}_b$, still obey the commutations (1.2.4).

Note 1.2.3 $\Delta(\det_q T) = \det_q T \otimes \det_q T$ so that the coproduct property $\Delta(I) = I \otimes I$ is compatible with $\det_q T = I$.

Note 1.2.4 The condition (1.2.9) can be relaxed. Then we have to include the central element $\zeta = (\det_q T)^{-1}$ in A, so as to be able to define the inverse of the

q-matrix $T^a{}_b$ as in (1.2.11), and the coinverse of the element $T^a{}_b$ as in (1.1.17). The q-group is then $GL_q(2)$. The reader can deduce the co-structures on ζ: $\Delta(\zeta) = \zeta \otimes \zeta$, $\varepsilon(\zeta) = 1$, $\kappa(\zeta) = \det_q T$.

Note 1.2.5 More generally, the quantum determinant of $n \times n$ q-matrices is defined by $\det_q T = \sum_\sigma (-q)^{l(\sigma)} T^1{}_{\sigma(1)} \cdots T^n{}_{\sigma(n)}$, where $l(\sigma)$ is the minimal number of inversions in the permutation σ. Then $\det_q T = 1$ restricts $GL_q(n)$ to $SL_q(n)$.

Note 1.2.6 The explicit expression of the R-matrices for the A,B,C,D q-groups will be given later. Here we recall the important relations [19] for the $\hat{R}$ matrix defined by $\hat{R}^{ab}{}_{cd} \equiv R^{ba}{}_{cd}$, whose $q = 1$ limit is the permutation operator $\delta^a_d \delta^b_c$:

$$\hat{R}^2 = (q - q^{-1})\hat{R} + I, \quad \text{for } A_{n-1} \quad \text{(Hecke condition)} \tag{1.2.17}$$

$$(\hat{R} - qI)(\hat{R} + q^{-1}I)(\hat{R} - q^{1-N}I) = 0, \quad \text{for } B_n, C_n, D_n, \tag{1.2.18}$$

with $N = 2n + 1$ for the series B_n and $N = 2n$ for C_n and D_n. Moreover for all A, B, C, D q-groups the R matrix is lower triangular ($R^{ab}{}_{cd} = 0$ if [$a = c$ and $b < d$] or $a < c$) and satisfies:

$$(R^{-1})^{ab}{}_{cd}(q) = R^{ab}{}_{cd}(q^{-1}) \tag{1.2.19}$$

$$R^{ab}{}_{cd} = R^{dc}{}_{ba}. \tag{1.2.20}$$

1.3 Duality and $*$-Structure

Duality

Consider a finite dimensional Hopf algebra A, the vector space A' dual to A is also a Hopf algebra with the following product, unit and costructures [we use the notation $\psi(a) = \langle \psi, a \rangle$ in order to stress the duality between A' and A]: $\forall \psi, \phi \in A'$, $\forall a, b \in A$

$$\langle \psi\phi, a \rangle = \langle \psi \otimes \phi, \Delta a \rangle \;, \quad \langle I, a \rangle = \varepsilon(a) \tag{1.3.1}$$

$$\langle \Delta'(\psi), a \otimes b \rangle = \langle \psi, ab \rangle \;, \quad \varepsilon'(\psi) = \langle \psi, I \rangle \tag{1.3.2}$$

$$\langle \kappa'(\psi), a \rangle = \langle \psi, \kappa(a) \rangle \tag{1.3.3}$$

where $\langle \psi \otimes \phi \;,\; a \otimes b \rangle \equiv \langle \psi, a \rangle \langle \phi, b \rangle$. Obviously $(A')' = A$ and A and A' are dual Hopf algebras.

In the infinite dimensional case the definition of duality between Hopf algebras is more delicate because the coproduct on A' might not take values in the subspace $A' \otimes A'$ of $(A \otimes A)'$ and therefore is ill defined. However, the space A^0 spanned by the matrix elements of all finite-dimensional representations of A is a subalgebra of A' and obviously $\Delta'(A^0) \subset A^0 \otimes A^0$, $\kappa(A^0) \subset A^0$ [indeed $\Delta'(M^i{}_j) = \sum_{k=1}^{dim(M)} M^i{}_k \otimes M^k{}_j$, $\kappa(M^i{}_j) = (M^{-1})^i{}_j$]. Then A^0 is a Hopf algebra: the Hopf dual of A. In general $(A^0)^0 \neq A$, for example if g is semisimple $U(g)^0 = Fun(G)$ while the vector space underlying $Fun(G)^0$ is $U(g) \otimes \mathbf{C}(G)$ where $\mathbf{C}(G)$ is the vector space on $\mathbf{C}$ freely generated by the elments of G [83]. Later on we will consider the quantum groups of the series A_n, B_n, C_n, D_n and their quantized universal enveloping algebras (as the algebras of regualr functionals on the deformed A_n, B_n, C_n, D_n [19]); disregarding the topological aspects we will call these algebras dually paired or dual, the quantum group $SL_q(N)$ can indeed be considered as the Hopf dual of the deformed universal enveloping algebra of the A_n series.

For generic Hopf algebras we will use the notion of non-degenerate pairing: two Hopf algebras A and U are paired if there exists a bilinear map $\langle\ ,\ \rangle\ : U \otimes A \to \mathbf{C}$ satisfying (1.3.1) and (1.3.2), the pairing is non-degenerate if we also have

$$\forall\ \psi \in U\ \ \langle \psi, a\rangle = 0 \Rightarrow a = 0 \tag{1.3.4}$$

and

$$\forall\ a \in A\ \ \langle \psi, a\rangle = 0 \Rightarrow \psi = 0\ . \tag{1.3.5}$$

Condition (1.3.4) states that U separates the points (elements) of A and viceversa for (1.3.5). If U and A are finite dimensional then (1.3.4) and (1.3.5) are equivalent to $A' = U$; indeed (1.3.4) induces the injection $a \to \langle\ , a\rangle$ of A in U', similarly, by (1.3.5) $U \subseteq A'$ and therefore $A' = U$.

It is easy to prove that the Hopf algebras $Fun(G)$ and $U(g)$ described in Section 1.1 are paired when g is the Lie algebra of G. Indeed we realize g as left invariant vectorfields t on the group manifold and $U(g)$ as the algebra generated by composition of the operators t. Then the pairing is defined by

$$\forall t \in g, \forall f \in Fun(G),\ \ \langle t, f\rangle = t(f)|_{1_G}$$

where 1_G is the unit of G. Notice that t is left invariant if $TL_g(t|_{1_G}) = t|_g$, where TL_g is the tangent map induced by the left multiplication of the group on itself: $L_g g' = gg'$. We then have

$$t(f)|_g = \left(TL_g t|_{1_G}\right)(f) = t[f(g\tilde{g})]|_{\tilde{g}=1_G} = t[f_1(g)f_2(\tilde{g})]|_{\tilde{g}=1_G} = f_1(g)\, t(f_2)|_{1_G} \tag{1.3.6}$$

and therefore

$$\langle \tilde{t}\circ t, f\rangle = \tilde{t}(t(f))|_{1_G} = \tilde{t}f_1|_{1_G} t f_2|_{1_G} = \langle \tilde{t} \otimes t, \Delta f\rangle$$

and, in agreement with (1.1.19) and (1.1.21):

$$\langle t, fh\rangle = t(f)|_{1_G} h|_{1_G} + f|_{1_G} t(h)|_{1_G} = \langle \Delta'(t), f \otimes h\rangle\ ,$$

$$\langle t, \kappa(f)\rangle = t[f(g^{-1})]|_{g=1_G} = -t[f(g)]|_{g=1_G} = \langle \kappa'(t), f\rangle\ .$$

$*$-Structure

The Hopf algebra $SL_q(2)$ we have considered in the previous section can be interpreted as the deformation of the algebra of functions on a group manifold only introducing a $*$-structure on $SL_q(2)$ (the analogue of complex conjugation). This procedure leads to the quantum groups $SL_q(2,\mathbf{R}) = Fun_q(SL(2,\mathbf{R}))$ and $SU_q(2) = Fun_q(SU(2))$. We need a $*$-structure because the algebra of regular functions on $SU(2)$ is isomorphic to the algebra of regular functions on $SL(2,\mathbf{R})$ and to the algebra of analytic functions on the *complex* manifold $SL(2,\mathbf{C})$; indeed any $f \in Fun(SU(2))$ can be analytically continued in a unique function $\hat{f} \in Fun(SL(2,\mathbf{C}))$, then the restriction of $\hat{f}$ to the $SL(2,\mathbf{R})$ sub-manifold of $SL(2,\mathbf{C})$ belongs to $Fun(SL(2,\mathbf{R}))$. Therefore, without a $*$-structure we cannot understand if the polynomials in the symbols $T^i{}_j$ with the relation $\det T = 1$ generate functions on $SU(2)$ or on $SL(2,\mathbf{R})$ or analytic functions on $SL(2,\mathbf{C})$.

In general a $*$-algebra over the complex numbers is an algebra with an antilinear map $* : A \rightarrow A$ that is involutive, $*^2 = id$ and anti-multiplicative, $(ab)^* = b^*a^*$ $\forall a, b \in A$. A Hopf $*$-algebra A is a Hopf algebra A over $\mathbf{C}$ equipped with a $*$-algebra structure which is compatible with the costructures of A:

$$\Delta(a^*) = a_1^* \otimes a_2^* \ ; \qquad \varepsilon(a^*) = \overline{\varepsilon(a)} \ . \tag{1.3.7}$$

These two conditions imply, forall $a \in A$

$$[\kappa(a)]^* = \kappa^{-1}(a^*) \ ; \tag{1.3.8}$$

indeed the operator $* \circ \kappa^{-1} \circ *$ satisfies all the properties of the antipode and since (as the inverse of an element in a group) the antipode is unique, we have (1.3.8).

Let us clarify the interrelation between real forms of groups and $*$-structures on Hopf algebras.

Let $A = \mathcal{F}(G_\mathbf{C})$ be the algebra of analytic functions on a complex group $G_\mathbf{C}$. A $*$-structure on A determines the following real form $G_\mathbf{R}$ of $G_\mathbf{C}$: $G_\mathbf{R} = \{g \in G / \ f^*(g) = \overline{f(g)}\}$; viceversa $G_\mathbf{R}$ induces on $Fun(G_\mathbf{R}) = \mathcal{F}(G_\mathbf{C})$ the following $*$-structure: $f^* = h \ \Leftrightarrow \overline{f(g)} = h(g) \ \forall g \in G_\mathbf{R}$. Moreover a real form of $G_\mathbf{C}$ determines a real form $g_\mathbf{R}$ of its Lie algebra, i.e. $g = g_\mathbf{R} \oplus \sqrt{-1}\, g_\mathbf{R}$. The $*$-operation that acts as minus the identity on $g_\mathbf{R}$ satisfies $[\chi, \chi']^* = [\chi'^*, \chi^*]$ $\ \forall \chi, \chi' \in g_\mathbf{R}$ and is uniquely extended as an anti-linear, anti-multiplicative and involutive map on the Hopf algebra $U(g)$, the universal enveloping algebra of g. We have seen that a $*$-structure on $A = \mathcal{F}(G_\mathbf{C}) = Fun(G)$ determines a $*$-structure on $U(g)$, the viceversa is also obviously true. The explicit relation is $\forall \psi \in U(g)$, $\forall a \in Fun(G)$,

$$\langle \psi^*, a \rangle = \overline{\langle \psi, [\kappa(a)]^* \rangle} \quad \text{i.e.} \quad \langle \psi, a^* \rangle = \overline{\langle [\kappa'(\psi)]^*, a \rangle} \tag{1.3.9}$$

where we have used the pairing $\langle \ , \rangle$ between the two Hopf algebras $Fun(G)$ and $U(g)$, and $\bar{}$ denotes complex conjugation.

More in general two Hopf $*$-algebras A and U are paired if, in addition to (1.3.1) and (1.3.2), relation (1.3.9) holds $\forall \psi \in U$ and $\forall a \in A$.

In a functional-analytic context, the $*$-operation becomes the hermitian conjugation. For example, Hopf $*$-algebras that are deformations of compact matrix groups can be canonically realized (using the Gelfand-Naimark-Segal (GNS) construction, since they are dense in a C^*-algebra and since they have a Haar measure) as bounded operators on a Hilbert space. Then the $*$-operation is realized as the usual adjoint map † on operators in Hilbert space.

We end this section listing the $*$-structures that define $SU_q(2)$ and $SL_q(2,R)$; these $*$-involutions are well defined because they are compatible with the RTT relations ($RT_1T_2 = T_2T_1R \Leftrightarrow \overline{R}T_2^*T_1^* = T_1^*T_2^*\overline{R}$) and with the determinat condition.

i) $T^* = T^{-1^t} \Rightarrow \alpha^* = \delta,\ \beta^* = -q\gamma,\ \gamma^* = -q^{-1}\beta,\ \delta^* = \alpha$, where q is a real number. Gives the Hopf $*$-algebra $SU_q(2)$.

ii)$T^* = T \Rightarrow \alpha^* = \alpha,\ \beta^* = \beta,\ \gamma^* = \gamma,\ \delta^* = \delta,\ |q| = 1$. Gives the Hopf $*$-algebra $SL_q(2, \mathbf{R})$.

Chapter 2

Differential Geometry on Quantum Groups

In this chapter we study basic notions of differential geometry on a Hopf algebra. We first give an introductory review of the q-differential calculus studied by [21]. In the first section, following [21], [27], we define the conditions an exterior differential d has to satisfy and we explain them as natural generalizations of the conditions on a classical Lie group. The space of 1-forms and then that of n-forms is fully characterized. Then we introduce the left invariant vectorfields and deduce their q-Lie algebra properties from the properties of the exterior differential d. The theory is exemplified on the quantum group $GL_q(N)$ in Section 2.2. In Section 2.3 we reconsider the basic postulates of a differential calculus and describe equivalent ones. This section is complementary to Section 2.1 because emphasizes the space of vectorfields rather than the space of one forms and the exterior differential. We start from a q-Lie algebra that closes under a quadratic q-Lie bracket and then we derive the properties of the exterior differential.

The last section of this chapter is a deeper study of the geometric structures on Hopf algebras. We analize the duality between the space of vectorfields and the space of 1-forms (tangent and the cotangent bundle) and generalize the construction to tensorfields. An inner derivative or contraction operator naturally arises from the above duality. We then introduce a Lie derivative and analize its properties. Finally we show how these operators and the exterior differential form a graded quantum Lie algebra of differantial operators.

2.1 Bicovariant differential calculus

In this section we review the bicovariant differential calculus on q-groups as developed by Woronowicz [21]. The $q \to 1$ limit will constantly appear in our discussion, so as to make clear which classical structure is being q-generalized.

The calculus can be developed on a generic Hopf algebra A with invertible antipode. Since we will constantly compare the construction with the classical one on Lie groups, we will think of A as the algebra of the preceding section, i.e. the algebra freely generated by the matrix entries $T^a{}_b$, modulo the relations (1.2.1) and possibly some reality or orthogonality conditions. However, as said, the construction can be applied to more general cases, for example it gives differential calculi on finite groups, where one cannot apply the usual techniques of differential geometry.

A *first-order differential calculus* on A is defined by

i) a linear map d: $A \to \Gamma$, satisfying the Leibniz rule

$$d(ab) = (da)b + a(db), \quad \forall a, b \in A; \tag{2.1.1}$$

Γ is an appropriate bimodule on A, which essentially means that its elements can be multiplied on the left and on the right by elements of A. [More precisely A is a left-module if $\forall a, b \in A \forall \rho, \rho' \in \Gamma$ we have: $a(\rho + \rho') = a\rho + a\rho'$, $(a + b)\rho = a\rho + b\rho$, $a(b\rho) = (ab)\rho$, $I\rho = \rho$. Similarly one defines a right-module. A left- and right-module is a bimodule if we also have $a(\rho b) = (a\rho)b$]. The space Γ q-generalizes the space of 1-forms on a Lie group.

ii) the possibility of expressing any $\rho \in \Gamma$ as

$$\rho = \sum_k a_k db_k \tag{2.1.2}$$

for some a_k, b_k belonging to A.

Bicovariance

The first-order differential calculus (Γ, d) is said to be *bicovariant* if it is both left- and right-covariant, i.e. if we can consistently define a left and right action of the q-group on Γ as follows

$$\Delta_\Gamma(adb) = \Delta(a)(id \otimes d)\Delta(b), \quad \Delta_\Gamma : \Gamma \to A \otimes \Gamma \quad \text{(left covariance)} \tag{2.1.3}$$
$$_\Gamma\Delta(adb) = \Delta(a)(d \otimes id)\Delta(b), \quad _\Gamma\Delta : \Gamma \to \Gamma \otimes A \quad \text{(right covariance)} \tag{2.1.4}$$

How can we understand these left and right actions on Γ in the $q \to 1$ limit ? The first observation is that the coproduct Δ on A is directly related, for $q = 1$, to the pullback induced by left multiplication of the group on itself

$$L_x y \equiv xy, \quad \forall x, y \in G. \tag{2.1.5}$$

This induces the left action (pullback) L_x^* on the functions on G:

$$L_x^* f(y) \equiv f(xy)|_y, \quad L_x^* : Fun(G) \to Fun(G) \tag{2.1.6}$$

where $f(xy)|_y$ means $f(xy)$ seen as a function of y. Let us introduce the mapping L^* defined by

$$(L^*f)(x,y) \equiv (L_x^*f)(y) = f(xy)|_y$$
$$L^* : Fun(G) \to Fun(G \times G) \approx Fun(G) \otimes Fun(G). \tag{2.1.7}$$

The coproduct Δ on A, when $q = 1$, reduces to the mapping L^*. Indeed, considering $T^a{}_b(y)$ as a function on G, we have:

$$L^*(T^a{}_b)(x,y) = L_x^*T^a{}_b(y) = T^a{}_b(xy) = T^a{}_c(x)T^c{}_b(y), \tag{2.1.8}$$

since $T^a{}_b$ is a representation of G. Therefore

$$L^*(T^a{}_b) = T^a{}_c \otimes T^c{}_b \tag{2.1.9}$$

and L^* is seen to coincide with Δ, cf. (1.1.15).

The pullback L_x^* can also be defined on 1-forms ρ as

$$(L_x^*\rho)(y) \equiv \rho(xy)|_y \tag{2.1.10}$$

and here too we can define L^* as

$$(L^*\rho)(x,y) \equiv (L_x^*\rho)(y) = \rho(xy)|_y. \tag{2.1.11}$$

In the $q = 1$ case we are now discussing, the left action Δ_Γ coincides with this mapping L^* for 1-forms. Indeed for $q = 1$

$$\begin{aligned}\Delta_\Gamma(adb)(x,y) &= [\Delta(a)(id \otimes d)\Delta(b)](x,y) = [(a_1 \otimes a_2)(id \otimes d)(b_1 \otimes b_2)](x,y)\\ &= [a_1b_1 \otimes a_2db_2](x,y) = a_1(x)b_1(x)a_2(y)db_2(y) = a_1(x)a_2(y)d_y[b_1(x)b_2(y)]\\ &= L^*(a)(x,y)d_y[L^*(b)(x,y)] = a(xy)db(xy)|_y.\end{aligned} \tag{2.1.12}$$

On the other hand:

$$L^*(adb)(x,y) = a(xy)db(xy)|_y, \tag{2.1.13}$$

so that $\Delta_\Gamma \to L^*$ when $q \to 1$. In the last equation we have used the well-known property $L_x^*(adb) = L_x^*(a)L_x^*(db) = L_x^*(a)dL_x^*(b)$ of the classical pullback. A similar discussion holds for ${}_\Gamma\Delta$, and we have ${}_\Gamma\Delta \to R^*$ when $q \to 1$, where R^* is defined via the pullback R_x^* on functions (0-forms) or on 1-forms induced by the right multiplication:

$$R_xy = yx, \quad \forall x,y \in G \tag{2.1.14}$$

$$(R_x^*\rho)(y) = \rho(yx)|_y \tag{2.1.15}$$

$$(R^*\rho)(y,x) \equiv (R_x^*\rho)(y). \tag{2.1.16}$$

These observations explain why Δ_Γ and ${}_\Gamma\Delta$ are called left and right actions of the quantum group on Γ when $q \neq 1$.

From the definitions (2.1.3) and (2.1.4) one deduces the following properties [21]:

$$(\varepsilon \otimes id)\Delta_\Gamma(\rho) = \rho, \quad (id \otimes \varepsilon)_\Gamma\Delta(\rho) = \rho \tag{2.1.17}$$

$$(\Delta \otimes id)\Delta_\Gamma = (id \otimes \Delta_\Gamma)\Delta_\Gamma, \quad (id \otimes \Delta)_\Gamma\Delta = ({}_\Gamma\Delta \otimes id)_\Gamma\Delta \tag{2.1.18}$$

$$(id \otimes {}_\Gamma\Delta)\Delta_\Gamma = (\Delta_\Gamma \otimes id)_\Gamma\Delta, \tag{2.1.19}$$

this last condition is the q-analogue of the fact that left and right actions commute for $q = 1$ ($L^*_x R^*_y = R^*_y L^*_x$).

Left- and right-invariant ω

An element ω of Γ is said to be *left-invariant* if

$$\Delta_\Gamma(\omega) = I \otimes \omega \tag{2.1.20}$$

and *right-invariant* if

$${}_\Gamma\Delta(\omega) = \omega \otimes I. \tag{2.1.21}$$

This terminology is easily understood: in the classical limit,

$$L^*\omega = I \otimes \omega \tag{2.1.22}$$

$$R^*\omega = \omega \otimes I \tag{2.1.23}$$

indeed define respectively left- and right-invariant 1-forms.
Proof: the classical definition of left-invariance is

$$(L^*_x\omega)(y) = \omega(y) \tag{2.1.24}$$

or, in terms of L^*,

$$(L^*\omega)(x,y) = L^*_x\omega(y) = \omega(y). \tag{2.1.25}$$

But

$$(I \otimes \omega)(x,y) = I(x)\omega(y) = \omega(y), \tag{2.1.26}$$

so that

$$L^*\omega = I \otimes \omega \tag{2.1.27}$$

for left-invariant ω. A similar argument holds for right-invariant ω.

Consequences

For any bicovariant first-order calculus one can prove the following [21] [statements **i)**, **ii)**, **iii)** and formulae (2.1.85) and (2.1.115)-(2.1.117) holds also for a calculus that is only left covariant)]:

i) Any $\rho \in \Gamma$ can be uniquely written in the form:

$$\rho = a_i \omega^i \tag{2.1.28}$$
$$\rho = \omega^i b_i \tag{2.1.29}$$

with a_i, $b_i \in A$, and ω^i a basis of ${}_{\text{inv}}\Gamma$, the linear subspace of all left-invariant elements of Γ. Thus, as in the classical case, the whole of Γ is generated by a basis of left invariant ω^i. An analogous theorem holds with a basis of right invariant elements $\eta^i \in {}_{\text{inv}}\Gamma$. Note that in the quantum case we have $a\omega^i \neq \omega^i a$, the bimodule structure of Γ being non-trivial for $q \neq 1$. [This is a consequence of associativity: $\forall \rho \in \Gamma, \forall a, a' \in A,\ \rho a = a\rho \Rightarrow (aa')\rho = \rho(aa') = (\rho a)a' = a'(\rho a) = a'a\rho \Rightarrow aa' = a'a$]

ii) There exist linear functionals $f^i{}_j$ on A such that, for any $a, b \in A$:

$$\omega^i b = (f^i{}_j * b)\omega^j \equiv (id \otimes f^i{}_j)\Delta(b)\omega^j \tag{2.1.30}$$
$$a\omega^i = \omega^j[(f^i{}_j \circ \kappa^{-1}) * a] \tag{2.1.31}$$

Once we have the functionals $f^i{}_j$, we know how to commute elements of A through elements of Γ. The $f^i{}_j$ are uniquely determined by (2.1.30) and for consistency must satisfy the conditions:

$$f^i{}_j(ab) = f^i{}_k(a) f^k{}_j(b) \tag{2.1.32}$$
$$f^i{}_j(I) = \delta^i_j \tag{2.1.33}$$
$$(f^k{}_j \circ \kappa) f^j{}_i = \delta^k_i\, \varepsilon; \quad f^k{}_j (f^j{}_i \circ \kappa) = \delta^k_i\, \varepsilon, \tag{2.1.34}$$

so that their coproduct, counit and coinverse are given by:

$$\Delta'(f^i{}_j) = f^i{}_k \otimes f^k{}_j \tag{2.1.35}$$
$$\varepsilon'(f^i{}_j) = \delta^i_j \tag{2.1.36}$$
$$\kappa'(f^k{}_j) f^j{}_i = \delta^k_i \varepsilon = f^k{}_j \kappa'(f^j{}_i) \tag{2.1.37}$$

cf. (1.3.1)-(1.3.3). Note that in the $q = 1$ limit $f^i{}_j \to \delta^i_j \varepsilon$, i.e. $f^i{}_j$ becomes proportional to the identity functional $\varepsilon(a) = a(1_G)$, and formulas (2.1.30), (2.1.31) become trivial, e.g. $\omega^i b = b\omega^i$ [use $\varepsilon * a = a$ from (1.1.5)].

iii) There exists an *adjoint representation* $M_j{}^i$ of the quantum group, defined by the right action on the (left invariant) ω^i:

$$\Delta_\Gamma(\omega^i) = \omega^j \otimes M_j{}^i, \quad M_j{}^i \in A. \tag{2.1.38}$$

It is easy to show that $\Delta_\Gamma(\omega^i)$ belongs to ${}_{\text{inv}}\Gamma \otimes A$, which proves the existence of $M_j{}^i$. In the classical case, $M_j{}^i$ is indeed the adjoint representation of the group. We recall that in this limit the left invariant 1-form ω^i can be constructed as

$$\omega^i(y) T_i = (y^{-1} dy)^i T_i, \qquad y \in G. \tag{2.1.39}$$

Under right multiplication by a (constant) element $x \in G : y \to yx$ we have, [1]

$$\omega^i(yx)T_i = [x^{-1}y^{-1}d(yx)]^i T_i = [x^{-1}(y^{-1}dy)x]^i T_i \quad (2.1.40)$$
$$= [x^{-1}T_j x]^i (y^{-1}dy)^j T_i = M_j{}^i(x)\omega^j(y)T_i, \quad (2.1.41)$$

so that

$$\omega^i(yx) = \omega^j(y)M_j{}^i(x) \quad (2.1.42)$$

or

$$R^*\omega^i(y,x) = \omega^j \otimes M_j{}^i(y,x), \quad (2.1.43)$$

which reproduces (2.1.38) for $q = 1$.

The co-structures on the $M_j{}^i$ can be deduced [21]:

$$\Delta(M_j{}^i) = M_j{}^l \otimes M_l{}^i \quad (2.1.44)$$
$$\varepsilon(M_j{}^i) = \delta^i_j \quad (2.1.45)$$
$$\kappa(M_i{}^l)M_l{}^j = \delta^j_i = M_i{}^l\kappa(M_l{}^j). \quad (2.1.46)$$

For example, in order to find the coproduct (2.1.44) it is sufficient to apply $(id \otimes \Delta)$ to both members of (2.1.38) and use the second of eqs.(2.1.18).

The elements $M_j{}^i$ can be used to build a right-invariant basis of Γ. Indeed the η^i defined by

$$\eta^i \equiv \kappa^{-1}(M_j{}^i)\omega^j \quad (2.1.47)$$

are right invariant [use $\kappa^{-1}(a_2)a_1 = \varepsilon(a)$]:

$$_\Gamma\Delta(\eta^i) = \Delta[\kappa^{-1}(M_j{}^i)]_\Gamma\Delta(\omega^j) =$$
$$[\kappa^{-1}(M_s{}^i) \otimes \kappa^{-1}(M_j{}^s)][\omega^k \otimes M_k{}^j] = \kappa^{-1}(M_s{}^i)\omega^k \otimes \delta^s_k I = \eta^i \otimes I \quad (2.1.48)$$

moreover every element of Γ can be written as $\rho = a_i\eta^i$ or $\rho = \eta^i b_i$ where a_i and b_i are uniquely determined.

It can be shown that the functionals $f^i{}_j$ previously defined satisfy:

$$\eta^i b = (b * f^i{}_j)\eta^j \quad (2.1.49)$$
$$a\eta^i = \eta^j[a * (f^i{}_j \circ \kappa)], \quad (2.1.50)$$

where $a * f \equiv (f \otimes id)\Delta(a), \ f \in A'$.

From (2.1.30), using (2.1.47) i.e. $\omega^i = M_l{}^i\eta^l$ and from (2.1.49) one immediately prove the relation

$$M_i{}^j(a * f^i{}_k) = (f^j{}_i * a)M_k{}^i, \quad (2.1.51)$$

with $a * f^i{}_j \equiv (f^i{}_k \otimes id)\Delta(a)$.

[1] Recall the $q = 1$ definition of the adjoint representation $x^{-1}T_j x \equiv M_j{}^i(x)T_i$.

Note 2.1.1 Given a first order differential calculus, the space Γ is a bicovariant bimodule i.e. Γ is a bimodule with left and right actions Δ_Γ and ${}_\Gamma\Delta$ that satisfy (2.1.17), (2.1.18), (2.1.19) and that are compatible with the bimodule structure:

$$\Delta_\Gamma(a\rho b) = \Delta(a)\Delta_\Gamma(\rho)\Delta(b) \quad ; \quad {}_\Gamma\Delta(a\rho b) = \Delta(a){}_\Gamma\Delta(\rho)\Delta(b) \; . \tag{2.1.52}$$

Points **i), ii), iii)**, and **iv)** below, hold not only for a first order differential calculus but also for a generic bicovariant bimodule (where Δ_Γ and ${}_\Gamma\Delta$ are not defined via d). Any bicovariant bimodule is uniquely characterized by the functionals $f^i{}_j$ and the elements $M_i{}^j$ satisfying (2.1.32), (2.1.33), (2.1.44), (2.1.45) and the fundamental condition (2.1.51). Indeed, cf. Theorem 2.4.3, Δ_Γ is well defined via (2.1.20) while ${}_\Gamma\Delta$, defined by (2.1.38), is compatible with the right product in Γ:

$${}_\Gamma\Delta(\omega^i a) = {}_\Gamma\Delta(\omega^i)\Delta(a) \tag{2.1.53}$$

if and only if (2.1.51) holds. *Proof.* The projection $P \; : \; \Gamma \rightarrow {}_{\mathrm{inv}}\Gamma$ defined by $\forall \rho \in \Gamma \; , P(\rho) = m(\kappa \otimes id)\Delta_\Gamma(\rho)$ [where m is the multiplication in the bimodule Γ] maps $a\omega^i$ in $P(a\omega^i) = \varepsilon(a)\omega^i$ and $\omega^i a$ in $P(\omega^i a) = f^i{}_j(a)\omega^j$ and is an epimorphism between the two bimodules Γ and ${}_{\mathrm{inv}}\Gamma$. We then have:

$$\begin{aligned}(P \otimes id){}_\Gamma\Delta(\omega^i a) &= (P \otimes id){}_\Gamma\Delta[(f^i{}_j * a)\omega^j] = \omega^k \otimes (f^i{}_j * a)M_k{}^j \; ; \\ (P \otimes id){}_\Gamma\Delta(\omega^i a) &= (P \otimes id)(\omega^k a_1 \otimes M_k{}^j a_2) = (P \otimes id)[(f^i{}_k * a_1)\omega^k \otimes M_k{}^j a_2] \\ &= \omega^k \otimes M_k{}^j(a * f^i{}_k) \; .\end{aligned}$$

This proves the implication (2.1.53) $\Rightarrow$ (2.1.51); the viceversa is also true since $(a_1 \otimes id)(P \otimes id){}_\Gamma\Delta(\omega^i a_2) = {}_\Gamma\Delta(\omega^i a)$. □□□

iv) An *exterior product*, compatible with the left and right actions of the q-group, can be defined by a bimodule automorphism Λ in $\Gamma \otimes \Gamma$ that generalizes the ordinary permutation operator:

$$\Lambda(\omega^i \otimes \eta^j) = \eta^j \otimes \omega^i, \tag{2.1.54}$$

where ω^i and η^j are respectively left and right invariant elements of Γ. Bimodule automorphism means that

$$\Lambda(a\tau) = a\Lambda(\tau) \tag{2.1.55}$$

$$\Lambda(\tau b) = \Lambda(\tau)b \tag{2.1.56}$$

for any $\tau \in \Gamma \otimes \Gamma$ and $a, b \in A$. The tensor product between elements $\rho, \rho' \in \Gamma$ is defined to have the properties $\rho a \otimes \rho' = \rho \otimes a\rho'$, $a(\rho \otimes \rho') = (a\rho) \otimes \rho'$ and $(\rho \otimes \rho')a = \rho \otimes (\rho' a)$. Left and right actions on $\Gamma \otimes \Gamma$ are defined by:

$$\Delta_\Gamma(\rho \otimes \rho') \equiv \rho_1\rho'_1 \otimes \rho_2 \otimes \rho'_2, \quad \Delta_\Gamma : \Gamma \otimes \Gamma \rightarrow A \otimes \Gamma \otimes \Gamma \tag{2.1.57}$$

$${}_\Gamma\Delta(\rho \otimes \rho') \equiv \rho_1 \otimes \rho'_1 \otimes \rho_2\rho'_2, \quad {}_\Gamma\Delta : \Gamma \otimes \Gamma \rightarrow \Gamma \otimes \Gamma \otimes A \tag{2.1.58}$$

where as usual ρ_1, ρ_2, etc., are defined by

$$\Delta_\Gamma(\rho) = \rho_1 \otimes \rho_2, \quad \rho_1 \in A, \ \rho_2 \in \Gamma \tag{2.1.59}$$

$$_\Gamma\Delta(\rho) = \rho_1 \otimes \rho_2, \quad \rho_1 \in \Gamma, \ \rho_2 \in A. \tag{2.1.60}$$

More generally, we can define the action of Δ_Γ on $\Gamma^{\otimes n} \equiv \underbrace{\Gamma \otimes \Gamma \otimes \cdots \otimes \Gamma}_{n\text{-times}}$ as

$$\Delta_\Gamma(\rho \otimes \rho' \otimes \cdots \otimes \rho'') \equiv \rho_1 \rho'_1 \cdots \rho''_1 \otimes \rho_2 \otimes \rho'_2 \otimes \cdots \otimes \rho''_2$$

$$\Delta_\Gamma : \Gamma^{\otimes n} \to A \otimes \Gamma^{\otimes n} \tag{2.1.61}$$

$$_\Gamma\Delta(\rho \otimes \rho' \otimes \cdots \otimes \rho'') \equiv \rho_1 \otimes \rho'_1 \cdots \otimes \rho''_1 \otimes \rho_2 \rho'_2 \cdots \rho''_2$$

$$_\Gamma\Delta : \Gamma^{\otimes n} \to \Gamma^{\otimes n} \otimes A \ . \tag{2.1.62}$$

Left invariance on $\Gamma \otimes \Gamma$ is naturally defined as $\Delta_\Gamma(\rho \otimes \rho') = I \otimes \rho \otimes \rho'$ (similar definition for right-invariance), so that, for example, $\omega^i \otimes \omega^j$ is left invariant, and is in fact a left invariant basis for $\Gamma \otimes \Gamma$.

— In general $\Lambda^2 \neq 1$, since $\Lambda(\eta^j \otimes \omega^i)$ is not necessarily equal to $\omega^i \otimes \eta^j$. By linearity, Λ can be extended to the whole of $\Gamma \otimes \Gamma$.

— Λ is invertible and commutes with the left and right action of the q-group, i.e. $\Delta_\Gamma \Lambda(\rho \otimes \rho') = (id \otimes \Lambda)\Delta_\Gamma(\rho \otimes \rho') = \rho_1 \rho'_1 \otimes \Lambda(\rho_2 \otimes \rho'_2)$, and similar for $_\Gamma\Delta$. Then we see that $\Lambda(\omega^i \otimes \omega^j)$ is left invariant, and therefore can be expanded on the left invariant basis $\omega^k \otimes \omega^l$:

$$\Lambda(\omega^i \otimes \omega^j) = \Lambda^{ij}{}_{kl} \omega^k \otimes \omega^l. \tag{2.1.63}$$

— From the definition (2.1.54) one can prove that [21]:

$$\Lambda^{ij}{}_{kl} = f^i{}_l(M_k{}^j); \tag{2.1.64}$$

thus the functionals $f^i{}_l$ and the elements $M_k{}^j \in A$ characterizing the bimodule Γ are dual in the sense of eq. (2.1.64) and determine the exterior product:

$$\rho \wedge \rho' \equiv W(\rho \otimes \rho') \equiv \rho \otimes \rho' - \Lambda(\rho \otimes \rho') \tag{2.1.65}$$

$$\omega^i \wedge \omega^j \equiv W^{ij}_{kl} \omega^k \otimes \omega^l \equiv \omega^i \otimes \omega^j - \Lambda^{ij}{}_{kl} \omega^k \otimes \omega^l. \tag{2.1.66}$$

Notice that, given the tensor $\Lambda^{ij}{}_{kl}$, we can compute the exterior product of any $\rho, \rho' \in \Gamma$, since any $\rho \in \Gamma$ is expressible in terms of ω^i [cf. (2.1.28), (2.1.29)]. The classical limit of $\Lambda^{ij}{}_{kl}$ is

$$\Lambda^{ij}{}_{kl} \xrightarrow{q \to 1} \delta^i_l \delta^j_k \tag{2.1.67}$$

since $f^i{}_j \xrightarrow{q \to 1} \delta^i_l \varepsilon$ and $\varepsilon(M_j{}^k) = \delta^j_k$. Thus in the $q = 1$ limit the product defined in (2.1.66) coincides with the usual exterior product.

From the property (2.1.55) and (2.1.56) applied to the case $\tau = \omega^i \otimes \omega^j$, one can derive the relation

$$\Lambda^{nm}{}_{ij} f^i{}_p f^j{}_q = f^n{}_i f^m{}_j \Lambda^{ij}{}_{pq}. \tag{2.1.68}$$

Applying both members of this equation to the element $M_r{}^s$ yields the braid relation for Λ:

$$\Lambda^{nm}{}_{ij}\Lambda^{ik}{}_{rp}\Lambda^{js}{}_{kq} = \Lambda^{nk}{}_{ri}\Lambda^{ms}{}_{kj}\Lambda^{ij}{}_{pq}, \quad i.e. \quad \Lambda_{12}\Lambda_{23}\Lambda_{12} = \Lambda_{23}\Lambda_{12}\Lambda_{23} \tag{2.1.69}$$

which is sufficient for the consistency of (2.1.68). Taking $a = M_p{}^q$ in (2.1.51), and using (2.1.64), we find the relation dual to (2.1.69):

$$M_i{}^j M_r{}^q \Lambda^{ir}{}_{pk} = \Lambda^{jq}{}_{ri} M_p{}^r M_k{}^i \ . \tag{2.1.70}$$

This last formula explicitly shows that W commutes with the coaction Δ_Γ [the commutation of W with ${}_\Gamma\Delta$ is implicit in (2.1.63)]; the new tensor $\omega^i \wedge \omega^j \equiv W^{ij}{}_{kl}\omega^k \otimes \omega^l$ transforms covariantly according to its index structure:

$${}_\Gamma\Delta(\omega^i \wedge \omega^j) \equiv {}_\Gamma\Delta(\omega^k \otimes \omega^l) W_{kl}{}^{ij} = \omega^k \wedge \omega^l \otimes M_k{}^i M_l{}^j \ . \tag{2.1.71}$$

Using also (2.1.68) we conclude that the action of Δ_Γ and ${}_\Gamma\Delta$ on the tensor $\rho \wedge \rho'$ has the same expression as in (2.1.57) and (2.1.58) with the tensor product replaced by the wedge product.

Defining

$$R^{ji}{}_{kl} \equiv \Lambda^{ij}{}_{kl}, \tag{2.1.72}$$

we see that $R^{ij}{}_{kl}$ satisfies the quantum Yang–Baxter equation (1.2.3), sufficient for the consistency of (2.1.70). Notice that the quantum Yang–Baxter equation is typically associated to a quasitriangular Hopf algebra with universal $\mathcal{R}$-matrix. In this case, as shown in [64] and [46], $\Lambda^{ij}{}_{pq}$ is a representation of the universal $\mathcal{R}$-matrix of the quantum double associated to the *generic* Hopf algebra A. In other words, (2.1.70) does not rely on the existence of "RTT" equations (1.2.1) used to define specific examples of Hopf algebras.

Generalizing equation (2.1.66), wedge products of n forms are again expressed in terms of tensorfields:

$$\omega_1 \wedge \ldots \wedge \omega_n = W_{1\ldots n}\,\omega_1 \otimes \ldots \otimes \omega_n. \tag{2.1.73}$$

The numerical coefficients $W_{1\ldots n}$ are given through a recursion relation

$$W_{1\ldots n} = \mathcal{I}_{1\ldots n} W_{1\ldots n-1}, \tag{2.1.74}$$

where

$$\mathcal{I}_{1\ldots n} = 1 - \Lambda_{n-1,n} + \Lambda_{n-2,n-1}\Lambda_{n-1,n} \ldots - (-1)^n \Lambda_{12}\Lambda_{23}\cdots\Lambda_{n-1,n} \tag{2.1.75}$$

and $W^i{}_j = \mathcal{I}^i{}_j = \delta^i_j$. The space of n-forms $\Gamma^{\wedge n}$ is therefore defined as in the classical case but with the quantum permutation operator Λ.
As is easily seen writing

$$\mathcal{I}_{s\ldots n} = 1 - \Lambda_{n-1,n} + \Lambda_{n-2,n-1}\Lambda_{n-1,n} \ldots - (-1)^{n-s+1}\Lambda_{s,s+1}\Lambda_{s+1,s+2}\ldots\Lambda_{n-1,n} \ ,$$

$\mathcal{I}$ has the following decomposition property that we will use later on:

$$\mathcal{I}_{1\ldots n} = \mathcal{I}_{s\ldots n} + (-1)^{n-s+1}\mathcal{I}_{1\ldots s-1}\Lambda_{s-1,s}\Lambda_{s,s+1}\cdots\Lambda_{n-1,n} \ . \tag{2.1.76}$$

Due to (2.1.70) and (2.1.68), the action of Δ_Γ and ${}_\Gamma\Delta$ on the tensor $\vartheta \in \Gamma^{\wedge n} \subset \Gamma^{\otimes n}$ has the same expression as in (2.1.61) and (2.1.62) with the tensor product replaced by the wedge product. Following Note 2.1.1, $\Gamma^{\wedge n}$ is a bicovariant bimodule with left and right coactions Δ_Γ and ${}_\Gamma\Delta$. We call the algebra $\Gamma^\wedge$ of exterior forms, $\Gamma^\wedge \equiv \Gamma^0 \oplus \Gamma \oplus \Gamma^{\wedge 2} \oplus \Gamma^{\wedge 3} \oplus \ldots$ (with $A \equiv \Gamma^0$), a bicovariant graded algebra because it is a graded algebra with Δ_Γ and ${}_\Gamma\Delta$ that are grade preserving.

v) Having the exterior product we can define the *exterior differential*

$$d \ : \ \Gamma \to \Gamma \wedge \Gamma \tag{2.1.77}$$

$$d(a_k db_k) = da_k \wedge db_k, \tag{2.1.78}$$

which can easily be extended to $\Gamma^{\wedge n}$:

$$d : \Gamma^{\wedge n} \to \Gamma^{\wedge(n+1)} \tag{2.1.79}$$

$$d(a_{k_1 k_2 \ldots k_n} db_{k_1} \wedge db_{k_2} \wedge \ldots db_{k_n}) = da_{k_1 k_2,\ldots k_n} \wedge db_{k_1} \wedge db_{k_2} \wedge \ldots db_{k_n} \tag{2.1.80}$$

and has the following properties:

$$d(\theta \wedge \theta') = d\theta \wedge \theta' + (-1)^k \theta \wedge d\theta' \tag{2.1.81}$$

$$d(d\theta) = 0 \tag{2.1.82}$$

$$\Delta_\Gamma(d\theta) = (id \otimes d)\Delta_\Gamma(\theta) \tag{2.1.83}$$

$${}_\Gamma\Delta(d\theta) = (d \otimes id){}_\Gamma\Delta(\theta), \tag{2.1.84}$$

where $\theta \in \Gamma^{\wedge k}$, $\theta' \in \Gamma^{\wedge n}$. The last two properties express the fact that d commutes with the left and right action of the quantum group, as in the classical case.

vi) The q-tangent space T, dual to the left invariant subspace ${}_{\mathrm{inv}}\Gamma$ can be introduced as a linear subspace of A', whose basis elements $\chi_i \in T \subset A'$ are defined by

$$da = (\chi_i * a)\omega^i, \quad \forall a \in A. \tag{2.1.85}$$

In the commutative case we write

$$da = \frac{\partial}{\partial y^\mu}a(y)dy^\mu = \left(\frac{\partial}{\partial y^\mu}a\right) e^\mu{}_i(y)e^i{}_\nu(y)dy^\nu = \left(\frac{\partial}{\partial y^\mu}a\right) e^\mu{}_i(y)\omega^i(y) = t_i|_y\omega^i \tag{2.1.86}$$

where $e^i{}_\nu(y)$ is the vielbein of the group manifold and $e^\mu{}_i$ is its inverse. Now recalling (1.3.6) we write

$$da = (t_i|_{1_G} * a)\omega^i \tag{2.1.87}$$

where t_i are left invariant vectorfields. Therefore $\chi_i *$ is the q-analogue of left invariant vectorfields, while χ_i is the q-analogue of the tangent vector at the origin 1_G of G:

$$\chi_i * a \xrightarrow{q\to 1} \frac{\partial}{\partial y^\mu} a(y) e^\mu{}_i \equiv \partial_i a(y) \ , \qquad \chi_i(a) \xrightarrow{q\to 1} \frac{\partial}{\partial x^i} a(x)|_{x=1_G} \tag{2.1.88}$$

i.e. $T \xrightarrow{q\to 1} g$ where g is the Lie algebra of G (and here the Hopf algebra A is the q-deformation of $Fun(G)$).

vii) Given the basis $\{\chi_i\}$ of the q-tangent space T, we can introduce the "coordinates" $\{x^i\}$ via the following definition: consider the linear space R (Woronowicz right ideal) given by $R = \{a \in A \ / \ \varepsilon(a) = 0 \ \text{ and } \ T(a) = 0\}$, define the linear space X by the relation

$$A = X \oplus R \oplus \{I\} \tag{2.1.89}$$

i.e. X is maximal in the (ordered) set of all linear subspaces of A disjoint from $R \oplus \{I\}$. From (2.1.89) it follows that the dual vector space X' is isomorphic to T and therefore there are n elements $x^i \in X \subset \ker\varepsilon$ uniquely defined by the duality

$$\langle \chi_i \, , x^j \rangle = \delta_i^j \ . \tag{2.1.90}$$

Note that $\varepsilon(x^i) = 0$ since $X \subset \ker\varepsilon$ because $A = \ker\varepsilon \oplus \{I\}$. In the classical limit, in a neighbourhood of the identity $1_G \in G$, we have $\forall g \in G, \ g = \prod_i e^{x^i(g)\chi_i}$, see for ex. [79], the x^i are called canonical coordinates of the second kind on the group G (while those of the first kind are given by $g = e^{\sum_i y^i(g)\chi_i}$).

viii) The χ_i functionals close on the q-Lie algebra:

$$\chi_i\chi_j - \Lambda^{kl}{}_{ij}\chi_k\chi_l = \mathbf{C}_{ij}{}^k\chi_k, \tag{2.1.91}$$

with $\Lambda^{kl}{}_{ij}$ as given in (2.1.64). The product $\chi_i\chi_j$ is defined by [cf. (1.3.1)]

$$\chi_i\chi_j \equiv (\chi_i \otimes \chi_j)\Delta \tag{2.1.92}$$

and sometimes indicated by $\chi_i * \chi_j$. Note that this $*$ product (called also convolution product) is associative:

$$\chi_i * (\chi_j * \chi_k) = (\chi_i * \chi_j) * \chi_k \tag{2.1.93}$$

$$\chi_i * (\chi_j * a) = (\chi_i * \chi_j) * a, \quad a \in A. \tag{2.1.94}$$

We leave the easy proof to the reader. The q-structure constants $\mathbf{C}_{ij}{}^k$ are given by

$$\mathbf{C}_{ij}{}^k = \chi_j(M_i{}^k). \tag{2.1.95}$$

This last equation is easily seen to hold in the $q=1$ limit, since the $(\chi_j)_i{}^k \equiv \mathbf{C}_{ij}{}^k$ are indeed in this case the infinitesimal generators of the adjoint representation:

$$M_i{}^k = \delta_i^k + \mathbf{C}_{ij}{}^k x^j + O(x^2). \tag{2.1.96}$$

Using $\chi_j \stackrel{q\to 1}{\longrightarrow} \frac{\partial}{\partial x^j}|_{x=1_G}$ indeed yields (2.1.95).

By applying both sides of (2.1.91) to $M_r{}^s \in A$, we find the q-Jacobi identities:

$$\mathbf{C}_{ri}{}^n \mathbf{C}_{nj}{}^s - \Lambda^{kl}{}_{ij} \mathbf{C}_{rk}{}^n \mathbf{C}_{nl}{}^s = \mathbf{C}_{ij}{}^k \mathbf{C}_{rk}{}^s, \tag{2.1.97}$$

which give an explicit matrix realization (the adjoint representation) of the generators χ_i:

$$(\chi_i)_k{}^l = \chi_i(M_k{}^l) = \mathbf{C}_{ki}{}^l. \tag{2.1.98}$$

Note that the q-Jacobi identities (2.1.97) can also be given in terms of the q-Lie algebra generators χ_i as :

$$[[\chi_r, \chi_i], \chi_j] - \Lambda^{kl}{}_{ij}[[\chi_r, \chi_k], \chi_l] = [\chi_r, [\chi_i, \chi_j]], \tag{2.1.99}$$

where

$$[\chi_i, \chi_j] \equiv \chi_i \chi_j - \Lambda^{kl}{}_{ij} \chi_k \chi_l \tag{2.1.100}$$

is the deformed commutator of eq. (2.1.91).

ix) The left invariant ω^i satisfy the q-analogue of the *Cartan-Maurer equations*:

$$d\omega^i + \frac{1}{2} C_{jk}{}^i \omega^j \wedge \omega^k = 0, \tag{2.1.101}$$

where

$$C_{jk}{}^i \equiv 2\chi_j \chi_k(x^i) \tag{2.1.102}$$

The structure constants C satisfy the Jacobi identities obtained by taking the exterior derivative of (2.1.101):

$$(C_{jk}{}^i C_{rs}{}^j - C_{rj}{}^i C_{sk}{}^j)\omega^r \wedge \omega^s \wedge \omega^k = 0. \tag{2.1.103}$$

In the $q=1$ limit, $\omega^j \wedge \omega^k$ becomes antisymmetric in j and k, and we have

$$C_{jk}{}^i \stackrel{q\to 1}{\longrightarrow} = (\chi_j \chi_k - \chi_k \chi_j)(x^i) = \mathbf{C}_{jk}{}^l \, \chi_l(x^i) = \mathbf{C}_{jk}{}^i, \tag{2.1.104}$$

where $\mathbf{C}_{jk}{}^l$ are now the classical structure constants. Thus when $q=1$ we have $C_{jk}{}^i = \mathbf{C}_{jk}{}^i$ and (2.1.101) reproduces the classical Cartan-Maurer equations.

For $q \neq 1$, we find the following relation:

$$\mathbf{C}_{jk}{}^i = \frac{1}{2} C_{jk}{}^i - \frac{1}{2} \Lambda^{rs}{}_{jk} C_{rs}{}^i \tag{2.1.105}$$

after applying both members of eq. (2.1.91) to x^i. Note that, using (2.1.105), the Cartan-Maurer equations (2.1.101) can also be written as:

$$d\omega^i + \mathbf{C}_{jk}{}^i \omega^j \otimes \omega^k = 0. \qquad (2.1.106)$$

x) Finally, we derive two operatorial identities that become trivial in the limit $q \to 1$. From the formula

$$d(h * \theta) = h * d\theta, \quad h \in A', \ \theta \in \Gamma^{\wedge n} \qquad (2.1.107)$$

[a direct consequence of (2.1.84)] with $h = f^n{}_l$, we find

$$\chi_k f^n{}_l = \Lambda^{ij}{}_{kl} f^n{}_i \chi_j \ . \qquad (2.1.108)$$

By requiring consistency between the external derivative and the bimodule structure of Γ, i.e. requiring that

$$d(\omega^i a) = d[(f^i{}_j * a)\omega^j], \qquad (2.1.109)$$

one finds the identity

$$\mathbf{C}_{mn}{}^i f^m{}_j f^n{}_k + f^i{}_j \chi_k = \Lambda^{pq}{}_{jk} \chi_p f^i{}_q + \mathbf{C}_{jk}{}^l f^i{}_l. \qquad (2.1.110)$$

See Appendix A for the derivation of (2.1.108) and (2.1.110); see Subsection 2.3.5 for an alternative derivation.

In summary, a bicovariant calculus on a Hopf algebra A ("the algebra of functions on the quantum group") is characterized by functionals χ_i and $f^i{}_j$ on A satisfying, cf. [22],

$$\chi_i \chi_j - \Lambda^{kl}{}_{ij} \chi_k \chi_l = \mathbf{C}_{ij}{}^k \chi_k \qquad (2.1.111)$$

$$\Lambda^{nm}{}_{ij} f^i{}_p f^j{}_q = f^n{}_i f^m{}_j \Lambda^{ij}{}_{pq} \qquad (2.1.112)$$

$$\mathbf{C}_{mn}{}^i f^m{}_j f^n{}_k + f^i{}_j \chi_k = \Lambda^{pq}{}_{jk} \chi_p f^i{}_q + \mathbf{C}_{jk}{}^l f^i{}_l \qquad (2.1.113)$$

$$\chi_k f^n{}_l = \Lambda^{ij}{}_{kl} f^n{}_i \chi_j, \qquad (2.1.114)$$

where the q-structure constants are given by $\mathbf{C}_{jk}{}^i = \chi_k(M_j{}^i)$ and the braiding matrix by $\Lambda^{ij}{}_{kl} = f^i{}_l(M_k{}^j)$.

The co-structures on the quantum Lie algebra generators χ_i are:

$$\Delta'(\chi_i) = \chi_j \otimes f^j{}_i + \varepsilon \otimes \chi_i \qquad (2.1.115)$$

$$\varepsilon'(\chi_i) = 0 \qquad (2.1.116)$$

$$\kappa'(\chi_i) = -\chi_j \kappa'(f^j{}_i), \qquad (2.1.117)$$

which q-generalize the ones given in (1.1.19)-(1.1.21). These co-structures derive from the duality relations (1.3.2) and (1.3.3). For example, using the Leibniz rule for the exterior differential and (2.1.30), we have $\chi_i(ab) = \chi_j(a)f^j{}_i(b)+\varepsilon(a)\chi_i(b)$ i.e. (2.1.115); a straighforward way to obtain (2.1.117) is to apply $m(id\otimes\kappa)$ to (2.1.115). The co-structures on the functionals $f^i{}_j$ have been given in (2.1.35)-(2.1.37) and can be easily derived from (2.1.115) and (2.1.116) using the coassociativity of the coproduct [eq. (1.1.4)]. These costructures are consistent with the bicovariance conditions (2.1.111)-(2.1.114).

Relations (2.1.111), (2.1.114) and (2.1.115) are also *sufficient* to construct a bicovariant calculus on A:

Proposition 2.1.1 Consider a set $\{\chi_i, f^i{}_j\}$ of functionals on A that satisfies:

$$\chi_i\chi_j - \Lambda^{ef}_{ij}\chi_e\chi_f \in T \tag{2.1.118}$$

$$\chi_k f^n{}_l = \Lambda^{ij}{}_{kl} f^n{}_i \chi_j \tag{2.1.119}$$

$$\chi_i(ab) = \chi_j(a)f^j{}_i(b) + \varepsilon(a)\chi_i(b) \quad \forall a,b \in A \tag{2.1.120}$$

where T is the vector space spanned by the χ_i, and assume that the algebra U of polynomials in χ_i and $f^i{}_j$ separates the points of A. Then these data determine a bicovariant differential calculus on A.

Proof This proposition is easily proven in the framework of Section 2.3. Formula (2.1.114) can also be written

$$ad_{f^k{}_j}\chi_i = \Lambda^{kh}_{ij}\chi_h \tag{2.1.121}$$

where ad is the adjoint action: $\forall\psi,\phi \in U$, $ad_\psi\phi \equiv \kappa'(\psi_1)\phi\psi_2$. We similarly have [see the first three terms in (2.3.47)]:

$$ad_{\chi_j}\chi_i \equiv \kappa'(\chi_{j_1})\chi_i\chi_{j_2} = \chi_i\chi_j - \Lambda^{ef}_{ij}\chi_e\chi_f \in T \tag{2.1.122}$$

Notice that the ad action is a right representation of U on U: $ad_{\psi\zeta}\varphi = ad_\zeta(ad_\psi\varphi)$ and therefore we conclude that, $\forall\psi \in U$, $ad_\psi\chi_k$ is a linear combination of χ_i elements. This last condition and (2.1.120) are formulae (2.3.18) and (2.3.7). In Section 2.3 the differential calculus is explicitly constructed out of these two conditions. □□□

By applying (2.1.111)-(2.1.114) to the element $M_r{}^s$ we express these relations in the adjoint representation, thus obtaining a set of numerical equations necessary for the existence of a bicovariant calculus:

$$\mathbf{C}_{ri}{}^n\mathbf{C}_{nj}{}^s - \Lambda^{kl}{}_{ij}\mathbf{C}_{rk}{}^n\mathbf{C}_{nl}{}^s = \mathbf{C}_{ij}{}^k\mathbf{C}_{rk}{}^s \quad \text{(q-Jacobi identities)} \tag{2.1.123}$$

$$\Lambda^{nm}{}_{ij}\Lambda^{ik}{}_{rp}\Lambda^{js}{}_{kq} = \Lambda^{nk}{}_{ri}\Lambda^{ms}{}_{kj}\Lambda^{ij}{}_{pq} \quad \text{(Yang–Baxter)} \tag{2.1.124}$$

$$\mathbf{C}_{mn}{}^i\Lambda^{ml}{}_{rj}\Lambda^{ns}{}_{lk} + \Lambda^{il}{}_{rj}\mathbf{C}_{lk}{}^s = \Lambda^{pq}{}_{jk}\Lambda^{il}{}_{rq}\mathbf{C}_{lp}{}^s + \mathbf{C}_{jk}{}^m\Lambda^{is}{}_{rm} \tag{2.1.125}$$

$$\mathbf{C}_{rk}{}^m\Lambda^{ns}{}_{ml} = \Lambda^{ij}{}_{kl}\Lambda^{nm}{}_{ri}\mathbf{C}_{mj}{}^s \tag{2.1.126}$$

In the next section, we describe a constructive procedure due to Jurčo [23] for a bicovariant differential calculus on any q-group of the A, B, C, D series considered in [19]. The procedure is illustrated on the example of $GL_q(2)$, for which all the objects $f^i{}_j$, $M_r{}^s$, $\Lambda^{ij}{}_{kl}$, $C_{jk}{}^i$ and $\mathbf{C}_{jk}{}^i$ are explicitly computed.

2.2 Constructive procedure and the example of $GL_q(2)$

The q-groups discussed in Section 1.2 are characterized by the matrix $R^{ab}{}_{cd}$. In terms of this matrix, it is possible to construct a bicovariant differential calculus on these q-groups [23], see also [24], [25]. The general procedure is described in this section, and the results for the specific case of $GL_q(2)$ are collected in the table. For a detailed study of the $GL_q(3)$ case see [29].

The $L^{\pm}$ functionals

We start by introducing the linear functionals $L^{\pm a}{}_b$, defined by their value on the elements $T^a{}_b$:

$$L^{\pm a}{}_b(T^c{}_d) = (R^{\pm})^{ac}{}_{bd}, \tag{2.2.1}$$

where

$$(R^{+})^{ac}{}_{bd} \equiv c^{+} R^{ca}{}_{db} \tag{2.2.2}$$

$$(R^{-})^{ac}{}_{bd} \equiv c^{-} (R^{-1})^{ac}{}_{bd}, \tag{2.2.3}$$

where c^{+}, c^{-} are free parameters (see later). The inverse matrix R^{-1} is defined by

$$(R^{-1})^{ab}{}_{cd} R^{cd}{}_{ef} \equiv \delta^a_e \delta^b_f \equiv R^{ab}{}_{cd} (R^{-1})^{cd}{}_{ef}. \tag{2.2.4}$$

We see that the $L^{\pm a}{}_b$ functionals are dual to the $T^a{}_b$ elements (fundamental representation) in the same way the $f^i{}_j$ functionals are dual to the $M_i{}^j$ elements of the adjoint representation. To extend the definition (2.2.1) to the whole algebra A, we set:

$$L^{\pm a}{}_b(ab) = L^{\pm a}{}_g(a) L^{\pm g}{}_b(b), \quad \forall a, b \in A \tag{2.2.5}$$

so that, for example,

$$L^{\pm a}{}_b(T^c{}_d T^e{}_f) = (R^{\pm})^{ac}{}_{gd} (R^{\pm})^{ge}{}_{bf}. \tag{2.2.6}$$

In general, using the compact notation introduced in Section 1.2,

$$L_1^{\pm}(T_2 T_3 ... T_n) = R_{12}^{\pm} R_{13}^{\pm} ... R_{1n}^{\pm}. \tag{2.2.7}$$

As it is esily seen from (2.2.6), the quantum Yang-Baxter equation (2.1.69) is a necessary and sufficient condition for the compatibility of (2.2.1) and (2.2.5) with the RTT relations: $L_1^{\pm}(R_{23} T_2 T_3 - T_3 T_2 R_{23}) = 0$

Finally, the value of $L^{\pm}$ on the unit I is defined by

$$L^{\pm a}{}_b(I) = \delta^a_b. \tag{2.2.8}$$

Thus the functionals $L^{\pm a}{}_b$ have the same properties as their adjoint counterpart $f^i{}_j$, and not surprisingly the latter will be constructed in terms of the former.

From (2.2.7) we can also find the action of $L^{\pm a}{}_b$ on $a \in A$, i.e. $L^{\pm a}{}_b * a$. Indeed

$$\begin{aligned}
L^{\pm a}{}_b * (T^{c_1}{}_{d_1} T^{c_2}{}_{d_2} \cdots T^{c_n}{}_{d_n}) = [id \otimes L^{\pm a}{}_b]\Delta(T^{c_1}{}_{d_1} T^{c_2}{}_{d_2} \cdots T^{c_n}{}_{d_n}) = \\
[id \otimes L^{\pm a}{}_b]\Delta(T^{c_1}{}_{d_1}) \cdots \Delta(T^{c_n}{}_{d_n}) = \\
[id \otimes L^{\pm a}{}_b](T^{c_1}{}_{e_1} \cdots T^{c_n}{}_{e_n} \otimes T^{e_1}{}_{d_1} \cdots T^{e_n}{}_{d_n}) \\
T^{c_1}{}_{e_1} \cdots T^{c_n}{}_{e_n} L^{\pm a}{}_b (T^{e_1}{}_{d_1} \cdots T^{e_n}{}_{d_n}) = \\
T^{c_1}{}_{e_1} \cdots T^{c_n}{}_{e_n} (R^{\pm})^{ae_1}{}_{g_1 d_1} (R^{\pm})^{g_1 e_2}{}_{g_2 d_2} \cdots (R^{\pm})^{g_{n-1} e_n}{}_{b d_n}
\end{aligned} \tag{2.2.9}$$

or, more compactly,

$$L_1^{\pm} * T_2 ... T_n = T_2 ... T_n R_{12}^{\pm} R_{13}^{\pm} ... R_{1n}^{\pm}, \tag{2.2.10}$$

which can also be written as the cross-commutation relation

$$L_1^{\pm} T_2 = T_2 R_{12}^{\pm} L_1^{\pm}. \tag{2.2.11}$$

It is not difficult to find the commutations between $L^{\pm a}{}_b$ and $L^{\pm c}{}_d$:

$$R_{12} L_2^{\pm} L_1^{\pm} = L_1^{\pm} L_2^{\pm} R_{12} \tag{2.2.12}$$

$$R_{12} L_2^{+} L_1^{-} = L_1^{-} L_2^{+} R_{12}, \tag{2.2.13}$$

where as usual the product $L_2^{\pm} L_1^{\pm}$ is the convolution product: $L_2^{\pm} L_1^{\pm}(a) \equiv (L_2^{\pm} \otimes L_1^{\pm})\Delta(a)\ \forall a \in A$. Consider

$$R_{12}(L_2^+ L_1^+)(T_3) = R_{12}(L_2^+ \otimes L_1^+)\Delta(T_3) = R_{12}(L_2^+ \otimes L_1^+)(T_3 \otimes T_3) = (c^+)^2\ R_{12} R_{32} R_{31} \tag{2.2.14}$$

and

$$L_1^+ L_2^+ (T_3) R_{12} = (c^+)^2\ R_{31} R_{32} R_{12} \tag{2.2.15}$$

so that the equation (2.2.12) is proven for L^+ by virtue of the quantum Yang–Baxter equation (1.2.3), where the indices have been renamed $2 \to 1, 3 \to 2, 1 \to 3$. Similarly, one proves the remaining "RLL" relations.

Note 2.2.1 As mentioned in [19], L^+ is upper triangular, L^- is lower triangular (this is due to the upper and lower triangularity of R^+ and R^-, respectively). From (2.2.12) and (2.2.13) we have

$$L^{\pm A}{}_A L^{\pm B}{}_B = L^{\pm B}{}_B L^{\pm A}{}_A\ ;\quad L^{+A}{}_A L^{-B}{}_B = L^{-B}{}_B L^{+A}{}_A \tag{2.2.16}$$

Note 2.2.2 A determinant can be defined for the matrix ${L^{\pm}}^A{}_B$ as in Note 1.2.5, with $q \to q^{-1}$. Indeed the "RLL" relations are identical to the "RTT" with $R \to R^{-1}$ (which means $q \to q^{-1}$, $r \to r^{-1}$, cf. eq. (3.1.9)). Then , because of the upper or lower triangularity of L^+ and L^- respectively, we have

$$\det_q L^{\pm} = {L^{\pm}}^1{}_1 {L^{\pm}}^2{}_2 \cdots {L^{\pm}}^N{}_N \tag{2.2.17}$$

Note 2.2.3 From (2.2.1) we deduce:

$${L^{\pm}}^A{}_B(\det_q T) = \delta^A_B (c^{\pm})^N r^{\pm 1} \tag{2.2.18}$$

Proof: observe that ${L^{\pm}}^A{}_B(\det_q T) = {L^{\pm}}^A{}_B(T^1{}_1 T^2{}_2 \cdots T^N{}_N)$ since all the other permutations do not contribute, due to the structure of the $R^{\pm}$ matrix. Then it is easy to see that

$${L^{\pm}}^A{}_B(T^1{}_1 T^2{}_2 \cdots T^N{}_N) = \tag{2.2.19}$$

$$\delta^A_B (c^{\pm})^N (R^{\pm})^{A1}{}_{A1} (R^{\pm})^{A2}{}_{A2} \cdots (R^{\pm})^{AN}{}_{AN} = \delta^A_B (c^{\pm})^N r^{\pm 1}\ . \tag{2.2.20}$$

If we set $\det_q T = I$, then ${L^{\pm}}^A{}_B(\det_q T) = \delta^A_B (c^{\pm})^N r^{\pm 1}$ must be equal to δ^A_B, or $c^{\pm} = r^{\mp \frac{1}{N}} \alpha^{\pm}$ with $(\alpha^{\pm})^N = 1$. In this case $[\det_q L^{\pm}](T^A{}_B) = \delta^A_B$ so that $\det L^{\pm} = \varepsilon$ [Proof: $\det_q L^{\pm}(T^A{}_B) = \delta^A_B (c^{\pm})^N (R^{\pm})^{1A}{}_{1A} \cdots (R^{\pm})^{NA}{}_{NA} = \delta^A_B (c^{\pm})^N r^{\pm 1}$]. Thus for $(c^{\pm})^N r^{\pm 1} = 1$, the functionals $L^{\pm}$and ε generate the Hopf algebra $U(sl_q(N))$. In the case of $GL_q(n)$, $c^{\pm}$ are extra free parameters. In fact, they appear only in the combination $s = (c^+)^{-1} c^-$. They do not enter in the Λ matrix, nor in the structure constants or the Cartan-Maurer equations, they however enter the $\omega - T$ commutation relations (see the table), so that different values of s give different bimodules of 1-forms and different bicovariant differential calculi on $GL_q(n)$. (This accounts for the one parameter family of differential calculi found in the classification of $GL_q(n)$ calculi [32]).

The co-structures are defined by the duality (2.2.1):

$$\Delta'({L^{\pm}}^a{}_b)(T^c{}_d \otimes T^e{}_f) \equiv {L^{\pm}}^a{}_b(T^c{}_d T^e{}_f) = {L^{\pm}}^a{}_g(T^c{}_d) {L^{\pm}}^g{}_b(T^e{}_f) \tag{2.2.21}$$

$$\varepsilon'({L^{\pm}}^a{}_b) \equiv {L^{\pm}}^a{}_b(I) \tag{2.2.22}$$

$$\kappa'({L^{\pm}}^a{}_b)(T^c{}_d) \equiv {L^{\pm}}^a{}_b(\kappa(T^c{}_d)) \tag{2.2.23}$$

cf. [(1.3.3), (1.3.3)], so that

$$\Delta'({L^{\pm}}^a{}_b) = {L^{\pm}}^a{}_g \otimes {L^{\pm}}^g{}_b \tag{2.2.24}$$

$$\varepsilon'({L^{\pm}}^a{}_b) = \delta^a_b \tag{2.2.25}$$

$$\kappa'({L^{\pm}}^a{}_b) = {L^{\pm}}^a{}_b \circ \kappa \tag{2.2.26}$$

The matrix $\kappa'(L^{\pm}) = (L^{\pm})^{-1}$ is a polynomial in the $L^{\pm a}{}_b$ elements and therefore the $L^{\pm a}{}_b$ generate a Hopf algebra, the Hopf algebra $U_q(gl(n))$ paired to the quantum group $GL_q(n)$[2]. Note that

$$L^{\pm a}{}_b(\kappa(T^c{}_d)) = ((R^{\pm})^{-1})^{ac}{}_{bd}, \tag{2.2.27}$$

since

$$L^{\pm a}{}_b(\kappa(T^c{}_d)T^d{}_e) = \delta^c_d L^{\pm a}{}_b(I) = \delta^c_e \delta^a_b \tag{2.2.28}$$

and

$$\begin{aligned} L^{\pm a}{}_b(\kappa(T^c{}_d)T^d{}_e) &= L^{\pm a}{}_f(\kappa(T^c{}_d))L^{\pm f}{}_b(T^d{}_e) \\ &= L^{\pm a}{}_f(\kappa(T^c{}_d))(R^{\pm})^{fd}{}_{be}. \end{aligned} \tag{2.2.29}$$

The space of quantum 1-forms

The bimodule Γ ("space of quantum 1-forms") can be constructed as follows. First we define $\omega_a{}^b$ to be a basis of left invariant quantum 1-forms. The index pairs ${}_a{}^b$ or ${}^a{}_b$ will replace in the sequel the indices ^{i} or ${}_i$ of the previous section. The dimension of ${}_{\text{inv}}\Gamma$ is therefore N^2 at this stage. Since the $\omega_a{}^b$ are left invariant, we have:

$$\Delta_\Gamma(\omega_a{}^b) = I \otimes \omega_a{}^b, \quad a,b = 1,...,N. \tag{2.2.30}$$

The left action Δ_Γ on the whole of Γ is then defined by (2.2.30), since $\omega_a{}^b$ is a basis for Γ. The bimodule Γ is further characterized by the commutations between $\omega_a{}^b$ and $a \in A$ [cf. eq. (2.1.30)]:

$$\omega_{a_1}{}^{a_2} b = (f_{a_1}{}^{a_2 b_1}{}_{b_2} * b)\omega_{b_1}{}^{b_2}, \tag{2.2.31}$$

where

$$f_{a_1}{}^{a_2 b_1}{}_{b_2} \equiv \kappa'(L^{+b_1}{}_{a_1})L^{-a_2}{}_{b_2}. \tag{2.2.32}$$

Finally, the right action ${}_\Gamma\Delta$ on Γ is defined by

$${}_\Gamma\Delta(\omega_{a_1}{}^{a_2}) = \omega_{b_1}{}^{b_2} \otimes M^{b_1}_{b_2 a_1}{}^{a_2}, \tag{2.2.33}$$

where $M^{b_1}_{b_2 a_1}{}^{a_2}$, the adjoint representation, is given by

$$M^{b_1}_{b_2 a_1}{}^{a_2} \equiv T^{b_1}{}_{a_1}\kappa(T^{a_2}{}_{b_2}). \tag{2.2.34}$$

It is easy to check that $f_{a_1}{}^{a_2 b_1}{}_{b_2}$ fulfill the consistency conditions (2.1.32)-(2.1.34), where the *i,j,...* indices stand for pairs of *a,b,...* indices. Also, the co-structures of

[2]The pairing between these two Hopf algebras is nondegenerate, indeed $U_q(gl((n))$ as a Hopf algebra is isomorphic [77, 19] to $\mathcal{U}_q(gl(n))$ (as defined by Jimbo [18]) then the Hopf isomorphism $\mathcal{U}_h^0(sl(n)) \cong SU_q(n)$ (where $\mathcal{U}_h^0(sl(n))$ is the Hopf dual of Drinfeld universal enveloping algebra $\mathcal{U}_h(sl(n))$ [16]) allows to conclude that the pairing between $GL_q(n)$ and $U_q(gl(n))$ is nondegenerate.

$M^{b_1}_{b_2a_1}{}^{a_2}$ are as given in (2.1.44)-(2.1.46). The last compatibility condition between the bimodule Γ and the action ${}_\Gamma\Delta$, as explained in Note 2.1.1, is (2.1.51). This relation is easily checked for $a = T^A{}_B$ since in this case it is implied by the RTT relations; it holds for a generic a because of property (2.1.32).

The Λ tensor and the exterior product

The Λ tensor defined in (2.1.72) can now be computed:

$$\begin{array}{rl} & \Lambda^{a_2}_{a_1}{}^{d_2}_{d_1}|^{c_1}_{c_2}{}^{b_1}_{b_2} \equiv f_{a_1}{}^{a_2b_1}{}_{b_2}(M^{c_1}_{c_2d_1}{}^{d_2}) = \kappa'(L^{+b_1}{}_{a_1})L^{-a_2}{}_{b_2}(T^{c_1}{}_{d_1}\kappa(T^{d_2}{}_{c_2})) \\ = & [\kappa'(L^{+b_1}{}_{a_1})\otimes L^{-a_2}{}_{b_2}]\Delta(T^{c_1}{}_{d_1}\kappa(T^{d_2}{}_{c_2})) \\ = & [\kappa'(L^{+b_1}{}_{a_1})\otimes L^{-a_2}{}_{b_2}](T^{c_1}{}_{e_1}\otimes T^{e_1}{}_{d_1})(\kappa(T^{f_2}{}_{c_2})\otimes\kappa(T^{d_2}{}_{f_2})) \\ = & [\kappa'(L^{+b_1}{}_{a_1})\otimes L^{-a_2}{}_{b_2}][T^{c_1}{}_{e_1}\kappa(T^{f_2}{}_{c_2})\otimes T^{e_1}{}_{d_1}\kappa(T^{d_2}{}_{f_2})] \\ = & L^{+b_1}{}_{a_1}(\kappa^2(T^{f_2}{}_{c_2})\kappa(T^{c_1}{}_{e_1}))\ L^{-a_2}{}_{b_2}(T^{e_1}{}_{d_1}\kappa(T^{d_2}{}_{f_2})) \\ = & d^{f_2}d^{-1}_{c_2}L^{+b_1}{}_{a_1}(T^{f_2}{}_{c_2}\kappa(T^{c_1}{}_{e_1}))\ L^{-a_2}{}_{b_2}(T^{e_1}{}_{d_1}\kappa(T^{d_2}{}_{f_2}) \\ = & d^{f_2}d^{-1}_{c_2}L^{+b_1}{}_{g_1}(T^{f_2}{}_{c_2})\ L^{+g_1}{}_{a_1}(\kappa(T^{c_1}{}_{e_1}))\ L^{-a_2}{}_{g_2}(T^{e_1}{}_{d_1})\ L^{-g_2}{}_{b_2}(\kappa(T^{d_2}{}_{f_2})) \\ = & d^{f_2}d^{-1}_{c_2}R^{f_2b_1}{}_{c_2g_1}(R^{-1})^{c_1g_1}{}_{e_1a_1}(R^{-1})^{a_2e_1}{}_{g_2d_1}R^{g_2d_2}{}_{b_2f_2} \end{array} \quad (2.2.35)$$

where we made use of relations (1.1.12), (1.3.3), (1.2.15), (2.2.1) and (2.2.27). The Λ tensor allows the definition of the exterior product as in (2.1.66). For future use we give here also the inverse Λ^{-1} of the Λ tensor, defined by:

$$(\Lambda^{-1})^{a_2}_{a_1}{}^{d_2}_{d_1}|^{b_1}_{b_2}{}^{c_1}_{c_2}\Lambda^{b_2}_{b_1}{}^{c_2}_{c_1}|^{e_1}_{e_2}{}^{f_1}_{f_2} = \delta^{a_2}_{e_2}\delta^{e_1}_{a_1}\delta^{f_1}_{d_1}\delta^{d_2}_{f_2}. \quad (2.2.36)$$

It is not difficult to see that

$$\begin{array}{l} (\Lambda^{-1})^{a_2}_{a_1}{}^{d_2}_{d_1}|^{b_1}_{b_2}{}^{c_1}_{c_2} = f_{d_1}{}^{d_2b_1}{}_{b_2}(T^{a_2}{}_{c_2}\kappa^{-1}(T^{c_1}{}_{a_1})) = \\ R^{f_1b_1}{}_{a_1g_1}(R^{-1})^{a_2g_1}{}_{e_2d_1}(R^{-1})^{d_2e_2}{}_{g_2c_2}R^{g_2c_1}{}_{b_2f_1}(d^{-1})^{c_1}d_{f_1} \end{array} \quad (2.2.37)$$

does the trick. Another useful relation gives a particular trace of the Λ matrix:

$$\Lambda^{c_2}_{c_1}{}^{b}_{b}|^{a_1}_{a_2}{}^{b_1}_{b_2} = \delta^{a_1}_{a_2}\delta^{b_1}_{c_1}\delta^{c_2}_{b_2}. \quad (2.2.38)$$

This identity is simply proven. Indeed:

$$\begin{array}{l} \Lambda^{c_2}_{c_1}{}^{b}_{b}|^{a_1}_{a_2}{}^{b_1}_{b_2} \equiv f_{c_1}{}^{c_2b_1}{}_{b_2}(M^{a_1}_{a_2b}{}^{b}) = \\ \kappa'(L^{+b_1}{}_{c_1})L^{-c_2}{}_{b_2}(T^{a_1}{}_b\kappa(T^b{}_{a_2})) = \kappa'(L^{+b_1}{}_{c_1})L^{-c_2}{}_{b_2}(\delta^{a_1}_{a_2}I) = \\ \delta^{a_1}_{a_2}[\kappa'(L^{+b_1}{}_{c_1})\otimes L^{-c_2}{}_{b_2}](I\otimes I) = \delta^{a_1}_{a_2}\delta^{b_1}_{c_1}\delta^{c_2}_{b_2}. \end{array} \quad (2.2.39)$$

The relations (1.2.17), (1.2.18) for the R matrix reflect themselves in relations for the Λ matrix (2.2.35). For example, the Hecke condition (1.2.17) implies:

$$(\Lambda + q^2)(\Lambda + q^{-2})(\Lambda - I) = 0 \quad (2.2.40)$$

for the A_{n-1} q-groups, and replaces the classical relation $(\Lambda-1)(\Lambda+1)=0$, Λ being for $q=1$ the ordinary permutation operator, cf. (2.1.67).

With the help of (2.2.40) we can give explicitly the commutations of the left invariant forms ω. Indeed, reverting to the $i,j...$ indices, relation (2.2.40) implies:

$$(\Lambda^{ij}{}_{kl}+q^2\delta^i_k\delta^j_l)(\Lambda^{kl}{}_{mn}+q^{-2}\delta^k_m\delta^l_n)(\Lambda^{mn}{}_{rs}-\delta^m_r\delta^n_s)\omega^r\otimes\omega^s=$$
$$(\Lambda^{ij}{}_{kl}+q^2\delta^i_k\delta^j_l)(\Lambda^{kl}{}_{mn}+q^{-2}\delta^k_m\delta^l_n)\omega^m\wedge\omega^n=0 \tag{2.2.41}$$

and it is easy to see that the last equality can be rewritten as

$$\omega^i\wedge\omega^j=-Z^{ij}{}_{kl}\omega^k\wedge\omega^l \tag{2.2.42}$$

$$Z^{ij}{}_{kl}\equiv\frac{1}{q^2+q^{-2}}[\Lambda^{ij}{}_{kl}+(\Lambda^{-1})^{ij}{}_{kl}]. \tag{2.2.43}$$

The exterior differential

The exterior differential on $\Gamma^{\wedge k}$ is defined by means of the bi-invariant (i.e. left and right invariant) element $\tau=\sum_a\omega_a{}^a\in\Gamma$ as follows:

$$d\theta\equiv\frac{1}{\lambda}[\tau\wedge\theta-(-1)^k\theta\wedge\tau], \tag{2.2.44}$$

where $\theta\in\Gamma^{\wedge k}$, and λ is a normalization factor depending on q, necessary in order to obtain the correct classical limit. It will be later determined to be $\lambda=q-q^{-1}$. Here we can only see that it has to vanish for $q=1$, since otherwise $d\theta$ would vanish in the classical limit. For $a\in A$ we have

$$da=\frac{1}{\lambda}[\tau a-a\tau]. \tag{2.2.45}$$

This linear map satisfies the Leibniz rule (2.1.1), and properties (2.1.81)-(2.1.84), as the reader can easily check (use the definition of exterior product and the bi-invariance of τ). A proof that also the property (2.1.2) holds can be obtained by considering the exterior differential of the adjoint representation:

$$dM_j{}^i=(\chi_k*M_j{}^i)\omega^k=M_j{}^l\mathbf{C}_{kl}{}^i\omega^k \tag{2.2.46}$$

or

$$\kappa(M_l{}^j)dM_j{}^i=\mathbf{C}_{kl}{}^i\omega^k. \tag{2.2.47}$$

Multiplying by $\mathbf{C}_{ni}{}^l$, we have:

$$\mathbf{C}_{ni}{}^l\kappa(M_l{}^j)dM_j{}^i=\mathbf{C}_{kl}{}^i\mathbf{C}_{ni}{}^l\omega^k\equiv g_{nk}\omega^k, \tag{2.2.48}$$

where g_{nk} is the q-Killing metric. The explicit example of this section being $GL_q(2)$, one may wonder what happens to the invertibility of the q-Killing metric, since its classical limit is no more invertible [$GL(2)$ being nonsemisimple]. The answer is that for $q \neq 1$ the q-Killing metric of $GL_q(2)$ *is* invertible, as can be checked explicitly from the values of the structure constants given in the table. Therefore $GL_q(2)$ could be said to be "q-semisimple". With an analogous procedure (using $T^a{}_b$ instead of $M_j{}^i$) we have derived in the table the explicit expression of the ω^i in terms of the $dT^a{}_b$ for $GL_q(2)$.

The q-Lie algebra

The "quantum generators" $\chi^{a_1}_{a_2}$ are introduced as in (2.1.85):

$$da = \frac{1}{\lambda}[\tau a - a\tau] = (\chi^{a_1}_{a_2} * a)\omega_{a_1}{}^{a_2}. \tag{2.2.49}$$

Using (2.2.31) we can find an explicit expression for the $\chi^{a_1}_{a_2}$ in terms of the $L^{\pm}$ functionals. Indeed

$$\tau a = \omega_b{}^b a = (f_b{}^{bc_1}{}_{c_2} * a)\omega_{c_1}{}^{c_2} = ([\kappa'(L^{+c_1}{}_b)L^{-b}{}_{c_2}] * a)\omega_{c_1}{}^{c_2}. \tag{2.2.50}$$

Therefore

$$da = \frac{1}{\lambda}[(\kappa'(L^{+c_1}{}_b)L^{-b}{}_{c_2} - \delta^{c_1}_{c_2}\varepsilon) * a]\omega_{c_1}{}^{c_2} \tag{2.2.51}$$

(recall $\varepsilon * a = a$), so that the q-generators take the explicit form

$$\chi^{c_1}_{c_2} = \frac{1}{\lambda}[\kappa'(L^{+c_1}{}_b)L^{-b}{}_{c_2} - \delta^{c_1}_{c_2}\varepsilon] = \frac{1}{\lambda}(f_b{}^{bc_1}{}_{c_2} - \delta^{c_1}_{c_2}\varepsilon)\ . \tag{2.2.52}$$

The commutations between the χ's can now be obtained by taking the exterior derivative of eq. (2.2.51). We find

$$\begin{aligned} d^2(a) = 0 = d[(\chi^{c_1}_{c_2} * a)\omega_{c_1}{}^{c_2}] &= (\chi^{d_1}_{d_2} * \chi^{c_1}_{c_2} * a)\omega_{d_1}{}^{d_2} \wedge \omega_{c_1}{}^{c_2} + (\chi^{c_1}_{c_2} * a)d\omega_{c_1}{}^{c_2} \\ &= (\chi^{d_1}_{d_2} * \chi^{c_1}_{c_2} * a)(\omega_{d_1}{}^{d_2} \otimes \omega_{c_1}{}^{c_2} - \Lambda_{d_1}{}^{d_2}{}_{c_1}{}^{c_2}|^{e_1}{}_{e_2}{}^{f_1}{}_{f_2}\omega_{e_1}{}^{e_2} \otimes \omega_{f_1}{}^{f_2}) \\ &+ \frac{1}{\lambda}(\chi^{c_1}_{c_2} * a)(\omega_b{}^b \wedge \omega_{c_1}{}^{c_2} + \omega_{c_1}{}^{c_2} \wedge \omega_b{}^b). \end{aligned} \tag{2.2.53}$$

Now we use the fact that $\tau = \omega_b{}^b$ is bi-invariant, and therefore also right-invariant, so that we can write

$$\begin{aligned} &\omega_b{}^b \wedge \omega_{c_1}{}^{c_2} + \omega_{c_1}{}^{c_2} \wedge \omega_b{}^b \equiv \\ &\quad \omega_b{}^b \otimes \omega_{c_1}{}^{c_2} - \Lambda(\omega_b{}^b \otimes \omega_{c_1}{}^{c_2}) + \omega_{c_1}{}^{c_2} \otimes \omega_b{}^b - \Lambda(\omega_{c_1}{}^{c_2} \otimes \omega_b{}^b) = \\ &\quad \omega_{c_1}{}^{c_2} \otimes \omega_b{}^b - \Lambda(\omega_b{}^b \otimes \omega_{c_1}{}^{c_2}) = \\ &\quad \omega_{c_1}{}^{c_2} \otimes \omega_b{}^b - \Lambda_b{}^b{}_{c_1}{}^{c_2}|^{e_1}{}_{e_2}{}^{f_1}{}_{f_2}\omega_{e_1}{}^{e_2} \otimes \omega_{f_1}{}^{f_2}, \end{aligned} \tag{2.2.54}$$

where we have used $\Lambda(\omega_{c_1}{}^{c_2} \otimes \tau) = \tau \otimes \omega_{c_1}{}^{c_2}$, cf. (2.1.54). After substituting (2.2.54) in (2.2.53), and factorizing $\omega_{d_1}{}^{d_2} \otimes \omega_{c_1}{}^{c_2}$, we arrive at the q-Lie algebra relations:

$$\chi^{d_1}_{d_2}\chi^{c_1}_{c_2} - \Lambda_{e_1}{}^{e_2}{}_{f_1}{}^{f_2}|^{d_1}{}_{d_2}{}^{c_1}{}_{c_2}\ \chi^{e_1}_{e_2}\chi^{f_1}_{f_2} = \frac{1}{\lambda}[-\delta^{c_1}_{c_2}\chi^{d_1}_{d_2} + \Lambda_b{}^b{}_{e_1}{}^{e_2}|^{d_1}{}_{d_2}{}^{c_1}{}_{c_2}\ \chi^{e_1}_{e_2}]. \tag{2.2.55}$$

The structure constants are then explicitly given by:

$$\mathbf{C}^{a_1}{}_{a_2}{}^{b_1}{}_{b_2}|_{c_1}^{c_2} = \frac{1}{\lambda}[-\delta^{b_1}_{b_2}\delta^{a_1}_{c_1}\delta^{c_2}_{a_2} + \Lambda_b{}^b{}_{c_1}{}^{c_2}|^{a_1}{}_{a_2}{}^{b_1}{}_{b_2}]. \tag{2.2.56}$$

Here we determine λ. Indeed we first observe that

$$\Lambda^{a_2}{}_{a_1}{}^{d_2}{}_{d_1}|^{c_1}{}_{c_2}{}^{b_1}{}_{b_2} = \delta^{b_1}_{a_1}\delta^{a_2}_{b_2}\delta^{c_1}_{d_1}\delta^{d_2}_{c_2} + (q-q^{-1})U^{a_2}{}_{a_1}{}^{d_2}{}_{d_1}|^{c_1}{}_{c_2}{}^{b_1}{}_{b_2}, \tag{2.2.57}$$

where the matrix U is finite and different from zero in the limit $q=1$. This can be proven by considering the explicit form of the R and R^{-1} matrices. In the case of the A_{n-1} q-groups, for example, these matrices have the form [19]:

$$R^{ab}{}_{cd} = \delta^a_c\delta^b_d + (q-q^{-1})\left[\frac{q-1}{q-q^{-1}}\delta^a_c\delta^b_d\delta^{ab} + \delta^b_c\delta^a_d\theta(a-b)\right] \tag{2.2.58}$$

$$(R^{-1})^{ab}{}_{cd} = \delta^a_c\delta^b_d - (q-q^{-1})\left[\frac{1-q^{-1}}{q-q^{-1}}\delta^a_c\delta^b_d\delta^{ab} + \delta^b_c\delta^a_d\theta(a-b)\right], \tag{2.2.59}$$

where $\theta(x)=1$ for $x>0$ and vanishes for $x\leq 0$. Substituting these expressions in the formula for Λ (2.2.35) we find (2.2.57). Using (2.2.57) in the expression (2.2.56) for the q-structure constants $\mathbf{C}$, we find that the terms proportional to $\frac{1}{\lambda}$ do cancel, and we are left with:

$$\mathbf{C}^{a_1}{}_{a_2}{}^{b_1}{}_{b_2}|_{c_1}^{c_2} = -\frac{1}{\lambda}(q-q^{-1})U_b{}^b{}_{c_1}{}^{c_2}|^{a_1}{}_{a_2}{}^{b_1}{}_{b_2}. \tag{2.2.60}$$

A simple choice for λ is therefore $\lambda = q-q^{-1}$, ensuring that $\mathbf{C}$ remains finite in the limit $q\to 1$; moreover with this normalization the differential d reduces for $q\to 1$ to the classical differential, cf. Section 4.6.

The Cartan-Maurer equations

The Cartan-Maurer equations are found as follows:

$$d\omega_{c_1}{}^{c_2} = \frac{1}{\lambda}(\omega_b{}^b \wedge \omega_{c_1}{}^{c_2} + \omega_{c_1}{}^{c_2}\wedge\omega_b{}^b) \equiv -\frac{1}{2}C^{a_1}{}_{a_2}{}^{b_1}{}_{b_2}|_{c_1}^{c_2}\ \omega_{a_1}{}^{a_2}\wedge\omega_{b_1}{}^{b_2}. \tag{2.2.61}$$

In order to obtain an explicit and, for $q\to 1$, well defined expression for the C structure constants in (2.2.61), we must use the relation (2.2.42) for the commutations of $\omega_{a_1}{}^{a_2}$ with $\omega_{b_1}{}^{b_2}$. Then the term $\omega_{c_1}{}^{c_2}\wedge\omega_b{}^b$ in (2.2.61) can be written as $-Z\omega\omega$ via formula (2.2.42), and we find the C-structure constants to be:

$$\begin{aligned}
C^{a_1}{}_{a_2}{}^{b_1}{}_{b_2}|_{c_1}^{c_2} &= -\frac{2}{\lambda}(\delta^{a_1}_{a_2}\delta^{b_1}_{c_1}\delta^{c_2}_{b_2} - \frac{1}{q^2+q^{-2}}[\Lambda_{c_1}{}^{c_2}{}_b{}^b|^{a_1}{}_{a_2}{}^{b_1}{}_{b_2} + (\Lambda^{-1})_{c_1}{}^{c_2}{}_b{}^b|^{a_1}{}_{a_2}{}^{b_1}{}_{b_2}])\\
&= -\frac{2}{\lambda}(\delta^{a_1}_{a_2}\delta^{b_1}_{c_1}\delta^{c_2}_{b_2} - \frac{1}{q^2+q^{-2}}[\delta^{a_1}_{a_2}\delta^{b_1}_{c_1}\delta^{c_2}_{b_2} + (\Lambda^{-1})_{c_1}{}^{c_2}{}_b{}^b|^{a_1}{}_{a_2}{}^{b_1}{}_{b_2}]),
\end{aligned} \tag{2.2.62}$$

where we have also used eq. (2.2.38). By considering the analogue of (2.2.57) for Λ^{-1}, it is not difficult to see that the terms proportional to $\frac{1}{\lambda}$ cancel, and the $q \to 1$ limit of (2.2.62) is well defined. For a similar result on the B_n, C_n and D_n q-groups see (4.5.16) and ref. [30].

In the table we summarize the results of this section for the case of $GL_q(2)$. The composite indices ${}_a{}^b$ are translated into the corresponding indices ^{i}, $i = 1, +, -, 2$, according to the convention:

$$ {}_1{}^1 \to {}^1, \quad {}_1{}^2 \to {}^+, \quad {}_2{}^1 \to {}^-, \quad {}_2{}^2 \to {}^2. \tag{2.2.63} $$

A similar convention holds for ${}^a{}_b \to {}_i$.

2.2.1 Table of $GL_q(2)$

The bicovariant $GL_q(2)$ algebra

R and D-matrices:

$$R^{ab}{}_{cd} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

$$(R^-)^{ab}{}_{cd} \equiv c^-(R^{-1})^{ab}{}_{cd} = c^- \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -(q-q^{-1}) & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}$$

$$(R^+)^{ab}{}_{cd} \equiv c^+ R^{ba}{}_{dc} = c^+ \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q-q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad D^a{}_b = \begin{pmatrix} q & 0 \\ 0 & q^3 \end{pmatrix}$$

Non-vanishing components of the Λ matrix:

$\Lambda^{11}{}_{11} = 1$	$\Lambda^{1+}{}_{+1} = q^{-2}$	$\Lambda^{1-}{}_{-1} = q^2$	$\Lambda^{12}{}_{21} = 1$
$\Lambda^{+1}{}_{1+} = 1$	$\Lambda^{+1}{}_{+1} = 1 - q^{-2}$	$\Lambda^{++}{}_{++} = 1$	$\Lambda^{+-}{}_{11} = 1 - q^2$
$\Lambda^{+-}{}_{-+} = 1$	$\Lambda^{+-}{}_{21} = 1 - q^{-2}$	$\Lambda^{+2}{}_{+1} = -1 + q^{-2}$	$\Lambda^{+2}{}_{2+} = 1$
$\Lambda^{-1}{}_{1-} = 1$	$\Lambda^{-1}{}_{-1} = 1 - q^2$	$\Lambda^{-+}{}_{11} = -1 + q^2$	$\Lambda^{-+}{}_{+-} = 1$
$\Lambda^{-+}{}_{21} = -1 + q^{-2}$	$\Lambda^{--}{}_{--} = 1$	$\Lambda^{-2}{}_{-1} = -1 + q^2$	$\Lambda^{-2}{}_{2-} = 1$
$\Lambda^{21}{}_{11} = (q^2-1)^2$	$\Lambda^{21}{}_{12} = 1$	$\Lambda^{21}{}_{+-} = q^2 - 1$	$\Lambda^{21}{}_{-+} = 1 - q^2$
$\Lambda^{21}{}_{21} = 2 - q^2 - q^{-2}$	$\Lambda^{2+}{}_{1+} = -q^2 + q^4$	$\Lambda^{2+}{}_{+2} = q^2$	$\Lambda^{2+}{}_{2+} = 1 - q^2$
$\Lambda^{2-}{}_{1-} = 1 - q^2$	$\Lambda^{2-}{}_{-1} = q^{-2} - 1 - q^2 + q^4$	$\Lambda^{2-}{}_{-2} = q^{-2}$	$\Lambda^{2-}{}_{2-} = 1 - q^{-2}$
$\Lambda^{22}{}_{11} = -(q^2-1)^2$	$\Lambda^{22}{}_{+-} = 1 - q^2$	$\Lambda^{22}{}_{-+} = q^2 - 1$	$\Lambda^{22}{}_{21} = (q^{-1} - q)^2$
$\Lambda^{22}{}_{22} = 1$			

Non-vanishing components of the **C** *structure constants:*

$\mathbf{C}_{11}{}^{1} = q(q^2-1)$	$\mathbf{C}_{11}{}^{2} = -q(q^2-1)$	$\mathbf{C}_{1+}{}^{+} = q^3$	$\mathbf{C}_{1-}{}^{-} = -q$
$\mathbf{C}_{21}{}^{1} = q^{-1} - q$	$\mathbf{C}_{21}{}^{2} = q - q^{-1}$	$\mathbf{C}_{2+}{}^{+} = -q$	$\mathbf{C}_{2-}{}^{-} = q^{-1}$
$\mathbf{C}_{+1}{}^{+} = -q^{-1}$	$\mathbf{C}_{+2}{}^{+} = q$	$\mathbf{C}_{+-}{}^{1} = q$	$\mathbf{C}_{+-}{}^{2} = -q$
$\mathbf{C}_{-1}{}^{-} = q(q^2+1) - q^{-1}$	$\mathbf{C}_{-2}{}^{-} = -q^{-1}$	$\mathbf{C}_{-+}{}^{1} = -q$	$\mathbf{C}_{-+}{}^{2} = q$

Non-vanishing components of the C structure constants:

$$
\begin{array}{llll}
C_{11}{}^{1} = \frac{q(q^2-1)^2}{1+q^4} & C_{11}{}^{2} = \frac{q^3(1-q^2)}{1+q^4} & C_{1+}{}^{+} = \frac{q^5}{1+q^4} & C_{1-}{}^{-} = \frac{-q^3}{1+q^4} \\
C_{12}{}^{1} = \frac{q(1-q^2)}{1+q^4} & C_{+1}{}^{+} = \frac{-q^3}{1+q^4} & C_{+-}{}^{1} = \frac{q^3}{1+q^4} & C_{+-}{}^{2} = \frac{-q^3}{1+q^4} \\
C_{+2}{}^{+} = \frac{q}{1+q^4} & C_{-1}{}^{-} = \frac{q^5}{1+q^4} & C_{-+}{}^{1} = \frac{-q^3}{1+q^4} & C_{-+}{}^{2} = \frac{q^3}{1+q^4} \\
C_{-2}{}^{-} = \frac{-q^3}{1+q^4} & C_{21}{}^{1} = \frac{q(1-q^2)}{1+q^4} & C_{2+}{}^{+} = \frac{-q^3}{1+q^4} & C_{2-}{}^{-} = \frac{q}{1+q^4} \\
C_{22}{}^{2} = \frac{q(1-q^2)}{1+q^4} & & &
\end{array}
$$

Cartan-Maurer equations:

$$
\begin{aligned}
d\omega^1 + q\omega^+ \wedge \omega^- &= 0 \\
d\omega^+ + q\omega^+(-q^2\omega^1 + \omega^2) &= 0 \\
d\omega^- + q(-q^2\omega^1 + \omega^2)\omega^- &= 0 \\
d\omega^2 - q\omega^+ \wedge \omega^- &= 0
\end{aligned}
$$

The q-Lie algebra:

$$
\begin{gathered}
\chi_1\chi_+ - \chi_+\chi_1 - (q^4 - q^2)\chi_2\chi_+ = q^3\chi_+ \\
\chi_1\chi_- - \chi_-\chi_1 + (q^2 - 1)\chi_2\chi_- = -q\chi_- \\
\chi_1\chi_2 - \chi_2\chi_1 = 0 \\
\chi_+\chi_- - \chi_-\chi_+ + (1 - q^2)\chi_2\chi_1 - (1 - q^2)\chi_2\chi_2 = q(\chi_1 - \chi_2) \\
\chi_+\chi_2 - q^2\chi_2\chi_+ = q\chi_+ \\
\chi_-\chi_2 - q^{-2}\chi_2\chi_- = -q^{-1}\chi_-
\end{gathered}
$$

Commutation relations between left invariant ω^i and ω^j:

$$
\begin{gathered}
\omega^1 \wedge \omega^+ + \omega^+ \wedge \omega^1 = 0 \\
\omega^1 \wedge \omega^- + \omega^- \wedge \omega^1 = 0 \\
\omega^1 \wedge \omega^2 + \omega^2 \wedge \omega^1 = (1 - q^2)\omega^+ \wedge \omega^- \\
\omega^+ \wedge \omega^- + \omega^- \wedge \omega^+ = 0 \\
\omega^2 \wedge \omega^+ + q^2\omega^+ \wedge \omega^2 = q^2(q^2 - 1)\omega^+ \wedge \omega^1 \\
\omega^2 \wedge \omega^- + q^{-2}\omega^- \wedge \omega^2 = (1 - q^2)\omega^- \wedge \omega^1 \\
\omega^2 \wedge \omega^2 = (q^2 - 1)\omega^+ \wedge \omega^- \\
\omega^1 \wedge \omega^1 = \omega^+ \wedge \omega^+ = \omega^- \wedge \omega^- = 0
\end{gathered}
$$

Commutation relations between ω^i and the basic elements of A ($s = (c^+)^{-1}c^-$):

$$\begin{array}{ll} \omega^1\alpha = sq^{-2}\alpha\omega^1 & \omega^+\alpha = sq^{-1}\alpha\omega^+ \\ \omega^1\beta = s\beta\omega^1 & \omega^+\beta = sq^{-1}\beta\omega^+ + s(q^{-2}-1)\alpha\omega^1 \\ \omega^1\gamma = sq^{-2}\gamma\omega^1 & \omega^+\gamma = sq^{-1}\gamma\omega^+ \\ \omega^1\delta = s\delta\omega^1 & \omega^+\delta = sq^{-1}\delta\omega^+ + s(q^{-2}-1)\gamma\omega^1 \end{array}$$

$$\begin{array}{ll} \omega^-\alpha = sq^{-1}\alpha\omega^- + s(q^{-2}-1)\beta\omega^1 & \omega^2\alpha = s\alpha\omega^2 + s(q^{-1}-q)\beta\omega^+ \\ \omega^-\beta = sq^{-1}\beta\omega^- & \omega^2\beta = sq^{-2}\beta\omega^2 + s(q^{-1}-q)\alpha\omega^- + s(q^{-1}-q)^2\beta\omega^1 \\ \omega^-\gamma = sq^{-1}\gamma\omega^- + s(q^{-2}-1)\delta\omega^1 & \omega^2\gamma = s\gamma\omega^2 + s(q^{-1}-q)\delta\omega^+ \\ \omega^-\delta = sq^{-1}\delta\omega^- & \omega^2\delta = sq^{-2}\delta\omega^2 + s(q^{-1}-q)\gamma\omega^- + s(q^{-1}-q)^2\delta\omega^1 \end{array}$$

Values and action of the generators on the q-group elements:

$$\begin{array}{llll} \chi_1(\alpha) = \frac{s-q^2}{q^3-q} & \chi_+(\alpha) = 0 & \chi_-(\alpha) = 0 & \chi_2(\alpha) = \frac{s-1}{q-q^{-1}} \\ \chi_1(\beta) = 0 & \chi_+(\beta) = 0 & \chi_-(\beta) = -s & \chi_2(\beta) = 0 \\ \chi_1(\gamma) = 0 & \chi_+(\gamma) = -s & \chi_-(\gamma) = 0 & \chi_2(\gamma) = 0 \\ \chi_1(\delta) = \frac{-q^2+s(1-q^2+q^4)}{q^3-q} & \chi_+(\delta) = 0 & \chi_-(\delta) = 0 & \chi_2(\delta) = \frac{s-q^2}{q^3-q} \end{array}$$

$$\begin{array}{llll} \chi_1 * \alpha = \frac{s-q^2}{q^3-q}\ \alpha & \chi_+ * \alpha = -s\beta & \chi_- * \alpha = 0 & \chi_2 * \alpha = \frac{s-1}{q-q^{-1}}\ \alpha \\ \chi_1 * \beta = \frac{-q^2+s(1-q^2+q^4)}{q^3-q}\ \beta & \chi_+ * \beta = 0 & \chi_- * \beta = -s\alpha & \chi_2 * \beta = \frac{(s-q^2)}{q^3-q}\ \beta \\ \chi_1 * \gamma = \frac{s-q^2}{q^3-q}\ \gamma & \chi_+ * \gamma = -s\delta & \chi_- * \gamma = 0 & \chi_2 * \gamma = \frac{s-1}{q-q^{-1}}\ \gamma \\ \chi_1 * \delta = \frac{-q^2+s(1-q^2+q^4)}{q^3-q}\ \delta & \chi_+ * \delta = 0 & \chi_- * \delta = -s\gamma & \chi_2 * \delta = \frac{s-q^2}{q^3-q}\ \delta \end{array}$$

Exterior derivatives of the basic elements of A:

$$\begin{aligned} d\alpha &= \tfrac{s-q^2}{q^3-q}\alpha\omega^1 - s\beta\omega^+ + \tfrac{s-1}{q-q^{-1}}\alpha\omega^2 \\ d\beta &= \tfrac{-q^2+s(1-q^2+q^4)}{q^3-q}\beta\omega^1 - s\alpha\omega^- + \tfrac{s-q^2}{q^3-q}\beta\omega^2 \\ d\gamma &= \tfrac{s-q^2}{q^3-q}\gamma\omega^1 - s\delta\omega^+ + \tfrac{s-1}{q-q^{-1}}\gamma\omega^2 \\ d\delta &= \tfrac{-q^2+s(1-q^2+q^4)}{q^3-q}\delta\omega^1 - s\gamma\omega^- + \tfrac{s-q^2}{q^3-q}\delta\omega^2 \end{aligned}$$

The ω^i in terms of the exterior derivatives on $\alpha, \beta, \gamma, \delta$:

$$\begin{aligned} \omega^1 &= \tfrac{q}{s(-q^2-q^4+s+sq^4)}[(q^2-s)(\kappa(\alpha)d\alpha + \kappa(\beta)d\gamma) + q^2(s-1)(\kappa(\gamma)d\beta + \kappa(\delta)d\delta)] \\ \omega^+ &= -\tfrac{1}{s}[\kappa(\gamma)d\alpha + \kappa(\delta)d\gamma] \\ \omega^- &= -\tfrac{1}{s}[\kappa(\alpha)d\beta + \kappa(\beta)d\delta] \\ \omega^2 &= \tfrac{q}{s(-q^2-q^4+s+sq^4)}[(s-q^2-sq^2+sq^4)(\kappa(\alpha)d\alpha + \kappa(\beta)d\gamma) + (q^2-s)(\kappa(\gamma)d\beta + \kappa(\delta)d\delta)] \end{aligned}$$

Lie derivative on ω^i*:* (See Subsection 2.4.5)

$$\begin{array}{ll}
\chi_1 * \omega^1 = q(q^2-1)\omega^1 + (q^{-1}-q)\omega^2 & \chi_+ * \omega^1 = -q\omega^- \\
\chi_1 * \omega^+ = -q^{-1}\omega^+ & \chi_+ * \omega^+ = -q\omega^2 + q^3\omega^1 \\
\chi_1 * \omega^- = [q(q^2+1) - q^{-1}]\omega^- & \chi_+ * \omega^- = 0 \\
\chi_1 * \omega^2 = -q(q^2-1)\omega^1 - (q^{-1}-q)\omega^2 & \chi_+ * \omega^2 = q\omega^-
\end{array}$$

$$\begin{array}{ll}
\chi_- * \omega^1 = q\omega^+ & \chi_2 * \omega^1 = 0 \\
\chi_- * \omega^+ = 0 & \chi_2 * \omega^+ = q\omega^+ \\
\chi_- * \omega^- = q^{-1}\omega^2 - q\omega^1 & \chi_2 * \omega^- = -q^{-1}\omega^- \\
\chi_- * \omega^2 = -q\omega^+ & \chi_2 * \omega^2 = 0
\end{array}$$

2.3 Differential calculus from the q-Lie Algebra. (A more intuitive presentation of the differential calculus on q-groups)

In the previous sections we have analized the differential calculus on q-groups starting from the properties of the exterior differential and the space of 1-forms. The left invariant vectorfields χ_i and their q-Lie algebra were then introduced at the very end. Here we would like to invert the exposition procedure and following [45], [44] and in the spirit of [36, 34, 35], [37, 38], [42], we derive differential calculi on q-groups from basic properties of q-Lie algebras.

This clarify the important role played by the adjoint action in the q-Lie algebra and in the construction of a bicovariant differential calculus. In this way we also give an alternative proof of the Woronowicz theorems that we stated in Section 2.1.

This approach is also suitable for the study of generalizations of the Woronowicz theory. As is evident from Subsection 2.3.3 bicovariant calculi that do not satisfy the undeformed Leibniz rule can be found studying quantum Lie algebras that are closed under the adjoint action [44].

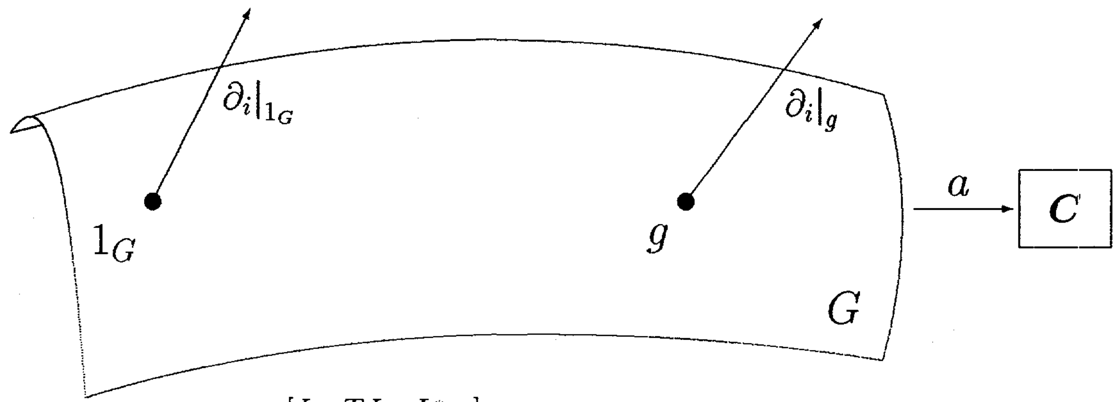
In this section we consider a generic Hopf algebra A and a Hopf algebra U paired to A, we also consider the pairing non-degenerate. Intuitively A and U are the quantum analogue of $Fun(G)$ and of the universal enveloping algebra $U(g)$. The differential calculus on A and the quantum Lie algebra structure can be formulated, as in Section 2.1, without the introduction of U, however to gain a better geometrical understanding of the structures we are condisering, the universal enveloping algebra U is helpful. We think that this presentation is more intuitive than the one in Section 2.1, because it is closer to the classical case, where the exterior differential on a group manifold can be introduced via a basis of left invariant vectorfields and the dual basis of 1-forms. In this way we emphasize the role played by left invariant vectorfields i.e. the q-tangent space, a more intuitive and basic concept than that of left invariant 1-forms (q-cotangent space). The q-tangent bundle of general vectorfields will be studied in the next section.

2.3.1 Left invariant Vectorfields

Classically the differential calculus on a group is uniquely determined by the Lie algebra of the tangent vectors to the origin of the group. Locally we write a basis as $\{\partial_i|_{1_G}\}$. Once we have this basis, using the tangent map (namely TL_g) induced by the left multiplication of the group on itself: $L_g g' = gg'$, $\forall g, g' \in G$ we can construct a basis of left invariant vectorfields $\{t_i\}$. The action of these vectorfields on a generic function a on the group manifold is

$$t_i(a) = a_1(\partial_i a_2|_{1_G}) \equiv \partial_i|_{1_G} * a \qquad (2.3.1)$$

in compliance with the following picture:



$$(1_G, \partial_i|_{1_G}, a) \xrightarrow{[L_g, TL_g, L^*_{g^{-1}}]} (g, \partial_i|_g, a_1(g^{-1})a_2)$$

where $L^*_g(a)(h) \equiv a(gh) = a_1(g)a_2(h)$ [cf. (2.1.10)]. Explicitly, t is left invariant if $TL_g(t|_{1_G}) = t|_g$, then we have

$$t(a)|_g = \left(TL_g t|_{1_G}\right)(a) = t[a(g\tilde{g})]|_{\tilde{g}=1_G} = t[a_1(g)a_2(\tilde{g})]|_{\tilde{g}=1_G} = a_1(g)t(a_2)|_{1_G}$$

and 2.3.1 follows [cf. (1.3.6)].
In the commutative case, since the space ${}_{\text{inv}}\Xi$ of left invariant vectorfields provides a trivialization of the tangent bundle of the group manifold the space Ξ of vectorfields is isomorphic to $C^\infty(G) \otimes {}_{\text{inv}}\Xi \simeq {}_{\text{inv}}\Xi \otimes C^\infty(G)$, i.e., a generic vectorfield V can be written as $V = b^i t_i$ where b^i are functions on the group manifold. Similarly a generic 1-form can be written $\rho = b_i \omega^i$ $[b_i \in C^\infty(G)]$ where $\{\omega^i\}$ is the dual basis of $\{t_i\}$. Finally, the exterior differential on a generic function b is

$$db = t_i(b)\omega^i \tag{2.3.2}$$

and is compatible with the left and right action of the group on the space of 1-forms: $L^*_x(adb) = L^*_x(a)L^*_x(db) = L^*_x(a)dL^*_x(b)$ and $R^*_x(adb) = R^*_x(a)R^*_x(db) = R^*_x(a)dR^*_x(b)$.

Following this classical construction, in this section we show that a differential calculus on a q-group A, with universal enveloping algebra U ($U \equiv U_q(\mathrm{g})$), is determined by a q-Lie algebra T: the q-deformation of g. The exterior differential is then given by (2.3.2) where now t_i are left invariant vectorfield on A and $\{\omega^i\}$ is the dual basis of $\{t_i\}$.

It is natural to look for a linear space T, $T \subset \ker\varepsilon \subset U$ satisfying the following three conditions:

$$T \text{ generates } U \tag{2.3.3}$$

$$\Delta'(T) \subset T \otimes U + \varepsilon \otimes T, \tag{2.3.4}$$

$$[T, T] \subset T \tag{2.3.5}$$

where the bracket is the adjoint action defined by

$$\forall \chi, \tilde{\chi} \in T \ , \quad [\tilde{\chi}, \chi] \equiv \kappa'(\chi_1)\tilde{\chi}\chi_2 \ . \tag{2.3.6}$$

Conditions (2.3.3) and (2.3.5) encode the quantum group properties of the q-tangent space because they involve the product and the antipode κ' of U. Condition (2.3.4) states that the elements of T are generalized tangent vectors, and in fact, if $\{\chi_i\}$ is a basis of the linear space T, we have

$$\Delta(\chi_i) = \chi_j \otimes f^j{}_i + \varepsilon \otimes \chi_j \tag{2.3.7}$$

that is equivalent to

$$\chi_i(ab) = \chi_j(a)\, f^j{}_i(b) + \varepsilon(a)\, \chi_j(b) \tag{2.3.8}$$

where $f^j{}_i \in U$ and $\varepsilon'(f^j{}_i) \equiv f^j{}_i(I) = \delta^j_i$. [Hint: apply $(id \dot{\otimes} \varepsilon')$ to (2.3.4)]. In the commutative limit we expect $f_i{}^j \to \varepsilon$. Notice that we follow the historical convention which consider derivative operators acting from the right to the left, as is seen from their deformed Leibniz rule (2.3.7). Of course one can consider deformed derivative operators $\check{\chi}_i$ acting from the left to the right, they are for example given by $\check{\chi}_i = -\kappa'^{-1}(\chi_i)$, similarly $\check{f}_j{}^i = \kappa'^{-1}(f^i{}_j)$ and then $\Delta(\check{\chi}_i) = \check{\chi}_i \otimes \varepsilon + \check{f}_i{}^j \otimes \check{\chi}_j$.

Following (2.3.1) we can also consider the q-deformed left invariant vectorfields

$$t_i \equiv \chi_i \, * \tag{2.3.9}$$

defined by $\chi_i * a \equiv a_1 \chi_i(a_2)$, then (2.3.4) states that the $\chi_i *$ are generalized derivations

$$\chi_i * (ab) = (\chi_j * a)(f^j{}_i * b) + a\, \chi_j * b \ , \tag{2.3.10}$$

where we have also defined $f^j{}_i * b \equiv b_1 f^j{}_i(b_2)$, and $\varepsilon * a \equiv a_1 \varepsilon(a_2) = a$.

There is a one-to-one correspondence $\chi_i \leftrightarrow t_i = \chi_i *$. In order to obtain χ_i from $\chi_i *$ we simply apply ε :

$$(\varepsilon \circ t_i)(a) = \varepsilon(id \otimes \chi_i)\Delta(a) = \varepsilon(a_1\chi_i(a_2)) = \varepsilon(a_1)\chi_i(a_2) = \chi_i(\varepsilon \otimes id)\Delta(a) = \chi_i(a) \ .$$

2.3.2 Adjoint action

In this subsection we examine and study the consequences of conditions (2.3.3) and (2.3.5). We see that they define the adjoint representation $M_i{}^j$ that we will later identify with the one studied in Section 2.1; we rewrite (2.3.3) and (2.3.5) in a more geometric language using left and right invariant vectorfields and finally, in Note 2.3.1, we establish the equivalence between (2.3.3)-(2.3.5) and Woronowicz theory.

Condition (2.3.5) is the closure of T under the adjoint action, in the classical case, if χ is a tangent vector: $\Delta(\chi) = \chi \otimes \varepsilon + \varepsilon \otimes \chi$, $\kappa'(\chi) = -\chi$ and the adjoint action of χ on $\tilde{\chi}$ is given by the commutator $\tilde{\chi}\chi - \chi\tilde{\chi}$. Expression (2.3.6) is a natural and simple way to write, both at the quantum and at the classical level, the adjoint action. We can derive (2.3.6) from the classical conjugation on the group elements. First we notice that the conjugation $Conj^{-1} \; : \; G \times G \to G$, $Conj_{\tilde{g}}^{-1}(g) = \tilde{g}^{-1}(g)\tilde{g}$ induces the following action on $A = Fun(G)$:

$$ad(a) = a_2 \otimes \kappa(a_1) a_3 \; . \tag{2.3.11}$$

Proof: $ad(a)(g, \tilde{g}) = a_2(g)(a_1)(\tilde{g}^{-1}) a_3(\tilde{g}) = a(\tilde{g}^{-1} g \tilde{g})$. Expression (2.3.11) is independent from the points g of G and therefore holds also for a generic Hopf algebra A. Now we use the pairing between A and $U \subseteq A'$, discussed in Section 1.3, to deduce the adjoint action of the universal enveloping algebra U on itself: $\forall \psi, \varphi \in U$, $\forall a \in A$

$$\begin{aligned} \langle ad_\psi \varphi, a \rangle &\equiv \langle \varphi \otimes \psi, ada \rangle & (2.3.12)\\ &= \langle \varphi \otimes \kappa'(\psi_1) \otimes \psi_2 \, , \, a_2 \otimes a_1 \otimes a_3 \rangle & (2.3.13)\\ &= \langle \kappa'(\psi_1) \varphi \psi_2 \, , \, a \rangle & (2.3.14) \end{aligned}$$

so that

$$ad_\psi \varphi = \kappa'(\psi_1) \varphi \psi_2 \; . \tag{2.3.15}$$

Notice that ad is a right action of U on U, $\forall \varphi, \psi, \zeta \in U$:

$$ad_{\psi\zeta} \varphi = ad_\zeta (ad_\psi \varphi) \; ; \tag{2.3.16}$$

in particular, for $\chi, \tilde{\chi}, \chi', ... \chi'' \in T$, formulae (2.3.15) and (2.3.16) read:

$$ad_\chi \tilde{\chi} = [\tilde{\chi}, \chi] \quad \text{and} \quad ad_{(\chi\chi' \cdots \chi'')} \tilde{\chi} = [...[[\tilde{\chi}, \chi], \chi'], \ldots \chi''] \; . \tag{2.3.17}$$

The first expression proves that the bracket in (2.3.6) is indeed the adjoint action; from the second expression, (2.3.3) and (2.3.5), we have

$$\forall \psi \in U, \forall \chi \in T \; , \qquad ad_\psi \chi \in T \qquad \text{i.e.} \qquad ad_\psi \chi_i = M_i{}^j(\psi) \chi_j \tag{2.3.18}$$

where we have introduced the basis $\{\chi_i\}_{i=1,...n}$ of T and $M_i{}^j(\psi)$ are complex numbers depending on ψ. The linearity of the adjoint map imply that the functionals $M_i{}^j$ are linear: $M_i{}^j(\alpha\psi + \phi) = \alpha M_i{}^j(\psi) + M_i{}^j(\phi)$ while, due to (2.3.16), they are a representation of U: $M_i{}^j(\psi\phi) = M_i{}^k(\psi) M_k{}^j(\phi)$. Since the pairing $A \leftrightarrow U$ is nondegenerate and ψ is a generic element of U, the second expression in (2.3.18) uniquely defines the functionals $M_i{}^j$ as elements of A. They are the adjoint representation, in Hopf algebra notations [see (2.1.46)]:

$$\varepsilon(M_i{}^j) = \delta_i^j \quad , \qquad \Delta(M_i{}^j) = M_i{}^l \otimes M_l{}^j . \tag{2.3.19}$$

We can also obtain an explicit expression for the elements $M_i{}^j \in A$; since A separates the points of U, and therefore of T, we can consider n elements $y^i \in A$ such that $\langle \chi_i \, , y^j \rangle = \chi_i(y^j) = \delta_i^j$. Then

$$M_i^j = (\chi_i \otimes id) ad \, y^j = \chi_i(y_2^j) \kappa(y_1^j) y_3^j \tag{2.3.20}$$

Proof: apply a generic element $\psi \in U$ to (2.3.20) and recall (2.3.12). □
In particular, formula (2.3.20) holds if we consider the coordinates x^i on the quantum group, as defined in (2.1.89) and (2.1.90), we therefore have:

$$M_i{}^j = (\chi_i \otimes id)ad\, x^j = \chi_i(x_2^j)\kappa(x_1^j)x_3^j \;. \tag{2.3.21}$$

There is an equivalent expression for (2.3.18) that shows its q-group geometric content. We have shown that $t = \chi *$ is a left-invariant vectorfield, similarly

$$h \equiv *\chi \qquad i.e. \quad \forall a \in A \quad h(a) \equiv \chi(a_1)a_2 \tag{2.3.22}$$

is a right invariant vectorfield.

Theorem 2.3.1 Relation (2.3.18) is equivalent to, $\forall a \in A \;\; t_j(a)M_i{}^j = h_i(a)$ i.e.

$$(\chi_j * a)M_i{}^j = a * \chi_i \qquad \text{i.e.} \quad a_1\chi_j(a_2)M_i{}^j = \chi_i(a_1)a_2 \tag{2.3.23}$$

Proof Multiplying (2.3.18) i.e. $ad_\psi \chi_i = \kappa'(\psi_1)\chi_i\psi_2 = \psi(M_i{}^j)\chi_j$ by ψ_0, where $(\Delta' \otimes id)\Delta'(\psi) = \psi_0 \otimes \psi_1 \otimes \psi_2$, we obtain the equivalent expression

$$\forall \psi \in U \qquad \chi_i\psi = \psi_1\chi_j\;\psi_2(M_i{}^j) \tag{2.3.24}$$

This relation gives the q-commutation relations between any $\psi \in U$ and the χ_j elements. We apply a generic element $a \in A$ to (2.3.24) and rewrite the right hand side as, $\forall \psi \in U$, $\forall a \in A$

$$\begin{aligned} \langle \psi_1\chi_j\psi_2(M_i{}^j)\,,\,a\rangle &= \langle \psi_1\chi_i \otimes \psi_2\,,\,a \otimes M_j{}^i\rangle &(2.3.25)\\ &= \langle \psi \otimes \chi_j\,,\,a_1 M_j{}^i \otimes a_2\rangle &(2.3.26)\end{aligned}$$

Since $\psi \in U$ and $a \in A$ are arbitrary elements and since the pairing $U \leftrightarrow A$ is nondegenerate (we actually only need U to separate the points of A) we conclude

$$\langle \chi_i \otimes \psi\,,\,a_1 \otimes a_2\rangle = \langle \psi \otimes \chi_j\,,\,a_1 M_j{}^i \otimes a_2\rangle \;\Leftrightarrow\; a * \chi_i = (\chi_j * a)M_i{}^j \tag{2.3.27}$$

that proves the theorem. □□□

Formula (2.3.23) relates the left invariant vectorfields $t_i = \chi_i *$ to the right invariant ones $h_i = *\chi$ via the adjoint representation $M_i{}^j$. We can write $h_i = t_j \square M_i{}^j$ where $(t_j \square M_i{}^j)(a) \equiv t_j(a)M_i{}^j \;\forall a \in A$. This formula is the analogue of (2.1.47). The space of vectorfields is analized in the next section.

Note 2.3.1 In [21] Woronowicz has shown that bicovariant differential calculi are in one-to-one correspondence with ad-invariant right ideals R of A : $Ra \subset R \; \forall a \in A$ $ad(R) \subset R \otimes A$. These two conditions are slightly weaker than (2.3.3)-(2.3.5). Relation (2.3.4) can also be written $\Delta(T \oplus \{\varepsilon\}) \subset (T \oplus \{\varepsilon\}) \otimes U$, where $T \oplus \{\varepsilon\}$

is the vector space spanned by χ_i and ε; therefore $(T \oplus \{\varepsilon\})$ is a right *co-ideal*, it is the space orthogonal to the Woronowicz [21] right *ideal* $R \equiv \{a \in A \;/\; \varepsilon(a) = 0 \;\text{ and }\; T(a) = 0\}$. We have seen that relations (2.3.3) and (2.3.5) imply (2.3.18) this condition is then equivalent to the ad invariance of R: $ad(R) \subset R \otimes A$.
Proof: $\forall r \in R, \forall \chi \in T, \forall \psi \in U$,

$$0 = \langle ad_\psi \chi \,,\, r \rangle = \langle \chi \otimes \psi \,,\, ad\, r \rangle \Rightarrow ad\, r \in (R \oplus \{I\}) \otimes A \Rightarrow ad\, r \in R \otimes A \quad (2.3.28)$$

where the last implication holds because $\langle \varepsilon \otimes id \,,\, ad\, r \rangle = 0$. Viceversa $ad\, R \subset R \otimes A \Rightarrow ad_\psi \chi \in T \oplus \{\varepsilon\} \Rightarrow ad_\psi \chi \in T$ since $\langle ad_\psi \chi \,,\, I \rangle = 0$ □

Notice that a Woronowicz type bicovariant differential calculus is given by a set $\{\chi_i\}$ of linear functionals on A satisfying (2.3.23) and (2.3.8), the full structure of the dual Hopf algebra U and the nondegeneracy of the $A \leftrightarrow U$ pairing is not needed to formulate the calculus. In particular (2.3.19) can be derived from (2.3.23).[3]

2.3.3 The space of 1-forms and the exterior differential

We now proceed in the construction of the differential calculus introducing the space Γ of 1-forms. The space of left-invariant 1-forms ${}_{\mathrm{inv}}\Gamma$ is defined as the space dual to that of the tangent vectors T, let $\{\omega^i\}$ be the base of ${}_{\mathrm{inv}}\Gamma$ dual to $\{\chi_i\}$, we use the notation

$$\langle \chi_i \,,\, \omega^j \rangle = \delta_i^j \;. \quad (2.3.29)$$

By definition a generic 1-form is then uniquely written as [see (2.1.28)] $\rho = a_i \omega^i$ i.e. the space of 1-forms is the left A-module freely generated by the elements ω^i. This corresponds to the classical property that the cotangent bundle of a group manifold is trivial. The differential is defined by

$$\forall a \in A \qquad da = (\chi_i * a)\, \omega^i \;. \quad (2.3.30)$$

Note 2.3.2 We can rewrite the exterior differential using right-invariant vector-fields:

$$da = (a * \chi_i) \kappa^{-1}(M_j{}^i) \omega^j = (a * \chi_i) \eta^i \quad (2.3.31)$$

where we have defined the 1-forms $\eta^i = \kappa^{-1}(M_j{}^i)\omega^j$. It is easy to check that the η^i are right invariant, see (2.1.47). Using (2.1.50) we also have:

$$da = -\eta^i (a * \kappa'(\chi_i)) \;. \quad (2.3.32)$$

---

[3] $$\begin{aligned} \Delta M_i{}^j &= \chi_i(x^j_2)\Delta(\kappa(x^j_1)x^j_3) = \kappa(x^j_1)\chi_i(x^j_2)x^j_3 \otimes \kappa(x^j_0)x^j_4 \\ &= \kappa(x^j_1)x^j_2\chi_n(x^j_3)M_i{}^n \otimes \kappa(x^j_0)x^j_4 = M_i{}^n \otimes \chi_n(x^j_3)\kappa(x^j_1)\varepsilon(x^j_2)x^j_4 \\ &= M_i{}^n \otimes M_n{}^j \;. \end{aligned}$$

Any 1-form $\rho \in \Gamma$ can be written as $\rho = \sum_k a_k \, db_k$ for some $a_k, b_k \in A$ because we have the following expression for the left invariant 1-forms:

$$\omega^i = \kappa(y^i_1)dy^i_2 \; ; \quad \text{in particular} \;\; \omega^i = \kappa(x^i_1)dx^i_2 \tag{2.3.33}$$

where y^i and x^i are the same elements that appear in (2.3.20) and (2.3.21).
Proof: $\kappa(y^i_1)dy^i_2 = \kappa(y^i_1)y^i_2\chi_j(y^i_3)\omega^j = \chi_j(y^i)\omega^j = \omega^i$ □

On the space Γ of 1-forms we can introduce a left and a right coaction of A [the analogue of the left and right pullback on 1-forms, see (2.1.10)-(2.1.11)] by the following definitions, [see (2.1.20) and (2.1.38)]:

$$\Delta_\Gamma(a_i\omega^i) = \Delta(a_i)(I \otimes \omega^i) \quad ; \quad {}_\Gamma\Delta(a_i\omega^i) = \Delta(a)(\omega^j \otimes M_j{}^i) \tag{2.3.34}$$

notice that the right A coaction on the left-invariant 1-forms ${}_{\rm inv}\Gamma$ corresponds to the (left) adjoint action of U on ${}_{\rm inv}\Gamma$: $\forall \psi \in U;\; ad_\psi\omega^i \equiv \omega^j\psi(M_j{}^i)$. We say that Γ is a *left and right covariant* left-module because properties (2.1.17), (2.1.18) and (2.1.19) are satisfied. [Hint: use (2.3.19)]. We are now able to prove that the differential calculus is left and right covariant i.e. it is bicovariant on the left module Γ:

Proposition 2.3.1 The exterior differential defined in (2.3.30) is bicovariant on the left module Γ:

$$\Delta_\Gamma(adb) = \Delta(a)(id \otimes d)\Delta(b), \quad \Delta_\Gamma : \Gamma \to A \otimes \Gamma \quad \text{(left covariance)} \tag{2.3.35}$$
$${}_\Gamma\Delta(adb) = \Delta(a)(d \otimes id)\Delta(b), \quad {}_\Gamma\Delta : \Gamma \to \Gamma \otimes A \quad \text{(right covariance)} \tag{2.3.36}$$

Proof: since $\Delta_\Gamma(a\rho) = \Delta(a)\Delta_\Gamma(\rho)$ and ${}_\Gamma\Delta(a\rho) = \Delta(a){}_\Gamma\Delta(\rho)$ where ρ is a generic 1-form it is sufficient to prove:

$$\Delta_\Gamma(db) = \Delta(\chi_i * b)I \otimes \omega^i = b_1 \otimes b_2\chi_i(b_3)\omega^i = (id \otimes d)\Delta(b) \; ; \tag{2.3.37}$$

$$\begin{aligned} {}_\Gamma\Delta(db) &= \Delta(\chi_i * b)\omega^j \otimes M_j{}^i = b_1\omega^j \otimes b_2\chi_i(b_3)M_j{}^i = b_1\omega^j \otimes \chi_j(b_2)b_3 \\ &= (d \otimes id)\Delta(b) \end{aligned} \tag{2.3.38}$$

□□□

We have seen that from the closure of the q-Lie algebra T under the adjoint action of U on T, equation (2.3.18) [or from (2.3.3) and (2.3.5)] or equivalently from the relation (2.3.23) between left and right invariant vectorfields, one can construct an exterior differential $d \; : \; A \to \Gamma$; where Γ is the left A-module of 1-forms freely generated by the space of left-invariant one forms ${}_{\rm inv}\Gamma$. We have introduced a left and a right coaction of the quantum group A on Γ and proved that the exterior differential is compatible with these coactions, see (2.3.35)-(2.3.36). This clarify the importance of the adjoint action in the construction of a differential calculus on a quantum group.

We now analize the consequences of (2.3.4) that, so far, we have never used in this section. We show that (2.3.4) is equivalent to the Leibniz rule for the exterior differential and that it implies the q-antisymmetry of the q-Lie algebra.

2.3.4 The Leibniz rule and the bicovariant bimodule of 1-forms

Lemma The deformed Leibniz rule (2.3.4) and (2.3.23) imply [see (2.1.51)]:

$$M_i{}^j(a * f^i{}_k) = (f^j{}_i * a)M_k{}^i, \tag{2.3.39}$$

Proof: from (2.3.23) we have, $\forall\, b \in A$:

$$\begin{aligned}
\kappa(x_1^l)(\chi_j * x_2^l b)M_i{}^j &= \kappa(x_1^l)(x_2^l b * \chi_i) \quad \Leftrightarrow \\
\kappa(x_1^l)x_2^l b_1 \chi_j(x_3^l b_2)M_i{}^j &= \kappa(x_1^l)\chi_i(x_2^l b_1)(x_3^l b_2) \quad \Leftrightarrow \\
(f^l{}_j * b)M_i{}^j &= \kappa(x_1^l)[\chi_n(x_2^l)f^n{}_i(b_1) + \varepsilon(x_2^l)\chi_i(b_1)]x_3^l b_2 \quad \Leftrightarrow \\
(f^l{}_j * b)M_i{}^j &= \kappa(x_1^l)\chi_n(x_2^l)f^n{}_i(b_1)x_3^l b_2 \quad \Leftrightarrow \\
(f^l{}_j * b)M_i{}^j &= M_n{}^l(b * f^n{}_i)
\end{aligned}$$

where in the left hand side of the second passage we have used $f^l{}_j(b) = \chi_j(x^l b)$ that is obtained from (2.3.7) when $x^i = a$. □□□

The Leibniz rule on the left A-module Γ can be introduced if we know how to multiply 1-forms with functions from the right, i.e. if Γ is also a right module. Consider the functionals $f^i{}_j$ given in (2.3.7), we define the following right product:

Definition

$$\omega^i c = (f^i{}_j * c)\omega^j \quad , \quad (a_i\omega^i)c = a_i(f^i{}_j * c)\omega^j \tag{2.3.40}$$

the definition is well given because

$$\Delta'(f^i{}_j) = f^i{}_k \otimes f^k{}_j \ , \quad \varepsilon'(f^i{}_j) = \delta^i_j \ , \tag{2.3.41}$$

[see (2.1.35)-(2.1.36)]; these two properties immediately follow, respectively, from the coassociativity of the coproduct on the χ_i elements, and from $\chi_i(x^j) = \delta_i^j$.

We now prove the compatibility of (2.3.40) with the left and right coactions Δ_Γ and ${}_\Gamma\Delta$; i.e. we prove that Δ_Γ and ${}_\Gamma\Delta$ are also, respectively, left and right coactions on Γ seen as a right module:

$$\forall \rho \in \Gamma, \forall a \in A, \qquad \Delta_\Gamma(\rho a) = \Delta_\Gamma(\rho)\Delta(a) \quad , \quad {}_\Gamma\Delta(\rho a) = {}_\Gamma\Delta(\rho)\Delta(a) \ . \tag{2.3.42}$$

Since any $\rho \in \Gamma$ is of the form $\rho = a_i\omega^i$ and since $\Delta_\Gamma\omega^i = I \otimes \omega^i$, the only nontrivial espression in (2.3.42) is ${}_\Gamma\Delta(a_i\omega^i a) = {}_\Gamma\Delta(a_i\omega^i)\Delta(a)$. As proven in Note 2.1.1, this is equivalent to (2.3.39) and we conclude that Γ is a bicovariant bimodule, i.e. that Δ_Γ and ${}_\Gamma\Delta$ are compatible with the bimodule structure of Γ.

From the deformed Leibniz rule for the tangent vectors χ_i, see (2.3.7) or (2.3.10), and from (2.3.40), the Leibniz rule for the exterior differential immediately follows.

Viceversa, suppose that on Γ a right module structure can be introduced such that d satisfies the Leibniz rule and is well defined in the following sense:

$$adb = a'db' \;\Rightarrow\; (adb)c = (a'db')c \quad \text{i.e.} \quad a\,d(bc) - ab\,dc = a'd(b'c) - a'b'dc$$

(adb is a shorthand notation for $\sum_k a_k db_k$; a, b, c are generic elements). Then we can use the Leibniz rule to express the right module structure on Γ as : $(adb)c = a\,d(bc) - ab\,dc$. In the particular case $adb = \omega^i$ we have, see (2.3.33),

$$\begin{aligned} \omega^i\, c &= \kappa(x^i_1)dx^i_2\, c = \kappa(x^i_1)d(x^i_2c) = \kappa(x^i_1)x^i_2c_1\chi_j(x^i_3c_2)\omega^j &\quad (2.3.43)\\ &= c_1\chi_j(x^ic_2)\omega^j \equiv c_1 f^j{}_i(c_2)\omega^j = (f^j{}_i * c)\,\omega^j &\quad (2.3.44) \end{aligned}$$

where, in the last but one passage we have defined $\forall c \in A \;\; f^j{}_i(c) \equiv \chi_j(x^ic)$. Now from $d(ab) = d(a)\,b + adb$ and (2.3.44) we obtain:

$$\chi_i * ab = (\chi_j * a)(f^j{}_i * b) + a(\chi_i * b) \qquad (2.3.45)$$

that is equivalent to (2.3.10).

2.3.5 q-antisymmetry of the q-Lie algebra bracket

The coproduct (2.3.4) implies that the espression $[\chi_i, \chi_j]$ is quadratic and q-antisymmetric. We first write

$$[\chi_i, \chi_j] = \kappa'(\chi_{j_1})\chi_i\chi_{j_2} = \kappa'(\chi_l)\chi_i f^l{}_j + \chi_i\chi_j\;; \qquad (2.3.46)$$

now we apply $m(id \otimes \kappa')$ to $\Delta(\chi_l) = \chi_n \otimes f^n{}_l + \varepsilon \otimes \chi_l$ to obtain $\kappa'(\chi_l) = -\chi_n\kappa'(f^n{}_l)$ and therefore

$$\kappa'(\chi_l)\chi_i f^l{}_j = -\chi_n\kappa'(f^n{}_l)\chi_i f^l{}_j = -\chi_n ad_{(f^n{}_j)}\chi_i = -\chi_n f^n{}_j(M_i{}^l)\chi_l \qquad (2.3.47)$$

so that

$$[\chi_i, \chi_j] = \chi_i\chi_j - f^n{}_j(M_i{}^l)\chi_n\chi_l\;. \qquad (2.3.48)$$

In the above framework it is easy to derive the bicovariance conditions (2.1.111)-(2.1.114); indeed recalling equations (2.1.64): $\Lambda^{ij}{}_{kl} = f^i{}_l(M_k{}^j)$ and (2.1.95): $\mathbf{C}_{ij}{}^k = \chi_j(M_i{}^k)$ we immediately derive from (2.3.48) the bicovariance condition (2.1.111) and from [recall (2.3.18)] $ad_{(f^n{}_j)}\chi_i = f^n{}_j(M_i{}^l)\chi_l$ the bicovariance condition (2.1.114). Relation (2.1.113) can be derived applying to $(x^i \otimes id)$ the coproduct of (2.1.111) and then using (2.1.114) and (2.1.105). Finally (2.1.112) can be obtained applying the functionals $f^l{}_n$ to (2.3.39).

2.3.6 q-Jacoby identities

We end this section briefly commenting on the q-Jacoby identities. We have seen that the ad map given in (2.3.15) is a right action of U on U [see (2.3.16)]: $\forall \varphi, \psi, \zeta$ $ad_{\psi\zeta}\varphi = ad_\zeta(ad_\psi \varphi)$. We use the identity $\psi\zeta = \zeta_1 ad_{\zeta_2}\psi$ to find

$$ad_\zeta(ad_\psi\varphi) = ad_{\psi\zeta}\varphi = ad_{\zeta_1 ad_{\zeta_2}\psi}\varphi = ad_{(ad_{\zeta_2}\psi)}(ad_{\zeta_1}\varphi) \ . \tag{2.3.49}$$

The above relation, written for elements χ_i, χ_j, χ_l of the q-Lie algebra T, is the q-Jacobi identity (with abuse of notation we define, $\forall \chi \in T$, $[\chi, f^i{}_j] \equiv ad_{(f^i{}_j)}\chi$, $[\chi, \varepsilon] \equiv ad_\varepsilon \chi = \chi$):

$$[[\chi_i, \chi_j], \chi_l] = [[\chi_i, \chi_{l_1}], [\chi_j, \chi_{l_2}]] \ ; \tag{2.3.50}$$

it express the property that the bracket operation is a derivation of the q-Lie algebra T (i.e. ad_ζ is a generalized derivation with respect to the product in U given by the ad map). Using the explicit coproduct expression $\Delta(\chi_i) = \chi_j \otimes f^j{}_i + \varepsilon \otimes \chi_j$ in (2.3.50), we obtain (2.1.99). There is also a second Jacobi identity : $ad_{ad_{(\zeta}\psi)}\varphi = ad_{\kappa(\zeta_1)\psi\zeta_2}\varphi = ad_{\zeta_2}(ad_\psi(ad_{\kappa(\zeta_1)}\varphi))$. On the χ elements it reads: $[\chi_i, [\chi_j, \chi_l]] = [[[\chi_i, \kappa(\chi_{l_1})], \chi_j], \chi_{l_2}]$. The two Jacobi identities are not independent because $ad_\zeta(ad_\psi\varphi) = ad_{[ad_{\zeta_2}\kappa^2(\zeta_1)\psi]}\varphi$.

Notice also that the map ad is compatible with the product of U in the sense that: $ad_\zeta(\psi\varphi) = ad_{\zeta_1}(\psi) ad_{\zeta_2}(\varphi)$ (i.e. ad_ζ is a generalized derivation with respect to the product of U); in particular $ad_{\chi_l}(\chi_i\chi_j) \equiv [\chi_i\chi_j, \chi_l] = [\chi_i, \chi_s] f^s{}_l(M_j{}^n)\chi_n + \chi_i[\chi_j, \chi_l]$.

2.3.7 $*$-Structure

Given a $*$-Hopf algebra A we have a canonical $*$-structure on the dual $\mathcal{U}$. This is compatible with the quantum Lie algebra T if $T^* \subseteq T$, i.e., if the tangent vectors $(\chi_i)^*$ are linear combination of the χ_i, so that we have a real form of the q-Lie algebra. In this case we can construct a differential calculus that is real:

$$(db)^* = db^* \ \text{ and more in general } \ (adb)^* = db^* \, a^* \ . \tag{2.3.51}$$

We now explicitly perform this construction. There is a canonical $*$-structure on the space of 1-forms. We first define a $*$-involution on ${}_{\text{inv}}\Gamma$ via the expression:

$$\forall \chi \in T \, , \ \forall \omega \in {}_{\text{inv}}\Gamma \, , \quad \langle \omega^* \, , \, \chi \rangle \equiv -\overline{\langle \omega \, , \, \chi^* \rangle} \tag{2.3.52}$$

and generalize it to Γ as

$$\forall a \in A \, , \ \forall \omega \in {}_{\text{inv}}\Gamma \, , \qquad (a\omega)^* = \omega^* a^* \ . \tag{2.3.53}$$

We have to check that these definitions are consistent with the bicovariant bimodule structure on Γ. We recall from Section 1.3 that the $*$ operation becomes the hermitian conjugation $\dagger$ when we realize the elements of A and U as operators on Hilbert space (in the $q \to 1$ limit the elements of A commute and correspond to diagonal operators, so that the $*$-operation becomes complex conjugation). It is then natural to consider a basis of antihermitian q-Lie algebra generators χ_i: $(\chi_i)^* = -\chi_i$, then the dual basis of 1-forms is real: $\omega^{i*} = \omega^i$. From $\Delta(\chi_j) = -\Delta(\chi_j^*) = -(\Delta\chi_j)^{*\otimes *}$ it follows that the $f^i{}_j$ are real and we have

$$\omega^{i*}a = (f^i{}_j * a)\omega^{j*} \tag{2.3.54}$$

that is compatible with $\omega^{i*} = \omega^i$. Proof of (2.3.54): $(\omega^{i*}a)^* = a^*\omega^i = \omega^j f^i{}_j \circ\kappa^{-1} * (a^*) = \omega^j a_1^* f^i{}_j(\kappa^{-1}(a_2^*)) = \omega^j a_1^* \overline{f^i{}_j(a_2)} = [(f^i{}_j * a)\omega^{j*}]^*$, where we have used (1.3.9).

Also the elements $M_i{}^j$ are real so that $_\Gamma\Delta(\omega^i) = \omega^j \otimes M_i{}^j$ is well defined.
Proof: $M_i{}^{j*}(\psi)\chi_j = \overline{M_i{}^j(\kappa'^{-1}(\psi^*))}\chi_j = -[M_i{}^j(\kappa'^{-1}(\psi^*))\chi_j]^* = -[ad_{\kappa'^{-1}(\psi^*)}\chi_i]^* = -[\psi_2^*\chi_i\kappa'^{-1}(\psi_1^*)]^* = M_i{}^j(\psi)\chi_j$.

We are now able to show that the differential, given by $da = (\chi_i * a)\omega^i$ is real:

$$\begin{aligned}(da^*)^* &= \omega^i(\chi_i * a^*)^* = [f^i{}_j * (\chi_i * a^*)^*]\omega^j = (f^i{}_j * a_1)\overline{\chi_i(a_2^*)}\omega^j \\ &= (f^i{}_j * a_1)(\kappa'^{-1}\chi_i^*)(a_2) = -[f^i{}_j(\kappa'^{-1}\chi_i) * a]\omega^j = (\chi_j * a)\omega^j \\ &= da\end{aligned}$$

where we have used $m[\tau(\kappa'^{-1} \otimes id)\Delta'(\chi)] = \varepsilon'(\chi) = 0$, a consequence of (A.4).

Conclusions

We have seen, from (2.3.3)–(2.3.5), or more in general from (2.3.4) and (2.3.18), or from (2.3.4) and (2.3.23), that the construction of the differential calculus associated to the q-Lie algebra T spanned by the χ_i elements is quite straighforward, the main ingredients are

i) the left invariant vectorfields $t_i = \chi_i *$, with deformed Leibniz rule: $t_i(ab) = t_j(a)f^j{}_i(b) + at_i(b)$

ii) the adjoint representation $M_i{}^j$ defined via (2.3.18) [or explicilty via the coordinates x^i (2.3.21)]. The adjoint representation satisfies $\Delta(M_i{}^j) = M_i{}^k \otimes M_k{}^j$ and $\varepsilon(M_i{}^j) = \delta_j^i$.

iii) the space of left invariant 1-forms, defined as the space dual to that of the tangent vectors: $\langle \chi_i\ ,\ \omega^j\rangle = \delta_i^j$. A generic 1-form is then given by $\rho = a_i\omega^i$. [The space of 1-forms is the bicovariant bimodule freely generated by the ω^i with $\omega^i a = (f^i{}_j * a)\omega^j$, $\Delta_L\omega^i \equiv I \otimes \omega^i$, $\Delta_R\omega^i \equiv \omega^j \otimes M_j{}^i$].

iv) The differential, defined by $da = (\chi_i * a)\omega^i$; it satisfies the undeformed Leibniz rule.

Note 2.3..3 Following [46] we here briefly characterize in a cohomological context the bicovariant differential calculus on quantum groups. For any $\omega \in {}_{\rm inv}\Gamma$, $a \in A$, we define the left and right products

$$a.\omega = \varepsilon(a)\omega \qquad ; \qquad \omega.a = \kappa(a_1)\omega a_2 \ .$$

In particular $\omega^i.a = \kappa(a_1)\omega^i a_2 = \kappa(a_1)(f^i{}_j * a_2)\omega^j = f^i{}_j(a)\omega^j$, this shows that the right product is well defined and, using the property (2.1.35) and (2.1.36) of the f functionals, that ${}_{\mathrm{inv}}\Gamma$ is a left and right A-module. The projection $P\ :\ \Gamma \to {}_{\mathrm{inv}}\Gamma$ defined in Note 2.1.1 is an epimorphism between the two bimodules Γ and ${}_{\mathrm{inv}}\Gamma$.

We now characterize the differential d through a 1-cocycle of the Hochschild coboundary operator δ relative to the A-bimodule ${}_{\mathrm{inv}}\Gamma$. Given an algebra $\mathcal{A}$ and a bimodule M over $\mathcal{A}$, a Hochschild k-cochain $C \in C^k(\mathcal{A}, M)$ is a k-multilinear map from $\mathcal{A}\otimes\mathcal{A}\otimes ...\mathcal{A}$ (k-times) to M, with $C^0(\mathcal{A}, M) = M$. The coboundary operator $\delta\ :\ C^k(\mathcal{A}, M) \to C^{k+1}(\mathcal{A}, M)$ is defined by

$$\begin{aligned}\delta C(a_1, \dots a_{k+1}) = {} & a_1.C(a_2, \dots a_{k+1}) \\ & + \textstyle\sum_{i=1}^{k}(-1)^i C(a_1, \dots a_i a_{i+1}, \dots a_{k+1}) + (-1)^{k+1}C(a_1, \dots a_k).a_{k+1}\end{aligned}$$

and satisfies $\delta^2 = 0$. [We have denoted by "." the multiplication in the bimodule M]

To a bicovariant differential calculus with differential d, we associate the map

$$\begin{aligned} c\ :\quad & A \to {}_{\mathrm{inv}}\Gamma \\ & a \mapsto P(da) = \kappa(a_1)da_2 \ . \end{aligned}$$

It is easy to see that $\delta c = 0$, i.e., c is a 1-cocycle : $c(ab) = c(a)b + ac(b)$. Viceversa, given a 1-cocycle c we immediately obtain a left covariant differential calculus defining

$$da \equiv a_1 c(a_2) \ .$$

[Proof of the left covariance: $\Delta_\Gamma(da) = a_1 \otimes a_2 c(a_3) = (id \otimes d)\Delta a$].

The right covariance (2.1.4) of a differential calculus is equivalent to the following property for the cocycle c:

$$\forall \psi \in U \qquad (id \otimes \psi_2)_\Gamma\Delta[c(a * \psi_1)] = c(\psi * a) \tag{2.3.55}$$

[Hint: (2.1.4) is equivalent to $(\kappa(a_1) \otimes id)_\Gamma\Delta(da_2) = \kappa(a_1)da_2 \otimes a_3$; apply $(id \otimes \psi)$ to this last expression]. Therefore 1-cocycles satisfying (2.3.55) are in one-to-one correspondence with bicovariant differential calculi. In the notations of Subsection 2.4.5 (2.3.55) reads $\ell_{\psi_2} c(a * \psi_1) = c(\psi * a)$. In [46] it is shown that there is a one-to-one correspondence between bicovariant A-bimodules on Γ and $\mathcal{D}$-bimodules on ${}_{\mathrm{inv}}\Gamma$ where $\mathcal{D}$ is the quantum double of A; moreover the cocycles satisfying (2.3.55) correspond to cocycles c in the set of the Hochschild cochains $C^1(\mathcal{D}, {}_{\mathrm{inv}}\Gamma)$ that have the simple property $c(U) = 0$.

Notice also that the 0-cochains are the left invariant one forms: $C^0(A, {}_{\mathrm{inv}}\Gamma) = {}_{\mathrm{inv}}\Gamma$, it can be checked that the coboundary of any bi-invariant 1-form, i.e. of any right invariant 0-cochain satisfies (2.3.55) and therefore defines a bicovariant differential calculus. The differential studied in (2.2.44) and (2.2.45) corresponds to the 1-cocycle $\delta(\frac{-1}{\lambda}\tau)$. Indeed $\delta(\frac{-1}{\lambda}\tau)(a) = \frac{-1}{\lambda}(a.\tau - \tau.a) = \frac{-1}{\lambda}\varepsilon(a)\tau + \frac{1}{\lambda}\kappa(a_1)\tau a_2$ and $da = a_1\,\delta(\frac{-1}{\lambda}\tau)(a_2) = \frac{1}{\lambda}(\tau a - a\tau)$ as in (2.2.45).

The differential calculus on classical Lie groups corresponds to a nontrivial 1-cocycle since in the commutative case $\delta\omega = 0$ for any $\omega \in {}_{\mathrm{inv}}\Gamma$. All the differential calculi we will examine correspond to 1-cocycles that are coboundaries, the only exception being those on the twisted homogeneous and inhomogeneous orthogonal groups of sections 4.6 and 4.7.

The existence of a bi-invariant 1-form trivializes the calculus from the Hochschild cohomology viewpoint (it is associated to a coboundary) but it is interesting geometrically and for physical speculations because introduces a discretized geometry; indeed d given by (2.2.45) is a finite difference operator as well as the partial derivatives χ_i in (2.2.52).

2.4 More q-Geometry: vectorfields, inner derivative and Lie derivative

In the previous section we have studied the space of left invariant vectorfields i.e. the q-tangent space, we now construct, for a generic quantum group, the space of vectorfields. Its elements are products of elements of the quantum group itself with left invariant vectorfields. We study the duality between vectorfields and 1-forms and generalize the construction to tensorfields. As in the classical case, using the duality between covariant and contravariant tensorfields, we can introduce the contraction operator. This is defined on the space of covariant tensorfields, and therefore acts in particular on forms; indeed the algebra of forms, as defined in (2.1.73), is a subalgebra of the algebra of covariant tensorfields. We then prove that the contraction operator is a (inner) derivation in the space of forms. On the other hand the right action of the q-group on the space of 1-forms naturally define the Lie derivative along left invariant vectorfields. The Cartan identity $\ell_{t_i} = i_{t_i}d + di_{t_i}$ is proven and the Cartan calculus of inner derivatives, Lie derivatives and the exterior derivative generalized to q-group geometry. Not all properties of the classical Cartan Calculus can however be generalized, while the contraction operator is defined for general vectorfields, there is no completely satisfactory expression for the Lie derivative along general vectorfields V. We propose the definition $\ell_V \equiv i_V d + di_V$ and analize and discuss its properties. The topics discussed in this sections have been studied in [26], [27] and more extensively in [38], [34], [39], [40], [37], [41]. We follow [37] and [27]. Here we give a self-contained exposition, and all the theorems are proved starting from only one data: a bicovariant differential calculus on a generic Hopf algebra.

2.4.1 From Left invariant Vectorfields to general Vectorfields

In this subsection we study the space Ξ of vectorfields over the generic Hopf algebra A defining a right product between left invariant vectorfields and elements of A.

In the commutative case a generic vectorfield can be written in the form $f^i t_i$ where $\{t_i\}$ $i = 1, \ldots, n$ is a basis of left invariant vectorfields and f^i are n smooth functions on the group manifold. In the commutative case $f^i t_i = t_i \square f^i$ i.e. left and right products (that we have denoted with $\square$) are the same, indeed $(t_i \square f^i)(h) \equiv t_i(h) f^i = f^i t_i(h)$. These considerations lead to the following definition.

Let $t_i = \chi_i *$ be a basis in ${}_{\rm inv}\Xi$, the space of left invariant vectorfields, and let a^i, $i = 1, \ldots, n$ be generic elements of A:

Definition

$$\Xi \equiv \{V \ / \ V : A \longrightarrow A \ ; \ V = t_i \square a^i\} \ , \qquad (2.4.1)$$

where the definition of the right product $\square$ is given below:

Definition

$$\forall a, b \in A, \forall t \in {}_{inv}\Xi \qquad (t \square a) b \equiv t(b) a = (\chi * b) a \ . \qquad (2.4.2)$$

The product $\square$ has a natural generalization to the whole Ξ :

$$\square \ : \quad \begin{array}{ccc} \Xi \times A & \longrightarrow & \Xi \\ (V, a) & \longmapsto & V \square a \end{array} \quad \text{where} \quad \forall b \in A \quad (V \square a)(b) \equiv V(b) a \ . \qquad (2.4.3)$$

It is easy to prove that $(\Xi, \square)$ is a right A-module:

$$V \square (a + b) = V \square a + V \square b \ ; \quad V \square (ab) = (V \square a) \square b \ ; \quad V \square (a + b) = V \square a + V \square b \quad (2.4.4)$$

(we have also $V \square \lambda a = \lambda V \square a$ with $\lambda \in \boldsymbol{C}$).
For example $V \square (ab) = (V \square a) \square b$ because

$$\forall c \in A \quad [(V \square a) \square b] c = [(V \square a)(c)] b = (V(c) a) b = V(c) ab = [V \square ab] c.$$

Notice that to distinguish the elements $V \square (ab) \in \Xi$ and $V(ab) \in A$ we have not omitted the simbol $\square$ representing the right product.

Ξ is the analogue of the space of derivations on the ring $C^\infty(G)$ of the smooth functions on the group G. Indeed we have:

$$V(a + b) = V(a) + V(b) \ , \qquad V(\lambda a) = \lambda V(a) \quad \text{Linearity} \qquad (2.4.5)$$

$$V(ab) \equiv (t_i \square c^i)(ab) = t_j(a)(f^j{}_i * b) c^i + a V(b) \qquad \text{Leibniz rule} \qquad (2.4.6)$$

in the classical case $t_j(a)(f^j{}_i * b) c^i = V(a) b$ (recall $f^j{}_i = \delta^j_i \varepsilon$; $\varepsilon * b = b$).

We have seen the duality between ${}_{\mathrm{inv}}\Gamma$ and ${}_{\mathrm{inv}}\Xi$. We now extend it to Γ and Ξ, where Γ is seen as a left A-module (not necessarily a bimodule) and Ξ is our right A-module.

Theorem 2.4.1 There exists a unique map

$$\langle \ , \ \rangle : \quad \Gamma \times \Xi \longrightarrow A$$

such that:

1) $\forall\, V \in \Xi$; the application

$$\langle \ , V\rangle : \quad \Gamma \longrightarrow A$$

is a left A-module morphism, i.e. is linear and $\langle a\rho, V\rangle = a\langle\rho, V\rangle$.
2) $\forall \rho \in \Gamma$; the application

$$\langle \rho, \ \rangle : \quad \Xi \longrightarrow A$$

is a right A-module morphism, i.e. is linear and $\langle \rho, Vb\rangle = \langle\rho, V\rangle b$.
3) Given $\rho \in \Gamma$

$$\langle \rho, \ \rangle = 0 \ \Rightarrow \ \rho = 0 \ , \tag{2.4.7}$$

where $\langle \rho, \ \rangle = 0$ means $\langle \rho, V\rangle = 0 \ \ \forall V \in \Xi$.
4) Given $V \in \Xi$

$$\langle \ , V\rangle = 0 \ \Rightarrow \ V = 0 \ , \tag{2.4.8}$$

where $\langle \ , V\rangle = 0$ means $\langle \rho, V\rangle = 0 \ \ \forall \rho \in \Gamma$.
5) On ${}_{\mathrm{inv}}\Gamma \times {}_{\mathrm{inv}}\Xi$ the bracket $\langle \ , \ \rangle$ acts as the one introduced in the previous section.

Remark Properties 3) and 4) state that Γ and Ξ are dual A-moduli, in the sense that they are dual with respect to A.

Proof
Properties 1), 2) and 5) uniquely characterize this map . To prove the existence of such a map we show that the following bracket

Definition

$$\langle \rho, V\rangle = \langle a_\alpha db_\beta \, , \, V\rangle \equiv a_\alpha V(b_\alpha) \ , \tag{2.4.9}$$

where a_α, b_α are elements of A such that $\rho = a_\alpha db_\alpha$, satisfies 1),2) and 5).

We first verify that the above definition is well given, that is:

$$\text{Let } \rho = a_\alpha db_\alpha = a'_\beta db'_\beta \quad \text{then} \quad a_\alpha V(b_\alpha) = a'_\beta V(b'_\beta) \ .$$

Indeed, since

$$a_\alpha db_\alpha = a'_\beta db'_\beta \ \Leftrightarrow \ a_\alpha t_i(b_\alpha)\omega^i = a'_\beta t_i(b'_\beta)\omega^i \ \Leftrightarrow \ a_\alpha t_i(b_\alpha) = a'_\beta t_i(b'_\beta)$$

[we used the uniqueness of the decomposition (2.1.28)] the definition is consistent because

$$a_\alpha V(b_\alpha) = a'_\beta V(b'_\beta) \;\Leftrightarrow\; a_\alpha t_i(b_\alpha)c^i = a'_\beta t_i(b'_\beta)c^i$$

where $V = t_i \square c^i$.

Property 1) is trivial since $a\rho = a(a_\alpha db_\alpha) = (aa_\alpha)db_\alpha$.
Property 2) holds since

$$\langle \rho, V\square c\rangle = a_\alpha (V\square c)(b_\alpha) = a_\alpha V(b_\alpha)c = \langle \rho, V\rangle c\ .$$

Property 5). Let $\{\omega^i\}$ and $\{t_i\}$ be dual bases in ${}_{\rm inv}\Gamma$ and ${}_{\rm inv}\Xi$. Since $\omega^i \in \Gamma$, $\omega^i = a_\alpha db_\alpha$ for some a_α and b_α in A. We can also write $\omega^i = a_\alpha db_\alpha = a_\alpha t_k(b_\alpha)\omega^k$, so that, due to the uniqueness of the decomposition (2.1.28), we have

$$a_\alpha t_k(b_\alpha) = \delta^i_k I \qquad\qquad (I \text{ unit of} A);$$

we then obtain

$$\langle \omega^i, t_j\rangle = a_\alpha t_j(b_\alpha) = \delta^i{}_j I\ .$$

Property 3). Let $\rho = a_i\omega^i \in \Gamma$.
If $\langle \rho, V\rangle = 0 \;\; \forall V \in \Xi$, in particular $\langle \rho, t_j\rangle = 0 \;\; \forall j = 1, ..., n$; then $a_i\langle \omega^i, t_j\rangle = 0 \Leftrightarrow a_j = 0$, and therefore $\rho = 0$.

Property 4). Let $V = t_i\square a^i \in \Xi$.
If $\langle \rho, V\rangle = 0 \;\; \forall \rho \in \Gamma$, in particular $\langle \omega^j, V\rangle = 0 \;\; \forall j = 1, ..., n$; then $\langle \omega^j, t_i\rangle a^i = 0 \Leftrightarrow a^j = 0$, and therefore $V = 0$. □□□

By construction every V is of the form

$$V = t_i\square a^i.$$

We can now show the unicity of such a decomposition.

Theorem 2.4.2 Any $V \in \Xi$ can be uniquely written in the form

$$V = t_i\square a^i$$

Proof
Let $V = t_i\square a^i = t_i\square a'^i$ then

$$\forall i = 1, \ldots, n \qquad a^i = \langle \omega^i, t_j\rangle a^j = \langle \omega^i, V\rangle = \langle \omega^i, t_j\rangle a'^j = a'^i\ .$$

□□□

Notice that once we know the decomposition of ρ and V in terms of ω^i and t_i, the evaluation of $\langle\ ,\ \rangle$ is trivial:

$$\langle \rho, V\rangle = \langle a_i\omega^i, t_j\square b^j\rangle = a_i\langle \omega^i, t_j\rangle b^j = a_i b^i\ .$$

Viceversa from the previous theorem $V = t_i \,\text{\tiny$\square$}\, \langle\omega^i, V\rangle$ and $\rho = \langle\rho, t_i\rangle\omega^i$.

We conclude this section by remarking the three different ways of looking at Ξ.
(I) Ξ as the set of all deformed derivations over A [see (2.4.1), (2.4.5) and (2.4.6)].
(II) Ξ as the right A-module freely generated by the *elements* t_i , $i = 1, \ldots, n$. The latter is the set of all the *formal* products and sums of the type $t_i a^i$, where a^i are generic elements of A. Indeed, by virtue of Theorem 2.4.2, the map that associates to each $V = t_i \,\text{\tiny$\square$}\, a^i$ in Ξ the corresponding element $t_i a^i$ is an isomorphism between right A-moduli.
(III) Ξ as $\Xi' = \{U \;:\; \Gamma \longrightarrow A,\ U$ linear and $U(a\rho) = aU(\rho)\ \forall a \in A\}$, i.e. Ξ as the dual (with respect to A) of the space of 1-forms Γ. The space Ξ' has a trivial right A-module structure: $(Ua)(\rho) \equiv U(\rho)a$. Ξ and Ξ' are isomorphic right A-moduli because of property (2.4.8) which states that to each $\langle\ \ , V\rangle \;:\; \Gamma \rightarrow A$ there corresponds one and only one $V.[\langle\ \ , V\rangle = \langle\ \ , V'\rangle \Rightarrow V = V']$. Every $U \in \Xi'$ is of the form $U = \langle\ \ , V\rangle$; more precisely, if a^i is such that $U(\omega^i) = a^i$ then $U = \langle\ \ , t_i \,\text{\tiny$\square$}\, a^i\rangle$.

These three ways of looking at Ξ will correspond to different aspects of the Cartan Calculus: the Lie derivatives ℓ_V will generalize (I), inner derivations i_V will correspond to (III), while the transformation properties of ℓ_V and i_V are governed by (II).

2.4.2 Bicovariant Bimodule Structure

In Section 2.1 we have studied the space Γ of 1-forms, we have seen that Γ is a bimodule over A because there is a right and a left product between elements of Γ and of A. The left and the right product are related by $\omega^i a = (f^i{}_j * a)\omega^j$. Since the coactions Δ_Γ and ${}_\Gamma\Delta$ are compatible with the bimodule structure and since they commute:

$$(id \otimes {}_\Gamma\Delta)\Delta_\Gamma = (\Delta_\Gamma \otimes id)_\Gamma\Delta$$

the bimodule Γ is a bicovariant bimodule (cf. Note 2.1.1).

In the previous subsection we have studied the right product $\text{\tiny$\square$}$ and we have seen that Ξ is a right module over A [see (2.4.4)]. Here we introduce a left product and a left and right coaction of the Hopf algebra A on Ξ. The left and right coactions Δ_Ξ and ${}_\Xi\Delta$ are the q-analogue of the push-forward of tensorfields on a group manifold. Similarly to Γ also Ξ is a bicovariant bimodule.

The construction of the left product on Ξ, of the right coaction ${}_\Xi\Delta$ and of the left coaction Δ_Ξ will be effected along the lines of Woronowicz' Theorem 2.5 in [21], whose statement can be explained in the following steps (cf. Note 2.1.1):

Theorem 2.4.3 Consider the *symbols* t_i $(i = 1, \ldots, n)$ and let Ξ be the right A-module freely generated by them:

$$\Xi \equiv \{t_i a^i \ / \ a^i \in A\}$$

Consider functionals $O_i{}^j : A \longrightarrow \boldsymbol{C}$ satisfying [see (2.1.32) and (2.1.33)]

$$O_i{}^j(ab) = O_i{}^k(a) O_k{}^j(b) \tag{2.4.10}$$
$$O_i{}^j(I) = \delta^i_j \quad . \tag{2.4.11}$$

Introduce a left product via the definition [see (2.1.31)]

Definition

$$b(t_i a^i) \equiv t_j[(O_i{}^j \circ \kappa^{-1}) * b] a^i \ . \tag{2.4.12}$$

It is easy to prove that
i) Ξ is a bimodule over A. (A proof of this first statement as well as of the following ones is contained in [21]).

□

Introduce an action (push-forward) of the Hopf algebra A on Ξ

Definition

$$\Delta_\Xi(t_i a^i) \equiv (I \otimes t_i)\Delta(a^i) \ . \tag{2.4.13}$$

It follows that
ii) (Ξ, Δ_Ξ) is a left covariant bimodule over A, that is

$$\Delta_\Xi(aVb) = \Delta(a)\Delta_\Xi(V)\Delta(b) \ ; \quad (\varepsilon \otimes id)\Delta_\Xi(V) = V \ ; \quad (\Delta \otimes id)\Delta_\Xi = (id \otimes \Delta_\Xi)\Delta_\Xi \ .$$

□

Introduce n^2 elements $N^i{}_j \in A$ satisfying [see (2.1.51),(2.1.44) and (2.1.45)]

$$N^i{}_j(a * O_i{}^k) = (O_j{}^i * a) N^k{}_i \tag{2.4.14}$$
$$\Delta(N^j{}_i) = N^j{}_l \otimes N^l{}_i \tag{2.4.15}$$
$$\varepsilon(N^j{}_i) = \delta^i_j \quad , \tag{2.4.16}$$

and introduce ${}_\Xi\Delta$ such that [see (2.1.38)]

Definition

$${}_\Xi\Delta(a^i t_i) \equiv \Delta(a^i) t_j \otimes N^j{}_i \ . \tag{2.4.17}$$

Then it can be proven that
iii) The elements [see (2.1.47)]

$$h_i \equiv t_j \kappa(N^j{}_i) \tag{2.4.18}$$

are right invariant: ${}_{\Xi}\Delta(h_i) = h_i \otimes I$. Moreover any $V \in \Xi$ can be expressed in a unique way respectively as $V = h_i a^i$ and as $V = b^i h_i$, where $a^i, b^i \in A$.

□

iv) $(\Xi, {}_{\Xi}\Delta)$ is a right covariant bimodule over A, that is

$${}_{\Xi}\Delta(aVb) = \Delta(a){}_{\Xi}\Delta(V)\Delta(b)\ ; \quad (id\otimes\varepsilon){}_{\Xi}\Delta(V) = V\ ; \quad (id\otimes\Delta){}_{\Xi}\Delta = ({}_{\Xi}\Delta\otimes id){}_{\Xi}\Delta\ .$$

□

v) The left and right covariant bimodule $(\Xi, \Delta_{\Xi}, {}_{\Xi}\Delta)$ is a bicovariant bimodule, that is left and right coactions are compatible:

$$(id \otimes {}_{\Xi}\Delta)\Delta_{\Xi} = (\Delta_{\Xi} \otimes id){}_{\Xi}\Delta\ .$$

□□□

In the previous section we have seen [remark (II)] that the space of vectorfields Ξ is the free right A-module generated by the symbols t_i, so that the above theorem applies to our case.

There are many bimodule structures (i.e. choices of $O_i{}^j$) Ξ can be endowed with. Using the fact that Ξ is dual to Γ we request compatibility with the Γ bimodule. In the commutative case $\langle f\omega^i, t_j\rangle = \langle \omega^i f, t_j\rangle = \langle \omega^i, ft_j\rangle = \langle \omega^i, t_j f\rangle$.
In the quantum case we know that $\langle a\omega^i, t_j\rangle = \langle \omega^i, t_j \square a\rangle$ and we require

$$\langle \omega^i a, t_j\rangle = \langle \omega^i, at_j\rangle\ ; \tag{2.4.19}$$

this condition uniquely determines the bimodule structure of Ξ. Indeed we have

$$\begin{aligned}\langle \omega^i, at_j\rangle &= \langle \omega^i a, t_j\rangle = \langle (f^i{}_k * a)\omega^k, t_j\rangle = (f^i{}_k * a)\langle \omega^k, t_j\rangle = f^i{}_k * a\,\delta^k_j = \delta^i_l f^l{}_j * a \\ &= \langle \omega^i, t_l \square (f^l{}_j * a)\rangle \end{aligned} \tag{2.4.20}$$

so that

$$at_i = t_j \square (f^j{}_i * a\ ). \tag{2.4.21}$$

We then define

$$O_i{}^j \equiv f^j{}_i \circ \kappa \tag{2.4.22}$$

it follows that $O_i{}^j \circ \kappa^{-1} = f^j{}_i$ and (2.4.21) can be rewritten [see (2.1.31) and (2.4.12)]

$$at_j = t_j \square [(O_i{}^j \circ \kappa^{-1}) * a]\ . \tag{2.4.23}$$

Theorem 2.4.4 The functionals $O_i{}^j$ satisfy conditions (2.4.10) and (2.4.11).

Proof The first condition $O_i{}^j(I) = \delta^j_i$ holds trivially.
The second one is also easily checked:

$$\begin{aligned}O_i{}^j(ab) &= (f^j{}_i \circ \kappa)(ab) = f^j{}_i[\kappa(b)\kappa(a)] = f^j{}_k[\kappa(b)]f^k{}_i[\kappa(a)] = f^k{}_i[\kappa(a)]f^j{}_k[(\kappa(b)] \\ &= O_i{}^k(a)O_k{}^j(b)\end{aligned}$$

□□□

So far Ξ has a bimodule structure. We now define a left coaction Δ_Ξ so that Ξ becomes a left covariant bimodule. The left invariant vectorfields were characterized in (2.3.9) and (2.3.1) through their action $t_i(a) = \chi_i * a$ on functions; following the same derivation as in (2.1.20)–(2.1.27) the left invariant property of the t_i can also be expressed via the coaction Δ_Ξ as defined in (2.4.13):

$$\Delta_\Xi(t_i a^i) \equiv (I \otimes t_i)\Delta(a^i) . \tag{2.4.24}$$

Similarly the right coaction ${}_\Gamma\Delta$ is defined to act trivially on the right invariant vectorfields $h_i = *\chi_i$. This uniquely defines the elements $N^l{}_k \in A$; indeed we want relation (2.4.18) and (2.3.23) to coincide and therefore:

$$N^l{}_k = \kappa^{-1}(M_k{}^l) . \tag{2.4.25}$$

Notice that (2.4.25) implies

$$\langle t_i, \omega^j \rangle = \delta_i^j = \langle h_i, \eta^j \rangle, \tag{2.4.26}$$

where t_i and ω^j are left-invariant and h_i and η^j are the canonically associated right-invariant objects; see (2.4.18) and (2.1.47). Notice also that $M_k{}^l$ and $f^i{}_j$ are dual, and likewise $N^l{}_k$ and $O_j{}^i$, in the sense that $f^i{}_j(M_k{}^l) = O_j{}^i(N^l{}_k) = \Lambda^{il}{}_{kj}$ with $\Lambda^{il}{}_{kj} = \delta^i_j \delta^l_k$ when $q = 1$.

Theorem 2.4.5 The $N^l{}_k$ elements defined above satisfy relations (2.4.16), (2.4.15) and (2.4.14):

$$1)\ \varepsilon(N^j{}_i) = \delta^i_j \qquad 2)\ \Delta(N^j{}_i) = N^j{}_l \otimes N^l{}_i \qquad 3)\ N^i{}_k(a * O_i{}^j) = (O_k{}^i * a)N^j{}_i$$

Proof

1) This expression is trivial.

2) Use $N^i{}_j = \kappa^{-1} M_j{}^i$ and $\Delta \circ \kappa^{-1} = \sigma \circ (\kappa^{-1} \otimes \kappa^{-1}) \circ \Delta$, where σ_A is the flip map in $A \otimes A$.

3) We know that [see (2.1.51)]

$$\forall a \in A \qquad M_i{}^j(a * f^i{}_k) = (f^j{}_i * a)M_k{}^i$$

or equivalently,

$$M_i{}^j[\kappa(a) * f^i{}_k] = [f^j{}_i * \kappa(a)]M_k{}^i .$$

Now

$$\begin{aligned}[\kappa(a) * f^i{}_k] &= (f^i{}_k \otimes id)\Delta[\kappa(a)] = (id \otimes f^i{}_k)(\kappa \otimes \kappa)\Delta(a) = \kappa(id \otimes f^i{}_k \circ \kappa)\Delta(a) \\ &= \kappa(O_k{}^i * a) .\end{aligned}$$

Similarly,

$$[f^j{}_i * \kappa(a)] = \kappa(a * O_i{}^j) \ .$$

So we can write

$$\kappa(a * O_i{}^j)M_k{}^i = M_i{}^j \kappa(O_k{}^i * a)$$

for all $a \in A$. Applying κ^{-1} to both members of this last expression we obtain relation 3).

□□□

Following Theorem 2.4.3 the construction of the bicovariant bimodule Ξ is now easy and straightforward, and we can conclude that $(\Xi, \Delta_\Xi, {}_\Xi\Delta)$ is a bicovariant bimodule.

We end this subsection observing that in the expression (2.3.30) for the exterior differential, elements of Ξ and Γ make a joint appearance. To be still able to talk about transformation properties of such expressions we need to combine the previously introduced coactions into one object, Δ_A, simply by putting $\Delta_A \equiv {}_\Xi\Delta$ on Ξ and $\Delta_A \equiv {}_\Gamma\Delta$ on Γ and requiring Δ_A to be an algebra homomorphism. From this definition we get the following important corollary:

Corollary. The expression $\omega^i t_i$ in (2.3.2) is invariant in the sense that

$$\Delta_A(t_i\,\omega^i) = {}_\Xi\Delta(t_i){}_\Gamma\Delta(\omega^i) = t_k\omega^j \otimes N^k{}_i M_j{}^i = t_i\omega^i \otimes 1.$$

Similar statements apply to ${}_A\Delta$.

Notice that, since Theorem 2.4.3 completely characterizes a bicovariant bimodule all the formulas containing the symbols $f^i{}_j$ or $M_k{}^l$ or elements of Γ are still valid under the substitutions $f^i{}_j \to O_i{}^j$, $\ M_k{}^l \to N^k{}_l$ and $\Gamma \to \Xi$.

2.4.3 Tensorfields

The construction completed for vectorfields is readily generalized to p-times contravariant tensorfields. We proceed as in (2.1.57)–(2.1.62) and define $\Xi \otimes \Xi$ to be the space of all elements that can be written as finite sums of the kind $\sum_i V_i \otimes V_i'$ with $V_i, V_i' \in \Xi$. The tensor product (in the algebra A) between V_i and V_i' has the following properties:

$V \square a \otimes V' = V \otimes aV'$, $a(V \otimes V') = (aV) \otimes V'$ and $(V \otimes V')\square a = V \otimes (V' \square a)$ so that $\Xi \otimes \Xi$ is naturally a bimodule over A.

Left and right coactions on $\Xi \otimes \Xi$ are defined by:

$$\Delta_\Xi(V \otimes V') \equiv V_1 V'_1 \otimes V_2 \otimes V'_2, \quad \Delta_\Xi : \Xi \otimes \Xi \to A \otimes \Xi \otimes \Xi \qquad (2.4.27)$$

$${}_\Xi\Delta(V \otimes V') \equiv V_1 \otimes V'_1 \otimes V_2 V'_2, \quad {}_\Xi\Delta : \Xi \otimes \Xi \to \Xi \otimes \Xi \otimes A \qquad (2.4.28)$$

where as usual V_1, V_2 etc. are defined by

$$\Delta_\Xi(V) = V_1 \otimes V_2, \quad V_1 \in A,\ V_2 \in \Xi \tag{2.4.29}$$

$$_\Xi\Delta(V) = V_1 \otimes V_2, \quad V_1 \in \Xi,\ V_2 \in A\ . \tag{2.4.30}$$

More generally, we can introduce the coaction of Δ_Ξ on $\Xi^{\otimes p} \equiv \underbrace{\Xi \otimes \Xi \otimes \cdots \otimes \Xi}_{p\text{-times}}$ as

$$\Delta_\Xi(V \otimes V' \otimes \cdots \otimes V'') \equiv V_1 V'_1 \cdots V''_1 \otimes V_2 \otimes V'_2 \otimes \cdots \otimes V''_2$$

$$\Delta_\Xi \ :\ \Xi^{\otimes p} \longrightarrow A \otimes \Xi^{\otimes p}\ ; \tag{2.4.31}$$

$$_\Xi\Delta(V \otimes V' \otimes \cdots \otimes V'') \equiv V_1 \otimes V'_1 \otimes \cdots \otimes V''_1 \otimes V_2 V'_2 \cdots V''_2$$

$$_\Xi\Delta \ :\ \Xi^{\otimes p} \longrightarrow \Xi^{\otimes p} \otimes A\ \ . \tag{2.4.32}$$

Left invariance on $\Xi \otimes \Xi$ is naturally defined as $\Delta_\Xi(V \otimes V') = I \otimes V \otimes V'$ (similar definition for right invariance), so that for example $t_i \otimes t_j$ is left invariant, and is in fact a left invariant basis for $\Xi \otimes \Xi$: each element can be written as $t_i \otimes t_j \Box a^{ij}$ in a unique way.

It is not difficult to show that $\Xi \otimes \Xi$ is a bicovariant bimodule. In the same way also $(\Xi^{\otimes p}, \Delta_\Xi, {}_\Xi\Delta)$ is a bicovariant bimodule.

Any element $v \in \Xi^{\otimes p}$ can be written as $v = t_{i_1} \otimes \ldots t_{1_p} \Box b^{i_1 \cdots i_p}$ in a unique way, similarly any element $\tau \in \Gamma^{\otimes n}$, the n-times tensor product of 1-forms, can be written as $\tau = a_{i_n \ldots i_1} \omega^{i_n} \otimes \ldots \omega^{i_1}$ in a unique way.

It is now possible to generalize the previous bracket $\langle\ ,\ \rangle : \Gamma \times \Xi \to A$ to $\Gamma^{\otimes n}$ and $\Xi^{\otimes p}$:

$$\begin{array}{llll} \langle\ ,\ \rangle : & \Gamma^{\otimes n} \times \Xi^{\otimes p} & \longrightarrow & A \\ & (\tau, v) & \longmapsto & \langle \tau, v \rangle = a_{i_n \ldots i_1} \langle \omega^{i_n} \otimes \ldots \omega^{i_1}\ ,\ t_{j_1} \otimes \ldots t_{j_p} \rangle b^{j_1 \cdots j_p} \\ & & & = a_{i_n \ldots i_1} \omega^{i_n} \otimes \ldots \omega^{i_{p+1}} b^{i_1 \cdots i_p} \end{array} \tag{2.4.33}$$

where $\Gamma^{\otimes 0} \equiv A$, $\Gamma^{\otimes 1} \equiv \Gamma$ and we have defined

$$\begin{aligned} \langle \omega^{i_n} \otimes \ldots \omega^{i_1}\ ,\ t_{j_1} \otimes \ldots t_{j_p} \rangle &\equiv \omega^{i_n} \otimes \ldots \omega^{i_{p+1}} \langle \omega^{i_p} \otimes \ldots \omega^{i_1}\ ,\ t_{j_1} \otimes \ldots t_{j_p} \rangle \\ &\equiv \omega^{i_n} \otimes \ldots \omega^{i_{p+1}} \langle \omega^{i_1}, t_{j_1} \rangle \ldots \langle \omega^{i_p}, t_{j_p} \rangle \\ &= \delta^{i_1}_{j_1} \ldots \delta^{i_p}_{j_p} \omega^{i_n} \otimes \ldots \omega^{i_{p+1}}\ \ . \end{aligned} \tag{2.4.34}$$

Using definition (2.4.34) it is easy to prove that

$$\langle \tau a, v \rangle = \langle \tau, a v \rangle\ , \tag{2.4.35}$$

namely

$$\langle \omega^{i_n} \otimes \ldots \omega^{i_{p+1}} \otimes \omega^{i_p} \otimes \ldots \omega^{i_1} a \,, t_{j_1} \otimes \ldots t_{j_p} \rangle =$$
$$= \omega^{i_n} \otimes \ldots \omega^{i_{p+1}} ({f^{i_p}}_{k_p} * \ldots {f^{i_1}}_{k_1} * a) \langle \omega^{k_p} \otimes \ldots \omega^{k_1}, t_{j_1} \otimes \ldots t_{j_p} \rangle$$

$$\langle \omega^{i_n} \otimes \ldots \omega^{i_{p+1}} \otimes \omega^{i_p} \otimes \ldots \omega^{i_1} \,, a t_{j_1} \otimes \ldots t_{j_p} \rangle =$$
$$= \omega^{i_n} \otimes \ldots \omega^{i_{p+1}} \langle \omega^{i_p} \otimes \ldots \omega^{i_1}, t_{l_1} \otimes \ldots t_{l_p} \rangle ({f^{l_p}}_{j_p} * \ldots {f^{l_1}}_{j_1} * a)$$

and these last two expressions are equal if and *only if* (2.4.34) holds.

Therefore we have also shown that definition (2.4.34) is the only one compatible with property (2.4.35), i.e. property (2.4.35) uniquely determines the coupling between $\Xi^{\otimes}$ and $\Gamma^{\otimes}$.

It is easy to prove that the bracket $\langle \ , \ \rangle$ extends to $\Gamma^{\otimes p}$ and $\Xi^{\otimes p}$ the duality between Γ and Ξ.

More generally we can define $\Xi^{\otimes} \equiv A \oplus \Xi \oplus \Xi^{\otimes 2} \oplus \Xi^{\otimes 3} \ldots$ to be the algebra of contravariant tensorfields. The coactions Δ_{Ξ} and ${}_{\Xi}\Delta$ have a natural generalization to $\Xi^{\otimes}$ so that we can conclude that $(\Xi^{\otimes}, \Delta_{\Xi}, {}_{\Xi}\Delta)$ is a bicovariant graded algebra, the graded algebra of tensorfields over the ring "of functions on the group" A, with the left and right "push-forward" Δ_{Ξ} and ${}_{\Xi}\Delta$. Similarly $\Gamma^{\otimes}$ is the bicovariant graded algebra of covariant tensorfields on A.

2.4.4 Contraction operator

In this subsection we study the contraction operator i_V along a generic vectorfield $V \in \Xi$ and we prove that it acts as a (deformed) derivative operator on the space of 1-forms. The definition of the contraction operator i_V with $V \in \Xi$ is based on equation (2.4.33). For a generic vectorfield $V = b^j t_j$ we define:

Definition of right inner derivative

$$(\vartheta) \overleftarrow{i_V} \equiv \langle \vartheta, V \rangle \qquad \qquad \forall \vartheta \in \Gamma^{\otimes} \ .$$

this definition applies when ϑ is a generic covariant tensorfield and in particular when ϑ is a generic form.

Theorem 2.4.6 The contraction operator $\overleftarrow{i_V}$ satisfies the following properties: $a, a_{i_1 \ldots i_n} \in A$; $V = t_i \square b^i$; $\lambda \in \mathbf{C}$, property d) holds only if ϑ, ϑ' are forms;

a) $(\vartheta) \overleftarrow{i_V} = (\vartheta) \overleftarrow{i_{t_j \square b^j}} = (\vartheta) \overleftarrow{i_j} \ b^j$

b) $(a) \overleftarrow{i_V} = 0$

c) $(\omega^j) \overleftarrow{i_V} = b^j$

d) $(a_{i_1 \ldots i_n} \omega_1 \wedge \ldots \wedge \omega_n) \overleftarrow{i_V} = (a_{i_1 \ldots i_n} \omega_1 \wedge \ldots \wedge \omega_{s-1}) \wedge (\omega_s \wedge \ldots \wedge \omega_n) \overleftarrow{i_{t_i}}$
$+ (-1)^{n-s+1} (a_{i_1 \ldots i_n} \omega_1 \wedge \ldots \wedge \omega_{s-1}) \overleftarrow{i_{t_j}} \wedge {f^j}_i * (\omega_s \wedge \omega_{s+1} \ldots \wedge \omega_n) b^i$

e) $(a\vartheta + \vartheta')\overleftarrow{i_V} = a\,(\vartheta)\overleftarrow{i_V} + (\vartheta')\overleftarrow{i_V}$

f) $(\vartheta a)\overleftarrow{i_V} = (\vartheta)\overleftarrow{i_{t_j}}\,(f^j{}_i * a)\,b^i$

g) $\overleftarrow{i_{\lambda V}} = \lambda \overleftarrow{i_V}$

Proof

Properties a), b), c), f), g) are direct consequences of (2.4.33), e) follows from (2.4.35). To proof d) we have to use the definition of the wedge product (2.1.73)–(2.1.75): first we note that (in tensor product notation)

$$\begin{aligned}(\omega_1 \wedge \ldots \wedge \omega_n)\overleftarrow{i_V} &= W_{1\ldots n}(\omega_1 \otimes \ldots \otimes \omega_n)\overleftarrow{i_V} \\ &= W_{1\ldots n}\,\omega_1 \otimes \ldots \otimes \omega_{n-1}(\omega_n)\overleftarrow{i_V} \\ &= \mathcal{I}_{1\ldots n}\,\omega_1 \wedge \ldots \wedge \omega_{n-1}(\omega_n)\overleftarrow{i_V},\end{aligned}$$

where we have used (2.1.74) in the last step — in index notation:

$$(\omega^{i_1} \wedge \ldots \wedge \omega^{i_n})\overleftarrow{i_{t_i}} = \mathcal{I}^{i_1\ldots i_n}_{j_1\ldots j_{n-1}i}\,\omega^{j_1} \wedge \ldots \wedge \omega^{j_{n-1}}.$$

Next we can show

$$\begin{aligned}f^{i_{s-1}}{}_i * (\omega^{i_s} \wedge \ldots \wedge \omega^{i_n}) &= \omega^{k_{s-1}} \wedge \ldots \wedge \omega^{k_{n-1}} f^{i_{s-1}}{}_i (M_{k_{s-1}}{}^{i_s} \cdots M_{k_{n-1}}{}^{i_n}) \\ &= \Lambda^{i_{s-1}i_s}{}_{k_{s-1}l_s}\Lambda^{l_s i_{s+1}}{}_{k_s l_{s+1}} \ldots \Lambda^{l_n i_n}{}_{k_{n-1}i}\omega^{k_{s-1}} \wedge \omega^{k_s} \wedge \ldots \omega^{k_{n-1}}\end{aligned}$$

that in tensor product notation can be written:

$$(\omega_{s-1})\overleftarrow{i_{t_j}}\, f^j{}_i * (\omega_s \wedge \omega_{s+1} \wedge \ldots \omega_n) = \Lambda_{s-1,s}\Lambda_{s,s+1} \ldots \Lambda_{n-1,n}(\omega_{s-1} \wedge \omega_s \wedge \ldots \omega_{n-1})\,(\omega_n)\overleftarrow{i_{t_i}}$$

Finally we utilize the decomposition property (2.1.76) and associativity of the wedge product (in tensor product notation)

$$\begin{aligned}(\omega_1 \wedge \ldots \wedge \omega_n)\overleftarrow{i_V} &= \\ &= \mathcal{I}_{1\ldots n}\omega_1 \wedge \ldots \omega_{n-1}(\omega_n)\overleftarrow{i_V} \\ &= [\mathcal{I}_{s\ldots n} + (-1)^{n-s+1}\mathcal{I}_{1\ldots s-1}\Lambda_{s-1,s} \cdots \Lambda_{n-1,n}](\omega_1 \wedge \ldots \omega_{s-1}) \wedge (\omega_s \wedge \ldots \omega_{n-1})\,(\omega_n)\overleftarrow{i_V} \\ &= (\omega_1 \wedge \ldots \wedge \omega_{s-1}) \wedge (\omega_s \wedge \ldots \wedge \omega_n)\overleftarrow{i_V} \\ &\quad +(-1)^{n-s+1}\mathcal{I}_{1\ldots s-1}(\omega_1 \wedge \ldots \omega_{s-2}) \wedge \Lambda_{s-1,s}\Lambda_{s,s+1} \cdots \Lambda_{n-1,n}(\omega_{s-1} \wedge \ldots \omega_{n-1})(\omega_n)\overleftarrow{i_{t_i}}\, b^i \\ &= (\omega_1 \wedge \ldots \wedge \omega_{s-1}) \wedge (\omega_s \wedge \ldots \wedge \omega_n)\overleftarrow{i_{t_i}} \\ &\quad +(-1)^{n-s+1}\mathcal{I}_{1\ldots s-1}(\omega_1 \wedge \ldots \wedge \omega_{s-2})\,(\omega_{s-1})\overleftarrow{i_{t_j}} \wedge f^j{}_i * (\omega_s \wedge \omega_{s+1} \ldots \wedge \omega_n)b^i \\ &= (\omega_1 \wedge \ldots \wedge \omega_{s-1}) \wedge (\omega_s \wedge \ldots \wedge \omega_n)\overleftarrow{i_{t_i}} \\ &\quad +(-1)^{n-s+1}(\omega_1 \wedge \ldots \wedge \omega_{s-1})\overleftarrow{i_{t_j}} \wedge f^j{}_i * (\omega_s \wedge \omega_{s+1} \ldots \wedge \omega_n)b^i\end{aligned}$$

With property e) this proves d). □□□

Remark A slight generalization of property d) for two generic forms ϑ and ϑ' is also true [use f)]:

$$(\vartheta \wedge \vartheta')\overleftarrow{i_V} = \vartheta \wedge (\vartheta')\overleftarrow{i_V} + (-1)^{deg(\vartheta')}(\vartheta')\overleftarrow{i_{t_j}} \wedge (f^j{}_i * \vartheta')b^i\ . \qquad (2.4.36)$$

We have defined the exterior differential as an operator acting from the left to the right, indeed we have the following behaviour under grading, as opposed to the one in (2.4.36):

$$d(\vartheta \wedge \vartheta') = d\vartheta \wedge \vartheta' + (-1)^{deg(\vartheta)} \vartheta \wedge d\vartheta' \ . \tag{2.4.37}$$

In order to find the Cartan expression for the Lie derivative : $\ell_V = i_V d + d i_V$, we therefore have to introduce an inner derivation i_V that has the same behaviour as in (2.4.37). This motivates the following

Definition of inner derivative

$$i_V(\vartheta) \equiv (-1)^{deg(\vartheta-1)} (\vartheta) \overleftarrow{i_V} \qquad \forall \vartheta \in \Gamma^{\otimes} \ . \tag{2.4.38}$$

this definition applies when ϑ is a generic covariant tensorfield and in particular when ϑ generic form. We immediately have:

Theorem 2.4.7 The i_V contraction operator satisfies the following properties: $a, a_{i_1 \ldots i_n} \in A$; $V = t_i \square b^i$; $\lambda \in \mathbf{C}$, property d) holds only if ϑ, ϑ' are forms

a′) $i_V(\vartheta) = i_{t_j \square b^j}(\vartheta) = i_{t_j}(\vartheta) b^j$

b′) $i_V(a) = 0$

c′) $i_V(\omega^j) = b^j$

d′) $i_V(\vartheta \wedge \vartheta') = i_{t_j}(\vartheta) \wedge (f^j{}_i * \vartheta') b^i + (-1)^{deg(\vartheta)} \vartheta \wedge i_V(\vartheta')$

e′) $i_V(a\vartheta + \vartheta') = a i_V(\vartheta) + i_V(\vartheta')$

f′) $i_V(\vartheta a) = i_{t_j}(\vartheta)(f^j{}_i * a) b^i$

g′) $i_{\lambda V} = \lambda i_V$

□□□

Notice that properties a′) e′) and f′) reduce in the commutative case to the familiar formulae:

$$i_{fV}\vartheta = f\, i_V \vartheta \quad \text{and} \quad i_V(f\vartheta) = f i_V \vartheta \ .$$

It is also straightforward to see that

$$(id \otimes i_t) \Delta_\Gamma(\vartheta) = \Delta_\Gamma i_t(\vartheta) \qquad \forall \vartheta \in \Gamma^{\otimes} \ . \tag{2.4.39}$$

This formula q-generalizes the classical commutativity of i_t with the left coaction Δ_Γ, when t is a left invariant vectorfield.

2.4.5 Lie Derivative and Cartan identity

In (2.3.8), or in (2.1.88), we have seen that the χ_i are the quantum analogues of the tangent vectors at the origin of the group :

$$\chi_i \stackrel{q\to 1}{\longrightarrow} \frac{\partial}{\partial x^i}\,|_{x=0} \qquad (2.4.40)$$

and that the left-invariant vectorfields t_i constructed from the χ_i are :

$$t_i = \chi_i * = (id \otimes \chi_i)\Delta \qquad (2.4.41)$$

In the commutative case, the Lie derivative along a generic vectorfield V is given by:

$$\ell_V \tau = \lim_{\varepsilon \to 1} \frac{1}{\varepsilon}[\varphi_\varepsilon^{V*}(\tau) - \tau] \qquad \forall \tau \in \Gamma^{\otimes} \qquad (2.4.42)$$

where φ_ε^V is the flow of the vectorfield V and φ_ε^{V*} the pullback. If t is a left invariant vectorfield then

$$\varphi_\varepsilon^t = R_{e^{\varepsilon t}} \qquad \text{i.e.} \qquad \varphi_\varepsilon^t(g) = g e^{\varepsilon t} \quad \forall\, g \in G\ . \qquad (2.4.43)$$

We have $\ell_t \vartheta = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[R_{e^{\varepsilon t}}{}^*(\vartheta) - \vartheta]$ and therefore the Lie derivative ℓ_t is given by the right action $R_{e^{\varepsilon t}}{}^*$ of the group on covariant tensorfields. At the quantum level, recalling (2.1.16) and that ${}_\Gamma\Delta \to R^*$ when $q \to 1$, it is natural to define:

Definition The quantum Lie derivative along the left-invariant vectorfield $t = (id \otimes \chi)\Delta$ is the operator:

$$\ell_t \equiv (id \otimes \chi)_\Gamma\Delta \qquad (2.4.44)$$

that is

$$\forall \tau \in \Gamma^{\otimes} \qquad \ell_t(\tau) \equiv (id \otimes \chi)_\Gamma\Delta(\tau) \qquad \ell_t\ :\ \Gamma^{\otimes n} \longrightarrow \Gamma^{\otimes n}\ .$$

For example we have :

$$\ell_t(a) = t(a), \quad a \in A, \qquad (2.4.45)$$

$$\ell_{t_i}(\omega^j) = (id \otimes \chi_i)_\Gamma\Delta(\omega^j) = \omega^k \chi_i(M_k{}^j) = \mathbf{C}_{ki}{}^j \omega^k, \qquad (2.4.46)$$

the classical limit being evident.

It is useful to define the $*$ product of a functional with any $\tau \in \Gamma^{\otimes n}$ as

$$\chi * \tau \equiv (id \otimes \chi)_\Gamma\Delta(\tau), \qquad (2.4.47)$$

where the ${}_\Gamma\Delta$ acts on a generic element $\tau = \rho^1 \otimes \rho^2 \otimes \cdots \rho^n \ \in\ \Gamma^{\otimes n}$ as in (2.1.62). In these notations we then have

$$\ell_t = \chi * \qquad (2.4.48)$$

The quantum Lie derivative has properties analogous to that of the ordinary Lie derivative:

i) it is linear in τ:

$$\ell_t(\lambda\tau + \tau') = \lambda\ell_t(\tau) + \ell_t(\tau'); \tag{2.4.49}$$

ii) it is linear in t:

$$\ell_{\lambda t+t'} = \lambda\ell_t + \ell_{t'}, \quad \lambda \in \mathbf{C}. \tag{2.4.50}$$

By virtue of this last property we can just study ℓ_{t_i}, where $\{t_i\}$ is a basis of ${}_{\mathrm{inv}}\Xi$.

Theorem 2.4.8 The following relation holds:

$$\ell_{t_i}(\tau \otimes \tau') = \ell_{t_j}(\tau) \otimes \ f^j{}_i * \tau' + \tau \otimes \ell_{t_i}(\tau') \tag{2.4.51}$$

Proof

$$\begin{aligned}
\ell_{t_i}(\tau \otimes \tau') &= (id \otimes \chi_i)_\Gamma\Delta(\tau \otimes \tau') \\
&= (id \otimes \chi_i)(\tau_1 \otimes \tau_1' \otimes \tau_2\tau_2') \\
&= (\tau_1 \otimes \tau_1')\chi_i(\tau_2\tau_2') = (\tau_1 \otimes \tau_1')[\chi_j(\tau_2)f^j_i(\tau_2') + \varepsilon(\tau_2)\chi_i(\tau_2')] \\
&= \tau_1\chi_j(\tau_2) \otimes \tau_1' f^j{}_i(\tau_2') + \tau_1\varepsilon(\tau_2) \otimes \tau_1'\chi_i(\tau_2') \\
&= \ell_{t_j}(\tau) \otimes (id \otimes f^j{}_i) * \tau' + \tau \otimes \ell_{t_i}(\tau')
\end{aligned}$$

[remember that $\chi_j(a)$ and $f^j{}_i(a)$ are $\mathbf{C}$ numbers]. The same argument leads to:

$$\ell_{t_i}(a\omega^j) = \ell_{t_k}(a)(f^k{}_i * \omega^j) + a\ell_{t_i}(\omega^j) \tag{2.4.52}$$

$$\ell_{t_i}(\omega^j a) = \ell_{t_k}(\omega^j)(f^k{}_i * a) + \omega^j\ell_{t_i}(a). \tag{2.4.53}$$

The classical limit of (2.4.51) is easy to recover if we remember that $\varepsilon * \tau = \tau$. Formulas (2.4.51), (2.4.45) and (2.4.46) uniquely define the quantum ℓ_t, which reduces, for $q \to 1$, to the classical Lie derivative.

Theorem 2.4.9 The Lie derivative commutes with the exterior derivative:

$$\ell_{t_i}(d\vartheta) = d(\ell_{t_i}\vartheta), \qquad \vartheta \in \Gamma^\wedge \subset \Gamma^\otimes : \ \text{generic form.} \tag{2.4.54}$$

Proof:

$$\ell_{t_i}(d\vartheta) = (id \otimes \chi_i)_\Gamma\Delta(d\vartheta) = (id \otimes \chi_i)(d \otimes id)_\Gamma\Delta(\vartheta) =$$

$$(d \otimes \chi_i)_\Gamma\Delta(\vartheta) = d\vartheta_1 \underbrace{\chi_i(\vartheta_2)}_{\in\mathbf{C}} = d[\vartheta_1\chi_i(\vartheta_2)] = d(\ell_{t_i}\vartheta),$$

where in the second equality we have used property (2.1.84).

Theorem 2.4.10 The Lie derivative commutes with the left and right coactions Δ_Γ and ${}_\Gamma\Delta$, $\forall\tau \in \Gamma^\otimes$:

$$(id \otimes \ell_t)\Delta_\Gamma(\tau) = \Delta_\Gamma(\ell_t\tau) \tag{2.4.55}$$

$$(id \otimes \ell_t)_\Gamma\Delta(\tau) = {}_\Gamma\Delta(\ell_t\tau) \ . \tag{2.4.56}$$

The proof is easy and relies on the fact that left and right coactions commute, cf. eq. (2.1.19). In the classical limit, eq. (2.4.55) becomes :

$$\ell_t(L_g^*\theta) = L_g^*(\ell_t\theta). \tag{2.4.57}$$

Note 2.4.1 It is not difficult to prove the associativity of the generalized $*$ product, for example that $(\chi * \chi') * \tau = \chi * (\chi' * \tau)$. From this property it follows that the q-Lie derivative is a representation of the q-Lie algebra:

$$[\ell_t, \ell_{t'}](\tau) = \ell_{[t,t']}(\tau),$$

where the left hand side is defined via the adjoint action :

$$[\ell_t, \ell_{t'}](\tau) \equiv \ell_{\kappa'(\chi_1')*} \circ \ell_t \circ \ell_{t_2'} \ .$$

In the $\{t_i\}$ basis: $[\ell_{t_i}, \ell_{t_k}] = \ell_{t_i} \circ \ell_{t_k} - \Lambda^{ef}{}_{ik} \ell_{t_e} \circ \ell_{t_f}$.

We can now prove the Cartan identity:

Theorem 2.4.11 The contraction operator i_t defined in (2.4.38) , the Lie derivative and the exterior differential satisfy (we omit the composition product $\circ$) :

$$\ell_{t_i} = i_{t_i} d + d i_{t_i}. \tag{2.4.58}$$

A proof of this theorem is given in Appendix B. □□□

Led by Theorem 2.4.11, it is natural to introduce the Lie derivative along a generic vectorfield V through the following

Definition

$$\ell_V = i_V d + d i_V \tag{2.4.59}$$

Theorem 2.4.12 The Lie derivative satisfies the following properties:

1) $\ell_V a = V(a)$

2) $\ell_V d\vartheta = d\ell_V \vartheta$

3) $\ell_V(\lambda\vartheta + \vartheta') = \lambda\ell_V(\vartheta) + \ell_V(\vartheta')$

4) $\ell_{V \square b}(\vartheta) = (\ell_V \vartheta) b - (-1)^p i_V(\vartheta) \wedge db$

5) $\ell_V(\vartheta \wedge \vartheta') = \vartheta \wedge \ell_V(\vartheta') + \ell_{t_k}(\vartheta) \wedge (f^k{}_j * \vartheta') b^j + (-1)^{deg(\vartheta')} i_{t_k}(\vartheta) \wedge (f^k{}_j * \vartheta') \wedge db^j$

where ϑ and ϑ' are generic forms and $V = t_j \square b^j$.

Proof

Properties 1), 2), 3) and 4) follow directly from the definition (2.4.59).
Property 5) is also a consequence of definition (2.4.59); the proof uses relation d') and the identity $d(f^k{}_j * \vartheta) = f^k{}_j * d\vartheta$ [see (2.1.107)]. □□□

Note 2.4.2 It is natural to define a Lie derivative $\ell^{\mathcal{R}}_{h_i}$ of a generic covariant tensorfield $\tau \in \Gamma^{\otimes}$ along a *right*-invariant vectorfield h_i in terms of the *left* coaction Δ_Γ :

$$\ell^{\mathcal{R}}_h(\tau) \equiv (\chi \otimes id)\Delta_\Gamma(\tau) \equiv \tau * \chi ,$$

just like it was natural that we used the *right* coaction, when we defined ℓ_{t_i} in (2.4.44). In this note we compare the two definitions.

From the above definition we find

$$\begin{aligned} \ell^{\mathcal{R}}_{h_i}(\vartheta \wedge \vartheta') &= \chi_i(\vartheta_1\vartheta'_1)\vartheta_2 \wedge \vartheta'_2 \\ &= \chi_j(\vartheta_1)f^j{}_i(\vartheta'_1)\vartheta_2 \wedge \vartheta'_2 + \varepsilon(\vartheta_1)\chi_i(\vartheta'_1)\vartheta_2 \wedge \vartheta'_2 \\ &= \ell^{\mathcal{R}}_{h_j}(\vartheta) \wedge (\vartheta' * f^j{}_i) + \vartheta \wedge \ell^{\mathcal{R}}_{h_i}(\vartheta') \end{aligned} \tag{2.4.60}$$

where $\vartheta' * f^j{}_i \equiv (f^j{}_i \otimes id)\Delta_\Gamma(\vartheta')$. In particular:

$$\ell^{\mathcal{R}}_{h_i}(adb) = h_i(a)d(b * f^j{}_i) + ad(h_j(b)) . \tag{2.4.61}$$

where we have used (2.1.83). On the other hand, since $h_i = t_j \square M_i{}^j$, we can give an alternative expression for the Lie derivative along the right invariant vectorfield h_i:

$$\begin{aligned} \ell_{h_i}(adb) = \ell_{t_j \square M_i{}^j}(adb) &= a\ell_{t_j \square M_i{}^j}(db) + \ell_{t_k}(a) \wedge (f^k{}_j * db)M_i{}^j \\ &= a\, d(h_i(b)) + t_k(a)d(f^k{}_j * b)\, M_i{}^j . \end{aligned} \tag{2.4.62}$$

The difference between expressions (2.4.61) and (2.4.62) is a good index for the "defect" between left and right transports on a quantum group:

$$\begin{aligned} (\ell_{h_i} - \ell^{\mathcal{R}}_{h_i})(adb) &= t_k(a)[(f^k{}_j * db)M_i{}^j - M_j{}^k(db * f^j{}_i)] \\ &= -t_k(a)DI^k{}_i(b); \end{aligned} \tag{2.4.63}$$

where

$$DI^k{}_i(b) \equiv [(f^k{}_j * b)d(M_i{}^j) - d(M_j{}^k)(b * f^j{}_i)] \qquad \text{(Defect Index)}. \tag{2.4.64}$$

In the last passage we have used the Leibniz rule for d combined with the bicovariance condition (2.1.51). The term in the square brackets is always zero in the classical (undeformed) case. Note that $(\ell_{h_i} - \ell^{\mathcal{R}}_{h_i})$ vanishes on a and db separately but not necessarily on adb. The case of "a" confirms (2.3.23):

$$\ell_{h_i}(a) = t_j(a)M_i{}^j = h_i(a) = \ell^{\mathcal{R}}_{h_i}(a) \tag{2.4.65}$$

and shows that we will not encounter any ambiguities or inconsistencies as long as we deal with general vectorfields and functions alone. Problems can occur however when we start to introduce forms. For example in the $GL_q(2)$ differential calculus of Section 2.2 we have, sum over a understood, $\kappa(x_a{}^a{}_1)(\ell_{h^i{}_j} - \ell^{\mathcal{R}}_{h^i{}_j})(x_a{}^a{}_2 db) = -(f_k{}^{ke}{}_f * b)dM^i{}_{je}{}^f = \lambda t^e{}_f(b)dM^i{}_{je}{}^f$ for any b such that $\varepsilon(b) = 0$. This expression is clearly $\neq 0$ in general.

2.4.6 Algebra of Differential Operators

In the previous sections, given a Woronowicz differential calculus on a generic Hopf algebra A, we have defined the quantum analogue of Lie derivative and of inner derivative by a natural generalization of their defining classical formulae. The Lie derivative and contraction operators act on the space $\Gamma^{\otimes}$ of covariant tensorfields, we have in particular studied their properties on the space $\Gamma^{\wedge} \subset \Gamma^{\otimes}$ of forms where the exterior differential is also present.

These operators form a graded quantum Lie algebra

$$\{d, d\} = 0 \tag{2.4.66}$$
$$[d, \ell_V] = 0 \tag{2.4.67}$$
$$\{d, i_V\} = \ell_V \tag{2.4.68}$$

which is supplemented by two more relations

$$[\ell_{t_i}, \ell_{t_k}] = \ell_{[t_i,t_k]} = \mathbf{C}_{ik}{}^{l}\ell_{t_l} \tag{2.4.69}$$
$$[i_{t_i}, \ell_{t_k}] = i_{[t_i,t_k]} = \mathbf{C}_{ik}{}^{l}i_{t_l} \tag{2.4.70}$$

where the definition of the brackets in the left hand side of (2.4.69) and (2.4.70) is the generalization of the adjoint action:

$$[\ell_{t_i}, \ell_{t_k}] \equiv \ell_{\kappa'(\chi_{k1})*}\circ\ell_{t_i}\circ\ell_{\chi_{k2}*} = \ell_{t_i}\circ\ell_{t_k} + \ell_{\kappa'(\chi_e)*}\circ\ell_{t_i}\circ f^e{}_k* = \ell_{t_i}\circ\ell_{t_k} - \Lambda^{ef}_{ik}\ell_{t_e}\circ\ell_{t_f}$$

(this last equality is explained in Note 2.4.1)

$$[i_{t_i}, \ell_{t_k},] \equiv \ell_{\kappa'(\chi_{k1})*}\circ i_{t_i}\circ\ell_{\chi_{k2}*} = i_{t_i}\circ\ell_{t_k} - \Lambda^{ef}_{ik}\ell_{t_e}\circ i_{t_f} \tag{2.4.71}$$

The proof of (2.4.70) and of the last equality in (2.4.71), similarly to the proof of the Cartan identity, is by induction. It is given in Appendix B.

The cross-commutation relations between forms, exterior derivative, Lie derivative and inner derivative, that we have derived from the actions of i_V and ℓ_t on generic tensors $\tau \in \Gamma^{\otimes}$ and essentially (see the definition of i_V) from the $\Gamma^{\otimes} \leftrightarrow \Xi^{\otimes}$ duality –i.e. the bicovariant bimodule structure of $\Gamma^{\otimes}$ and $\Xi^{\otimes}$– can be formally derived also from the cross product algebra $\Gamma^{\wedge}\rtimes\Gamma^{\wedge *}$ [40, 41]. Here $\Gamma^{\wedge}$ is seen as a graded Hopf algebra: the product in $\Gamma^{\wedge}\otimes\Gamma^{\wedge}$ is given by $(I \otimes \mu)(\nu \otimes I) = (-1)^{deg(\mu)deg(\nu)}(\nu\otimes\mu)$, the costructures generalize those of A and are: $\Delta(\omega^i) = ({}_\Gamma\Delta + \Delta_\Gamma)(\omega^i) = \omega^i \otimes M_j{}^i + I \otimes \omega^i$, $\varepsilon(\omega^i) = 0$, $\kappa(\omega^i) = -\omega^j\kappa(M_j{}^i)$ [47]. $\Gamma^{\wedge *}$ is the graded Hopf algebra dual to $\Gamma^{\wedge}$, $\Gamma^{\wedge *} = U \oplus \Gamma^* \oplus \Gamma^{\wedge 2*} \oplus \ldots$. For example $\chi_i\vartheta = \vartheta_1\langle\chi_{i1},\vartheta_2\rangle\chi_{i2} = (\chi_j * \vartheta)f^j{}_i + \vartheta\chi_i$ corresponds to $\ell(\vartheta\wedge\vartheta') = \ell_{\chi_i}(\vartheta)\wedge(f^j{}_i * \vartheta') + \vartheta\wedge\ell_{\chi_i}(\vartheta')$. As shown in [37] the graded q-Lie algebra of the operators i_V, d, ℓ can also be interpreted as a braided tensor algebra.

Chapter 3

Geometry of the quantum Inhomogeneous Linear Groups $IGL_{q,r}(N)$

In this chapter, following [60], we analize the geometry of the inhomogeneous quantum linear groups $IGL_{q,r}(N)$. Quantum deformations of inhomogeneous Lie groups have been studied in [65] [58] [57]. An R-matrix approach has been independently proposed for $IGL_q(N)$ in ref.s [58] and [57].

We construct the *multiparametric* $IGL_{q,r}(N)$ q-groups, their universal enveloping algebra and their bicovariant differential calculus using a projection $P : GL_{q,r}(N+1) \to IGL_{q,r}(N)$; this projection procedure was first introduced in [59].

All the quantities relevant to the $IGL_{q,r}(N)$ (bicovariant) differential calculus are given explicitly: exterior derivatives, left-invariant 1-forms, Cartan-Maurer equations, tangent vectors and their q-Lie algebra and so on. The method is illustrated in the case of $IGL_{q,r}(2)$: the general formulas are applied and tested on this example.

In this framework we construct the differential geometry of the (multiparametric) quantum plane in a novel and easy way.

Deformations of inhomogeneous Lie groups and Lie algebras usually include a dilatation generator, moreover the determinant of the fundamental representation of the q-group is in general not central. It is studying the most general (multiparametric) deformation that we understand the interplay between the absence or presence of the dilatation and the properties of the determinant. This also clarify the relation between the non-commutativity of the quantum plane coordinates x^a discussed in Section 3.7 (due to the auxiliary deformation parameters q_i), the non-comutativity of the generators $T^a{}_b$ of the homogeneous linear subgroup and the finite difference structure of the differential calculus, that is due to the main deformation parameter r (called q in the previous chapters), cf. [75].

In Section 3.1 we recall the basics of the linear quantum groups and in Section 3.2 we discuss their duals in some detail. In fact, Sections 3.1 and 3.2 are a short review of the multiparametric deformations of $GL_{q,r}(N)$, where q indicates a set of parameters q_i, and of their universal enveloping algebras. The usual uniparametric case is recovered for $r = q_i = q$. For references on multiparametric deformations, see [72, 73, 74].

In Section 3.3, we first present the quantum group $IGL_{q,r}(N)$ as a Hopf algebra with given generators, commutation relations and co-structures. We then reobtain it as the image of a projection P from $GL_{q,r}(N+1)$, and show how the "mother" Hopf algebra $GL_{q,r}(N+1)$ determines the Hopf algebra structure on $IGL_{q,r}(N)$. In the language of Hopf algebra ideals $IGL_{q,r}(N)$ is seen as the quotient of $GL_{q,r}(N+1)$ with respect to a suitable Hopf ideal.

The fundamental representation of $IGL_{q,r}(N)$ contains the $GL_{q,r}(N)$ elements $T^a{}_b$ and the "coordinates" x^a as in the classical case, in addition, there is also an element u playing the role of a dilatation. By fixing some of the parameters q, we find that this element u can be made central, and hence consistently set equal to the identity I.

A quantum determinant can be defined, and is central only in a subclass of the multiparametric deformations. In this subclass, however, the element u is not central. We end the section analizing the semidirect product structure of $IGL_{q,r}(N)$ given by $GL_{q,r}(N)$ and the quantum plane: this construction is based on the observation that $GL_{q,r}(N)$ is both a Hopf subalgebra in $IGL_{q,r}(N)$ and a quotient of $IGL_{q,r}(N)$ obtained projecting to zero the quantum plane coordinates x^a.

The explicit construction of the bicovariant differential calculus for $GL_{q,r}(N)$, in terms of the dual algebra, is given in Section 3.4. In Section 3.5 we project the bicovariant differential calculus of $GL_{q,r}(N+1)$ to $IGL_{q,r}(N)$ and study the bicovariant bimodules of 1-forms and tangent vectors on $IGL_{q,r}(N)$. In particular, the q-Lie algebra is given explicitly. We also study in detail the exterior algebra and the exterior derivative, and find the Cartan-Maurer equations. In Section 3.6 we then study the universal enveloping algebra $U_{q,r}(igl(N))$ its semidirect product structure [given by $U_{q,r}(gl(N))$ and the translation generators] and the duality with $IGL_{q,r}(N)$. The Universal enveloping algebra $U_{q,r}(igl(N))$ is the natural setting where to study q-Lie algebras and therefore differential calculi. Using the general theory of Section 2.3, we easily obtain another differential calculus on $IGL_{q,r}(N)$ that differs from the previous one by the presence of a dilatation generator corresponding to the dilatation $u \in IGL_{q,r}(N)$.

In Section 3.7 we discuss the multiparametric quantum plane, i.e. the quantum coset space $IGL_{q,r}(N)/GL_{q,r}(N)$ spanned by the coordinates x^a, and find a generalization of the differential geometry of the q-plane of [48], [50], see also Schirrmacher in [74].

In the Table at the end of the chapter we specialize our general treatment to $IGL_{q,r}(2)$ and collect all the relevant formulas for its bicovariant differential calculus.

3.1 $GL_{q,r}(N)$ and its real forms

We here introduce the multiparametric q-group $GL_{q,r}(N)$, where now the index $q \equiv q_{ab}$ represents a set of parameters, and r is the parameter we called q in the previous chapters. $GL_{q,r}(N)$ is the algebra (over the complex field) freely generated by the non-commuting matrix elements $T^A{}_B$, (A,B=1,..N), the identity I and the inverse Ξ of the q-determinant of T defined in (3.1.6), modulo the "RTT" relations:

$$R^{AB}{}_{EF}T^E{}_CT^F{}_D = T^B{}_FT^A{}_ER^{EF}{}_{CD} \qquad (3.1.1)$$

where the R-matrix is given by [72, 73]:

$$R^{AB}{}_{CD} = \delta^A_C\delta^B_D[\frac{r}{q_{AB}} + (r-1)\delta^{AB}] + (r - r^{-1})\ \delta^A_D\delta^B_C\theta^{AB} \qquad (3.1.2)$$

with $\theta^{AB} = 1$ for $A > B$ and zero otherwise, and

$$q_{AB} = \frac{r^2}{q_{BA}},\ \ q_{AA} = r \qquad (3.1.3)$$

It is useful to list the nonzero complex components of the R matrix (no sum on repeated indices):

$$\begin{array}{ll} R^{AA}{}_{AA} = r & \\ R^{AB}{}_{AB} = \frac{r}{q_{AB}}, & A \neq B \\ R^{BA}{}_{AB} = r - r^{-1}, & B > A \end{array} \qquad (3.1.4)$$

The R matrix in (3.1.2) satisfies the quantum Yang-Baxter equation.

The standard uniparametric R matrix [19] is obtained from (3.1.2) by setting all deformation parameters q_{AB}, r equal to a single parameter q. For a further insight about the relationship between the multiparametric and the uniparametric R-matrix see Section 4.1.

The quantum determinant of T and its inverse Ξ are defined by:

$$\Xi\ \det T = \det T\ \Xi = I \qquad (3.1.5)$$

$$\det T \equiv \sum_\sigma \left[\prod_{A<B,\sigma(A)>\sigma(B)} (-\frac{r^2}{q_{\sigma(B)\sigma(A)}})\right]\ T^1{}_{\sigma(1)}\cdots T^N{}_{\sigma(N)} \qquad (3.1.6)$$

Note 3.1.1 In the uniparametric case $r = q_{AB} = q$ we recover the usual formula

$$\det T \equiv \sum_\sigma (-q)^{l(\sigma)} T^1{}_{\sigma(1)} \cdots T^N{}_{\sigma(N)} \tag{3.1.7}$$

where $l(\sigma)$ is the minimum number of transpositions in the permutation σ.

Note 3.1.2 In more mathematical terms, the algebra $GL_{q,r}(N)$ is the quotient of the non-commuting algebra $\mathbf{C}\langle T^A{}_B, I, \Xi\rangle$ freely generated by the elements $T^A{}_B, I, \Xi$ with respect to the two-sided ideal in $\mathbf{C}\langle T^A{}_B, I, \Xi\rangle$ generated by the RTT relations (3.1.1).

Note 3.1.3 The inverse matrix R^{-1}, defined as

$$(R^{-1})^{AB}{}_{CD} R^{CD}{}_{EF} \equiv \delta^A_E \delta^B_F \equiv R^{AB}{}_{CD} (R^{-1})^{CD}{}_{EF} \tag{3.1.8}$$

is given by

$$R^{-1}_{q,r} = R_{q^{-1},r^{-1}} \tag{3.1.9}$$

Note 3.1.4 The $\hat{R}$ matrix defined by $\hat{R}^{AB}{}_{CD} \equiv R^{BA}{}_{CD}$ satisfies the spectral decomposition (Hecke condition):

$$(\hat{R} - rI)\ (\hat{R} + r^{-1}I) = 0 \tag{3.1.10}$$

Note 3.1.5 The determinant in (3.1.6) is central if and only if the following conditions on the parameters are satisfied (see ref. [72]):

$$q_{1,A} q_{2,A} \cdots q_{A-1,A} \frac{r^2}{q_{A,A+1}} \frac{r^2}{q_{A,A+2}} \cdots \frac{r^2}{q_{A,N}} = const. \tag{3.1.11}$$

for all A=1,...N. This results in N-1 conditions among the q_{AB} and determines $const = r^{N-1}$. Using (3.1.3), and defining

$$Q_A \equiv \prod_{C=1}^{N} (\frac{q_{CA}}{r}) \tag{3.1.12}$$

the centrality conditions (3.1.11) become:

$$Q_A = 1 \tag{3.1.13}$$

We have used also $const = r^{N-1}$, so that only $N-1$ of the above conditions are independent. Indeed the Q_A satisfy the relation

$$Q_1 Q_2 \cdots Q_N = 1 \tag{3.1.14}$$

In general we have:

$$(\mathrm{det}T)T^A{}_B = \frac{Q_A}{Q_B}T^A{}_B(\mathrm{det}T), \quad \Xi T^A{}_B = \frac{Q_B}{Q_A}T^A{}_B\Xi \tag{3.1.15}$$

When (3.1.13) holds[1], we can consistently set $\mathrm{det}T^A{}_B = I = \Xi$, and obtain the multiparametric deformations $SL_{q,r}(N)$.

The algebra $GL_{q,r}(N)$ becomes a Hopf algebra with the following coproduct Δ, counit ε and coinverse κ:

$$\Delta(T^A{}_B) = T^A{}_B \otimes T^B{}_C \tag{3.1.16}$$
$$\varepsilon(T^A{}_B) = \delta^A_B \tag{3.1.17}$$
$$\kappa(T^A{}_B) = (T^{-1})^A{}_B \tag{3.1.18}$$
$$\Delta(\mathrm{det}T) = \mathrm{det}T \otimes \mathrm{det}T, \quad \Delta(\Xi) = \Xi \otimes \Xi, \quad \Delta(I) = I \otimes I \tag{3.1.19}$$
$$\varepsilon(\mathrm{det}T) = 1, \quad \varepsilon(\Xi) = 1, \quad \varepsilon(I) = 1 \tag{3.1.20}$$
$$\kappa(\mathrm{det}T) = \Xi, \quad \kappa(\Xi) = \mathrm{det}T, \quad \kappa(I) = I \tag{3.1.21}$$

The quantum inverse of $T^A{}_B$ in (3.1.18) is given by:

$$(T^{-1})^A{}_B = \Xi\ \Pi^{(1,N)}_{AB}\ t^{\ A}_B \tag{3.1.22}$$

where $t^{\ A}_B$ is the quantum minor, i.e. the quantum determinant of the submatrix of T obtained by removing the B-th row and the A-th column, and $\Pi^{(1,N)}_{AB}$ is a function of the parameters q:

$$\Pi^{(1,N)}_{AB} \equiv \frac{\prod^N_{C=B+1}(-q_{BC})}{\prod^N_{D=A+1}(-q_{AD})} \tag{3.1.23}$$

The superscript (1,N) reminds the range of the indices A,B,C,... In the uniparametric case, the quantum inverse has the simpler expression:

$$(T^{-1})^A{}_B = \Xi\ (-q)^{A-B} t^{\ A}_B \tag{3.1.24}$$

Note 3.1.6 As in Note 1.2.1, we recall that in general $\kappa^2 \neq 1$ and

$$\kappa^2(T^A{}_B) = D^A{}_C T^C{}_D (D^{-1})^D{}_B = d^A d^{-1}_B T^A{}_B, \tag{3.1.25}$$

where D is a diagonal matrix, $D^A_B = d^A\delta^A_B$, given by $d^A = r^{2A-1}$ for $GL_{q,r}(N)$. This matrix satisfies:

$$d^A d^{-1}_C (R^{-1})^{BA}{}_{DC} R^{EC}{}_{BF} = \delta^A_F \delta^E_D, \quad d^A d^{-1}_C R^{AB}{}_{CD}(R^{-1})^{CE}{}_{FB} = \delta^A_F\delta^E_D \tag{3.1.26}$$

$$d^B d^{-1}_D (R^{-1})^{AB}{}_{CD} R^{CE}{}_{FB} = \delta^A_F \delta^E_D, \quad d^B d^{-1}_D R^{BA}{}_{DC}(R^{-1})^{EC}{}_{BF} = \delta^A_F\delta^E_D \tag{3.1.27}$$

[1]We disregard the solutions $\forall A \in a,\ Q_A = \sqrt[N]{1}$ because we want a continuous deformation of the classical limit.

$$R^{AC}{}_{CB}d_C^{-1} = \delta^A_B = (R^{-1})^{AC}{}_{CB}d_C \qquad (3.1.28)$$

This last condition fixes the normalization of D. Relations (3.1.26) and (3.1.27) define a second inverse $R^{\sim 1}$ of the R matrix and a second inverse $(R^{-1})^{\sim 1}$ of the R^{-1} matrix as:

$$(R^{\sim 1})^{AB}{}_{CD} \equiv d^B d_D^{-1} (R^{-1})^{AB}{}_{CD} \qquad (3.1.29)$$

$$((R^{-1})^{\sim 1})^{AB}{}_{CD} \equiv d^A d_C^{-1} R^{AB}{}_{CD} \qquad (3.1.30)$$

Using (3.1.28) we can relate the D matrix to this second inverse:

$$(D^{-1})^A{}_B = (R^{\sim 1})^{AC}{}_{CB}, \quad D^A{}_B = ((R^{-1})^{\sim 1})^{AC}{}_{CB} \qquad (3.1.31)$$

This generalizes the analogous discussion for the uniparametric D matrix given in [19].

We turn now to the real forms of $GL_{q,r}(N)$, that are defined by $*$-conjugations of the $GL_{q,r}(N)$ Hopf algebra; see Section 1.3. These conjugations must be compatible with the RTT relations: this restricts the range of the parameters q, r. Three such conjugations are known (cf. [72]):

i) $T^* = T$, i.e. the elements $T^A{}_B$ are "real". Applying the $*$-conjugation to the RTT equations (3.1.1) yields again the RTT relations if the R matrix satisfies $\bar{R} = R^{-1}$. This happens for $|q_{AB}| = |r| = 1$, i.e. for deformation parameters lying on the unit circle in $\mathbf{C}$ (cf. eq. (3.1.9)). The quantum group is then denoted by $GL_{q,r}(N; \mathbf{R})$.

ii) $(T^A{}_B)^* = T^{A'}{}_{B'}$ with primed indices defined as $A' = N + 1 - A$. Here compatibility with the RTT relations requires $\bar{R}^{AB}{}_{CD} = R^{B'A'}{}_{D'C'}$, satisfied when $\bar{q}_{AB} = q_{B'A'}$, $r \in \mathbf{R}$.

iii) $(T^A{}_B)^* = \kappa(T^B{}_A)$, the generalization of the unitarity condition for the matrix T. In this case (left as an exercise in [72]) the restriction on the R matrix is $\bar{R}^{AB}{}_{CD} = R^{DC}{}_{BA}$, leading to the conditions $\bar{q}_{AB} = q_{BA}$, $r \in \mathbf{R}$. The corresponding quantum groups are denoted by $U_{q,r}(N)$.

Imposing also $\det T = I$ yields the quantum groups $SL_{q,r}(N; \mathbf{R})$ or $SU_{q,r}(N)$.

3.2 The universal enveloping algebra of $GL_{q,r}(N)$

We construct the universal enveloping algebra of $GL_{q,r}(N)$ as the algebra of regular functionals [19] on $GL_{q,r}(N)$: it is generated by the functionals $L^{\pm}, \varepsilon$ and Φ defined below.

The $L^{\pm}$ functionals are defined as in Section 2.2 where the uniparametric R-matrix is now replaced by the multiparametric one.

A determinant can be defined for the matrix ${L^{\pm A}}_B$, as in Note 2.2.1, this is given by:

$$\det L^{\pm} = {L^{\pm 1}}_1 {L^{\pm 2}}_2 \cdots {L^{\pm N}}_N \ . \tag{3.2.1}$$

A quantum inverse for ${L^{\pm A}}_B$ can be found, using an expression analogous to (3.1.22) with $q_{AB} \to q_{AB}^{-1}$. For this we need to introduce the element Φ defined by:

$$\Phi \det L^+ \det L^- = \det L^+ \det L^- \Phi = \varepsilon \ . \tag{3.2.2}$$

Then the quantum inverse of ${L^{\pm A}}_B$ is given by:

$$({L^{\pm A}}_B)^{-1} = \Phi \ \det L^{\mp} \ \Pi^{(1,N)}_{BA} \ \ell_B^A \tag{3.2.3}$$

where ℓ_B^A is the quantum minor and $\Pi^{(1,N)}_{BA}$ is given in (3.1.23). Notice that $\Phi \ \det L^{\mp}$ is the inverse of $\det L^{\pm}$ because of property (3.2.13) below.

The co-structures of the algebra generated by the functionals $L^{\pm}$, ε and Φ are as in Section 2.2 :

$$\Delta'({L^{\pm A}}_B) = {L^{\pm A}}_G \otimes {L^{\pm G}}_B \tag{3.2.4}$$

$$\varepsilon'({L^{\pm A}}_B) = \delta^A_B \tag{3.2.5}$$

$$\kappa'({L^{\pm A}}_B) = {L^{\pm A}}_B \circ \kappa \tag{3.2.6}$$

$$\Delta'(\det L^{\pm}) = \det L^{\pm} \otimes \det L^{\pm}, \tag{3.2.7}$$

$$\Delta'(\Phi) = \Phi \otimes \Phi, \Delta'(\varepsilon) = \varepsilon \otimes \varepsilon \tag{3.2.8}$$

$$\varepsilon'(\det L^{\pm}) = 1, \ \varepsilon'(\Phi) = 1, \ \varepsilon'(\varepsilon) = 1 \tag{3.2.9}$$

$$\kappa'(\det L^{\pm}) = \Phi \ \det L^{\mp}, \tag{3.2.10}$$

$$\kappa'(\Phi) = \det L^+ \det L^-, \ \kappa'(\varepsilon) = \varepsilon \tag{3.2.11}$$

Note 3.2.1 In (3.2.6) we have defined κ' using κ, we now prove that $\kappa'({L^{\pm A}}_B) = ({L^{\pm A}}_B)^{-1}$ as defined in (3.2.3). This shows that $\kappa'({L^{\pm A}}_B)$ is expressible by polynomials in ${L^{\pm A}}_B, \Phi$.

Proof : From $(L^{\pm})^{-1} L^{\pm} = \varepsilon$ we have $1 = [(L_1^{\pm})^{-1} L_1^{\pm}](T) = (L_1^{\pm})^{-1}(T_2) L_1^{\pm}(T_2) = (L_1^{\pm})^{-1}(T_2) R_{12}^{\pm}$ so that $(L_1^{\pm})^{-1}(T_2) = {R_{12}^{\pm}}^{-1}$.
From $\kappa(T)T = 1$ we similarly have $[\kappa'(L_1^{\pm})](T_2) = {R_{12}^{\pm}}^{-1}$ and therefore $\kappa'({L^{\pm A}}_B) = ({L^{\pm A}}_B)^{-1}$.

Since κ' is an inner operation in the algebra generated by the functionals ${L^{\pm A}}_B$, ε and ϕ we conclude that these elements generate the Hopf algebra $U_{q,r}(gl(N))$ of the regular functionals on the quantum group $GL_{q,r}(N)$.

In the following we list some useful properties of the $L^{\pm}$ functionals.

Properties of $L^{\pm}$

i) Similarly to the uniparametric case, cf. Note 2.2.1 – Note 2.2.3, we have

$$L^{\pm A}{}_{A} L^{\pm B}{}_{B} = L^{\pm B}{}_{B} L^{\pm A}{}_{A} \; ; \quad L^{+A}{}_{A} L^{-B}{}_{B} = L^{-B}{}_{B} L^{+A}{}_{A} \tag{3.2.12}$$

As a consequence:

$$\det L^{+} \det L^{-} = \det L^{-} \det L^{+}. \tag{3.2.13}$$

We also have

$$L^{\pm A}{}_{B}(\det T) = \delta^{A}_{B} (c^{\pm})^{N} r^{\pm 1} Q_{A}^{-1} \tag{3.2.14}$$

$$\det L^{\pm}(T^{A}{}_{B}) = \delta^{A}_{B} (c^{\pm})^{N} r^{\pm 1} Q_{A} \tag{3.2.15}$$

$$\det L^{\pm}(T^{A}{}_{B}) = L^{\pm A}{}_{B}(\det T) Q_{A}^{2}. \tag{3.2.16}$$

From (3.2.1) it is easy to see that $\det L^{\pm}(I) = 1$.

ii) Since the RLL relations are the same as the RTT relations with $q_{AB} \to (q_{AB})^{-1}$, $r \to r^{-1}$, we obtain a formula analogous to (3.1.15):

$$(\det L^{\pm}) L^{\pm A}{}_{B} = \frac{Q_B}{Q_A} L^{\pm A}{}_{B} (\det L^{\pm}), \tag{3.2.17}$$

moreover

$$(\det L^{\mp}) L^{\pm A}{}_{B} = \frac{Q_B}{Q_A} L^{\pm A}{}_{B} (\det L^{\mp}) \; . \tag{3.2.18}$$

iii) From (3.2.17) and (3.2.18) the following element:

$$\det L^{+} (\det L^{-})^{-1} = (\det L^{-})^{-1} \det L^{+} \tag{3.2.19}$$

is seen to be central. Notice that it is also group-like since

$$\Delta'(\det L^{\pm}) = \det L^{\pm} \otimes \det L^{\pm}. \tag{3.2.20}$$

In general even if $\det L^{+}(\det L^{-})^{-1}$ is central and group-like it is not equal to ε because

$$\det L^{+} (\det L^{-})^{-1} (T^{A}{}_{B}) = (c^{+})^{N} (c^{-})^{-N} \; r^{2} \delta^{A}_{B}. \tag{3.2.21}$$

iv) The elements $L^{+A}{}_{A} L^{-A}{}_{A}$ (no sum on A) play a special role for particular values of the deformation parameters q_{AB}, r; if we set

$$L^{+A}{}_{A} L^{-A}{}_{A} \equiv \varepsilon_{A} \tag{3.2.22}$$

we leave as an exercise to deduce that (no sum on repeated indices):

$$\varepsilon_{A}(T^{B}{}_{C}) \equiv c^{+} c^{-} \delta^{B}_{C} \, \frac{q_{AB}^{2}}{r^{2}}, \quad \varepsilon_{A}(I) = 1, \quad \varepsilon_{A}(\Xi) = [\varepsilon_{A}(\det T)]^{-1} \tag{3.2.23}$$

$$\varepsilon_A(ab) = \varepsilon_A(a)\varepsilon_A(b), \quad a, b \in GL_{q,r}(N) \tag{3.2.24}$$

$$\kappa'(L^{\pm A}{}_A) = L^{\mp A}{}_A \varepsilon_A^{-1} \tag{3.2.25}$$

$$\varepsilon_A \varepsilon_B = \varepsilon_B \varepsilon_A, \quad \varepsilon_A L^{\pm B}{}_B = L^{\pm B}{}_B \varepsilon_A \tag{3.2.26}$$

$$\det L^+ \det L^- = \varepsilon_1 \cdots \varepsilon_N \; ; \tag{3.2.27}$$

$$\kappa'(\det L^\pm) = \det L^\mp (\varepsilon_1 \cdots \varepsilon_N)^{-1} = (\varepsilon_1 \cdots \varepsilon_N)^{-1} \det L^\mp \tag{3.2.28}$$

Note 3.2.2 When $\det T$ is central ($Q_A = 1$) we also have that $\det L^\pm$ is central (cf. (3.2.17) and (3.2.18)). As in Note 2.2.2, for $Q_A = 1$ and $(c^\pm)^N r^{\pm 1} = 1$, the functionals $L^\pm$and ε generate the Hopf algebra $U(sl_{q,r}(N))$, and we have the simplified relations:

$$\det L^+ (\det L^-)^{-1} = \varepsilon \tag{3.2.29}$$

$$[L^{\pm A}{}_B](\det T) = \delta^A_B \quad \text{no sum on } A \tag{3.2.30}$$

$$[\det L^\pm](T^A{}_B) = \delta^A_B \tag{3.2.31}$$

$$[\det L^\pm](\det T) = 1 \tag{3.2.32}$$

Note 3.2.3 When $q_{AB} = r$ we recover the standard uniparametric R matrix, we have also $Q_A = 1$ and, for $c^+ c^- = 1$,

$$\forall A \;\; \varepsilon_A = \varepsilon \quad \text{i.e. } L^{+A}{}_A L^{-A}{}_A = L^{-A}{}_A L^{+A}{}_A = \varepsilon \; . \tag{3.2.33}$$

In this case the Hopf algebra of functionals $U_{q,r}(gl(N))$ is equivalent to the algebra generated by the symbols $L^\pm, \Phi$ and ε modulo relations (2.2.12),(2.2.13) and (3.2.33) [19].

Note 3.2.4 $GL_{q,r}(N)$ and $U_{q,r}(gl(N))$ are graded Hopf algebras: $T^A{}_B$ has grade $+1$, $\kappa(T^A{}_B)$ has grade -1, I has grade 0, $\det T$ has grade $+N$ etc., and similarly for $L^\pm$.

Conjugation

The canonical $*$-conjugation on $U_{q,r}(gl(N))$ induced by the $*$-conjugation on $GL_{q,r}(N)$ is given by:

$$\psi^*(a) \equiv \overline{\psi(\kappa^{-1}(a^*))} \tag{3.2.34}$$

where $\psi \in U_{q,r}(gl(N))$, $a \in GL_{q,r}(N)$, and the overline denotes the usual complex conjugation. It is not difficult to determine the action on the basis elements $L^{\pm A}{}_B$. The three $GL_{q,r}(N)$ $*$-conjugations i), ii), iii) of the previous section induce respectively the following conjugations on the $L^{\pm A}{}_B$:

$$\begin{array}{ll} i) & (L^{\pm A}{}_B)^* = \kappa'^2(L^{\pm A}{}_B) \\ ii) & (L^{\pm A}{}_B)^* = \kappa'^2(L^{\mp A'}{}_{B'}) \\ iii) & (L^{\pm A}{}_B)^* = \kappa'(L^{\mp B}{}_A) \; . \end{array} \tag{3.2.35}$$

3.3 The quantum group $IGL_{q,r}(N)$

The q-inhomogeneous group $IGL_{q,r}(N)$ is freely generated by the non-commuting matrix elements $T^A{}_B$ $[A = (0,a); a : 1,..N]$, the identity I and the inverse ξ of the q-determinant of T as defined in (3.1.6), modulo the relations:

$$T^0{}_a = 0 \tag{3.3.1}$$

and the relations:

$$R^{ab}{}_{ef}T^e{}_cT^f{}_d = T^b{}_fT^a{}_eR^{ef}{}_{cd} \tag{3.3.2}$$
$$R^{ab}{}_{ef}T^e{}_cx^f = \frac{q_{0c}}{r}x^bT^a{}_c \tag{3.3.3}$$
$$R^{ab}{}_{ef}x^ex^f = rx^bx^a \tag{3.3.4}$$
$$q_{0a}T^a{}_cu = q_{0c}uT^a{}_c \tag{3.3.5}$$
$$q_{0a}x^au = ux^a \tag{3.3.6}$$

where $x^a \equiv T^a{}_0$ and $u \equiv T^0{}_0$.

It is not difficult to check that this algebra, endowed with the coproduct Δ, the counit ε and the coinverse κ defined by :

$$\Delta(T^A{}_B) = T^A{}_C \otimes T^C{}_A; \quad \varepsilon(T^A{}_B) = \delta^A_B; \quad \kappa(T) = T^{-1} \tag{3.3.7}$$
$$\Delta(\xi) = \xi \otimes \xi; \quad \varepsilon(\xi) = 1; \quad \kappa(\xi) = \det T \tag{3.3.8}$$
$$\Delta(I) = I \otimes I; \quad \varepsilon(I) = 1; \quad \kappa(I) = I \tag{3.3.9}$$

where the quantum inverse of $T^A{}_B$ is given by $(T^{-1})^A{}_B = \xi\ \Pi^{(0,N)}_{AB}\ t_B^A$ [see eq. (3.1.23): $t_B{}^A$ is the quantum minor], is a Hopf algebra. The proof goes as in uniparametric case (see the second ref. of [65]).

In the commutative limit it is the algebra of functions on $IGL(N)$ plus the dilatation $T^0{}_0$.

Relations (3.3.7)-(3.3.9) explicity read:

$$\Delta(T^a{}_b) = T^a{}_c \otimes T^c{}_b, \quad \Delta(I) = I \otimes I, \tag{3.3.10}$$
$$\Delta(x^a) = T^a{}_b \otimes x^b + x^a \otimes u \tag{3.3.11}$$
$$\Delta(u) = u \otimes u, \quad \Delta(\xi) = \xi \otimes \xi \tag{3.3.12}$$
$$\Delta(\det T^a{}_b) = \det T^a{}_b \otimes \det T^a{}_b \tag{3.3.13}$$

$$\varepsilon(T^a{}_b) = \delta^a_b, \quad \varepsilon(I) = 1, \tag{3.3.14}$$
$$\varepsilon(x^a) = 0 \tag{3.3.15}$$
$$\varepsilon(u) = \varepsilon(\xi) = 1 \tag{3.3.16}$$
$$\varepsilon(\det T^a{}_b) = 1 \tag{3.3.17}$$

$$\kappa(T^a{}_b) = (T^{-1})^a{}_b = \xi u \; \Pi^{(1,N)}_{ab} t_b{}^a \tag{3.3.18}$$
$$\kappa(I) = I, \tag{3.3.19}$$
$$\kappa(x^a) = -\kappa(T^a{}_b) x^b \kappa(u) \tag{3.3.20}$$
$$\kappa(u) = \det T^a{}_b \; \xi \tag{3.3.21}$$
$$\kappa(\xi) = u \; \det T^a{}_b, \quad \kappa(\det T^a{}_b) = \xi u \tag{3.3.22}$$

where for completeness we have included the expressions for the q-determinant of T. Note that $\kappa(u)u = I = u\kappa(u)$.

This procedure is very similar to that discussed for $GL_{q,r}(N+1)$ in Section 3.1: indeed both these Hopf algebrae are obtained from the algebra freely generated by $T^A{}_B, I, \Xi$ or ξ through the introduction of moduli relations i.e. as quotients of suitable two-sided ideals: the one generated by the RTT relations in the $GL_{q,r}(N+1)$ case, and the one generated by the (3.3.1)-(3.3.6) relations in the $IGL_{q,r}(N)$ case.

We now rederive the quantum group $IGL_{q,r}(N)$ as a quotient of $GL_{q,r}(N+1)$: all Hopf algebra properties of $IGL_{q,r}(N)$ will descend from those of $GL_{q,r}(N+1)$. The formalism employed will be useful in the next section to deduce the differential calculus on $IGL_{q,r}(N)$ from the one on $GL_{q,r}(N+1)$.

We start from the observation that the R-matrix of $GL_{q,r}(N+1)$ can be written as (A=(0,a)):

$$R^{AB}{}_{CD} = \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & \frac{r}{q_{0b}}\delta^b_d & 0 & 0 \\ 0 & (r-r^{-1})\delta^a_d & \frac{q_{0a}}{r}\delta^a_c & 0 \\ 0 & 0 & 0 & R^{ab}{}_{cd} \end{pmatrix} \tag{3.3.23}$$

where $R^{ab}{}_{cd}$ is the R-matrix of $GL_{q,r}(N)$, and the indices AB are ordered as $00, 0b, a0, ab$.

It is apparent that the $GL_{q,r}(N+1)$ R matrix contains the information on $GL_{q,r}(N)$. We will show that it also contains the information about the quantum group $IGL_{q,r}(N)$.

In the index notation $A=(0,a)$ the RTT relations explicity read :

$$R^{ab}{}_{ef} T^e{}_c T^f{}_d = T^b{}_f T^a{}_e R^{ef}{}_{cd} \tag{3.3.24}$$
$$T^a{}_c T^b{}_0 = \frac{q_{ab}}{q_{c0}} T^b{}_0 T^a{}_c \tag{3.3.25}$$
$$T^a{}_0 T^b{}_d = \frac{q_{ab}}{q_{0d}} T^b{}_d T^a{}_0 + \frac{r}{q_{0d}}(r-r^{-1}) T^a{}_d T^b{}_0 \tag{3.3.26}$$
$$T^a{}_c T^0{}_d = \frac{q_{a0}}{q_{cd}} T^0{}_d T^a{}_c \tag{3.3.27}$$
$$T^0{}_c T^b{}_d = \frac{q_{0b}}{q_{cd}} T^b{}_d T^0{}_c + \frac{r}{q_{cd}}(r-r^{-1}) T^0{}_d T^b{}_c \tag{3.3.28}$$

$$T^a{}_0 T^b{}_0 = q_{ab} T^b{}_0 T^a{}_0 \tag{3.3.29}$$
$$T^0{}_c T^b{}_0 = \frac{q_{0b}}{q_{c0}} T^b{}_0 T^0{}_c \tag{3.3.30}$$
$$T^0{}_c T^0{}_d = q_{dc} T^0{}_d T^0{}_c \tag{3.3.31}$$
$$T^0{}_0 T^b{}_d = \frac{q_{0b}}{q_{0d}} T^b{}_d T^0{}_0 + \frac{r}{q_{0d}}(r - r^{-1}) T^0{}_d T^b{}_0 \tag{3.3.32}$$
$$T^0{}_0 T^b{}_0 = q_{0b} T^b{}_0 T^0{}_0 \tag{3.3.33}$$
$$T^0{}_0 T^0{}_d = q_{d0} T^0{}_d T^0{}_0 \tag{3.3.34}$$

where $a < b$ and $c < d$.

Consider now in $GL_{q,r}(N)$ the space H of all sums of monomials containing at least an element of the kind $T^0{}_a$ (i.e. H is the ideal in $GL_{q,r}(N+1)$ generated by the elements $T^0{}_a$ as we will see). Notice that $T^0{}_0 T^b{}_d - \frac{q_{0b}}{q_{0d}} T^b{}_d T^0{}_0$ is an element of H because of relation (3.3.32).

We now prove that H is a Hopf ideal, i.e. an ideal in the $GL_{q,r}(N+1)$ algebra that is also compatible with the co-structures of $GL_{q,r}(N+1)$; this allows to structure $GL_{q,r}(N+1)/H$ as a Hopf algebra [81]. We denote by Δ_{N+1}, ε_{N+1} and κ_{N+1} the co-structures of $GL_{q,r}(N+1)$.

Theorem 3.3.1 The space H is a Hopf ideal in $GL_{q,r}(N+1)$, that is:

i) H is a two-sided ideal in $GL_{q,r}(N+1)$

ii) H is a co-ideal i.e.

$$\Delta_{N+1}(H) \subseteq H \otimes GL_{q,r}(N+1) + GL_{q,r}(N+1) \otimes H \ ; \quad \varepsilon_{N+1}(H) = 0 \tag{3.3.35}$$

iii) H is compatible with κ_{N+1} :

$$\kappa_{N+1}(H) \subseteq H \ . \tag{3.3.36}$$

Proof:

i) H is trivially a subalgebra of $GL_{q,r}(N+1)$. It is a right and left ideal since $\forall h \in H, \forall a \in GL_{q,r}(N+1)$ $ha \in H$ and $ah \in H$. This follows immediately from the definition of H as sums of monomials containing at least a factor $T^0{}_a$. H is the ideal in $GL_{q,r}(N+1)$ generated by the elements $T^0{}_a$.

ii) First notice that $\Delta_{N+1}(T^0{}_b) \in H \otimes GL_{q,r}(N+1) + GL_{q,r}(N+1) \otimes H$. Now by definition of H we have

$$\forall\, h \in H, \quad h = a\ T^0{}_b c, \qquad a, c \in GL_{q,r}(N+1). \tag{3.3.37}$$

where $a\ T^0{}_b c$ represents a sum of monomials. Then we find

$$\Delta_{N+1}(h) = \Delta_{N+1}(a)\Delta_{N+1}(T^0{}_b)\Delta_{N+1}(c) \in H \otimes GL_{q,r}(N+1) + GL_{q,r}(N+1) \otimes H\ . \tag{3.3.38}$$

Moreover, since ε_{N+1} vanishes on $T^0{}_b$ we have:

$$\varepsilon_{N+1}(h) = 0, \qquad \forall h \in H. \tag{3.3.39}$$

These relations ensure that (3.3.35) hold.

iii)

$$\kappa_{N+1}(T^0{}_b) = \Xi\ \Pi^{(0,N)}_{ob}\ t_b{}^0 \tag{3.3.40}$$

where $\Pi^{(0,N)}_{ob}$ is defined in (3.1.23) and it is easy to see that the quantum minor $t_b{}^0 \in H$ since it is the determinant of a matrix that has elements $T^0{}_a$ in the first row. Then

$$\kappa_{N+1}(h) = \kappa_{N+1}(a\ T^0{}_b c) = \kappa_{N+1}(c)\kappa_{N+1}(T^0{}_b)\kappa_{N+1}(a) \in H \tag{3.3.41}$$

and Theorem 3.3.1 is proved. □□□

We now consider the quotient

$$\frac{GL_{q,r}(N+1)}{H}\ , \tag{3.3.42}$$

and the canonical projection

$$P\ :\ GL_{q,r}(N+1) \longrightarrow GL_{q,r}(N+1)/H \tag{3.3.43}$$

Any element of $GL_{q,r}(N+1)/H$ is of the form $P(a)$. Also, $P(H) = 0$, i.e. $H = Ker(P)$.

Since H is a two-sided ideal, $GL_{q,r}(N+1)/H$ is an algebra with the following sum and products:

$$P(a) + P(b) \equiv P(a+b)\ ;\ \ P(a)P(b) \equiv P(ab)\ ;\ \ \mu P(a) \equiv P(\mu a), \quad \mu \in \mathbf{C} \tag{3.3.44}$$

We will use the following notation:

$$x^a \equiv P(T^a{}_0)\ ;\ \ u \equiv P(T^0{}_0)\ ;\ \ \xi \equiv P(\Xi) \tag{3.3.45}$$

and with abuse of symbols:

$$T^a{}_b \equiv P(T^a{}_b)\ ;\ \ I \equiv P(I)\ ;\ \ 0 \equiv P(0) \tag{3.3.46}$$

notice that $P(T^0{}_a) = P(0) = 0$. Using (3.3.44) it is easy to show that $T^a{}_b$, x^a, u, ξ and I generate the algebra $GL_{q,r}(N+1)/H$. From the RTT relations $R_{12}T_1T_2 = T_2T_1R_{12}$ in $GL_{q,r}(N+1)$ we find the "$P(RTT)$" relations in $GL_{q,r}(N+1)/H$:

$$P(R_{12}T_1T_2) = P(T_2T_1R_{12}) \quad i.e. \quad R_{12}P(T_1)P(T_2) = P(T_2)P(T_1)R_{12} \tag{3.3.47}$$

that are explicity given in (3.3.2)-(3.3.6).

Since H is a Hopf ideal then $GL_{q,r}(N+1)/H$ is also a Hopf algebra with co-structures:

$$\Delta(P(a)) \equiv (P \otimes P)\Delta_{N+1}(a)\ ; \quad \varepsilon(P(a)) \equiv \varepsilon_{N+1}(a)\ ; \quad \kappa(P(a)) \equiv P(\kappa_{N+1}(a)) \tag{3.3.48}$$

Indeed (3.3.35) and (3.3.36) ensure that Δ, ε, and κ are well defined. For example

$$(P \otimes P)\Delta_{N+1}(a) = (P \otimes P)\Delta_{N+1}(b) \quad \text{if} \quad P(a) = P(b)\ . \tag{3.3.49}$$

In order to prove the Hopf algebra axioms of Appendix A for Δ, ε, κ we just have to project those for Δ_{N+1}, ε_{N+1}, κ_{N+1} . For example, the first axiom is proved by applying $P \otimes P \otimes P$ to $(\Delta_{N+1} \otimes id)\Delta_{N+1}(a) = (id \otimes \Delta_{N+1})\Delta_{N+1}(a)$. The other axioms are proved in a similar way.

Notice that on the generators $T^a{}_b$, x^a, u, ξ and I the co-structures (3.3.48) act as in (3.3.7)-(3.3.9).

In conclusion: the elements $T^a{}_b$, x^a, u, ξ and I generate the Hopf algebra $GL_{q,r}(N+1)/H$ and satisfy the "$P(RTT)$" commutation rules (3.3.2)-(3.3.6). The co-structures act on them exactly as the co-structures defined in (3.3.7)-(3.3.9). Therefore the quotient $GL_{q,r}(N+1)/H$ is the q-inhomogeneous group defined at the beginning of this section:

$$IGL_{q,r}(N) = \frac{GL_{q,r}(N+1)}{H}\ . \tag{3.3.50}$$

The canonical projection $P\ :\ GL_{q,r}(N+1) \rightarrow IGL_{q,r}(N)$ is an epimorphism between these two Hopf algebrae.

Note 3.3.1 The consistency of the $P(RTT)$ relations with the co-structures Δ, ε and κ is easily proved. For example,

$$\Delta(P(R_{12}T_1T_2) - P(T_2T_1R_{12})) = 0 \tag{3.3.51}$$

is a particular case of eq. (3.3.49). Similarly for ε and κ.

We have thus obtained a R matrix formulation of the inhomogeneous $IGL_{q,r}(N)$ quantum groups. Indeed the results of this section can be summarized in the following theorem:

Theorem 3.3.2 The quantum inhomogeneous groups $IGL_{q,r}(N)$ are freely generated by the non-commuting matrix elements $T^A{}_B$ [A=$(0,a)$, with $a = 1,...N)$] and the identity I, modulo the relations:

$$T^0{}_b = 0 \tag{3.3.52}$$

and the RTT relations

$$R^{AB}{}_{EF}T^E{}_CT^F{}_D = T^B{}_FT^A{}_ER^{EF}{}_{CD} \tag{3.3.53}$$

The co-structures of $IGL_{q,r}(N)$ are simply given by:

$$\Delta(T^A{}_B) = T^A{}_C \otimes T^C{}_B \tag{3.3.54}$$
$$\kappa(T^A{}_B) = T^{-1}{}^D_C \tag{3.3.55}$$
$$\varepsilon(T^A{}_B) = \delta^A_B \tag{3.3.56}$$

□□□

Note 3.3.2 From the commutations (3.3.5) - (3.3.6) we see that one can set $u = I$ only when $q_{0a} = 1$ for all a.

Note 3.3.3 $P(\det T^A{}_B) = u \det T^a{}_b$ is central in $IGL_{q,r}(N)$ only when $Q_A = 1$, *A=0,1,..N* (apply the projection P to eq. (3.1.15)). Note that here we have $Q_A \equiv \prod_{C=0}^{N}(\frac{q_{CA}}{r})$.

Note 3.3.4 It is not difficult to see how the real forms of $GL_{q,r}(N+1)$ are inherited by $IGL_{q,r}(N)$.. In fact, only the conjugation i) of $GL_{q,r}(N+1)$, discussed in Section 3.1, is compatible with the coset structure of $IGL_{q,r}(N)$. More precisely, H is a $*$-Hopf ideal, i.e. $(H)^* \subseteq H$, only for $T^* = T$. Then we can define a $*$-structure on $IGL_{q,r}(N)$ as $[P(a)]^* \equiv P(a^*)$.

Theorem 3.3.3 The centrality of u is incompatible with the centrality of $\det T^a{}_b$.

Proof: Suppose that $q_{0a} = 1$ so that u is central. Then the centrality of $\det T^a{}_b$ is equivalent to the centrality of $P(\det T^A{}_B)$ and requires $Q_A = 1$ (Note 3.3.3); in particular $Q_0 \equiv \prod_{c=1}^{N} \frac{r}{q_{0c}} = 1$, which cannot be since for $q_{0a} = 1$ we find $Q_0 = r^N$.

□□□

The commutations of $\det T^a{}_b$ and ξ with all the generators are given by:

$$(\det T^c{}_d)T^a{}_b = \frac{Q_a}{Q_b}T^a{}_b(\det T^c{}_d), \qquad \zeta T^a{}_b = \frac{Q_b}{Q_a}T^a{}_b\zeta \tag{3.3.57}$$

$$(\det T^c{}_d)x^a = \frac{Q_a}{Q_0}x^a(\det T^c{}_d), \qquad \zeta x^a = \frac{Q_0}{Q_a}x^a\zeta \tag{3.3.58}$$

$$(\det T^c{}_d)u = u(\det T^c{}_d), \qquad \zeta u = u\zeta \tag{3.3.59}$$

where here $Q_a \equiv \prod_{c=1}^{N}(\frac{q_{ca}}{r})$ and ζ is the inverse of $\det T^c{}_d$, i.e. $\zeta \equiv u\xi$. We see that the commutations of $\det T^c{}_d$ with $T^a{}_b$ are the correct ones for $GL_{q,r}(N)$ (i.e. are identical to the ones deduced in Section 3.1). In the standard uniparametric case $Q_a = 1$, and the q-determinant $\det T^c{}_d$ becomes central (and likewise ζ), provided that also $Q_0 = 1$.

We have derived the properties of the quantum group $IGL_{q,r}(N)$ from those of $IGL_{q,r}(N+1)$, we now study the structure of $IGL_{q,r}(N)$ with respect to its Hopf subalgebra $GL_{q,r}(N)$; this is explicitly done in Theorem 3.3.4, while in Theorem 3.3.5 the same construction is seen in a more general and abstract setting.

We first notice that the $x^0 \equiv u$ and x^a elements generate a subalgebra of $IGL_{q,r}(N)$ because their commutation relations do not involve the $T^a{}_b$ elements. Moreover these elements can be ordered using (3.3.4) and (3.3.6), and the Poincaré series of this subalgebra is the same as that of the commutative algebra in $N+1$ indeterminates, indeed (3.3.4) and (3.3.6) read

$$x^A x^B = q_{AB} x^B x^A \qquad \forall\ A < B\ . \tag{3.3.60}$$

A linear basis of this subalgebra is therefore given by the ordered monomials: $\zeta^i \equiv u^{i_0}(x^1)^{i_1}...(x^N)^{i_N}$. Then, using (3.3.3) and (3.3.5), a generic element of $IGL_{q,r}(N)$ can be written as $\zeta^i a_i$ where $a_i \in GL_{q,r}(N)$ and we conclude that $IGL_{q,r}(N)$ is a right $GL_{q,r}(N)$–module generated by the ordered monomials ζ^i. Since the RTT relations of $IGL_{q,r}(N)$ (3.3.2)–(3.3.6) are homogeneous both in the x^a and in x^0 we can naturally introduce a (**Z**,**N**) grading : the generators x^a have grade $(0,1)$, x^0 has degree $(1,0)$, $(x^0)^{-1}$ has degree $(-1,0)$, the elements of $GL_{q,r}(N)$ have degree $(0,0)$. Then

$$IGL_{q,r}(N) \;=\; \sum_{(h,k)\in(\mathbf{Z},\mathbf{N})}^{\oplus} \Gamma^{(h,k)} \tag{3.3.61}$$

where $\Gamma^{(0,0)} = GL_{q,r}(N)$,

$$\Gamma^{(0,1)} = \{x^a b_a \ /\ b_a \in GL_{q,r}(N)\}\ , \qquad \Gamma^{(\pm 1,0)} = \{(x^0)^{\pm 1} b\ /\ b \in GL_{q,r}(N)\}$$
$$\Gamma^{(h,k)} = \{(x^0)^h x^{a_1} x^{a_2} \ldots x^{a_k} b_{a_1 a_2 \ldots a_k}\ /\ b_{a_1 a_2 \ldots a_k} \in GL_{q,r}(N)\} \qquad \forall\ h \in \mathbf{Z},\ k \in \mathbf{N}\ .$$

Therefore $IGL_{q,r}(N)$ is a direct sum of right $GL_{q,r}(N)$-modules; it is also a graded algebra with the product $\zeta^i b_i \cdot \zeta'^j b'_j$ trivially inherited from the $IGL_{q,r}(N)$ algebra structure (in the sequel we will omit the "$\cdot$").

We now show that each right module $\Gamma^{(h,k)}$ is a bicovariant bimodule on $GL_{q,r}(N)$, also $IGL_{q,r}(N) = \sum_{(h,k)\in(\mathbf{Z},\mathbf{N})}^{\oplus} \Gamma^{(h,k)}$ is a bicovariant bimodule with left and right coactions δ_L and δ_R that are multiplicative: forall a, $b \in IGL_{q,r}(N)$, $\delta_L(ab) = \delta_L(a)\delta_L(b)$, $\delta_R(ab) = \delta_R(a)\delta_R(b)$. This shows that the structure of a inhomogeneous quantum group is similar to that of the exterior algebra of a generic Hopf algebra [as discussed at the end of point **iv)**, Section 2.1]; also recall that, as noticed in the end of Subsection 2.4.6, the exterior algebra of forms is a Hopf algebra.

Theorem 3.3.4 $IGL_{q,r}(N)$, when $q_{a0} = const\ \forall a$, is a bicovariant graded algebra, i.e. it is a graded algebra with left and right coactions

$$\begin{aligned} \delta_L &: IGL_{q,r}(N) \to GL_{q,r}(N) \otimes IGL_{q,r}(N) \\ \delta_R &: IGL_{q,r}(N) \to IGL_{q,r}(N) \otimes GL_{q,r}(N) \end{aligned}$$

that commute, see (3.3.67), are multiplicative: $\delta_L(ab) = \delta_L(a)\delta_L(b)$, $\delta_R(ab) = \delta_R(a)\delta_R(b)$, $\forall$ a, $b \in IGL_{q,r}(N)$, and preserve the grading .

Proof Consider the linear map $\delta_R : IGL_{q,r}(N) \to IGL_{q,r}(N) \otimes GL_{q,r}(N)$ defined by

$$\delta_R(x^A) = x^A \otimes I \ ; \ \ \delta_R(a) = \Delta(a) \quad \forall\, a \in GL_{q,r}(N). \tag{3.3.62}$$

and extended multiplicatively on all $IGL_{q,r}(N)$. This grade preserving map is obviously well defined on $GL_{q,r}(N)$ because it coincides with the coproduct on $GL_{q,r}(N)$ [$GL_{q,r}(N)$ is the Hopf subalgebra of $IGL_{q,r}(N)$ with degree zero]; it is also well defined on all $IGL_{q,r}(N)$ since it is multiplicative and compatible with (3.3.2)-(3.3.6). We check for example (3.3.3) with $q_{a0} = const \equiv q_0\ \ \forall a$:

$$\delta_R(x^a T^b{}_d) = x^a T^b{}_c \otimes T^c{}_d = \frac{q_0}{r} R^{ba}{}_{ef} T^e{}_c x^f \otimes T^c{}_d = \delta_R\Big(\frac{q_0}{r} R^{ba}{}_{ef} T^e{}_d x^f\Big)\,.$$

This shows that $\delta_R : IGL_{q,r}(N) \to IGL_{q,r}(N) \otimes GL_{q,r}(N)$ is well defined.

To show that δ_R is a right coaction notice that

$$\forall\, \zeta^i a_i\,, \quad (\delta_R \otimes id)\delta_R(\zeta^i a_i) = (id \otimes \Delta)\delta_R(\zeta^i a_i)\ ; \quad (id \otimes \varepsilon)\delta_R(\zeta^i a_i) = \zeta^i a_i\ . \tag{3.3.63}$$

For the left coaction we proceed as in the previous case, defining the linear map $\delta_L : IGL_{q,r}(N) \to GL_{q,r}(N) \otimes IGL_{q,r}(N)$,

$$\delta_L(x^a) = T^a{}_b \otimes x^b\ ; \ \ \delta_L(x^0) = I \otimes x^0\ ; \ \ \delta_L(a) = \Delta(a) \quad \forall\, a \in GL_{q,r}(N) \tag{3.3.64}$$

which is extended multiplicatively on all $IGL_{q,r}(N)$. As was the case for δ_R, it is well defined on $GL_{q,r}(N)$ and it is also well defined on all $IGL_{q,r}(N)$ because it is multiplicative and compatible with (3.3.2)-(3.3.6).

To prove that δ_L is a left coaction notice that

$$(\varepsilon \otimes id)\delta_L(x^a) = x^a\ ,\ (\Delta \otimes id)\delta_L(x^a) = T^a{}_d \otimes T^d{}_b \otimes x^b = (id \otimes \delta_L)\delta_L(x^a) \tag{3.3.65}$$

and similarly for x^0. Now since $\delta_L(a) \doteq \Delta(a)$ if $a \in GL_{q,r}(N)$, and since δ_L is multiplicative, we have on all $IGL_{q,r}(N)$:

$$(\varepsilon \otimes id)\delta_L = id\ ; \ \ (\Delta \otimes id)\delta_L = (id \otimes \delta_L)\delta_L\ . \tag{3.3.66}$$

Finally, the compatibility of δ_L and δ_R:

$$(id \otimes \delta_R)\delta_L = (\delta_L \otimes id)\delta_R \tag{3.3.67}$$

follows directly from:

$$(id \otimes \delta_R)\delta_L(x^a) = T^a{}_b \otimes x^b \otimes I = (\delta_L \otimes id)\delta_R(x^a)$$
$$(id \otimes \delta_R)\delta_L(x^0) = I \otimes x^0 \otimes I = (\delta_L \otimes id)\delta_R(x^0)$$

□□□

Corollary 3.3.5 $IGL_{q,r}(N)$, for $q_{a0} = const\ \forall a$, is a bicovariant bimodule over $GL_{q,r}(N)$ freely generated, as a right module, by the elements ζ^i; also any submodule $\Gamma^{(h,k)}$ is a bicovariant bimodule freely generated by the elements ζ^i with degree (h,k).

Proof We immediately have that $IGL_{q,r}(N)$ and $\Gamma^{(h,k)}$ are bimodules with the left module structure trivially inherited from the algebra $IGL_{q,r}(N)$. $IGL_{q,r}(N)$ is a bicovariant bimodule because, since the left and right coactions δ_L and δ_R are multiplicative, they are compatible with the left and right product of $GL_{q,r}(N)$ on $IGL_{q,r}(N)$; moreover they satisfy (3.3.67). Also the submodules $\Gamma^{(h,k)}$ are bicovariant bimodules since the coactions δ_L and δ_R are grade preserving.

We now recall that a bicovariant bimodule is always freely generated by a basis of right invariant elements, [cf. the text after (2.1.47)]. We also know that the ζ^i are right invariant. Now, since they generate $IGL_{q,r}(N)$, they linearly span the space of right invariant elements $[IGL_{q,r}(N)]_{\text{inv}}$, and since they are linearly independent, they form a basis of $[IGL_{q,r}(N)]_{\text{inv}}$. We conclude that $IGL_{q,r}(N)$ is freely generated by the ζ^i: $\zeta^i a_i = 0 \Rightarrow a_i = 0\ \forall i$. The same arguments apply also to each submodule $\Gamma^{(h,k)}$. □□□

In conclusion, the Hopf algebra $IGL_{q,r}(N)$ is very rich because it is both a bicovariant and a graded algebra on $GL_{q,r}(N)$. The bicovariant structure of $IGL_{q,r}(N)$ can be seen as an example of a general theory by Radford [61] on the properties of Hopf algebras A with a Hopf subalgebra H that is also a quotient of A. On the structure of inhomogeneous quantum groups see also the last reference in [65]. We summarize some results of [61] in the following

Theorem 3.3.6 Let A and H be Hopf algebras and suppose there exist Hopf algebras homomorphisms $\pi : A \to H$ and $i : H \hookrightarrow A$ such that $\pi \circ i = id_H$. Consider the projection Π on A defined by:

$$\Pi(a) = a_1 i[\kappa\pi(a_2)] \quad a \in A \tag{3.3.68}$$

and let

$$B \equiv \Pi(A)\ . \tag{3.3.69}$$

Then:

a) B is a subalgebra of A, $\Delta(B) \subseteq A \otimes B$ and

$$B \equiv \Pi(A) = \{b\ /\ \ b_1 \otimes \pi(b_2) = b \otimes I\}. \tag{3.3.70}$$

b) B is also a coalgebra with counit $\underline{\varepsilon}$ that is the restriction to B of the counit ε of A, and with coproduct $\underline{\Delta}$ given by

$$\underline{\Delta}(\Pi(a)) = \Pi(a_1) \otimes \Pi(a_2) \ . \tag{3.3.71}$$

[Notice that $\underline{\Delta}$ is in general not compatible with the algebra structure of B, only (1.1.4) and (1.1.5) hold].

c) B is an H-bicovariant algebra with trivial right action and coaction and with left action given by the adjoint map $ad_h b = i(h_1) b i(\kappa(h_2))$ $\forall h \in H$, $\forall b \in B$, and left coaction given by $\delta_L(b) = \pi(b_1) \otimes b_2 \in H \otimes B$, $\forall b \in B$.

d) As a coalgebra B is compatible with the left action ad and with the left coaction δ_L (we use the notation $\underline{\Delta}(b) = b_{\underline{1}} \otimes b_{\underline{2}}$):

$$\begin{aligned} &\underline{\Delta}(ad_h b) = ad_{h_1} b_{\underline{1}} \otimes ad_{h_2} b_{\underline{2}} \ , \quad \underline{\varepsilon}(ad_h b) = \varepsilon(h)\underline{\varepsilon}(b) \ , \\ &(id \otimes \underline{\Delta})\delta_L(b) = (m_H \otimes id)(\delta_L \otimes \delta_L)\underline{\Delta}(b) \ , \\ &(id \otimes \underline{\varepsilon})\delta_L(b) = \underline{\varepsilon}(b) I_H \ . \end{aligned} \tag{3.3.72}$$

Moreover

$$\underline{\Delta}(bb') = b_{\underline{1}}\, ad_{b_{\underline{2}}{}^{(1)}} b'_{\underline{1}} \otimes b_{\underline{2}}{}^{(2)} b'_{\underline{2}} \tag{3.3.73}$$

where we have used the notations $\delta_L(b) = b^{(1)} \otimes b^{(2)}$.

e) $B \otimes H$ has a canonical Hopf algebra structure (cross-product and cross-coproduct construction) denoted $B \rtimes H$. The product is given by:

$$(b \otimes h)(b' \otimes h') = b(ad_{h_1} b') \otimes h_2 h' \quad \forall h \in H, \ b \in B \tag{3.3.74}$$

the counit is given by $\underline{\varepsilon} \otimes \varepsilon$ and the coproduct is given by

$$\Delta(b \otimes h) = (b_{\underline{1}} \otimes b_{\underline{2}}{}^{(1)} h_1) \otimes (b_{\underline{2}}{}^{(2)} \otimes h_2) \ . \tag{3.3.75}$$

d) $B \rtimes H$ and A are isomorphic Hopf algebras via the isomorphism ϑ:

$$\vartheta(b \otimes h) = b i(h) \quad , \quad \vartheta^{-1}(a) = \Pi(a_1) \otimes \pi(a_2) \tag{3.3.76}$$

□□□

Note 3.3.5 The algebra B has a braided Hopf algebra structure, and the quantum group A has a natural interpretation via Majid bosonization procedure [62], [63]. Since B is a bicovariant bimodule over H, with trivial right action and right coaction [i.e. B is a bimodule on the quantum double $\mathcal{D}(H)$, cf. Note 2.3.3] the braiding Ψ is given via the left action and the left coaction of H on B:

$$\Psi(b \otimes b') = ad_{b^{(1)}} b' \otimes b^{(2)} \tag{3.3.77}$$

then (3.3.73) states that $\underline{\Delta}$ as defined in (3.3.71) is braided multiplicative:

$$\underline{\Delta}(bb') = b_{\underline{1}}\Psi(b_{\underline{2}} \otimes b'_{\underline{1}})b'_{\underline{2}} \,. \tag{3.3.78}$$

Finally B is a braided Hopf algebra with antipode $\underline{\kappa}(b) = i[\pi(b_1)]\kappa(b_2)$.

In our case $A = IGL_{q,r}(N)$, the projection $\pi : IGL_{q,r}(N) \to GL_{q,r}(N)$, that is well defined only if $q_{a0} = const \equiv q_0 \;\forall a$, is given by

$$\pi(T^a{}_b) = T^a{}_b \,, \quad \pi(u) = I \,, \quad \pi(x^a) = 0 \,, \tag{3.3.79}$$

the subalgebra B is generated by the $x^0 \equiv u$ and x^a elements that satisfy (3.3.4) and (3.3.6) i.e. $x^A x^B = rR^{-1}{}^{AB}{}_{CD}x^D x^C$. The braiding is: $\Psi(x^a \otimes x^b) = \frac{r}{q_0}R^{-1}{}^{AB}{}_{CD}x^D x^C$, $\Psi(x \otimes u) = u \otimes x$, $\Psi(u \otimes x) = x \otimes u$, the coproduct (coaddition) is $\underline{\Delta}(u) = u \otimes u$, $\underline{\Delta}(x^a) = x^a \otimes I + I \otimes x^a$ and the braided antipode and counit are $\underline{\kappa}(u) = u^{-1}$, $\underline{\kappa}(x^a) = -x^a$ and $\underline{\varepsilon}(u) = 1$, $\underline{\varepsilon}(x^a) = 0$.

Note 3.3.6 We also have $IGL_{q,r}(N) = M \rtimes GL^u_{q,r}(N)$ where M is the subalgebra of $IGL_{q,r}(N)$ generated by the elements $x^a u^{-1}$ and $GL^u_{q,r}(N)$ is the quantum group generated by $GL_{q,r}(N)$ and the dilatation u.

3.4 Differential calculus on $GL_{q,r}(N)$

The bicovariant differential calculus on the uniparametric q-groups of the A, B, C, D series can be formulated in terms of the corresponding R-matrix and the associated $L^\pm$ functionals. This holds also for the multiparametric case. In fact all formulas are the same, modulo substituting the q parameter with r when it appears explicitly (typically as $\frac{1}{q-q^{-1}}$).

We list here some relevant formulae that do not appear in Section 2.2:

$$\omega_{A_1}^{\ A_2} T^R{}_S = s(R^{-1})^{TB_1}{}_{CA_1}(R^{-1})^{A_2C}{}_{B_2S}T^R{}_T\omega_{B_1}^{\ B_2} \tag{3.4.1}$$

$$\omega_{A_1}^{\ A_2} \det T = s^N r^{-2}\frac{Q_{A_1}}{Q_{A_2}}(\det T)\omega_{A_1}^{\ A_2} \tag{3.4.2}$$

$$\omega_{A_1}^{\ A_2}\Xi = s^{-N}r^2\frac{Q_{A_2}}{Q_{A_1}}(\Xi)\omega_{A_1}^{\ A_2} \tag{3.4.3}$$

where we recall that $s \equiv (c^+)^{-1}c^-$, cf. eq.s (2.2.2) and (2.2.3).

Notice that $\det T$ and Ξ commute with all the ω (and thus can be set to I) iff all Q_A are equal and for $s^N r^{-2} = 1$, or $s = r^{\frac{2}{N}}\alpha$ with $\alpha^N = 1$, which agrees with the condition found in Note 3.1.5.

Using

$$da = (\chi^{A_1}_{\ A_2} * a)\,\omega_{A_1}^{\ A_2} \tag{3.4.4}$$

we compute the exterior derivative on the basis elements of $GL_{q,r}(N)$, and on the q-determinant:

$$d\,T^A{}_B = \frac{1}{r-r^{-1}}[s\ (R^{-1})^{CR}{}_{ET}(R^{-1})^{TE}{}_{SB}T^A{}_C - \delta^S_R T^A{}_B]\ \omega_R^S \tag{3.4.5}$$

$$d\,\Xi = \frac{s^{-N}r^2-1}{r-r^{-1}}\ \Xi\,\tau \tag{3.4.6}$$

$$d\ \det T = \frac{s^N r^{-2}-1}{r-r^{-1}}\ (\det T)\tau \tag{3.4.7}$$

The reader can verify via the Leibniz rule, and with the help of eq. (3.4.2), that $d[(\det T)\Xi] = d[\Xi(\det T)] = 0$. From (3.4.7) we find that the bi-invariant 1-form τ that defines the exterior differential via $da = \frac{1}{r-r^{-1}}[\tau a - a\tau]$ is given by

$$\tau = \frac{r-r^1}{s^N r^{-2}-1}\Xi d\det T\ . \tag{3.4.8}$$

again, $\det T = I = \Xi$ requires $s^N r^{-2} = 1$.

The expression of the ω in terms of a linear combination of $\kappa(T)dT$, similar to the classical case is:

$$\begin{aligned}\omega_A{}^A &= \frac{r}{s(s-r^2-r^4+sr^4)}[(r^2-s)\ \kappa(T^A{}_B)dT^B{}_A + r^2(s-1)\ \kappa(T^C{}_B)dT^B{}_C\theta^{CA} + \\ &\quad +(-r^2-sr^2+s+sr^4)\ \kappa(T^C{}_B)dT^B{}_C\theta^{AC}], \qquad \text{no sum on A} \qquad (3.4.9)\\ \omega_A{}^B &= -s^{-1}\frac{r}{q_{BA}}\kappa(T^B{}_C)dT^C{}_A, \qquad A \neq B \qquad (3.4.10)\end{aligned}$$

When $s=1$, the classical limit $\omega_A{}^B \stackrel{q=r\to 1}{\longrightarrow} -\kappa(T^A{}_C)dT^C{}_B$ reproduces the familiar formula $\omega = -g^{-1}dg$ for the left-invariant 1-forms on the group manifold. More generally, for $s = r^\alpha, \alpha \in \mathbf{C}$, we have:

$$\omega_A{}^A \stackrel{r\to 1}{\longrightarrow} [\frac{2-\alpha}{2(\alpha-1)}\sum_B \kappa(T^A{}_B)dT^B{}_A + \frac{\alpha}{2(\alpha-1)}\sum_B\sum_{C\neq A}\kappa(T^C{}_B)dT^B{}_C], \quad \text{no sum on A,} \tag{3.4.11}$$

which shows that the inversion formula (3.4.9) diverges in the classical limit for $s = r$.

Conjugation

Compatibility of the conjugation defined in (3.2.34) with the differential calculus requires $(\chi_i)^*$ to be a linear combination of the χ_i. From (3.2.35) and (2.2.52), it is straightforward to find how the $*$-conjugation acts on the tangent vectors χ. Only the conjugations i) and iii) are compatible with the differential calculus:

$$\begin{aligned}&i)\quad (\chi^A{}_B)^* = -r^{-2N}d_C R^{DA}{}_{BC}(\chi^C{}_D)\\ &iii)\quad (\chi^A{}_B)^* = (\chi^B{}_A)\end{aligned} \tag{3.4.12}$$

Using the inversion formulae (3.4.10) and (3.4.5), or using (2.3.52), one finds the induced $*$-conjugation on the left-invariant 1-forms (here $s = r^\alpha,\ \alpha \in \mathbf{R}$):

$$\begin{aligned} i) \quad & ({\omega_A}^B)^* = r^{2N} (R^{-1})^{BC}{}_{DA} d_B^{-1} {\omega_C}^D \ , \\ iii) \quad & ({\omega_A}^B)^* = -{\omega_B}^A \ . \end{aligned} \tag{3.4.13}$$

3.5 Differential calculus on $IGL_{q,r}(N)$

In this section we present a bicovariant differential calculus on $IGL_{q,r}(N)$, based on the following set of functionals f and elements M :

$$\begin{aligned} f_{a_1}{}^{a_2 b_1}{}_{b_2} &= \kappa'(L^{+b_1}{}_{a_1}) L^{-a_2}{}_{b_2} \\ f_{a_1}{}^{a_2 0}{}_{b_2} &= \kappa'(L^{+0}{}_{a_1}) L^{-a_2}{}_{b_2} \\ f_0{}^{a_2 b_1}{}_{b_2} &= \kappa'(L^{+b_1}{}_0) L^{-a_2}{}_{b_2} \equiv 0 \\ f_0{}^{a_2 0}{}_{b_2} &= \kappa'(L^{+0}{}_0) L^{-a_2}{}_{b_2} \end{aligned} \tag{3.5.1}$$

$$\begin{aligned} M^{b_1}{}_{b_2 a_1}{}^{a_2} &= T^{b_1}{}_{a_1} \kappa(T^{a_2}{}_{b_2}) \\ M^{b_1}{}_{b_2 0}{}^{a_2} &= T^{b_1}{}_0 \kappa(T^{a_2}{}_{b_2}) \\ M^{0}{}_{b_2 a_1}{}^{a_2} &= 0 \\ M^{0}{}_{b_2 0}{}^{a_2} &= T^0{}_0 \kappa(T^{a_2}{}_{b_2}) \end{aligned} \tag{3.5.2}$$

The f in (3.5.1) are a subset of the f functionals of $GL_{q,r}(N+1)$, obtained by restricting the indices of $f^i{}_j$ to $i = ab$ and $i = 0b$. The third f is identically zero because of upper triangularity of L^+, i.e. $L^{+b_1}{}_0 = 0$.

The elements $M \in IGL_{q,r}(N)$ in (3.5.2) are obtained with the same restriction on the adjoint indices, and by projecting with P. The effect of the projection is to replace the coinverse in $GL_{q,r}(N+1)$, i.e. κ_{N+1} , with the coinverse κ of $IGL_{q,r}(N)$ (see the last of (3.3.48)). The third element in (3.5.2) becomes zero because of P.

Theorem 3.5.1 The functionals in (3.5.1) vanish when applied to elements of $Ker(P) \subset GL_{q,r}(N+1)$.

Proof: first one checks directly that the functionals (3.5.1) vanish when applied to $T^0{}_b$. This extends to any element of the form $T^0{}_b a$ $(a \in A)$, i.e. to any element of $Ker(P)$, because of the property (2.1.32) and the vanishing of the functionals in (3.5.7).

□□□

The theorem states that the f functionals are well defined on the quotient $IGL_{q,r}(N) = GL_{q,r}(N+1)/Ker(P)$, in the sense that the "projected" functionals

$$\bar{f} : IGL_{q,r}(N) \to \mathbf{C}, \quad \bar{f}(P(a)) \equiv f(a) \ , \quad \forall a \in GL_{q,r}(N+1) \tag{3.5.3}$$

are well defined. Indeed if $P(a) = P(b)$, then $f(a) = f(b)$ because $f(Ker(P)) = 0$. This holds for any functional f vanishing on $Ker(P)$, not only for the $f^i{}_j$ functionals.

The product fg of two generic functionals vanishing on $KerP$ also vanishes on $KerP$, because $KerP$ is a co-ideal (see Theorem 3.3.1): $fg(KerP) = (f \otimes g)\Delta_{N+1}(KerP) = 0$. Therefore $\overline{fg}$ is well defined on $IGL_{q,r}(N)$, and

$$\overline{fg}[P(a)] = fg(a) = (f \otimes g)\Delta_{N+1}(a) = (\bar{f}P \otimes \bar{g}P)\Delta_{N+1}(a) = (\bar{f} \otimes \bar{g})\Delta(P(a)) \equiv \bar{f}\bar{g}[P(a)] \tag{3.5.4}$$

(use the first of (3.3.48)) so that the product of $\bar{f}$ and $\bar{g}$ involves the coproduct Δ of $IGL_{q,r}(N)$.

There is a natural way to introduce a coproduct on the $\bar{f}$'s :

$$\Delta'\bar{f}[P(a) \otimes P(b)] \equiv \bar{f}[P(a)P(b)] = \bar{f}[P(ab)] = f(ab) = \Delta'_{N+1}f(a \otimes b)\,. \tag{3.5.5}$$

It is also easy to show that

$$\Delta'\bar{f}^i{}_j = \bar{f}^i{}_k \otimes \bar{f}^k{}_j \quad \text{i.e.} \quad \bar{f}^i{}_j[P(a)P(b)] = \bar{f}^i{}_k[P(a)]\bar{f}^k{}_j[P(b)] \tag{3.5.6}$$

with i, j, k running over the restricted set of indices $ab, 0b$. Indeed due to

$$f_0{}^{A_2 b_1}{}_{B_2} \equiv 0, \quad f_{A_1}{}^{0B_1}{}_{b_2} \equiv 0 \tag{3.5.7}$$

(a consequence of upper and lower triangularity of L^+ and L^- respectively), formulae (2.1.35) and (2.1.32) involve only the $f^i{}_j$ listed in (3.5.1), which annihilate $KerP$. Then

$$\bar{f}^i{}_j[P(a)P(b)] = \bar{f}^i{}_j[P(ab)] = f^i{}_j(ab) = f^i{}_k(a)f^k{}_j(b) = \bar{f}^i{}_k[P(a)]\bar{f}^k{}_j[P(b)] \tag{3.5.8}$$

and (3.5.6) is proved.

With abuse of notations we will simply write f instead of $\bar{f}$. Then the f in (3.5.1) will be seen as functionals on $IGL_{q,r}(N)$. Notice that with the same abuse of notations, the product, coproduct and antipode of the f and $\bar{f}$ functionals coincide.

Theorem 3.5.2 The right A-module ($A = IGL_{q,r}(N)$) Γ freely generated by $\omega^i \equiv \omega_{a_1}{}^{a_2}, \omega_0{}^{a_2}$ is a bicovariant bimodule over $IGL_{q,r}(N)$ with right multiplication:

$$\omega^i a = (f^i{}_j * a)\omega^j, \quad a \in IGL_{q,r}(N) \tag{3.5.9}$$

where the $f^i{}_j$ are given in (3.5.1), the $*$-product is computed with the co-product Δ of $IGL_{q,r}(N)$, and the left and right actions of $IGL_{q,r}(N)$ on Γ are given by

$$\Delta_L(a_i\omega^i) \equiv \Delta(a_i)I \otimes \omega^i \tag{3.5.10}$$

$$\Delta_R(a_i\omega^i) \equiv \Delta(a_i)\omega^j \otimes M_j{}^i \tag{3.5.11}$$

where the $M_j{}^i$ are given in (3.5.2) and from now on we use the notation $\Delta_L = \Delta_\Gamma$ and $\Delta_R = {}_\Gamma\Delta$.

Proof: we prove the theorem by showing that the functionals f and the elements M listed in (3.5.1) and (3.5.2) satisfy the properties (2.1.32)-(2.1.51) (cf. the theorem by Woronowicz discussed in the Section 2.1). It is straightforward to verify directly that the elements M in (3.5.2) do satisfy the properties (2.1.44) and (2.1.45). We have already shown that the functionals f in (3.5.1) satisfy (2.1.32), and property (2.1.33) obviously also holds for this subset.

Consider now the last property (2.1.51). We know that it holds for $GL_{q,r}(N+1)$. Take the free indices j and k as ab and $0b$, and apply the projection P on both members of the equation. It is an easy matter to show that only the f's in (3.5.1) and the M's in (3.5.2) enter the sums: this is due to the vanishing of some $P(M)$ and to (3.5.7). We still have to prove that the $*$ product in (2.1.51) can be computed via the coproduct Δ in $IGL_{q,r}(N)$. Consider the projection of property (2.1.51), written symbolically as:

$$P[M(f \otimes id)\Delta_{N+1}(a)] = P[(id \otimes f)\Delta_{N+1}(a)M] \ . \tag{3.5.12}$$

Now apply the definition (3.5.3) and the first of (3.3.48) to rewrite (3.5.12) as

$$P(M)(\bar{f} \otimes id)\Delta(P(a)) = (id \otimes \bar{f})\Delta(P(a))P(M) \ . \tag{3.5.13}$$

This projected equation then *becomes* property (2.1.51) for the $IGL_{q,r}(N)$ functionals f and adjoint elements M, with the correct coproduct Δ of $IGL_{q,r}(N)$□□□

Using the general formula (3.5.9) we can deduce the ω, T commutations for $IGL_{q,r}(N)$:

$$\omega_{a_1}{}^{a_2} T^r{}_s = s(R^{-1})^{tb_1}{}_{ca_1}(R^{-1})^{a_2c}{}_{b_2s} T^r{}_t \omega_{b_1}{}^{b_2} \tag{3.5.14}$$

$$\omega_{a_1}{}^{a_2} x^r = s\frac{q_{0a_1}}{q_{0a_2}} x^r \omega_{a_1}{}^{a_2} - (r - r^{-1})\frac{sr}{q_{0a_2}} T^r{}_{a_1}\omega_0{}^{a_2} \tag{3.5.15}$$

$$\omega_{a_1}{}^{a_2} u = s\frac{q_{0a_1}}{q_{0a_2}} u\ \omega_{a_1}{}^{a_2} \tag{3.5.16}$$

$$\omega_{a_1}{}^{a_2} \det T^a{}_b = s^N r^{-2}\frac{Q_{a_1}}{Q_{a_2}}(\det T^a{}_b)\omega_{a_1}{}^{a_2} \tag{3.5.17}$$

$$\zeta\omega_{a_1}{}^{a_2} = s^N r^{-2}\frac{Q_{a_1}}{Q_{a_2}}\omega_{a_1}{}^{a_2}\zeta \tag{3.5.18}$$

$$\omega_0{}^{a_2} T^r{}_s = s\frac{r}{q_{0t}}(R^{-1})^{a_2t}{}_{b_2s} T^r{}_t \omega_0{}^{b_2} \tag{3.5.19}$$

$$\omega_0{}^{a_2} x^r = \frac{s}{q_{0a_2}} x^r \omega_0{}^{a_2} \tag{3.5.20}$$

$$\omega_0{}^{a_2} u = \frac{s}{q_{0a_2}} u\omega_0{}^{a_2} \tag{3.5.21}$$

$$\omega_0{}^{a_2}\det T^a{}_b = s^N r^{-2}\frac{Q_0}{Q_{a_2}}(\det T^a{}_b)\omega_0{}^{a_2} \tag{3.5.22}$$

$$\zeta\omega_0{}^{a_2} = s^N r^{-2}\frac{Q_0}{Q_{a_2}}\omega_0{}^{a_2}\zeta \tag{3.5.23}$$

Note 3.5.1 u commutes with all ω 's only if $q_{0a} = 1$ (cf. Note 3.3.2) and $s = 1$. This means that $u = I$ is consistent with the differential calculus on $IGL_{q_{0a}=1,r}(N)$ only if the additional condition $s = 1$ is satisfied.

The 1-form $\tau \equiv \sum_a \omega_a{}^a$ is bi-invariant, as one can check by using (3.5.10)-(3.5.11). Then an exterior derivative on $IGL_{q,r}(N)$ can be defined as in eq. (2.2.45), and satisfies the Leibniz rule. The alternative expression $da = (\chi_i * a)\omega^i$ (cf. (3.4.4)) continues to hold, where

$$\begin{aligned}\chi^a{}_b &= \frac{1}{r - r^{-1}}[f_c{}^{ca}{}_b - \delta^a_b\varepsilon]\\ \chi^0{}_b &= \frac{1}{r - r^{-1}}[f_c{}^{c0}{}_b]\end{aligned} \tag{3.5.24}$$

are the left-invariant vectors dual to the left-invariant 1-forms $\omega_a{}^b$ and $\omega_0{}^b$. They are functionals on $IGL_{q,r}(N)$ and as a consequence of (3.5.6) we have

$$\Delta'(\chi^a{}_b) = \chi^c{}_d \otimes f_c{}^{da}{}_b + \varepsilon \otimes \chi^a{}_b \tag{3.5.25}$$

$$\Delta'(\chi^0{}_b) = \chi^c{}_d \otimes f_c{}^{d0}{}_b + \chi^0{}_d \otimes f_0{}^{d0}{}_b + \varepsilon \otimes \chi^0{}_b \tag{3.5.26}$$

The exterior derivative on the generators $T^a{}_b$ is given by formula (3.4.5) with lower case indices. For the other generators we find:

$$dx^a = -s\frac{r}{q_{0s}}T^a{}_s\omega_0{}^s + \frac{s-1}{r - r^{-1}}x^a\tau \tag{3.5.27}$$

$$du = \frac{s-1}{r - r^{-1}}u\tau \tag{3.5.28}$$

$$d\xi = \frac{s^{-N-1}r^2 - 1}{r - r^{-1}}\ \xi\ \tau \tag{3.5.29}$$

Moreover:

$$d(\det T^a{}_b) = \frac{s^N r^{-2} - 1}{r - r^{-1}}\ (\det T^a{}_b)\tau \tag{3.5.30}$$

$$d\zeta = \frac{s^{-N}r^2 - 1}{r - r^{-1}}\ \zeta\tau \tag{3.5.31}$$

($\zeta \equiv u\xi$). Again we find that $u = I$ implies $s = 1$, and $\det T^a{}_b = \zeta = I$ requires $s^N r^{-2} = 1$.

Every element ρ of Γ can be written as $\rho = a_k db_k$ for some a_k, b_k belonging to $IGL_{q,r}(N)$. In fact one has the same formula as in (3.4.9) for $\omega_m{}^n$, where all indices now are lower case. For $\omega_0{}^n$ we find:

$$\omega_0{}^n = -\frac{q_{0n}}{sr}[\kappa(T^n{}_a)dx^a + \kappa(x^n)du] \tag{3.5.32}$$

Finally, the two properties (2.1.3) and (2.1.4) hold also for $IGL_{q,r}(N)$, because of the bi-invariance of $\tau = \omega_a{}^a$. Thus all the axioms for a bicovariant first order differential calculus on $IGL_{q,r}(N)$ are satisfied.

The exterior product of left-invariant 1-forms is defined as

$$\omega^i \wedge \omega^j \equiv \omega^i \otimes \omega^j - \Lambda^{ij}{}_{kl}\omega^k \otimes \omega^l \tag{3.5.33}$$

where

$$\Lambda^{ij}{}_{kl} = f^i{}_l(M_k{}^j) \tag{3.5.34}$$

This Λ tensor can in fact be obtained from the one of $GL_{q,r}(N+1)$ by restricting its indices to the subset $ab, 0b$. This is true because when $i, l = ab$ or $0b$ we have $f^i{}_l(KerP) = 0$ so that $f^i{}_l$ is well defined on $IGL_{q,r}(N)$, and we can write $f^i{}_l(M_k{}^j) = \bar{f}^i{}_l[P(M_k{}^j)]$ (see discussion after Theorem 3.5.1). The non-vanishing components of Λ read:

$$\Lambda_{a_1}{}^{a_2}{}_{d_1}{}^{d_2}|^{c_1}{}_{c_2}{}^{b_1}{}_{b_2} = d^{f_2} d^{-1}_{c_2} R^{f_2 b_1}{}_{c_2 g_1} (R^{-1})^{c_1 g_1}{}_{e_1 a_1} (R^{-1})^{a_2 e_1}{}_{g_2 d_1} R^{g_2 d_2}{}_{b_2 f_2} \tag{3.5.35}$$

$$\Lambda_0{}^{a_2}{}_{d_1}{}^{d_2}|^{c_1}{}_{c_2}{}^{0}{}_{b_2} = \frac{q_{0c_2}}{q_{0c_1}} (R^{-1})^{a_2 c_1}{}_{g_2 d_1} R^{g_2 d_2}{}_{b_2 c_2} \tag{3.5.36}$$

$$\Lambda_{a_1}{}^{a_2}{}_0{}^{d_2}|^{c_1}{}_{c_2}{}^{0}{}_{b_2} = -(r - r^{-1}) \frac{q_{0c_2}}{q_{0a_2}} \delta^{c_1}_{a_1} R^{a_2 d_2}{}_{b_2 c_2} \tag{3.5.37}$$

$$\Lambda_{a_1}{}^{a_2}{}_0{}^{d_2}|^{0}{}_{c_2}{}^{b_1}{}_{b_2} = \frac{q_{0a_1}}{q_{0a_2}} d^{f_2} d^{-1}_{c_2} R^{f_2 b_1}{}_{c_2 a_1} R^{a_2 d_2}{}_{b_2 f_2} \tag{3.5.38}$$

$$\Lambda_0{}^{a_2}{}_0{}^{d_2}|^{0}{}_{c_2}{}^{0}{}_{b_2} = \frac{q_{0c_2}}{q_{0a_2}} r^{-1} R^{a_2 d_2}{}_{b_2 c_2} \tag{3.5.39}$$

These components still satisfy the characteristic equation (2.2.40), because the Λ tensor of $GL_{q,r}(N+1)$ does satisfy this equation, and if the free adjoint indices are taken as ab, $0b$, only the components in (3.5.35)-(3.5.39) enter in (2.2.40). To prove this, consider $\Lambda^{ij}{}_{kl}$ with k, l of the type ab or $0b$ and observe that it vanishes unless also i, j are of the type $ab, 0b$. (This can be checked directly via the formula (2.2.39)). Then equations (2.2.42) and (2.2.43) hold also for the ω's of $IGL_{q,r}(N)$.

Note that Λ^{-1} tensor of $IGL_{q,r}(N)$ can be obtained by specializing the indices in the Λ^{-1} tensor of $GL_{q,r}(N+1)$ given in (2.2.37), as we did for Λ. The reader can convince himself of this by i) observing that the $\Lambda^{-1}{}^{ij}{}_{kl}$ tensor of (2.2.37) also vanishes when $k, l = ab$ or $0b$ and i, j are not of the type $ab, 0b$; ii) considering the equation $\Lambda^{-1}{}^{ij}{}_{rs}\Lambda^{rs}{}_{kl} = \delta^i_k \delta^j_l$ for $k, l = ab$ or $0b$.

The exterior differential on $\Gamma^{\wedge n}$ can be defined as in Section 2.2 (eq. (2.2.44)), and satisfies all the properties (2.1.81)-(2.1.84). As for $GL_{q,r}(N+1)$ the last two hold because of the bi-invariance of τ.

The Cartan-Maurer equations are

$$d\omega^i = \frac{1}{r-r^{-1}}(\tau \wedge \omega^i + \omega^i \wedge \tau) = -\frac{1}{2} C_{jk}{}^i \omega^j \wedge \omega^k \tag{3.5.40}$$

with

$$C^{a_1}{}_{a_2}{}^{b_1}{}_{b_2}|_{c_1}^{c_2} = \frac{2}{r^2+r^{-2}}[-(r-r^{-1})\delta^{b_1}_{b_2}\delta^{a_1}_{c_1}\delta^{c_2}_{a_2} + \mathbf{C}^{a_1}{}_{a_2}{}^{b_1}{}_{b_2}|_{c_1}^{c_2}] \tag{3.5.41}$$

$$C^{a_1}{}_{a_2}{}^{0}{}_{b_2}|_{0}^{c_2} = \frac{2}{r^2+r^{-2}}\mathbf{C}^{a_1}{}_{a_2}{}^{0}{}_{b_2}|_{0}^{c_2} \tag{3.5.42}$$

$$C^{0}{}_{a_2}{}^{b_1}{}_{b_2}|_{0}^{c_2} = \frac{2}{r^2+r^{-2}}[-(r-r^{-1})\delta^{b_1}_{b_2}\delta^{c_2}_{a_2} + \mathbf{C}^{0}{}_{a_2}{}^{b_1}{}_{b_2}|_{0}^{c_2}] \tag{3.5.43}$$

The structure constants $\mathbf{C}$ (appearing in the q-Lie algebra of $IGL_{q,r}(N)$, see later) are given by

$$\begin{aligned}\mathbf{C}^{c_1}{}_{c_2}{}^{b_1}{}_{b_2}|_{d_1}^{d_2} &= \frac{1}{r-r^{-1}}[-\delta^{b_1}_{b_2}\delta^{c_1}_{d_1}\delta^{d_2}_{c_2} + \Lambda_a{}^a{}_{d_1}{}^{d_2}|^{c_1}{}_{c_2}{}^{b_1}{}_{b_2}] \\ &= \text{structure constants of } GL_{q,r}(N)\end{aligned} \tag{3.5.44}$$

$$\mathbf{C}^{c_1}{}_{c_2}{}^{0}{}_{b_2}|_{0}^{d_2} = -\frac{q_{0c_2}}{q_{0c_1}} R^{c_1 d_2}{}_{b_2 c_2} \tag{3.5.45}$$

$$\mathbf{C}^{0}{}_{c_2}{}^{b_1}{}_{b_2}|_{0}^{d_2} = \frac{1}{r-r^{-1}}[-\delta^{b_1}_{b_2}\delta^{d_2}_{c_2} + d^{f_2}d^{-1}_{c_2}R^{f_2 b_1}{}_{c_2 a}R^{a d_2}{}_{b_2 f_2}] \tag{3.5.46}$$

We conclude this section by observing that the functionals f and χ in (3.5.1) and (3.5.24) close on the algebra (2.1.111), (2.1.112)-(2.1.114), where the product of functionals is defined by the coproduct Δ in $IGL_{q,r}(N)$. This result is expected, since the functionals in (3.5.1) and (3.5.24) correspond to a bicovariant differential calculus on $IGL_{q,r}(N)$.[2]

[2] An explicit proof is also instructive. We first note that in $GL_{q,r}(N+1)$ the subset in (3.5.1) and (3.5.24) closes by itself on the bicovariant algebra (2.1.111), (2.1.112)-(2.1.114). This is due to the particular index structure of the tensors $\mathbf{C}$ and Λ, and to the vanishing of the f components in (3.5.7). The nonvanishing components of $\mathbf{C}$ and Λ that enter the operatorial bicovariance conditions (where the free adjoint indices are taken as $ab, 0b$), are given in (3.5.44)-(3.5.46) and (3.5.35)-(3.5.39). Finally, we know that the f functionals vanish on $KerP$, and so do the χ functionals (as can be deduced from their definition in terms of the f functionals, eq. (2.2.52)). From the discussion after Theorem 3.5.1 it follows that they are well defined on $IGL_{q,r}(N)$, and that their products involve the $IGL_{q,r}(N)$ coproduct Δ.

Thus the relations (2.1.111), (2.1.112)-(2.1.114) hold for the functionals (3.5.1) and (3.5.24) on $IGL_{q,r}(N)$. They are the bicovariance conditions corresponding to a consistent differential calculus on $IGL_{q,r}(N)$.

Using the values of the Λ and $\mathbf{C}$ tensors in (3.5.35)-(3.5.39) and (3.5.44)-(3.5.46), we can explicitly write the "q-Lie algebra" of $IGL_{q,r}(N)$ as:

$$\chi^{c_1}_{\ c_2}\chi^{b_1}_{\ b_2} - \Lambda_{a_1\ d_1}^{a_2\ d_2}\big|^{c_1\ b_1}_{c_2\ b_2}\ \chi^{a_1}_{\ a_2}\chi^{d_1}_{\ d_2} = \frac{1}{r-r^{-1}}[-\delta^{b_1}_{b_2}\delta^{c_1}_{d_1}\delta^{d_2}_{c_2} + \Lambda_{a\ d_1}^{a\ d_2}\big|^{c_1\ b_1}_{c_2\ b_2}]\chi^{d_1}_{\ d_2} \qquad (3.5.47)$$

$$\begin{aligned}
&\chi^{c_1}_{\ c_2}\chi^{0}_{\ b_2} + (r-r^{-1})\frac{q_{0c_2}}{q_{0a_2}}R^{a_2d_2}{}_{b_2c_2}\chi^{c_1}_{\ a_2}\chi^{0}_{\ d_2} - \\
&\qquad - \frac{q_{0c_2}}{q_{0c_1}}(R^{-1})^{a_2c_1}{}_{g_2d_1}R^{g_2d_2}{}_{b_2c_2}\chi^{0}_{\ a_2}\chi^{d_1}_{\ d_2} = -\frac{q_{0c_2}}{q_{0c_1}}R^{c_1d_2}{}_{b_2c_2}\ \chi^{0}_{\ d_2} &(3.5.48)\\
&\chi^{0}_{\ c_2}\chi^{b_1}_{\ b_2} - \frac{q_{0a_1}}{q_{0a_2}}d^{f_2}d^{-1}_{c_2}R^{f_2b_1}{}_{c_2a_1}R^{a_2d_2}{}_{b_2f_2}\chi^{a_1}_{\ a_2}\chi^{0}_{\ d_2} = \\
&\qquad \frac{1}{r-r^{-1}}[-\delta^{b_1}_{b_2}\delta^{d_2}_{c_2} + d^{f_2}d^{-1}_{c_2}R^{f_2b_1}{}_{c_2a}R^{ad_2}{}_{b_2f_2}]\chi^{0}_{\ d_2} &(3.5.49)\\
&\chi^{0}_{\ c_2}\chi^{0}_{\ b_2} - \frac{q_{0c_2}}{q_{0a_2}}r^{-1}\ R^{a_2d_2}{}_{b_2c_2}\ \chi^{0}_{\ a_2}\chi^{0}_{\ d_2} = 0 &(3.5.50)
\end{aligned}$$

where $\Lambda_{a_1\ d_1}^{a_2\ d_2}\big|^{c_1\ b_1}_{c_2\ b_2}$ is the braiding matrix of $GL_q(n)$, given in (3.5.35), so that the commutations in (3.5.47) are those of the q-subalgebra $GL_q(n)$. Note that the $r \to 1$ limit on the right hand sides of (3.5.47) and (3.5.49) is finite, since the terms in square parentheses are a (finite) series in $r - r^{-1}$ whose $0-th$ order part vanishes [see (2.2.55) and (2.2.57)].

3.6 The universal enveloping algebra of $IGL_{q,r}(N)$

In the previous section we have considered the Hopf algebra generated by the f functionals in (3.5.1). This, togheter with $f_0{}^{00}{}_0$, is the universal enveloping algebra $U_{q,r}(igl(N))$ of $IGL_{q,r}(N)$ [see later, after (3.6.66)]. We now briefly give an $L^{\pm}$ description of $U_{q,r}(igl(N))$. For all the details we refer to Section 4.3 where a similar construction is performed in the orthogonal case.

In the preceding section we have identified the $\overline{f}$ functionals on $IGL_{q,r}(N)$ with the corresponding f functionals on $GL_{q,r}(N)$, in the same perspective, we construct $U_{q,r}(igl(N))$ as a Hopf subalgebra of $U_{q,r}(gl(N+1))$. Let

$$IU \equiv [L^{+A}{}_B, L^{-a}{}_b, L^{-0}{}_0, \Phi, \varepsilon] \qquad (3.6.51)$$

be the subalgebra of $U_{q,r}(gl(N+1))$ generated by $L^{+A}{}_B, L^{-a}{}_b, L^{-0}{}_0, \Phi, \varepsilon$. ($\Phi$ is the inverse of $\det L^+ \det L^-$).

The remaining $U_{q,r}(gl(N+1))$ generators $L^{-b}{}_0$ are the only ones that do not annihilate $T^0{}_a$ (the generators of H) and are not included in (3.6.51): we construct the universal enveloping algebra of $IGL_{q,r}(N)$ as the Hopf subalgebra of $U_{q,r}(gl(N+1))$ that annihilates the ideal H.

Since $\Delta(IU) \subseteq IU \otimes IU$ and $\kappa'(IU) \subseteq IU$ (as can be immediately seen at the generators level) we have that IU is a Hopf subalgebra of $U_{q,r}(gl(N+1))$. Moreover one can also give the following R-marix formulation (cf. Theorem 4.4.2):

Theorem 3.6.1 The Hopf algebra IU is generated by ε, Φ and the matrix entries:

$$L^{-} = \left(L^{+A}{}_{B}\right) \, , \, \mathcal{L}^{-} = \begin{pmatrix} L^{-0}{}_{0} & 0 \\ 0 & L^{-a}{}_{b} \end{pmatrix}$$

these functionals satisfy the q-commutation relations:

$$R_{12}\mathcal{L}^{-}{}_{2}\mathcal{L}^{-}{}_{1} = \mathcal{L}^{-}{}_{1}\mathcal{L}^{-}{}_{2}R_{12} \quad \text{or equivalently} \quad \mathcal{R}_{12}\mathcal{L}^{-}{}_{2}\mathcal{L}^{-}{}_{1} = \mathcal{L}^{-}{}_{1}\mathcal{L}^{-}{}_{2}\mathcal{R}_{12} \quad (3.6.52)$$

$$R_{12}L_2^{+}L_1^{+} = L_1^{+}L_2^{+}R_{12} \, , \quad (3.6.53)$$

$$\mathcal{R}_{12}L_2^{+}\mathcal{L}^{-}{}_{1} = \mathcal{L}^{-}{}_{1}L_2^{+}\mathcal{R}_{12} \, , \quad (3.6.54)$$

where $\mathcal{R}_{12} \equiv c^{-}[\mathcal{L}^{-}{}_{1}(T_2)]^{-1}$ that is $\mathcal{R}^{ab}_{cd} = R^{ab}_{cd}$, $\mathcal{R}^{AB}_{AB} = R^{AB}_{AB}$ and otherwise $\mathcal{R}^{AB}_{CD} = 0$. □□□

Relations (3.6.52) and (3.6.53) explicitly read as in (3.3.24)–(3.3.34), just substitute T with $L^{\pm}$ and "read from right to left"; this is due to $R_{12}L_2^{\pm}L_1^{\pm} = L_1^{\pm}L_2^{\pm}R_{12}$ while we have $R_{12}T_1T_2 = T_2T_1R_{12}$. Relations (3.6.54) read

$$R^{ab}_{ef}L^{+f}{}_{d}L^{-e}{}_{c} = L^{-a}{}_{e}L^{+b}{}_{f}R^{ef}_{cd} \quad (3.6.55)$$

$$L^{+b}{}_{d}L^{-0}{}_{0} = \frac{q_{d0}}{q_{b0}}L^{-0}{}_{0}L^{+b}{}_{d} \quad (3.6.56)$$

$$L^{+0}{}_{d}L^{-a}{}_{c} = \frac{q_{a0}}{r}R^{ef}_{cd}L^{-a}{}_{e}L^{+0}{}_{f} \quad (3.6.57)$$

$$L^{+0}{}_{d}L^{-0}{}_{0} = \frac{q_{d0}}{r^2}L^{-0}{}_{0}L^{+0}{}_{d} \quad (3.6.58)$$

$$L^{+0}{}_{0}L^{-a}{}_{c} = \frac{q_{a0}}{q_{c0}}L^{-a}{}_{c}L^{+0}{}_{0} \quad (3.6.59)$$

$$L^{+0}{}_{0}L^{-0}{}_{0} = L^{-0}{}_{0}L^{+0}{}_{0} \quad (3.6.60)$$

Note 3.6.1 Apply the second espression in (3.6.52) to $T^{A}{}_{B}$ to obtain the quantum Yang-Baxter equation for the matrix $\mathcal{R}$. The need for a new R-matrix $\mathcal{R}$ can be seen as due to the impossibility of considering IU as a quotient of the algebra $U_{q,r}(GL(N+1))$. The commutation relation that prevents IU to be a quotient of $U_{q,r}(GL(N+1))$ with respect to the ideal generated by the $L^{-a}{}_{0}$ elements is: $L^{+0}{}_{d}L^{-a}{}_{0} = \frac{q_{ao}}{q_{0d}}L^{-a}{}_{0}L^{+0}{}_{d} + (1-r^{-2})q_{a0}(L^{-a}{}_{d}L^{+0}{}_{0} - L^{+a}{}_{d}L^{-0}{}_{0})$.

We stress that IU is a subalgebra of $U_{q,r}(N+1)$, so that (3.6.52), (3.6.53), (3.6.55)–(3.6.60) hold in $GL_{q,r}(N)$ as well. On the opposite side, the R-matrix of the $IGL_{q,r}(N)$ RTT relations is the same as the R-matrix of the $GL_{q,r}(N+1)$ RTT relations, but this last set of RTT relations does not contain as a subset the previous one.

We now briefly study the structure of IU with respect to $U_{q,r}(gl(N))$, that is easily seen to be a Hopf subalgebra of IU. It is also a quotient of IU via the Hopf algebra projection [well defined only if $q_{a0} = const \equiv q_0\ \forall a$ see (3.6.56) and (3.6.59)]:

$$\pi(L^{-0}{}_a) = 0\ ,\quad \pi(L^{-0}{}_0) = I\ ,\quad \pi(L^{\pm a}{}_b) = L^{\pm a}{}_b\ ,\quad \pi(L^{+a}{}_0) = L^{+a}{}_0\ ,\quad \pi(\varepsilon') = \varepsilon'\ .$$

Then the results of Theorem 3.3.6 apply to IU as well, and we can write the Hopf algebra isomorhism $IU \cong B' \rtimes U_{q,r}(gl(N))$ where B' is the algebra generated by $L^{+0}{}_0$ and $L^{+0}{}_a$. Also Theorem 3.3.4 hold for IU since we can introduce the following $(\mathbf{Z},\mathbf{N})$ grading: the elements $L^{\pm a}{}_b$ have grade $(0,0)$, the elements $L^{+0}{}_a$ have grade $(0,1)$, the elements $L^{\pm 0}{}_0$ have grade $(\pm 1, 0)$. This grading is compatible with the RLL commutation relations. Notice also that the elements $L^{+0}{}_0$ and $L^{-0}{}_0$ are not independent (see the text after (4.3.17) for a general discussion in the orthogonal case, the $GL_{q,r}(N)$ case is similar); here we give an easier argument that holds only if $q_{a0} = const = r\ \forall a$: we fix the coefficient c^- defined in (2.2.3) and studied after (2.2.20), to be $c^- = (c^+)^{-1}$ (notice that the parameter $s = (c^+)^{-1}c^-$ entering the differential calculus is still arbitrary, the parameter c^- is completely irrelevant). It follows that $L^{-0}{}_0$ has a simple dependence from $L^{+0}{}_0$: $(L^{-0}{}_0)^{-1} = L^{+0}{}_0$. [Proof: $\forall A, B\,,\ (L^{-0}{}_0)^{-1}(T^A{}_B) = L^{+0}{}_0(T^A{}_B)$, $\Delta'(L^{\pm 0}{}_0) = L^{\pm 0}{}_0 \otimes L^{\pm 0}{}_0$].

From the RLL relations a generic element of IU can be written as $\eta^i a_i$ (or $a_i \eta^i$) where $a_i \in U_{q,r}(gl(N))$ and η^i are ordered monomials in the $L^{+0}{}_0$ and $L^{+0}{}_a$ elements: $\eta^i = (L^{+0}{}_0)^{i_0}(L^{+0}{}_1)^{i_1} \ldots (L^{+0}{}_N)^{i_N}$. As in Corollary 3.1.1, we have that IU, for $q_{a0} = const\ \forall a$, is a bicovariant bimodule over $GL_{q,r}(N)$ freely generated, as a right module, by the elements η^i; moreover

$$U_{q,r}(igl(N)) \;=\; \sum_{(h,k)\in(\mathbf{Z},\mathbf{N})}^{\oplus} \Gamma'^{(h,k)} \tag{3.6.61}$$

where $\Gamma'^{(0,0)} = U_{q,r}(gl(N))$

$$\Gamma'^{(0,1)} = \{L^{+0}{}_a \varphi^a\ /\ \varphi^a \in U_{q,r}(gl(N))\}\ ,\qquad \Gamma'^{(\pm 1,0)} = \{(L^{+0}{}_0)^{\pm 1}\varphi\ /\ \varphi \in U_{q,r}(gl(N))\}$$
$$\Gamma'^{(h,k)} = \{(L^{+0}{}_0)^h L^{+0}{}_{a_1} L^{+0}{}_{a_2} \ldots L^{+0}{}_{a_k} \varphi^{a_1 a_2 \ldots a_k}\ /\ \varphi^{a_1 a_2 \ldots a_k} \in U_{q,r}(gl(N))\} \tag{3.6.62}$$

Any submodule $\Gamma'^{(h,k)}$ is a bicovariant bimodule freely generated by the elements η^i with degree $(h,k) \in (\mathbf{Z},\mathbf{N})$. We leave to the reader to reformulate Note 3.3.5 and Note 3.3.6 in this context.

Duality $IU \leftrightarrow IGL_{q,r}(N)$

We now show that IU is dually paired to $IGL_{q,r}(N)$. This is the fundamental step allowing to interpret IU as the universal enveloping algebra of $IGL_{q,r}(N)$.

Theorem 3.6.2 *IU annihilates H.*

Proof : This theorem has implicitly been proved in Theorem 3.5.1 and in the comments before (3.5.4). An explicit proof is given in Theorem 4.4.4. □□□

In virtue of Theorem 3.6.2 the following bracket is well defined:

$$\begin{aligned} \textit{Definition.} \quad & \langle \; , \; \rangle \; : \; IU \otimes IGL_{q,r}(N) \longrightarrow \boldsymbol{C} \\ & \langle a', P(a) \rangle \equiv a'(a) \\ & \forall a' \in IU \; , \; \forall a \in IGL_{q,r}(N) \end{aligned} \tag{3.6.63}$$

where $P \; : \; GL_{q,r}(N+1) \to GL_{q,r}(N+1)/H \equiv IGL_{q,r}(N)$ is the canonical projection, which is surjective. The bracket is well defined because two generic counter-images of $P(a)$ differ by an addend belonging to H.

Since IU is a Hopf subalgebra of $U_{q,r}(gl(N+1))$ and P is compatible with the structures and costructures of $GL_{q,r}(N+1)$ and $IGL_{q,r}(N)$, the following theorem is then easily shown [cf. (3.5.4), (3.5.8) and Theorem 4.4.5]

Theorem 3.6.3 The bracket (3.6.63) defines a pairing between IU and $IGL_{q,r}(N)$: $\forall a', b' \in IU \; , \; \forall P(a), P(b) \in IGL_{q,r}(N)$

$$\begin{aligned} & \langle a'b', P(a) \rangle = \langle a' \otimes b', \Delta(P(a)) \rangle \\ & \langle a', P(a)P(b) \rangle = \langle \Delta'(a'), P(a) \otimes P(b) \rangle \\ & \langle \kappa'(a'), P(a) \rangle = \langle a', \kappa(P(a)) \rangle \\ & \langle I, P(a) \rangle = \varepsilon(a) \; ; \; \; \langle a', P(I) \rangle = \varepsilon'(a') \end{aligned}$$

□□□

We now recall that IU and $IGL_{q,r}(N)$, besides being dually paired, are bicovariant algebras with the same graded structure (3.3.61) and (3.6.61), and can both be obtained as a cross-product cross-coproduct construction: $IGL_{q,r}(N) \cong B \rtimes\!\!\!\lhd GL_{q,r}(N)$, $IU \cong B' \rtimes\!\!\!\lhd U_{q,r}(gl(N))$. In particular $IGL_{q,r}(N)$ and IU are freely generated (as modules) by B and B' i.e. by the two isomorphic sets of the ordered monomials in the q-plane plus dilatation coordinates ${L^{+0}}_0$, ${L^{+0}}_a$ and u, x^a respectively. We then conclude that IU is the universal enveloping algebra of $IGL_{q,r}(N)$:

$$U_{q,r}(igl(N)) \equiv IU \; . \tag{3.6.64}$$

Projected differential calculus

We have found the inhomogeneous quantum group $IGL_{q,r}(N)$ by means of a projection from $GL_{q,r}(N+1)$; dually, its universal enveloping algebra is a given Hopf subalgebra of $U_{q,r}(gl(N+1))$. Using the same techniques we conclude this section presenting another differential calculus on $IGL_{q,r}(N)$, that is obtained from the previous one considering also the dilatation generator ${\chi^0}_0$. To derive this calculus one can follow the same steps of the Section 3.5, however the easiest way to derive it is to apply the results of Section 2.3.

From (2.3.4) and (2.3.5) it is immediate to see that $T' \equiv T \cap U_{q,r}(igl(N))$ satisfies

$$\Delta(T') \subset T' \otimes \varepsilon + U_{q,r}(igl(N)) \otimes T' \qquad (3.6.65)$$

$$[T', T'] \subseteq T \cap IU = T' \qquad (3.6.66)$$

indeed $U_{q,r}(igl(N))$ is a Hopf subalgebra of $U_{q,r}(gl(N+1))$. Also condition (2.3.3) is fulfilled since T' generates $U_{q,r}(igl(N))$ in the same way T generates $U_{q,r}(gl(N+1))$ [78], this is a consequence of the upper and lower triangularity of the L^+ and L^- matrices and of the dependence of the diagonal elements of L^+ from the diagonal elements of L^-. We therefore obtain an $IGL_{q,r}(N)$ bicovariant differential calculus with q-Lie algebra generators:

$$\chi^a{}_b \ , \quad \chi^0{}_b \ , \quad \chi^0{}_0 \ . \qquad (3.6.67)$$

Since these generators close on the subalgebra $T' \subset T$, we have that the structure constants that appear in the $IGL_{q,r}(N)$ Lie algebra are a subset of the structure constants that appear in the $GL_{q,r}(N+1)$ Lie algebra [cf. the text after (3.5.34)].

The exterior differential reads

$$da = (\chi^a{}_b * a)\omega_a{}^b + (\chi^a{}_0 * a)\omega_a{}^0 + (\chi^0{}_0 * a)\omega_0{}^0 \ ; \qquad (3.6.68)$$

where $\omega_a{}^b$, $\omega_a{}^0$, and $\omega_0{}^0$, following Section 2.3, are the 1-forms dual to the tangent vectors (3.6.67).

3.7 The multiparametric quantum plane as a quantum coset space

In this section we derive the differential calculus on the quantum plane

$$Fun_{q,r}\left(\frac{IGL(N)}{GL(N)}\right) \ , \qquad (3.7.1)$$

i.e. the subalgebra of $IGL_{q,r}(N)$ generated by the coordinates x^a. These coordinates satisfy the commutations (3.3.4):

$$R^{ab}{}_{ef} x^e x^f = r x^b x^a \qquad (3.7.2)$$

The main difference with the more conventional approach to the quantum plane is that now the coordinates do not trivially commute with the $GL_{q,r}(N)$ q-group elements, but q-commute according to relations (3.3.3):

$$R^{ab}{}_{ef} T^e{}_c x^f = \frac{q_{0c}}{r} x^b T^a{}_c \qquad (3.7.3)$$

From a more mathematical viewpoint the q-coset space $Fun_{q,r}(IGL(N)/GL(N))$ is the algebra B discussed in Theorem 3.3.6 and Note 3.3.5. B is generated by u and x^a. We then study the subalgebra of B generated only by the elements x^a. Expression (3.3.70) is the translation in Hopf algebra language of the classical property: $\forall g \in ISO(N)$, $\forall h \in SO(N)$ $\{gh\} = \{g\}$, where $\{g\}$ is an element of the coset $ISO(N)/SO(N)$.

Lemma 3.7.1 $\chi^b{}_c(a) = 0$ when a is a polynomial in x^a and u with all monomials containing at least one x^a. This is easily proved by observing that no tensor exists with the correct index structure. For $s = 1$ we can extend this lemma even to $u \cdots u$, since for example

$$\chi^b{}_c(u) = \frac{s-1}{r - r^{-1}} \delta^b_c \tag{3.7.4}$$

and using the coproduct rule (3.5.25) one finds that $\chi^b{}_c(u \cdots u)$ is always proportional to $s - 1$. □□□

Theorem 3.7.1 $\chi^b{}_c * a = 0$ when a is a polynomial in x^a and $s = 1$.

Proof: we have $\chi^b{}_c * a = (id \otimes \chi^b{}_c)(a_1 \otimes a_2) = a_1 \chi^b{}_c(a_2)$. (We use the notation $\Delta(a) \equiv a_1 \otimes a_2$). Since a_2 is a polynomial in x^a and u (use the coproduct rule (3.3.11)), and $\chi^b{}_c$ vanishes on such a polynomial when $s = 1$ (previous Lemma), the theorem is proved. □□□

Because of this theorem we will henceforth set $s = 1$: then we can write the exterior derivative of an element of the quantum plane as

$$da = (\chi_s * a) V^s \tag{3.7.5}$$

(with $\chi_s \equiv \chi^0{}_s$, $V^s \equiv \omega_0{}^s$), i.e. only in terms of the "q-vielbein" V^s. Notice also that $du = 0$.

The action and value of χ_s on the coordinates is easily computed [cf. the definition in (3.5.24)]:

$$\chi_s * x^a = -\frac{r}{q_{0s}} T^a{}_s, \quad \chi_s(x^a) = -\frac{r}{q_{0s}} \delta^a_s \tag{3.7.6}$$

so that the exterior derivative of x^a is:

$$dx^a = -\frac{r}{q_{0s}} T^a{}_s V^s \tag{3.7.7}$$

and gives the relation between the q-vielbein V^s and the differentials dx^a.

Using (3.5.26), the Leibniz rule for the "q-partial derivatives" χ_c is given by :

$$\chi_c * (ab) = (\chi_d * a) f^d{}_c * b + a \chi_c * b \tag{3.7.8}$$

where $f^d{}_c \equiv f_0{}^{d0}{}_c$.

The x^a and V^b q-commute as (cf. (3.5.20)):

$$V^a x^b = (q_{0a})^{-1} x^b V^a \tag{3.7.9}$$

and via eq. (3.7.7) and (3.7.3) we find the dx^a, x^b commutations :

$$dx^a x^b = r^{-1}(R^{-1})^{ab}{}_{ef} x^f dx^e \tag{3.7.10}$$

After acting on this equation with d we obtain:

$$dx^a \wedge dx^b = -r^{-1}(R^{-1})^{ab}{}_{ef} dx^f \wedge dx^e \tag{3.7.11}$$

which reproduce the known commutations between the differentials of the quantum plane, cf. ref. [48], [50].

The commutations between the partial derivatives are given in eq.(3.5.50).

All the relations of this section are covariant under the $IGL_{q,r}(N)$ action:

$$x^a \longrightarrow T^a{}_b \otimes x^b + x^a \otimes u \tag{3.7.12}$$

and in particular under the $GL_{q,r}(N)$ action $x^a \longrightarrow T^a{}_b \otimes x^b$.

Note 3.7.1 The partial derivatives χ_c, and in general all the tangent vectors χ of this chapter have "flat" indices. To compare them with the partial derivatives discussed in [48], which have "curved" indices, we need to define the operators $\overleftarrow{\partial}_s$:

$$\overleftarrow{\partial}_s (a) \equiv -\frac{q_{0a}}{r}(\chi_a * a)\kappa(T^a{}_s) \tag{3.7.13}$$

whose value and action on the coordinates is

$$\overleftarrow{\partial}_s (x^a) = \delta^a_s I \tag{3.7.14}$$

so that

$$da = \overleftarrow{\partial}_s (a)\ dx^s \tag{3.7.15}$$

which is equation (3.7.5) in "curved" indices [Note: ref. [48] adopts a definition of $\overleftarrow{\partial}_s$ such that $da = dx^s\ (\partial_s(a))$].

Using the Lie derivarive and the contraction operator defined on the full inhomogeneous quantum group one can also study the Cartan calculus on the quantum plane; this should provide an alternative approach to [52].

The results of this section are applied to the multiparametric quantum plane $IGL_{qr}(2)/GL_{qr}(2)$ at the end of the Table. The usual relations of the uniparametric case [48] are recovered after setting $q = r$.

3.8 Table of $IGL_{q,r}(2)$

The quantum group $IGL_{q,r}(2)$ and its differential calculus

Parameters: $q(\equiv q_{12}), q_{01}, q_{02}, r$

R and D-matrices of $GL_{q,r}(2)$*:*

$$R^{ab}{}_{cd} = \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & \frac{r}{q} & 0 & 0 \\ 0 & r - r^{-1} & \frac{q}{r} & 0 \\ 0 & 0 & 0 & r \end{pmatrix}, \quad D^a{}_b = \begin{pmatrix} r & 0 \\ 0 & r^3 \end{pmatrix}$$

$T^A{}_B$ *(A,B=0,1,2): fundamental representation of* $IGL_{q,r}(2)$

$$T^A{}_B = \begin{pmatrix} u & 0 & 0 \\ x^1 & \alpha & \beta \\ x^2 & \gamma & \delta \end{pmatrix}$$

Determinant of $IGL_{q,r}(2)$ *and definition of* ξ

$$\det T^A{}_B = u \det T^a{}_b, \quad \text{where } \det T^a{}_b = \alpha\delta - \frac{r^2}{q}\beta\gamma$$

$$\xi \det T^A{}_B = \det T^A{}_B \xi = I$$

Basis elements generating $IGL_{q,r}(2)$

$$\alpha, \beta, \gamma, \delta, \ x^1, \ x^2, \ u, \ \xi$$

Commutations of the basis elements

$$\alpha\beta = \frac{r^2}{q}\beta\alpha, \quad \alpha\gamma = q\gamma\alpha, \quad \beta\delta = q\delta\beta, \quad \gamma\delta = \frac{r^2}{q}\delta\gamma$$

$$\beta\gamma = \frac{q^2}{r^2}\gamma\beta, \quad \alpha\delta - \delta\alpha = \frac{r}{q}(r - r^{-1})\beta\gamma,$$

$$\alpha x^1 = \frac{q_{01}}{r^2}x^1\alpha, \quad \beta x^1 = \frac{q_{02}}{r^2}x^1\beta,$$

$$\alpha x^2 = q\frac{q_{01}}{r^2}x^2\alpha, \quad \beta x^2 = q\frac{q_{02}}{r^2}x^2\beta,$$

$$\gamma x^1 = \frac{q_{01}}{q}x^1\gamma - \frac{r}{q}(r-r^{-1})\alpha x^2, \quad \delta x^1 = \frac{q_{02}}{q}x^1\delta - \frac{r}{q}(r-r^{-1})\beta x^2,$$

$$\gamma x^2 = \frac{q_{01}}{r^2}x^2\gamma, \quad \delta x^2 = \frac{q_{02}}{r^2}x^2\delta$$

$$x^1x^2 = qx^2x^1$$

$$T^a{}_b u = \frac{q_{0b}}{q_{0a}}uT^a{}_b, \quad x^a u = (q_{0a})^{-1}ux^a$$

$$(\det T^A{}_B)T^A{}_B = \frac{q_{0A}q_{1A}q_{2A}}{q_{0B}q_{1B}q_{2B}}T^A{}_B(\det T^A{}_B), \; q_{AA} \equiv r, \; q_{AB} \equiv \frac{r^2}{q_{BA}}$$

$$T^A{}_B\xi = \frac{q_{0A}q_{1A}q_{2A}}{q_{0B}q_{1B}q_{2B}}\xi T^A{}_B$$

Conditions for centrality of $\det T^A{}_B = u\det T^a{}_b$, $\det T^a{}_b$ *and* u

centrality of $u\det T^a{}_b \iff q_{01}q_{02} = r^2, \; q_{01} = q$
centrality of $\det T^a{}_b \iff q_{01}q_{02} = r^2, \; q = r$
centrality of $u \iff q_{01} = q_{02} = 1$

Inverse of $T^A{}_B$

$$(T^{-1})^A{}_B = \begin{pmatrix} \det T^a{}_b\xi & -(T^{-1})^a{}_b x^b \det T^a{}_b\xi \\ 0 & (T^{-1})^a{}_b \end{pmatrix}$$

$$(T^{-1})^a{}_b = \xi u \begin{pmatrix} \delta & -q^{-1}\beta \\ -q\gamma & \alpha \end{pmatrix}$$

Commutations of the left-invariant 1*-forms* ω

Notations: $\omega^1 \equiv \omega_1{}^1, \omega^+ \equiv \omega_1{}^2, \omega^- \equiv \omega_2{}^1, \omega^2 \equiv \omega_2{}^2, \; V^1 \equiv \omega_0{}^1, V^2 \equiv \omega_0{}^2$

$$\omega^1 \wedge \omega^+ + \omega^+ \wedge \omega^1 = 0$$

$$\omega^1 \wedge \omega^- + \omega^- \wedge \omega^1 = 0$$

$$\omega^1 \wedge \omega^2 + \omega^2 \wedge \omega^1 = (1-r^2)\omega^+ \wedge \omega^-$$

$$\omega^+ \wedge \omega^- + \omega^- \wedge \omega^+ = 0$$

$$\omega^2 \wedge \omega^+ + r^2\omega^+ \wedge \omega^2 = r^2(r^2-1)\omega^+ \wedge \omega^1$$

$$\omega^2 \wedge \omega^- + r^{-2}\omega^- \wedge \omega^2 = (1-r^2)\omega^- \wedge \omega^1$$

$$\omega^2 \wedge \omega^2 = (r^2-1)\omega^+ \wedge \omega^-$$

$$\omega^1 \wedge \omega^1 = \omega^+ \wedge \omega^+ = \omega^- \wedge \omega^- = 0$$

$$\omega^1 \wedge V^1 + r^2 V^1 \wedge \omega^1 = 0$$

$$q^{-1}\frac{q_{02}}{q_{01}}\omega^+ \wedge V^1 + V^1 \wedge \omega^+ = (1 - r^{-2})\omega^1 \wedge V^2$$

$$\omega^- \wedge V^1 + \frac{r^2}{q}\frac{q_{02}}{q_{01}} V^1 \wedge \omega^- = 0$$

$$\omega^2 \wedge V^1 + V^1 \wedge \omega^2 = (1 - r^{-2}) q \frac{q_{01}}{q_{02}} \omega^- V^2$$

$$\omega^1 \wedge V^2 + V^2 \wedge \omega^1 = 0$$

$$q^{-1}\frac{q_{02}}{q_{01}}\omega^+ \wedge V^2 + V^2 \wedge \omega^+ = 0$$

$$\frac{q}{r^2}\frac{q_{01}}{q_{02}}\omega^- \wedge V^2 + V^2 \wedge \omega^- = (1 - r^2) V^1 \wedge \omega^1$$

$$\omega^2 \wedge V^2 + r^2 \wedge V^2 = (r^2 - 1)[(1 - r^2)\omega^1 \wedge V^2 + \frac{r^2}{q}\frac{q_{02}}{q_{01}}\omega^+ \wedge V^1]$$

Cartan-Maurer equations:

$$d\omega^1 + r\omega^+ \wedge \omega^- = 0$$

$$d\omega^+ + r\omega^+(-r^2\omega^1 + \omega^2) = 0$$

$$d\omega^- + r(-r^2\omega^1 + \omega^2)\omega^- = 0$$

$$d\omega^2 - r\omega^+ \wedge \omega^- = 0$$

$$dV^1 - \frac{q}{r}\frac{q_{01}}{q_{02}}\omega^- \wedge V^2 - r^{-1}\omega^1 \wedge V^1 = 0$$

$$dV^2 - \frac{r}{q}\frac{q_{02}}{q_{01}}\omega^+ \wedge V^1 - r^{-1}\omega^2 \wedge V^2 - (r - r^{-1}) V^2 \wedge \omega^1 = 0$$

The q-Lie algebra:

Notations: $\chi^1 \equiv \chi^1{}_1, \chi^+ \equiv \chi^1{}_2, \chi^- \equiv \chi^2{}_1, \chi^2 \equiv \chi^2{}_{,}\ P^1 \equiv \chi^0{}_1, P^2 \equiv \chi^0{}_2$

$$\chi_1\chi_+ - \chi_+\chi_1 - (r^4 - r^2)\chi_2\chi_+ = r^3\chi_+$$

$$\chi_1\chi_- - \chi_-\chi_1 + (r^2 - 1)\chi_2\chi_- = -r\chi_-$$

$$\chi_1\chi_2 - \chi_2\chi_1 = 0$$

$$\chi_+\chi_- - \chi_-\chi_+ + (1 - r^2)\chi_2\chi_1 - (1 - r^2)\chi_2\chi_2 = r(\chi_1 - \chi_2)$$

$$\chi_+\chi_2 - r^2\chi_2\chi_+ = r\chi_+$$

$$\chi_-\chi_2 - r^{-2}\chi_2\chi_- = -r^{-1}\chi_-$$

$$r^2\chi_1 P_1 - P_1\chi_1 + (r^2-1)P_2\chi_- = -rP_1$$

$$q\frac{q_{01}}{q_{02}}\chi_+ P_1 - P_1\chi_+ - r^2(1-r^2)\chi_2 P_2 = r^3 P_2$$

$$\chi_- P_1 - \frac{q}{r^2}\frac{q_{01}}{q_{02}} P_1\chi_- = 0$$

$$\chi_2 P_1 - P_1\chi_2 = 0$$

$$\chi_1 P_2 - P_2\chi_1 + (r^2-1)\frac{q}{r^2}\frac{q_{01}}{q_{02}}\chi_+ P_1 = 0$$

$$\chi_+ P_2 - q^{-1}\frac{q_{02}}{q_{01}} P_2\chi_+ = 0$$

$$\frac{r^2}{q}\frac{q_{02}}{q_{01}}\chi_- P_2 - P_2\chi_- + (1-r^2)\chi_2 P_1 = -rP_1$$

$$r^2\chi_2 P_2 - P_2\chi_2 = -rP_2$$

$$P_1 P_2 - \frac{q}{r^2}\frac{q_{01}}{q_{02}} P_2 P_1 = 0$$

The exterior derivative of the basis elements

$$\begin{aligned}
d\alpha &= \tfrac{s-r^2}{r^3-r}\alpha\omega^1 - s\tfrac{r}{q_{12}}\beta\omega^+ + \tfrac{s-1}{r-r^{-1}}\alpha\omega^2\\
d\beta &= \tfrac{-r^2+s(1-r^2+r^4)}{r^3-r}\beta\omega^1 - s\tfrac{q_{12}}{r}\alpha\omega^- + \tfrac{s-r^2}{r^3-r}\beta\omega^2\\
d\gamma &= \tfrac{s-r^2}{r^3-r}\gamma\omega^1 - s\tfrac{r}{q_{12}}\delta\omega^+ + \tfrac{s-1}{r-r^{-1}}\gamma\omega^2\\
d\delta &= \tfrac{-r^2+s(1-r^2+r^4)}{r^3-r}\delta\omega^1 - s\tfrac{q_{12}}{r}\gamma\omega^- + \tfrac{s-r^2}{r^3-r}\delta\omega^2\\
dx^1 &= -\tfrac{sr}{q_{01}}\alpha V^1 - \tfrac{sr}{q_{02}}\beta V^2 + \tfrac{s-1}{r-r^{-1}}x^1\tau\\
dx^2 &= -\tfrac{sr}{q_{01}}\gamma V^1 - \tfrac{sr}{q_{02}}\delta V^2 + \tfrac{s-1}{r-r^{-1}}x^2\tau\\
du &= \tfrac{s-1}{r-r^{-1}}u\tau\\
d\xi &= \tfrac{r^2 s^{-N-1}-1}{r-r^{-1}}\xi\tau, \quad d\zeta = \tfrac{r^2 s^{-N}-1}{r-r^{-1}}\zeta\tau\\
d(\det T^A{}_B) &= \tfrac{r^{-2}s^{N+1}-1}{r-r^{-1}}(\det T^A{}_B)\tau, \quad d(\det T^a{}_b) = \tfrac{r^{-2}s^{N}-1}{r-r^{-1}}(\det T^A{}_B)\tau
\end{aligned}$$

The ω^i in terms of the exterior derivatives on $\alpha, \beta, \gamma, \delta, x^1, x^2, u$:

$$\begin{aligned}
\omega^1 &= \tfrac{r}{s(-r^2-r^4+s+sr^4)}[(r^2-s)(\kappa(\alpha)da + \kappa(\beta)d\gamma) + r^2(s-1)(\kappa(\gamma)d\beta + \kappa(\delta)d\delta)]\\
\omega^+ &= -\tfrac{1}{s}\tfrac{q_{12}}{r}[\kappa(\gamma)d\alpha + \kappa(\delta)d\gamma]\\
\omega^- &= -\tfrac{1}{s}\tfrac{r}{q_{12}}[\kappa(\alpha)d\beta + \kappa(\beta)d\delta]\\
\omega^2 &= \tfrac{r}{s(-r^2-r^4+s+sr^4)}[(s-r^2-sr^2+sr^4)(\kappa(\alpha)d\alpha + \kappa(\beta)d\gamma) + (r^2-s)(\kappa(\gamma)d\beta + \kappa(\delta)d\delta)]\\
V^1 &= -\tfrac{q_{01}}{sr}[\kappa(\alpha)dx^1 + \kappa(\beta)dx^2 + \kappa(x^1)du]\\
V^2 &= -\tfrac{q_{02}}{sr}[\kappa(\gamma)dx^1 + \kappa(\delta)dx^2 + \kappa(x^2)du]
\end{aligned}$$

The multiparametric quantum plane $Fun_{q,r}(IGL(2)/GL(2))$

$$x^1x^2 = qx^2x^1$$

$$\begin{aligned}
dx^1x^1 &= r^{-2}x^1dx^1 \\
dx^1\ x^2 &= \tfrac{q}{r^2}x^2dx^1 \\
dx^2\ x^1 &= (r^{-2}-1)x^2dx^1 + q^{-1}x^1dx^2 \\
dx^2x^2 &= r^{-2}x^2dx^2
\end{aligned}$$

$$dx^1 \wedge dx^2 = -\tfrac{q}{r^2}dx^2 \wedge dx^1$$

Chapter 4

Geometry of the quantum Inhomogeneous Orthogonal and Symplectic Groups $ISO_{q,r}(N)$ and $ISp_{q,r}(N)$

In this chapter we study the inhomogeneous orthogonal and symplectic groups, their universal enveloping algebras and their differential calculi. We tush give a detailed analysis of the geometry of these inhomogeneous groups that are canonically associated (via a quotient procedure) to the orthogonal and symplectic quantum groups studied in [19].

The method used in the previous chapter to obtain $IGL_{q,r}(N)$, is here applied to obtain $ISO_{q,r}(N)$ and $ISp_{q,r}(N)$ and to give an R-matrix formulation of these q-groups. This method is based on a projection (consistent with respect to the Hopf structure) from the corresponding quantum groups of higher rank $A_{n+1}, B_{n+1}, C_{n+1}, D_{n+1}$.

In general the quantum inhomogeneous groups we analize do contain dilatations. There exists however a subclass of dilatation-free cases for special values of the deformation parameters. The important example of the q-Poincaré group is contained in our construction. In particular, we find a dilatation-free q-Poincaré group depending on one real parameter q.

We next present a detailed study of the universal enveloping algebra of the multiparametric homogeneous orthogonal and symplectic groups and find a suitable set of generators that can be ordered. This will clarify the structure of their inhomogeneous version. The projection procedure used to derive the $ISO_{q,r}(N)$ [$ISp_{q,r}(N)$] q-groups is then used to find their universal enveloping algebras as Hopf subalgebras of $U_{q,r}(so(N+2))$ [$U_{q,r}(sp(N+2))$]. An R-matrix formulation and the duality $ISO_{q,r}(N) \leftrightarrow U_{q,r}(iso(N))$ [$ISp_{q,r}(N) \leftrightarrow U_{q,r}(isp(N))$] are explicitly given.

The quantum Lie algebras of $ISO_{q,r}(N)$ $[ISP_{q,r}(N)]$ are subspaces (adjoint submodules) of $U_{q,r}(iso(N))$ $[U_{q,r}(isp(N))]$, and in the second part of the chapter we study these deformed Lie algebras and their associated differential calculi. This is again done using the projection or quotient structure of $ISO_{q,r}(N)$ and $ISp_{q,r}(N)$. Contrary to the $IGL_{q,r}(N)$ case, only for $r = 1$ we have a quantum differential calculus that is a continuous deformation of the commutative one. The necessity of taking $r = 1$ is discussed. In Section 4.5 we briefly introduce the multiparametric bicovariant calculus on the homogeneous orthogonal and symplectyc q-groups. In Section 4.6 we examine the case $r = 1$. We clarify some issues related to the classical limit and see how in this limit some tangent vectors become linearly dependent, thus providing the correct classical dimension of the tangent space. A similar mechanism occurs for the left-invariant 1-forms. In Section 4.7 the bicovariant calculi on $ISO_{q,r=1}(N)$ $[ISp_{q,r=1}(N)]$ are studied. We first consider a calculus that has one more generator than in the classical case, this generator correspond to the dilatation u of the quantum inhomogeneous group. Then we show how to restrict this calculus to one that has the same numbler of tangent vectors that appear in the classical case. All the quantities relevant to this differential calculus are explicitly constructed. The results are then directly applied to the q-Poincaré group $ISO_q(3,1)$.

4.1 B_n, C_n, D_n multiparametric quantum groups

The B_n, C_n, D_n multiparametric quantum groups are freely generated by the noncommuting matrix elements $T^a{}_b$ (fundamental representation) and the identity I, modulo the quadratic RTT and CTT relations discussed below. The noncommutativity is controlled by the R matrix:

$$R^{ab}{}_{ef}T^e{}_cT^f{}_d = T^b{}_fT^a{}_eR^{ef}{}_{cd} \tag{4.1.1}$$

which satisfies the quantum Yang-Baxter equation (2.1.69)

$$R^{a_1b_1}{}_{a_2b_2}R^{a_2c_1}{}_{a_3c_2}R^{b_2c_2}{}_{b_3c_3} = R^{b_1c_1}{}_{b_2c_2}R^{a_1c_2}{}_{a_2c_3}R^{a_2b_2}{}_{a_3b_3}, \tag{4.1.2}$$

a sufficient condition for the consistency of the "RTT" relations (4.1.1). The R-matrix components $R^{ab}{}_{cd}$ depend continuously on a (in general complex) set of parameters q_{ab}, r. For $q_{ab} = r$ we recover the uniparametric q-groups of ref. [19]. Then $q_{ab} \to 1, r \to 1$ is the classical limit for which $R^{ab}{}_{cd} \to \delta^a_c\delta^b_d$: the matrix entries $T^a{}_b$ commute and become the usual entries of the fundamental representation. The multiparametric R matrices for the A, B, C, D series can be found in [72] (other ref.s on multiparametric q-groups are given in [73, 74]). For the B, C, D case they read:

$$\begin{aligned} R^{ab}{}_{cd} = \ & \delta^a_c\delta^b_d[\tfrac{r}{q_{ab}} + (r-1)\delta^{ab} + (r^{-1}-1)\delta^{ab'}](1-\delta^{an_2}) + \delta^a_{n_2}\delta^b_{n_2}\delta^{n_2}_c\delta^{n_2}_d \\ & +(r-r^{-1})[\theta^{ab}\delta^b_c\delta^a_d - \epsilon_a\epsilon_c\theta^{ac}r^{\rho_a-\rho_c}\delta^{a'b}\delta_{c'd}] \end{aligned} \tag{4.1.3}$$

where $\theta^{ab} = 1$ for $a > b$ and $\theta^{ab} = 0$ for $a \langle b$; we define $n_2 \equiv \frac{N+1}{2}$ and primed indices as $a' \equiv N + 1 - a$. The indices run on N values (N=dimension of the fundamental representation $T^a{}_b$), with $N = 2n + 1$ for $B_n[SO(2n+1)]$, $N = 2n$ for $C_n[Sp(2n)]$, $D_n[SO(2n)]$. The terms with the index n_2 are present only for the B_n series. The ϵ_a and ρ_a vectors are given by:

$$\epsilon_a = \begin{cases} +1 & \text{for } B_n,\ D_n, \\ +1 & \text{for } C_n \text{ and } a < n, \\ -1 & \text{for } C_n \text{ and } a > n. \end{cases} \tag{4.1.4}$$

$$(\rho_1, ...\rho_N) = \begin{cases} (\frac{N}{2} - 1, \frac{N}{2} - 2, ..., \frac{1}{2}, 0, -\frac{1}{2}, ..., -\frac{N}{2} + 1) & \text{for } B_n \\ (\frac{N}{2}, \frac{N}{2} - 1, ...1, -1, ..., -\frac{N}{2}) & \text{for } C_n \\ (\frac{N}{2} - 1, \frac{N}{2} - 2, ..., 1, 0, 0, -1, ..., -\frac{N}{2} + 1) & \text{for } D_n \end{cases} \tag{4.1.5}$$

Moreover the following relations reduce the number of independent q_{ab} parameters [73], [72]:

$$q_{aa} = r, \quad q_{ba} = \frac{r^2}{q_{ab}}; \tag{4.1.6}$$

$$q_{ab} = \frac{r^2}{q_{ab'}} = \frac{r^2}{q_{a'b}} = q_{a'b'} \tag{4.1.7}$$

where (4.1.7) also implies $q_{aa'} = r$. Therefore the q_{ab} with $a < b < \frac{N}{2}$ give all the q's.

It is useful to list the nonzero complex components of the R matrix (no sum on repeated indices):

$$\begin{array}{lll} R^{aa}{}_{aa} = r, & a \neq n_2 & \\ R^{aa'}{}_{aa'} = r^{-1}, & a \neq n_2 & \\ R^{n_2 n_2}{}_{n_2 n_2} = 1 & & \\ R^{ab}{}_{ab} = \dfrac{r}{q_{ab}}, & a \neq b,\ a' \neq b & \\ R^{ab}{}_{ba} = r - r^{-1}, & a > b, a' \neq b & \\ R^{aa'}{}_{a'a} = (r - r^{-1})(1 - \epsilon r^{\rho_a - \rho_{a'}}) = (r - r^{-1})[1 - C^{a'a}C_{a'a}], & a > a' & \\ R^{aa'}{}_{bb'} = -(r - r^{-1})\epsilon_a \epsilon_b r^{\rho_a - \rho_b} = -(r - r^{-1})C^{a'a}C_{bb'}, & a > b,\ a' \neq b & \end{array} \tag{4.1.8}$$

where $\epsilon = \epsilon_a \epsilon_{a'}$, i.e. $\epsilon = 1$ for B_n, D_n and $\epsilon = -1$ for C_n.

Note 4.1.1 The matrix R is lower triangular, that is $R^{ab}{}_{cd} = 0$ if [$a = c$ and $b < d$] or $a < c$, and has the following properties:

$$R^{-1}_{q,r} = R_{q^{-1},r^{-1}} \quad ; \quad (R_{q,r})^{ab}{}_{cd} = (R_{q,r})^{c'd'}{}_{a'b'} \quad ; \quad (R_{q,r})^{ab}{}_{cd} = (R_{p,r})^{dc}{}_{ba} \tag{4.1.9}$$

where q, r denote the set of parameters q_{ab}, r, and $p_{ab} \equiv q_{ba}$.

The inverse R^{-1} is defined by $(R^{-1})^{ab}{}_{cd}R^{cd}{}_{ef} = \delta^a_e\delta^b_f = R^{ab}{}_{cd}(R^{-1})^{cd}{}_{ef}$. Eq. (4.1.9) implies that for $|q| = |r| = 1$, $\bar{R} = R^{-1}$.

Note 4.1.2 Let R_r be the uniparametric R matrix for the B, C, D q-groups. The multiparametric $R_{q,r}$ matrix is obtained from R_r via the transformation [73, 72]

$$R_{q,r} = F^{-1}R_rF^{-1} \tag{4.1.10}$$

where $(F^{-1})^{ab}{}_{cd}$ is a diagonal matrix in the index couples ab, cd:

$$F^{-1} \equiv diag(\sqrt{\frac{r}{q_{11}}}, \sqrt{\frac{r}{q_{12}}}, \cdots \sqrt{\frac{r}{q_{NN}}}) \tag{4.1.11}$$

and ab, cd are ordered as in the R matrix. Since $\sqrt{\frac{r}{q_{ab}}} = (\sqrt{\frac{r}{q_{ba}}})^{-1}$ and $q_{aa'} = q_{bb'}$, the non diagonal elements of $R_{q,r}$ coincide with those of R_r. The matrix F satisfies $F_{12}F_{21} = 1$ i.e. $F^{ab}{}_{ef}F^{fe}{}_{dc} = \delta^a_c\delta^b_d$, the quantum Yang-Baxter equation $F_{12}F_{13}F_{23} = F_{23}F_{13}F_{12}$ and the relations $(R_r)_{12}F_{13}F_{23} = F_{23}F_{13}(R_r)_{12}$. Note that for $r = 1$ the multiparametric R matrix reduces to $R = F^{-2}$.

Note 4.1.3 Let $\hat{R}$ the matrix defined by $\hat{R}^{ab}{}_{cd} \equiv R^{ba}{}_{cd}$. Then the multiparametric $\hat{R}_{q,r}$ is obtained from $\hat{R}_r$ via the similarity transformation

$$\hat{R}_{q,r} = F\hat{R}_rF^{-1} \tag{4.1.12}$$

The characteristic equation and the projector decomposition of $\hat{R}_{q,r}$ are therefore the same as in the uniparametric case:

$$(\hat{R} - rI)(\hat{R} + r^{-1}I)(\hat{R} - \epsilon r^{\epsilon-N}I) = 0 \tag{4.1.13}$$

$$\hat{R} - \hat{R}^{-1} = (r - r^{-1})(I - K) \tag{4.1.14}$$

$$\hat{R} = rP_S - r^{-1}P_A + \epsilon r^{\epsilon-N}P_0 \tag{4.1.15}$$

with

$$\begin{array}{l} P_S = \frac{1}{r+r^{-1}}[\hat{R} + r^{-1}I - (r^{-1} + \epsilon r^{\epsilon-N})P_0] \\ P_A = \frac{1}{r+r^{-1}}[-\hat{R} + rI - (r - \epsilon r^{\epsilon-N})P_0] \\ P_0 = Q_N(r)K \\ Q_N(r) \equiv (C_{ab}C^{ab})^{-1} = \frac{1-r^{-2}}{(1-\epsilon r^{-N-1+\epsilon})(1+\epsilon r^{N-1-\epsilon})}\ , \qquad K^{ab}{}_{cd} = C^{ab}C_{cd} \\ I = P_S + P_A + P_0 \end{array} \tag{4.1.16}$$

To prove (4.1.14) in the multiparametric case note that $F_{12}K_{12}F_{12}^{-1} = K_{12}$. Orthogonality (and symplecticity) conditions can be imposed on the elements $T^a{}_b$, consistently with the RTT relations (4.1.1):

$$C^{bc}T^a{}_bT^d{}_c = C^{ad}I\ , \quad C_{ac}T^a{}_bT^c{}_d = C_{bd}I \tag{4.1.17}$$

where the (antidiagonal) metric is :

$$C_{ab} = \epsilon_a r^{-\rho_a} \delta_{ab'} \tag{4.1.18}$$

and its inverse C^{ab} satisfies $C^{ab}C_{bc} = \delta^a_c = C_{cb}C^{ba}$. We see that for the orthogonal series, the matrix elements of the metric and the inverse metric coincide, while for the symplectic series there is a change of sign: $C^{ab} = \epsilon C_{ab}$. Notice also the symmetries $C_{ab} = C_{b'a'}$ and $C_{ba}(r) = \epsilon C_{ab}(r^{-1})$.

The consistency of (4.1.17) with the RTT relations is due to the identities:

$$C_{ab}\hat{R}^{bc}{}_{de} = (\hat{R}^{-1})^{cf}{}_{ad}C_{fe} \; , \tag{4.1.19}$$

$$\hat{R}^{bc}{}_{de}C^{ea} = C^{bf}(\hat{R}^{-1})^{ca}{}_{fd} \; . \tag{4.1.20}$$

These identities hold also for $\hat{R} \to \hat{R}^{-1}$ and can be proved using the explicit expression (4.1.8) of R.

We also note the useful relations, easily deduced from (4.1.15):

$$C_{ab}\hat{R}^{ab}{}_{cd} = \epsilon r^{\epsilon-N} C_{cd}, \quad C^{cd}\hat{R}^{ab}{}_{cd} = \epsilon r^{\epsilon-N} C^{ab} \tag{4.1.21}$$

and, from (4.1.8),

$$R^{ab}{}_{cc'} = -(r-r^{-1})C^{ba}C_{cc'} \; , \; R^{aa'}{}_{cd} = -(r-r^{-1})C^{a'a}C_{cd} \quad \text{for} \;\; a>c \; , \; a \neq c' \; . \tag{4.1.22}$$

Notice also that $\kappa^2(T^a{}_b) = D^a_e T^e{}_f D^{-1}{}^f_b$ where $D^a_e = C^{as}C_{es}$ and its inverse $D^{-1}{}^f_b = C^{sf}C_{sb}$ are diagonal.

The co-structures of the B_n, C_n, D_n multiparametric quantum groups have the same form as in the uniparametric case: the coproduct Δ, the counit ε and the coinverse κ are given by

$$\Delta(T^a{}_b) = T^a{}_b \otimes T^b{}_c \tag{4.1.23}$$

$$\varepsilon(T^a{}_b) = \delta^a_b \tag{4.1.24}$$

$$\kappa(T^a{}_b) = C^{ac}T^d{}_c C_{db} \tag{4.1.25}$$

Note 4.1.4 Using formula (4.1.3) or (4.1.8), we find that the $R^{AB}{}_{CD}$ matrix for the $SO_{q,r}(N+2)$ and $Sp_{q,r}(N+2)$ quantum groups can be decomposed in terms of $SO_{q,r}(N)$ and $Sp_{q,r}(N)$ quantities as follows (splitting the index A as A=$(\circ, a, \bullet)$, with $a = 1, \ldots N$):

$$
R^{AB}{}_{CD} = \left(\begin{array}{c|ccccccccc}
 & \circ\circ & \circ\bullet & \bullet\circ & \bullet\bullet & \circ d & \bullet d & c\circ & c\bullet & cd \\
\circ\circ & r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\circ\bullet & 0 & r^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bullet\circ & 0 & f(r) & r^{-1} & 0 & 0 & 0 & 0 & 0 & -\epsilon C_{cd}\lambda r^{-\rho} \\
\bullet\bullet & 0 & 0 & 0 & r & 0 & 0 & 0 & 0 & 0 \\
\circ b & 0 & 0 & 0 & 0 & \frac{r}{q_{\circ b}}\delta^b_d & 0 & 0 & 0 & 0 \\
\bullet b & 0 & 0 & 0 & 0 & 0 & \frac{r}{q_{\bullet b}}\delta^b_d & 0 & \lambda\delta^b_c & 0 \\
a\circ & 0 & 0 & 0 & 0 & \lambda\delta^a_d & 0 & \frac{r}{q_{a\circ}}\delta^a_c & 0 & 0 \\
a\bullet & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{r}{q_{a\bullet}}\delta^a_c & 0 \\
ab & 0 & -C^{ba}\lambda r^{-\rho} & 0 & 0 & 0 & 0 & 0 & 0 & R^{ab}{}_{cd}
\end{array} \right) \tag{4.1.26}
$$

where $R^{ab}{}_{cd}$ is the R matrix for $SO_{q,r}(N)$ or $Sp_{q,r}(N)$, C_{ab} is the corresponding metric, $\lambda \equiv r - r^{-1}$, $\rho = \frac{N+1-\epsilon}{2}$ ($\epsilon r^{\rho} = C_{\bullet\circ}$) and $f(r) \equiv \lambda(1 - \epsilon r^{-2\rho})$. The sign ϵ has been defined after eq. s (4.1.8).

4.1.1 Real forms: $SO_{q,r}(N, \mathbf{R})$, $SO_{q,r}(N-1,1)$, $SO_{q,r}(n,n)$, $SO_{q,r}(n+1, n-1)$, $SO_{q,r}(n+1,n)$

Following [19], a conjugation —i.e. an algebra antiautomorphism, coalgebra automorphism and involution, satisfying $\kappa(\kappa(T^*)^*) = T$— can be defined

• trivially as $T^* = T$. Compatibility with the RTT relations (1.2.1) requires $\bar{R}_{q,r} = R^{-1}_{q,r} = R_{q^{-1},r^{-1}}$, i.e. $|q| = |r| = 1$. Then the CTT relations are invariant under $*$-conjugation. The corresponding real forms are $SO_{q,r}(n,n;R)$, $SO_{q,r}(n+1,n;R)$ (for N even and odd respectively) and $Sp_{q,r}(2n;R)$. A conjugation on the quantum orthogonal (symplectic) plane (defined respectively by $P_A^{ab}{}_{cd}x^c x^d = 0$ and $x^a x^b = r^{-1}R^{ab}_{cd}x^c x^d$) that is compatible with the natural coaction δ of the q-group on the q-plane: $x^a \to T^a{}_b \otimes x^b$ is given by $(x^a)^* = x^a$, indeed we have $\delta(x^*) = T^* \otimes x^* \equiv \delta^*(x)$.

• via the metric as $T^* = (\kappa(T))^t$ i.e. $T^* = C^t T C^t$. The condition on R is $\bar{R}^{ab}{}_{cd} = R^{dc}{}_{ba}$, which happens for $q_{ab}\bar{q}_{ab} = r^2, r \in \mathbf{R}$. Again the CTT relations are $\star$-invariant. The metric on a "real" basis has compact signature $(+,+,\ldots+)$ so that the real form is $SO_{q,r}(N;R)$. The conjugation on the quantum orthogonal plane compatible with the coaction $x^a \to T^a{}_b \otimes x^b$ is: $(x^a)^* = C_{ba}x^b$, indeed $\delta(x^*) = \delta^*(x)$.

We now introduce, following [71], two other conjugations that give the real forms $SO_{q,r}(N-1,1)$, $SO_{q,r}(n+1,n-1)$, $SO_{q,r}(n+1,n)$. The real form $SO_{q,r}(n+1,n-1)$ has been found in [76], the real form $SO_{q,r}(2n-1,1)$, as far as we know, appears here for the first time.

We first notice that if we have an involution $\sharp$ that is a Hopf algebra automorphism (algebra and coalgebra morphism compatible with the antipode: $\kappa(a^\sharp) =$

$[\kappa(a)]^\sharp$) and that commutes with a conjugation $*$, then the composition of these two involutions: $*^\sharp \equiv \sharp\circ* = *\circ\sharp$ is again a conjugation. We now find an involution $\sharp$ that comutes with $*$ and $\star$ as defined above.

Define the map $\sharp$ on the generators as:

$$T^\sharp = \mathcal{D}T\mathcal{D}^{-1} \quad \text{i.e.} \quad (T^a{}_b)^\sharp = \mathcal{D}^a{}_e T^e{}_f \mathcal{D}^f{}_b{}^{-1} \tag{4.1.27}$$

and extend it by linearity and multiplicativity to all $SO_{q,r}(N)$. The entries of the N-dimensional $\mathcal{D}$ matrix are

$$\mathcal{D} = \begin{pmatrix} 1 & & & & & & \\ & \cdots & & & & & \\ & & 1 & & & & \\ & & & 0 & 1 & & & \\ & & & 1 & 0 & & & \\ & & & & & 1 & & \\ & & & & & & \cdots & \\ & & & & & & & 1 \end{pmatrix}, \quad \mathcal{D} = \begin{pmatrix} 1 & & & & & & \\ & \cdots & & & & & \\ & & 1 & & & & \\ & & & -1 & & & \\ & & & & 1 & & \\ & & & & & \cdots & \\ & & & & & & 1 \end{pmatrix}$$

for N even for N odd

(4.1.28)

In the $N = 2n$ case the $\mathcal{D}$ matrix exchanges the index n with the index $n+1$, in the $N = 2n+1$ case $\mathcal{D}$ change the sign of the entries of the T matrix as many times as the index $n_2 = (N+1)/2$ appears. Since $\mathcal{D}^2 = \mathbf{1}$ we immediately see that $\sharp$ is an involution.

We now prove that $\sharp$ is compatible with the algebra structure, i.e. it is compatible with the RTT and CTT relations; in the $N = 2n$ case this is true if $q_{ab} = r$ when at least one of the indices a, b is equal to n or $n+1$.

Use relation (4.1.8) and, if $N = 2n$, the above restriction on the q_{ab} parameters to prove that

$$\mathcal{D}_1\mathcal{D}_2 R\, \mathcal{D}_1\mathcal{D}_2 = R\ . \tag{4.1.29}$$

The compatibility with the RTT relations is then esily seen to hold. [Hint: multiply $(R_{12}T_1T_2)^\sharp = (T_2T_1R_{12})^\sharp$ by $\mathcal{D}_1\mathcal{D}_2$ from the left and from the right and use $\mathcal{D}^2 = \mathbf{1}$ to prove the equivalence with $R_{12}T_1T_2 = T_2T_1R_{12}$]. Similarly the compatibility of $\sharp$ with the orthogonality relations (4.1.17), that we rewite in matrix notation as:

$$TCT^t = C\,I\ , \quad T^tCT = C\,I\ , \tag{4.1.30}$$

is due to $\mathcal{D}^2 = \mathbf{1}$, $\mathcal{D}^t = \mathcal{D}$ and the commutativity of the C matrix with the $\mathcal{D}$ matrix:

$$\mathcal{D}C\mathcal{D} = C\ . \tag{4.1.31}$$

For example we have: $(TCT^t)^\sharp = T^\sharp C(T^t)^\sharp = \mathcal{D}T\mathcal{D}C\mathcal{D}T^t\mathcal{D} = \mathcal{D}(TCT^t)\mathcal{D}$ and using $\mathcal{D}^2 = \mathbf{1}$ and again (4.1.31) we conclude that $(TCT^t)^\sharp = C\,I$ is equivalent to $TCT^t = C\,I$.

Next we prove that $\sharp$ is compatible with the coalgebra structure. Compatibility with the coproduct is trivial, compatibility with the antipode is easily verified:

$$\kappa(T^\sharp) = \kappa(\mathcal{D}T\mathcal{D}) = \mathcal{D}\kappa(T)\mathcal{D} = \mathcal{D}CT^tC\mathcal{D} = C\mathcal{D}T^t\mathcal{D}C = [\kappa(T)]^\sharp . \qquad (4.1.32)$$

We now show that the two conjugations defined at the beginning of this subsection commutes with $\sharp$. For the second conjugation, defined by $T^\star = [\kappa(T)]^t = C^tTC^t$, we have, since $\overline{\mathcal{D}} = \mathcal{D}$:

$$\begin{aligned} (T^\sharp)^\star &= (\mathcal{D}T\mathcal{D})^\star = \mathcal{D}T^\star\mathcal{D} = \mathcal{D}[\kappa(T)]^t\mathcal{D} \\ &= \mathcal{D}C^tTC^t\mathcal{D} = C^t\mathcal{D}T\mathcal{D}C^t = (C^tTC^t)^\sharp = ([\kappa(T)]^t)^\sharp \qquad (4.1.33) \\ &= (T^\star)^\sharp \end{aligned}$$

The two maps $\star$ and $\sharp$ not only commute when applied to the $T^a{}_b$ matrix entries, they also commute when applied to any element of the q-group because they are respectively multiplicative and antimultiplicative. The proof that $\sharp\circ * = *\circ\sharp$ for the first conjugation, defined by $T^* = T$, is straighforward.

We restate the above results as a theorem:

Theorem 4.1.1 The map $\sharp$ is an automorphism and an involution of the quantum group $SO_{q,r}(N)$; in the $N = 2n$ case this holds with the restriction $q_{ab} = r$ when at least one of the indices a, b is equal to n or $n+1$. The compositions $*^\sharp \equiv \sharp\circ *$ and $\star^\sharp \equiv \sharp\circ\star$ of the conjugations $*$ and $\star$ with the automorphism $\sharp$ is again a conjugation. The restrictions on the parameters r, q_{ab} are obtained adding to the constraint imposed by $*$ ($\star$ respectively) the constaints imposed by $\sharp$. □□□

Associated to $*^\sharp$ and $\star^\sharp$ we have the conjugations that act on the quantum orthogonal plane and are compatible with the coaction $x^a \to T^a{}_b \otimes x^b$. These conjugations respectively are: $(x^a)^{*^\sharp} = (x^a)^\sharp$ and $(x^a)^{\star^\sharp} = C_{ba}\mathcal{D}^b{}_e x^e$.

We now study the real forms related to the $*^\sharp$ and $\star^\sharp$ conjugations.

- The conjugation $*^\sharp$ for the N-dimensional orthogonal quantum groups with N odd gives the real form $SO_{q,r}(n, n+1)$.

- The conjugation $\star^\sharp$ for the N-dimensional orthogonal quantum goups with N even has been studied (in the uniparametric case) in [76], it gives the real form $SO_{q,r}(n+1, n-1; R)$ and in particular the quantum Lorentz group $SO_r(3,1)$ with $|r| = 1$. For an explicit proof see formula (5.2.105).

If we require $*^\sharp$ to be a conjugaton but do not require $\sharp$ to be an automorphism and $*$ to be a conjugation, we can partially relax the constraints on the r, q parameters. Compatibility of $*^\sharp$ with the RTT relations is indeed easily seen to require

$$(\bar{R})_{n\leftrightarrow n+1} = R^{-1}, \quad \text{i.e.} \quad \mathcal{D}_1\mathcal{D}_2R_{12}\mathcal{D}_1\mathcal{D}_2 = \overline{R}_{12}^{-1} \qquad (4.1.34)$$

which implies

i) $|q_{ab}| = |r| = 1$ for a and b both different from n or $n+1$;

ii) $q_{ab}/r \in \mathbf{R}$ when at least one of the indices a, b is equal to n or $n+1$.

In the sequel we will denote simply by $*$ this conjugation. As discussed in ref.s [14, 67] and later in this chapter we will need this conjugation to obtain the inhomogeneous Lorentz group $ISO_{q,r}(3,1;R)$.

- The conjugation $\star^{\sharp}$ for $N = 2n+1$ gives the real form $SO(2n,1)$ and has been introduced in [19].

- The conjugation $\star^{\sharp}$ for $N = 2n$ as far as we know is not known in the literature, it gives the real form $SO_{q,r}(2n-1,1)$. In particular we obtain another quantum Lorentz group $SO_r(3,1)$, notice that here $r \in \mathbf{R}$.

4.2 The quantum inhomogeneous groups $ISO_{q,r}(N)$ and $ISp_{q,r}(N)$

Following the projection procedure described in Section 3.3 we here introduce the quantum inhomogeneous groups $ISO_{q,r}(N)$ and $ISp_{q,r}(N)$ and give an R-matrix formulation. The $ISO_{q,r}(N)$ quantum group has been independently studied – without an R-matrix formulation– in the first reference of [65]. The structure of $ISp_{q,r}(N)$ cannot be derived from $Sp_{q,r}(N)$ and the symplectic q-plane relations as is the case for $ISO_{q,r}(N)$, however it can be easily defined via the projection procedure. $ISp_{q,r}(N)$ and its R-matrix formualtion where first introduced in [67].

We define $ISO_{q,r}(N)$ and $ISp_{q,r}(N)$ as the quotients:

$$ISO_{q,r}(N) \equiv \frac{SO_{q,r}(N+2)}{H}\ ,\quad ISp_{q,r}(N) \equiv \frac{Sp_{q,r}(N+2)}{H} \tag{4.2.1}$$

where H is the Hopf ideal in $SO_{q,r}(N+2)$ or $Sp_{q,r}(N+2)$ of all sums of monomials containing at least an element of the kind $T^a{}_\circ, T^\bullet{}_b, T^\bullet{}_\circ$. The Hopf structure of the groups in the numerators of (4.2.1) is naturally inherited by the quotient groups.

We introduce the following convenient notations: $\mathcal{T}$ stands for $T^a{}_\circ$, $T^\bullet{}_b$ or $T^\bullet{}_\circ$, $S_{q,r}(N+2)$ stands for either $SO_{q,r}(N+2)$ or $Sp_{q,r}(N+2)$, and we indicate by Δ_{N+2}, ε_{N+2} and κ_{N+2} the corresponding co-structures.

We denote by P the canonical projection

$$P\ :\ S_{q,r}(N+2) \longrightarrow S_{q,r}(N+2)/H \tag{4.2.2}$$

It is a Hopf algebra epimorphism because $H = Ker(P)$ is a Hopf ideal. [The proof is as in Theorem 3.3.1, just use $\mathcal{T}$ instead of $T^0{}_b$. In order to show that $\kappa_{N+2}(H) \subseteq H$, notice that $\kappa_{N+2}(\mathcal{T}) \propto \mathcal{T}$ and therefore, $\forall h \in H$, $\kappa_{N+2}(h) = \kappa_{N+2}(b\mathcal{T}c) = \kappa_{N+2}(c)\kappa_{N+2}(\mathcal{T})\kappa_{N+2}(b) \in H$]. Then any element of $S_{q,r}(N+2)/H$

is of the form $P(a)$ and the Hopf algebra structure is given by:

$$P(a)+P(b)\equiv P(a+b)\ ;\ \ P(a)P(b)\equiv P(ab)\ ;\ \ \mu P(a)\equiv P(\mu a),\quad \mu\in\mathbf{C} \tag{4.2.3}$$

$$\Delta(P(a))\equiv(P\otimes P)\Delta_{N+2}(a)\ ;\ \ \varepsilon(P(a))\equiv\varepsilon_{N+2}(a)\ ;\ \ \kappa(P(a))\equiv P(\kappa_{N+2}(a))\ . \tag{4.2.4}$$

We can also give an R–matrix formulation of the inhomogeneous $ISO_{q,r}(N)$ and $ISp_{q,r}(N)$ q-groups. Indeed recall that $S_{q,r}(N+2)$ is the Hopf algebra freely generated by the non-commuting matrix elements $T^A{}_B$ modulo the ideal generated by the RTT and CTT relations [R matrix and metric C of $S_{q,r}(N+2)$]. This can be expressed as:

$$S_{q,r}(N+2)\equiv\frac{<T^A{}_B>}{[RTT,CTT]} \tag{4.2.5}$$

Therefore we have (recall that $H\equiv[T^a{}_\circ,T^\bullet{}_b,T^\bullet{}_\circ]\equiv[\mathcal{T}]$) :

$$IS_{q,r}(N)=\frac{S_{q,r}(N+2)}{[\mathcal{T}]}=\frac{<T^A{}_B>/[RTT,CTT]}{[\mathcal{T}]}=\frac{<T^A{}_B>}{[RTT,CTT,\mathcal{T}]} \tag{4.2.6}$$

so that we have shown the following:

Theorem 4.2.1 The quantum inhomogeneous groups $ISO_{q,r}(N)$ and $ISp_{q,r}(N)$ are freely generated by the non-commuting matrix elements $T^A{}_B$ [A=$(\circ,a,\bullet)$, with $a=1,...N)$] and the identity I, modulo the relations:

$$T^a{}_\circ=T^\bullet{}_b=T^\bullet{}_\circ=0, \tag{4.2.7}$$

the RTT relations

$$R^{AB}{}_{EF}T^E{}_CT^F{}_D=T^B{}_FT^A{}_ER^{EF}{}_{CD}, \tag{4.2.8}$$

and the orthogonality (symplecticity) relations

$$C^{BC}T^A{}_BT^D{}_C=C^{AD},\quad C_{AC}T^A{}_BT^C{}_D=C_{BD} \tag{4.2.9}$$

The co-structures of $ISO_{q,r}(N)$ and $ISp_{q,r}(N)$ are simply given by:

$$\Delta(T^A{}_B)=T^A{}_C\otimes T^C{}_B,\ \ \kappa(T^A{}_B)=C^{AC}T^D{}_CC_{DB},\ \ \varepsilon(T^A{}_B)=\delta^A_B\ . \tag{4.2.10}$$

□□□

After decomposing the indices A=$(\circ,a,\bullet)$, and defining:

$$u\equiv T^\circ{}_\circ,\quad v\equiv T^\bullet{}_\bullet,\quad z\equiv T^\circ{}_\bullet,\quad x^a\equiv T^a{}_\bullet,\quad y_a\equiv T^\circ{}_a \tag{4.2.11}$$

the relations (4.2.8) and (4.2.9) become [67]:

$$R^{ab}{}_{ef}T^e{}_cT^f{}_d=T^b{}_fT^a{}_eR^{ef}{}_{cd} \tag{4.2.12}$$

$$T^a{}_bC^{bc}T^d{}_c=C^{ad}I \tag{4.2.13}$$

$$T^a{}_bC_{ac}T^c{}_d=C_{bd}I \tag{4.2.14}$$

$$T^b{}_d x^a = \frac{r}{q_{d\bullet}} R^{ab}{}_{ef} x^e T^f{}_d \tag{4.2.15}$$

$$P_A^{ab}{}_{cd} x^c x^d = 0 \tag{4.2.16}$$

$$T^b{}_d v = \frac{q_{b\bullet}}{q_{d\bullet}} v T^b{}_d \tag{4.2.17}$$

$$x^b v = q_{b\bullet} v x^b \tag{4.2.18}$$

$$uv = vu = I \tag{4.2.19}$$

$$ux^b = q_{b\bullet} x^b u \tag{4.2.20}$$

$$uT^b{}_d = \frac{q_{b\bullet}}{q_{d\bullet}} T^b{}_d u \tag{4.2.21}$$

$$y_b = -r^\rho T^a{}_b C_{ac} x^c u \tag{4.2.22}$$

$$(r^{-\rho} + \epsilon r^{\rho-2})\, z = -x^b C_{ba} x^a u \tag{4.2.23}$$

where $q_{a\bullet}$ are N complex parameters related by $q_{a\bullet} = r^2/q_{a'\bullet}$, with $a' = N+1-a$. The matrix P_A in eq. (4.2.16) is the q-antisymmetrizer for the B, C, D q-groups given by (cf. (4.1.16)):

$$P_A^{ab}{}_{cd} = -\frac{1}{r+r^{-1}} (\hat{R}^{ab}{}_{cd} - r\delta^a_c \delta^b_d + \frac{r - r^{-1}}{\epsilon r^{N-1-\epsilon}+1} C^{ab} C_{cd}). \tag{4.2.24}$$

The last two relations (4.2.22) - (4.2.23) are constraints, showing that the $T^A{}_B$ matrix elements in eq. (4.2.8) are really a *redundant* set. This redundance is necessary if we want to express the q-commuations of the $ISO_{q,r}(N)$ and $ISp_{q,r}(N)$ basic group elements as $RTT = TTR$ (i.e. if we want an R-matrix formulation). Remark that, in the R-matrix formulation for $IGL_{q,r}(N)$, *all* the $T^A{}_B$ are independent. Here we can take as independent generators the elements

$$T^a{}_b, x^a, v, u \equiv v^{-1} \text{ and the identity } I \qquad (a = 1, ... N) \tag{4.2.25}$$

The co-structures on the $ISO_{q,r}(N)$ (or $ISp_{q,r}(N)$) generators can be read from (4.2.10) after decomposing the indices $A = \circ, a, \bullet$:

$$\Delta(T^a{}_b) = T^a{}_c \otimes T^c{}_b\ , \quad \Delta(x^a) = T^a{}_c \otimes x^c + x^a \otimes v\ , \tag{4.2.26}$$

$$\Delta(v) = v \otimes v\ , \quad \Delta(u) = u \otimes u\ , \tag{4.2.27}$$

$$\kappa(T^a{}_b) = C^{ac} T^d{}_c C_{db} = \epsilon_a \epsilon_b r^{-\rho_a + \rho_b}\, T^{b'}{}_{a'}\ , \tag{4.2.28}$$

$$\kappa(x^a) = -\kappa(T^a{}_c) x^c u\ , \quad \kappa(v) = u\ , \quad \kappa(u) = v\ , \tag{4.2.29}$$

$$\varepsilon(T^a{}_b) = \delta^a_b\ , \quad \varepsilon(x^a) = 0\ , \quad \varepsilon(u) = \varepsilon(v) = \varepsilon(I) = 1\ . \tag{4.2.30}$$

In the commutative limit $q \to 1, r \to 1$ we recover the algebra of functions on $ISO(N)$ (plus the dilatation v that can be set to the identity). In the $ISp(N)$ case, the $q \to 1, r \to 1$ limit of relation (4.2.23) implies $x^b C_{ba} x^a = 0$ i.e. $P_0^{ab}{}_{ef} x^e x^f = 0$, that with (4.2.16) gives the commutativity of the coordinates x^a (both P_A and P_0 are antisymmetrizers in the symplectic case). We then recover the algebra of functions on $ISp(N)$ plus the element z, that can be set to zero, and the dilatation v, that can be set to the identity (see also Note 4.2.4).

Note 4.2.1 In order to study the $ISO_{q,r}(N)$ and $ISp_{q,r}(N)$ differential calculus and universal enveloping algebras we will use the definition (4.2.1) rather then Theorem 4.2.1 : $IS_{q,r}(N) = \frac{<T^A{}_B>}{[RTT,CTT,T]}$. With abuse of notations we therefore identify [cf. (4.2.11)]: $u = P(T^\circ{}_\circ)$, $v = P(T^\bullet{}_\bullet)$, $z = P(T^\circ{}_\bullet)$, $x^a = P(T^a{}_\bullet)$, $y_a = P(T^\circ{}_a)$, $T^a{}_b = P(T^a{}_b)$; $I = P(I)$ where P : $S_{q,r}(N+2) \to IS_{q,r}(N) \equiv S_{q,r}(N+2)/H$.

Note 4.2.2 From the commutations (4.2.20) - (4.2.21) we see that one can set $u = I$ only when $q_{a\bullet} = 1$ for all a. From $q_{a\bullet} = r^2/q_{a'\bullet}$, cf. eq. (4.1.7), this implies also $r = 1$.

Note 4.2..3 Eq.s (4.2.16) are the multiparametric orthogonal quantum plane commutations. They follow from the $({}^a{}_\bullet{}^b{}_\bullet)$ RTT components and (4.2.23).

Note 4.2.4 In the symplectic case eq.s (4.2.16) alone are not sufficient to order in an arbitrary given way a monomial in the x elements; we can obtain an ordering only if we consider also the element zv (or z) besides the x elements. In other terms, the expression $x^a C_{ab} x^b$ appearing in (4.2.23) cannot be ordered as $\alpha_{ab} x^a x^b$ with $\alpha_{ab} \in \boldsymbol{C}$ and $\alpha_{ab} = 0$ if $a > b$.

To recover the symplectic q-plane commutations relations described in [19] one has to impose also the condition

$$P_0{}^{ab}{}_{cd} x^c x^d = 0 \quad \text{i.e.} \quad x^a C_{ab} x^b = 0 \ , \quad z = 0 \tag{4.2.31}$$

that arises naturally from the characteristic equation and the projector decomposition of the R-matrix: in the symplectic case P_0 is an antisymmetrizer. As a consequence the xx commutations (4.2.16) become

$$\hat{R}^{ab}{}_{cd} x^c x^d - r x^b x^a = 0 \ . \tag{4.2.32}$$

Notice however that (4.2.31) is not compatible with a deformation of the whole symplectic group. Condition (4.2.31) amounts to consider the Hopf quotient $ISp_{q,r}(N)/K$ where K is the Hopf ideal generated by z. From $\Delta(z) = y_a \otimes x^a + u \otimes z + z \otimes v$ $\in ISp_{q,r}(N) \otimes K + K \otimes ISp_{q,r}(N)$ we deduce that $y_a \otimes x^a \in ISp_{q,r}(N) \otimes K + K \otimes ISp_{q,r}(N)$ and applying $(m \otimes id)(id \otimes \Delta)$ to $y_a \otimes x^a$ we obtain that $x^b C_{bd} \otimes x^d$ $\in ISp_{q,r}(N) \otimes K + K \otimes ISp_{q,r}(N)$. Projecting on $ISp_{q,r}(N)/K$ yields $x^b C_{bd} \otimes x^d = 0$ since K is the kernel of the projection. Now classically $x^b C_{bd} \otimes x^d \neq 0$. This proves that the classical limit of $ISp_{q,r}(N)/K$ is not the algebra of functions over $ISp(N)$.

Note 4.2.5 We here briefly study the structure of $IS_{q,r}(N)$ with respect to $S_{q,r}(N)$, that is easily seen to be a Hopf subalgebra of $IS_{q,r}(N)$. It is also a quotient of

$IS_{q,r}(N)$ via the Hopf algebra projection [well defined only if $q_{a\bullet} = const\ \forall a$ i.e. $q_{a\bullet} = r\ \forall a$ see (4.2.21)]:

$$\pi(x^a) = 0 \ , \quad \pi(u) = I \ , \quad \pi(T^a{}_b) = T^a{}_b \ , \quad \pi(I) = I \ .$$

Then the results of Theorem 3.3.6 apply to $IS_{q,r}(N)$ as well, and we can write the Hopf algebra isomorhism

$$IS_{q,r}(N) \cong B \rtimes S_{q,r}(N) \tag{4.2.33}$$

where B is the subalgebra of $IS_{q,r}(N)$ generated by u and x^a (in the orthogonal case B is the quantum orthogonal plane with dilatation u). Also Theorem 3.3.4 hold for $IS_{q,r}(N)$. [This theorem, neglecting the graded structure, is a consequence of Theorem 3.3.6, an explicit proof for the $IS_{q,r}(N)$ case follows the same steps as for $IGL_{q,r}(N)$]. The $(\mathbf{Z},\mathbf{N})$ grading is introduced in the following way: the elements $T^a{}_b$ have grade $(0,0)$, the elements x^a have grade $(0,1)$, the elements u and $v = u^{-1}$ have grade $(1,0)$ and $(-1,0)$. This grading is compatible with the RTT commutation relations.

For $ISO_{q,r}(N)$, the generators u and x^a of B can be ordered using (4.2.16) and (4.2.20), and the Poincaré series of the subalgebra B is the same as that of the commutative algebra in the $N+1$ symbols u, x^a [19]. A linear basis of B is therefore given by the ordered monomials: $\zeta^i = u^{i_o}(x^1)^{i_1}...(x^N)^{i_N}$ with $i_o \in \boldsymbol{Z}$, $i_1,...i_N \in \boldsymbol{N} \cup \{0\}$. In the $ISp_{q,r}(N)$ case, if we also consider the elements z, a linear basis of $B \subset ISp_{q,r}(N)$ is given by the ordered monomials $\zeta^i = u^{i_o}(x^1)^{i_1}...(x^N)^{i_N}(zv)^{i_{N+1}}$ with $i_o \in \boldsymbol{Z}$, $i_1,...i_N, i_{N+1} \in \boldsymbol{N} \cup \{0\}$ (zv commutes with the coordinates x^a).

Using (3.3.76), or (4.2.15) and (4.2.21), a generic element of $IS_{q,r}(N)$ can be written as $\zeta^i a_i$ (and also $a_i \zeta^i$) where $a_i \in S_{q,r}(N)$. As in Corollary 3.1.1, we have that $IS_{q,r}(N)$, for $q_{a\bullet} = r\ \forall a$, is a bicovariant bimodule over $S_{q,r}(N)$ freely generated, as a right module, by the elements ζ^i; moreover

$$IS_{q,r}(N) \ = \ \sum_{(h,k)\in(\mathbf{Z},\mathbf{N})}^{\oplus} \Gamma^{(h,k)} \tag{4.2.34}$$

where $\Gamma^{(0,0)} = S_{q,r}(N)$

$$\Gamma^{(0,1)} = \{x^a b_a \ / \ b_a \in S_{q,r}(N)\} \ , \qquad \Gamma^{(\pm 1,0)} = \{u^{\pm 1} b \ / \ b \in S_{q,r}(N)\}$$
$$\Gamma^{(h,k)} = \{u^h x^{a_1} x^{a_2} \dots x^{a_k} b_{a_1 a_2 \dots a_k} \ / \ b_{a_1 a_2 \dots a_k} \in S_{q,r}(N)\} \tag{4.2.35}$$

Any submodule $\Gamma^{(h,k)}$ is a bicovariant bimodule freely generated by the ordered monomials ζ^i with degree $(h,k) \in (\mathbf{Z},\mathbf{N})$. We leave to the reader to reformulate Note 3.3.5 and Note 3.3.6 in this context.

Note 4.2.6 Among all the real forms of $S_{q,r}(N+2)$ mentioned in the previous section, only * and $^{*\sharp}$ are inherited by $IS_{q,r}(N)$, indeed only these two conjugations are compatible with the ideal H: $H^* \subseteq H$ and $H^{*\sharp} \subseteq H$ [or, more easily, are compatible with (4.2.15)]. The conditions on the parameters are:

• $|q_{ab}| = |q_{a\bullet}| = |r| = 1$ for $ISO_{q,r}(n,n;\mathbf{R})$, $ISO_{q,r}(n,n+1;\mathbf{R})$ and $ISp_{q,r}(n;\mathbf{R})$.

• For $ISO_{q,r}(n+1,n-1;\mathbf{R})$: $|r|=1$; $|q_{a\bullet}|=1$ for $a \neq n, n+1$; $|q_{ab}|=1$ for a and b both different from n or $n+1$; $q_{ab}/r \in \mathbf{R}$ when at least one of the indices a,b is equal to n or $n+1$; $q_{a\bullet}/r \in \mathbf{R}$ for $a=n$ or $a=n+1$.

In particular, the quantum Poincaré group $ISO_{q,r}(3,1;\mathbf{R})$ is obtained by setting $|q_{1\bullet}| = |r| = 1$, $q_{2\bullet}/r \in \mathbf{R}$, $q_{12}/r \in \mathbf{R}$.

According to Note 4.2.2, a dilatation-free q-Poincaré group is found after the further restrictions $q_{1\bullet} = q_{2\bullet} = r = 1$. The only free parameter remaining is then $q_{12} \in \mathbf{R}$.

4.3 Universal enveloping algebras $U_{q,r}(so(N+2))$ and $U_{q,r}(sp(N+2))$

We construct the universal enveloping algebra $U_{q,r}(s(N+2))$ of $S_{q,r}(N+2)$ as the algebra of regular functionals [19] on $S_{q,r}(N+2)$ (recall that S stands for SO and Sp).

$U_{q,r}(s(N+2))$ is the algebra over $\boldsymbol{C}$ generated by the counit ε and by the functionals $L^{\pm}$ defined by their value on the matrix elements $T^A{}_B$:

$$L^{\pm A}{}_B(T^C{}_D) = (R^{\pm})^{AC}{}_{BD}, \tag{4.3.1}$$

$$L^{\pm A}{}_B(I) = \delta^A_B \tag{4.3.2}$$

with

$$(R^+)^{AC}{}_{BD} \equiv R^{CA}{}_{DB} \;\; ; \;\; (R^-)^{AC}{}_{BD} \equiv (R^{-1})^{AC}{}_{BD} \; . \tag{4.3.3}$$

To extend the definition (4.3.1) to the whole algebra $S_{q,r}(N+2)$ we set

$$L^{\pm A}{}_B(ab) = L^{\pm A}{}_C(a) L^{\pm C}{}_B(b) \quad \forall a,b \in S_{q,r}(N+2) \; . \tag{4.3.4}$$

From (4.3.1), using the upper and lower triangularity of R^+ and R^-, we see that L^+ is upper triangular and L^- is lower triangular.

The commutations between $L^{\pm A}{}_B$ and $L^{\pm C}{}_D$ are induced by (4.1.2) :

$$R_{12} L_2^{\pm} L_1^{\pm} = L_1^{\pm} L_2^{\pm} R_{12} \; , \tag{4.3.5}$$

$$R_{12} L_2^{+} L_1^{-} = L_1^{-} L_2^{+} R_{12} \; , \tag{4.3.6}$$

where as usual the product $L_2^{\pm} L_1^{\pm}$ is the convolution product $L_2^{\pm} L_1^{\pm} \equiv (L_2^{\pm} \otimes L_1^{\pm})\Delta$.

The $L^{\pm A}{}_B$ elements satisfy orthogonality conditions analogous to (4.1.17):

$$C^{AB} L^{\pm C}{}_B L^{\pm D}{}_A = C^{DC} \varepsilon \tag{4.3.7}$$

$$C_{AB} L^{\pm B}{}_C L^{\pm A}{}_D = C_{DC} \varepsilon \tag{4.3.8}$$

as can be verified by applying them to the q-group generators and using (4.1.19), (4.1.20). They provide the inverse for the matrix $L^{\pm}$:

$$[(L^{\pm})^{-1}]^A{}_B = C^{DA} L^{\pm C}{}_D C_{BC} \tag{4.3.9}$$

The co-structures of the algebra generated by the functionals $L^{\pm}$ and ε are defined by the duality (4.3.4):

$$\Delta'(L^{\pm A}{}_B)(a \otimes b) \equiv L^{\pm A}{}_B(ab) = L^{\pm A}{}_G(a) L^{\pm G}{}_B(b) \tag{4.3.10}$$

$$\varepsilon'(L^{\pm A}{}_B) \equiv L^{\pm A}{}_B(I) \tag{4.3.11}$$

$$\kappa'(L^{\pm A}{}_B)(a) \equiv L^{\pm A}{}_B(\kappa(a)) \tag{4.3.12}$$

so that

$$\Delta'(L^{\pm A}{}_B) = L^{\pm A}{}_G \otimes L^{\pm G}{}_B \tag{4.3.13}$$

$$\varepsilon'(L^{\pm A}{}_B) = \delta^A_B \tag{4.3.14}$$

$$\kappa'(L^{\pm A}{}_B) = [(L^{\pm})^{-1}]^A{}_B = C^{DA} L^{\pm C}{}_D C_{BC} \tag{4.3.15}$$

From (4.3.15) we have that κ' is an inner operation in the algebra generated by the functionals $L^{\pm A}{}_B$ and ε, it is then easy to see that these elements generate a Hopf algebra, the Hopf algebra $U_{q,r}(s(N+2))$ of regular functionals on the quantum group $S_{q,r}(N+2)$.

Note 4.3.1 From the CLL relations $\kappa'(L^{\pm A}{}_B) L^{\pm B}{}_C = L^{\pm A}{}_B \kappa'(L^{\pm B}{}_C) = \delta^A_C \varepsilon$ we have, using upper-lower triangularity of $L^{\pm}$:

$$L^{\pm A}{}_A \kappa'(L^{\pm A}{}_A) = \kappa'(L^{\pm A}{}_A) L^{\pm A}{}_A = \varepsilon \quad \text{i.e.} \quad L^{\pm A}{}_A L^{\pm A'}{}_{A'} = L^{\pm A'}{}_{A'} L^{\pm A}{}_A = \varepsilon \tag{4.3.16}$$

As a consequence $\det L^{\pm} \equiv L^{\pm \circ}{}_{\circ} L^{\pm 1}{}_1 L^{\pm 2}{}_2 \ldots L^{\pm N}{}_N L^{\pm \bullet}{}_{\bullet} = \varepsilon$. In the B_n case we also have $L^{\pm n_2}{}_{n_2} = \varepsilon$.

Note 4.3.2 The RLL relations imply that the subalgebra U^0 generated by the elements $L^{\pm A}{}_A$ and ε is commutative (use upper triangularity of R). Moreover, from (4.3.13) the invertible elements $L^{\pm A}{}_A$ are also group like, and we conclude that U^0 is the group Hopf algebra of the abelian group generated by $L^{\pm A}{}_A$ and ε . In the classical limit U^0 is a maximal commutative subgroup of $S(N+2)$.

Note 4.3.3 When $q_{AB} = r$, the multiparametric R-matrix goes into the uniparametric R-matrix and we recover the standard uniparametric orthogonal (symplectic) quantum groups. Then the $L^{\pm}$ functionals satisfy the further relation:

$$\forall \, \text{A} \, , \qquad L^{+A}{}_A L^{-A}{}_A = \varepsilon \, , \tag{4.3.17}$$

indeed $L^{+A}{}_A L^{-A}{}_A(a) = \varepsilon(a)$ as can be easily seen when $a = T^A{}_B$ and generalized to any $a \in S_{q,r}(N+2)$ using (4.3.4). In this case [19] we can avoid to realize the Hopf algebra $U_r(s(N+2))$ as functionals on $S_r(N+2)$ and we can define it abstractly as the Hopf algebra generated by the *symbols* $L^{\pm}$ and the unit ε modulo relations (4.3.5),(4.3.6),(4.3.7),(4.3.8), and (4.3.17).

As discussed in [19] in the uniparametric case, the Hopf algebra $U_r(s(N+2))$ of regular functionals is a Hopf subalgebra of the orthogonal (symplectic) Drinfeld-Jimbo universal enveloping algebra U_h, where $r=e^h$. In the general multiparametric case, relation (4.3.17) does not hold any more. Here we discuss the generalization of (4.3.17) and the relation between $U_{q,r}(s(N+2))$ and the multiparametric orthogonal (symplectic) Drinfeld-Jimbo universal enveloping algebra $U_h^{(\mathcal{F})}$. This latter is the quasitriangular Hopf algebra $U_h^{(\mathcal{F})} = (U_h, \Delta^{(\mathcal{F})}, S, \mathcal{R}^{(\mathcal{F})})$ paired to the multiparametric q-group $S_{q,r}(N+2)$. It is obtained from $U_h = (U_h, \Delta, S, \mathcal{R})$ via a twist [73]. $U_h^{(\mathcal{F})}$ has the same algebra structure of U_h (and the same antipode S), while the coproduct $\Delta^{(\mathcal{F})}$ and the universal element $\mathcal{R}^{(\mathcal{F})}$ belonging to (a completion of) $U_h \otimes U_h$ are determined by the twisting element $\mathcal{F}$ that belongs to (a completion of) a maximal commutative subalgebra of $U_h \otimes U_h$. We have

$$\forall\, \phi \in U_h\,,\quad \Delta^{(\mathcal{F})}(\phi) = \mathcal{F}\Delta(\phi)\mathcal{F}^{-1}\;;\quad \mathcal{R}^{(\mathcal{F})} = \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1}\;;\quad \mathcal{R}^{(\mathcal{F})}(T\otimes T) = R_{q,r}\,. \tag{4.3.18}$$

The element $\mathcal{F}$ satisfies: $(\Delta^{(\mathcal{F})}\otimes id)\mathcal{F} = \mathcal{F}_{13}\mathcal{F}_{23}\,,\ (id\otimes\Delta^{(\mathcal{F})})\mathcal{F} = \mathcal{F}_{13}\mathcal{F}_{12}\,,\ \mathcal{F}_{12}\mathcal{F}_{21} = I\,,\ \mathcal{F}_{12}\mathcal{F}_{13}\mathcal{F}_{23} = \mathcal{F}_{23}\mathcal{F}_{13}\mathcal{F}_{12}\,,\ (\varepsilon\otimes id)\mathcal{F} = (id\otimes\varepsilon)\mathcal{F} = \varepsilon\,,\ (S\otimes id)\mathcal{F} = (id\otimes S)\mathcal{F} = \mathcal{F}^{-1},\ \cdot(id\otimes S)\mathcal{F} = \cdot(S\otimes id)\mathcal{F} = \cdot(id\otimes id)\mathcal{F} = \varepsilon$; we explicitly have

$$\mathcal{F}(T^A{}_B \otimes T^C{}_D) = F^{AC}_{BD} \tag{4.3.19}$$

where F^{AC}_{BD} is the diagonal matrix

$$F = diag(\sqrt{\frac{q_{11}}{r}}, \sqrt{\frac{q_{12}}{r}}, \ldots \sqrt{\frac{q_{NN}}{r}}) \tag{4.3.20}$$

It is easy to see that the definition of the $L^{\pm}$ functionals given in the beginning of this section is equivalent to the following one: $L^{+A}{}_B(a) = \mathcal{R}^{(\mathcal{F})}(a\otimes T^A{}_B)$ and $L^{-A}{}_B(a) = \mathcal{R}^{(\mathcal{F})^{-1}}(T^A{}_B\otimes a)$. From $(\Delta^{(\mathcal{F})}\otimes id)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}\,,\ (id\otimes\Delta^{(\mathcal{F})})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}\,,$ we have $\Delta^{(\mathcal{F})}(L^{\pm A}{}_B) = L^{\pm A}{}_C \otimes L^{\pm C}{}_B$ and therefore $\Delta^{(\mathcal{F})} = \Delta'$ on $U_{q,r}(s(N+2))$. From $(id\otimes S)(\mathcal{R}) = (S\otimes id)(\mathcal{R}) = \mathcal{R}^{-1}$ it is also easy to see that $S=\kappa'$ on $U_{q,r}(s(N+2))$ and we conclude that the algebra of regular functionals $U_{q,r}(s(N+2))$ is a realization [in terms of functionals on $S_{q,r}(N+2)$] of a Hopf subalgebra of $U_h^{(\mathcal{F})}$ with $r=e^h$. The generalization of (4.3.17) lies in $U_h^{(\mathcal{F})}$ and not in $U_{q,r}(s(N+2))$, and it is given by

$$\forall\, A \qquad L^{+A}{}_A L^{-A}{}_A = f_i(T^A{}_A)f^i \quad \text{where } \mathcal{F}^4 = f_i\otimes f^i\,. \tag{4.3.21}$$

This relation holds with $L^{\pm}$ considered as abstract symbols. It can easily be checked when $L^{\pm}$ are realized as functionals: indeed $L^{+A}{}_A L^{-A}{}_A(a) = \mathcal{F}^4(T^A{}_A\otimes a)$ as can be seen when $a = T^A{}_B$ [use $\mathcal{F}^2(T^A{}_A\otimes b) = \mathcal{F}(T^A{}_A\otimes b_1)\mathcal{F}(T^A{}_A\otimes b_2)$] and generalized to any $a\in S_{q,r}(N+2)$ using $\mathcal{F}(T^A{}_A\otimes ab) = \mathcal{F}(T^A{}_A\otimes a)\mathcal{F}(T^A{}_A\otimes b)$.

In order to characterize the relation between the Hopf algebra of regular functionals $U_{q,r}(s(N+2))$ and $U_h^{(\mathcal{F})}$, following [19], we extend the group Hopf algebra U^0 described in Note 4.3.2 to $\hat{U}^0$ by means of the elements [1] ${\ell^{\pm A}}_A = \ln {L^{\pm A}}_A$. Otherwise stated this means that in $\hat{U}^0$ we can write ${L^{\pm A}}_A = \exp({\ell^{\pm A}}_A)$ where ${\ell^{\pm A}}_A \in \hat{U}^0$. [Explicitly ${\ell^{\pm A}}_A({T^C}_D) = \ln(R^{\pm AC}_{AC})\,\delta^C_D$, ${\ell^{\pm A}}_A(I) = 0$, ${\ell^{\pm A}}_A(ab) = {\ell^{\pm A}}_A(a)\varepsilon(b) + \varepsilon(a){\ell^{\pm A}}_A(b)$ and $\kappa'({\ell^{\pm A}}_A) = -{\ell^{\pm A}}_A$]. It then follows that $\mathcal{F}$ belongs to (a completion of) $\hat{U}^0 \otimes \hat{U}^0$. The corresponding extension $\hat{U}_{q,r}(s(N+2))$ of $U_{q,r}(s(N+2))$, defined as the Hopf algebra generated by the *symbols* $L^{\pm}$ and $\ell^{\pm}$ modulo relations (4.3.5)-(4.3.8) and (4.3.21), is isomorphic – when $r = e^h$ – to $U_h^{(\mathcal{F})}$: $\hat{U}_{q,r}(s(N+2)) \cong U_h^{(\mathcal{F})}$. This relation holds because it is the twisted version of the known uniparametric analogue $\hat{U}_r(s(N+2)) \cong U_h$ [19, 77].

The elements $L^{\pm}$ [or $\frac{1}{r-r^{-1}}({L^{\pm A}}_B - \delta^A_B\varepsilon)$] may be seen as the quantum analogue of the tangent vectors; then the RLL relations are the quantum analogue of the Lie algebra relations, and we can use the orthogonal (symplectic) CLL conditions to reduce the number of the $L^{\pm}$ generators to $(N+2)(N+1)/2$, (orthogonal case) or $(N+2)(N+3)/2$ (symplectic case) i.e. the dimension of the classical group manifold.

This we proceed to do for $U_{q,r}(so(N+2))$, for $U_{q,r}(sp(N+2))$ one can proceed in a similar way[2]; we next study the $RL^{\pm}L^{\pm}$ commutation relations restricted to these $(N+2)(N+1)/2$ generators and find a set of ordered monomials in the reduced $L^{\pm}$ that linearly span all $\hat{U}_{q,r}(so(N+2))$.

We first observe that the commutative subalgebra $\hat{U}^0$ is generated by $(N+2)/2$ elements (N even, $N = 2n$) or $(N+1)/2$ elements (N odd, $N = 2n+1$), for example ${\ell^{-\circ}}_\circ$, ${\ell^{-1}}_1$ ... ${\ell^{-n}}_n$. For the off-diagonal $L^{\pm}$ elements, we can choose as free indices $(C,D) = (c,\circ)$ in relation (4.3.8), and using ${L^{-\circ}}_\circ {L^{-\bullet}}_\bullet = \varepsilon$, we find:

$$ {L^{-\bullet}}_c = -(C_{\circ\bullet})^{-1} C_{ab} {L^{-b}}_c {L^{-a}}_\circ {L^{-\bullet}}_\bullet \,. \tag{4.3.22} $$

If we choose $(C,D) = (\circ,\circ)$ we obtain

$$ {L^{-\bullet}}_\circ = -(r^{-2}C_{\bullet\circ} + C_{\circ\bullet})^{-1} C_{ab} {L^{-b}}_\circ {L^{-a}}_\circ {L^{-\bullet}}_\bullet \,. \tag{4.3.23} $$

Similar results hold for ${L^{+\circ}}_d$ and ${L^{+\circ}}_\bullet$. Iterating this procedure, from $C_{ab}{L^{-b}}_c{L^{-a}}_d = C_{dc}\varepsilon$ we find that ${L^{-N}}_i$ (with $i = 2, ...N-1$) and ${L^{-N}}_1$ are functionally dependent on ${L^{-i}}_1$ and ${L^{-N}}_N$. Similarly for ${L^{+1}}_i$ and ${L^{+1}}_N$. The final result is that the elements ${L^{-a}}_J$ with $J < a < J'$ and ${L^{+a}}_J$ with $J' < a < J$ – whose number in both $\pm$

[1] In the classical limit ${\ell^{\pm A}}_A$ are the tangent vectors to a maximal commutative subgroup of $S(N+2)$. They generate a Cartan subalgebra of the Lie algebra $s(N+2)$.

[2] In the $U_{q,r}(sp(N+2))$ case relations (4.3.26) [and more in general (4.3.25)] do not allow to order the ${L^{-a}}_\circ$ (and more in general the ${L^{\pm\alpha}}_J$) elements because we are missing the relation with the P_0 projector, cf. Note 4.2.4. However we can still order the ${L^{-a}}_\circ$ (or ${L^{\pm\alpha}}_J$) elements if we consider also ${L^{-\bullet}}_\circ$ (or ${L^{\pm J'}}_J$). This leads to the $(N+2)(N+3)/2$ generators of the symplectic case. Notice that Lemma 4.3.1 and 4.3.2 still hold; Theorem 4.3.1 holds as well provided that α can also be equal to J': $J' < \alpha \leq J$.

cases is $\frac{1}{4}N(N+2)$ for N even and $\frac{1}{4}(N+1)^2$ for N odd – and the elements $\ell^{-\circ}{}_{\circ}$, $\ell^{-1}{}_1 \ldots \ell^{-n}{}_n$ generate all $\hat{U}_{q,r}(so(N+2))$. The total number of generators is therefore $(N+2)(N+1)/2$.

Notice that in this derivation we have not used the RLL relations (i.e. the quantum analogue of the Lie algebra relations) to further reduce the number of generators. We therefore expect that, as in the classical case, monomials in the $(N+2)(N+1)/2$ generators can be ordered (in any arbitrary way). We begin by proving this for polynomials in $L^{+A}{}_A$, $L^{+\alpha}{}_J$ with $J' < \alpha < J$, and for polynomials in $L^{-A}{}_A$, $L^{-\alpha}{}_J$ with $J < \alpha < J'$.

Lemma 4.3.1 Consider the $RL^{\pm}L^{\pm}$ commutation relations

$$R^{AB}{}_{EF}L^{\pm F}{}_D L^{\pm E}{}_C = L^{\pm A}{}_E L^{\pm B}{}_F R^{EF}{}_{CD} \ . \tag{4.3.24}$$

For $C \neq D$ they close respectively on the subset of the $L^{+\alpha}{}_J$ with $J' < \alpha \leq J$ and on the subset of the $L^{-\alpha}{}_J$ with $J \leq \alpha < J'$. For $C = D$ they are equivalent to the q^{-1}-plane commutation relations:

$$[P_A(J'-J+1)]^{\alpha\beta}_{\gamma\delta} L^{\pm\delta}{}_J L^{\pm\gamma}{}_J = 0 \ , \tag{4.3.25}$$

where $P_A(J'-J+1)$ is the antisymmetrizer in dimension $J - J' + 1$ [compare with (4.1.16)]. In particular

$$P_A^{ab}{}_{cd} L^{-d}{}_{\circ} L^{-c}{}_{\circ} = 0 \tag{4.3.26}$$

or equivalently $[(P_A)_{q^{-1},r^{-1}}]^{ab}_{cd} L^{-c}{}_{\circ} L^{-d}{}_{\circ} = 0$ which coincide, for $r \to r^{-1}$ and $q \to q^{-1}$, with the N-dimensional quantum orthogonal plane relations (4.2.16).
Proof : The proof is a straightforward calculation based on (4.1.22) and on upper or lower triangularity of the R matrix and of the $L^{\pm}$ functionals. □□□

Lemma 4.3.2 $U_{q,r}(so(N))$ is a Hopf subalgebra of $U_{q,r}(so(N+2))$.
Proof: Choosing $SO_{q,r}(N)$ indices as free indices in (4.3.24) and using upper or lower triangularity of the $L^{\pm}$ matrices, and (4.1.8) or (4.1.26), we find that only $SO_{q,r}(N)$ indices appear in (4.3.24); similarly for relations (4.3.6)-(4.3.8), and for the costructures (4.3.13)-(4.3.15). □□□

Now we observe that in virtue of the RL^+L^+ relations the L^+ elements can be ordered; similarly we can order the L^- using the RL^-L^- relations. This statement can be proved by induction using that $U_{q,r}(so(N))$ is a subalgebra of $U_{q,r}(so(N+2))$, and splitting the $SO_{q,r}(N+2)$ index in the usual way [some of the resulting formulas are given in (4.4.9)-(4.4.12)].

It is then straightforward to prove that the elements $L^{+\alpha}{}_J$ with $J' < \alpha \leq J$ can be ordered; indeed we can always order the $L^{+\alpha}{}_J$ $L^{+\beta}{}_K$ with $J' < \alpha \leq J$, $K' < \beta \leq K$ and $J \neq K$ since their commutation relations are a closed subset of (4.3.24) [see Lemma 4.3.1]. Then there is no difficulty in ordering substrings composed by $L^{+\alpha}{}_J$ and

$L^{+\beta}{}_J$ elements because (4.3.25) are q^{-1}-plane commutation relations, that allow for any ordering of the quantum plane coordinates [19]. More in general the $L^{+A}{}_A$ and $L^{+\alpha}{}_J$ with $J' < \alpha < J$ can be ordered because of $L^{+A}{}_A L^{+B}{}_C = (q_{BA}/q_{CA}) L^{+B}{}_C L^{+A}{}_A$. Similarly we can order the $L^{-A}{}_A$ and $L^{-\alpha}{}_J$ with $J < \alpha < J'$. It is now easy to prove the following

Theorem 4.3.1 A set of elements spanning $\hat{U}_{q,r}(so(N+2))$ is given by the ordered monomials

$$Mon(L^{+\alpha}{}_J; J' < \alpha < J)\,(\ell^{-\circ}{}_\circ)^{p_\circ}(\ell^{-1}{}_1)^{p_1}\dots(\ell^{-n}{}_n)^{p_n}\;Mon(L^{-\alpha}{}_J; J < \alpha < J') \quad (4.3.27)$$

where $p_\circ, p_1, ...p_n \in \boldsymbol{N} \cup \{0\}$, $n = N/2$ (N even), $n = (N-1)/2$ (N odd) and $Mon(L^{+\alpha}{}_J; J' < \alpha < J)$, $[Mon(L^{-\alpha}{}_J; J < \alpha < J')]$ is a monomial in the off-diagonal elements $L^{+\alpha}{}_J$ with $J' < \alpha < J$ $[L^{-\alpha}{}_J$ with $J < \alpha < J']$ where an ordering has been chosen. □□□

Note 4.3.4 Conjecture: the above monomials are linearly independent and therefore form a basis of $\hat{U}_{q,r}(so(N+2))$.

Conjugation

The canonical $*$-conjugation on $U_{q,r}(s(N+2))$ induced by the $*$-conjugation on $S_{q,r}(N+2)$ is given by:

$$\psi^*(a) \equiv \overline{\psi(\kappa^{-1}(a^*))} \quad (4.3.28)$$

where $\psi \in U_{q,r}(s(N+2))$, $a \in S_{q,r}(N+2)$, and the overline denotes the usual complex conjugation. It is not difficult to determine the action on the basis elements $L^{\pm A}{}_B$. The two $S_{q,r}(N+2)$ $*$-conjugations that are compatible with $ISO_{q,r}N$ induce respectively the following conjugations on the $L^{\pm A}{}_B$ (we denote $*^\sharp$ simply by $*$):

$$(L^{\pm A}{}_B)^* = \kappa'^2(L^{\pm A}{}_B) \quad (4.3.29)$$

$$(L^{\pm A}{}_B)^* = \mathcal{D}^A{}_C \kappa'^2(L^{\pm C}{}_D)\mathcal{D}^D{}_B \quad (4.3.30)$$

Notice that $\kappa'^2(L^{\pm A}{}_B) = D^{-1}{}^A_E L^{\pm E}{}_F D^F_B$ where $D^a_e = C^{as}C_{es}$ and its inverse is $D^{-1}{}^f_b = C^{sf}C_{sb}$.

4.4 Universal enveloping algebras $U_{q,r}(iso(N))$ and $U_{q,r}(isp(N))$

Consider a generic functional $f \in U_{q,r}(s(N+2))$. It is well defined on the quotient $IS_{q,r}(N) = S_{q,r}(N+2)/H$ if and only if $f(H) = 0$. It is easy to see that the set $H^\perp$ of all these functionals is a subalgebra of $U_{q,r}(s(N+2))$: if $f(H) = 0$ and $g(H) = 0$ then $fg(H) = 0$ because $\Delta(H) \subseteq H \otimes S_{q,r}(N+2) + S_{q,r}(N+2) \otimes H$. Moreover [81] $H^\perp$ is a Hopf subalgebra of $U_{q,r}(s(N+2))$ since H is a Hopf ideal, cf.

sections 3.5-6. In agreement with these observations we will find the Hopf algebra $U_{q,r}(iso(N))$ [dually paired to $IS_{q,r}(N)$] as a subalgebra of $U_{q,r}(s(N+2))$ vanishing on the ideal H.

Let

$$IU \equiv [L^{-A}{}_B, L^{+a}{}_b, L^{+\circ}{}_\circ, L^{+\bullet}{}_\bullet, \varepsilon] \subseteq U_{q,r}(s(N+2)) \tag{4.4.1}$$

be the subalgebra of $U_{q,r}(s(N+2))$ generated by $L^{-A}{}_B, L^{+a}{}_b, L^{+\circ}{}_\circ, L^{+\bullet}{}_\bullet, \varepsilon$.

Note 4.4.1 These are all and only the functionals annihilating the generators of H: $T^a{}_\circ$, $T^\bullet{}_b$ and $T^\bullet{}_\circ$. The remaining $U_{q,r}(s(N+2))$ generators $L^{+\circ}{}_b$, $L^{+a}{}_\bullet$, $L^{+\circ}{}_\bullet$ do not annihilate the generators of H and are not included in (4.4.1).

We now proceed to study this algebra IU. We will show that it is a Hopf algebra and that $IU \subseteq H^\perp$; we will give an R-matrix formulation, and prove that IU is the semidirect product of $U_{q,r}(s(N))$ and the algebra B' generated by the elements $L^{-\circ}{}_\circ$ and $L^{-a}{}_\circ$. This is the analogue of $IS_{q,r}(N)$ being the semidirect product of $S_{q,r}(N)$ and the algebra B generated by the elements u and x^a, cf. Note 4.2.5.

We then show that IU is dually paired with $IS_{q,r}(N)$. These results lead to the conclusion that IU is the universal enveloping algebra of $IS_{q,r}(N)$.

Theorem 4.4.1 *IU is a Hopf subalgebra of $U_{q,r}(s(N+2))$.*
Proof : IU is by definition a subalgebra. The sub-coalgebra property $\Delta'(IU) \subseteq IU \otimes IU$ follows immediately from the upper triangularity of $L^{+A}{}_B$:

$$\Delta'(L^{+a}{}_b) = L^{+a}{}_c \otimes L^{+c}{}_b \; ; \; \Delta'(L^{+\circ}{}_\circ) = L^{+\circ}{}_\circ \otimes L^{+\circ}{}_\circ \; ; \; \Delta'(L^{+\bullet}{}_\bullet) = L^{+\bullet}{}_\bullet \otimes L^{+\bullet}{}_\bullet \tag{4.4.2}$$

and the compatibility of Δ' with the product. We conclude that IU is a Hopf-subalgebra because $\kappa'(IU) \subseteq IU$ as is easily seen using (4.3.15) and antimultiplicativity of κ'. □□□

We may wonder whether the RLL and CLL relations of $U_{q,r}(s(N+2))$ close in IU. In this case IU will be given by all and *only* the polynomials in the functionals $L^{-A}{}_B, L^{+a}{}_b, L^{+\circ}{}_\circ, L^{+\bullet}{}_\bullet, \varepsilon$. This check is done by writing explicitly all q-commutations between the generators of IU: they do not involve the functionals $L^{+\circ}{}_b$, $L^{+a}{}_\bullet$, $L^{+\circ}{}_\bullet$. Moreover one can also write them in a compact form using a new R-matrix $\mathcal{R}_{12} \equiv \mathcal{L}^+{}_2(t_1)$, where $\mathcal{L}^+$ is defined below. Similarly the orthogonality (symplecticity) conditions (4.3.7)-(4.3.8) do not relate elements of IU with elements not belonging to IU. We therefore conclude

Theorem 4.4.2 The Hopf algebra IU is generated by the unit ε and the matrix entries:

$$L^- = \left(L^{-A}{}_B\right) \; ; \; \mathcal{L}^+ = \begin{pmatrix} L^{+\circ}{}_\circ & 0 & 0 \\ 0 & L^{+a}{}_b & 0 \\ 0 & 0 & L^{+\bullet}{}_\bullet \end{pmatrix} \; ; \tag{4.4.3}$$

these functionals satisfy the q-commutation relations:

$$R_{12}\mathcal{L}^+{}_2\mathcal{L}^+{}_1 = \mathcal{L}^+{}_1\mathcal{L}^+{}_2 R_{12} \;\text{ or equivalently }\; \mathcal{R}_{12}\mathcal{L}^+{}_2\mathcal{L}^+{}_1 = \mathcal{L}^+{}_1\mathcal{L}^+{}_2\mathcal{R}_{12} \tag{4.4.4}$$

$$R_{12}L_2^- L_1^- = L_1^- L_2^- R_{12} \ , \tag{4.4.5}$$

$$\mathcal{R}_{12}\mathcal{L}^+{}_2 L_1^- = L_1^- \mathcal{L}^+{}_2 \mathcal{R}_{12} \ , \tag{4.4.6}$$

where

$$\mathcal{R}_{12} \equiv \mathcal{L}^+{}_2(t_1) \quad \text{that is} \quad \mathcal{R}^{ab}_{cd} = R^{ab}_{cd} \ ; \ \ \mathcal{R}^{AB}_{AB} = R^{AB}_{AB} \ \text{ and otherwise } \ \mathcal{R}^{AB}_{CD} = 0$$

and the orthogonality (symplecticity) conditions :

$$C^{AB}\mathcal{L}^{+C}{}_B\mathcal{L}^{+D}{}_A = C^{DC}\varepsilon \ ; \ \ C_{AB}\mathcal{L}^{+B}{}_C\mathcal{L}^{+A}{}_D = C_{DC}\varepsilon \ ; \tag{4.4.7}$$

$$C^{AB}L^{-C}{}_B L^{-D}{}_A = C^{DC}\varepsilon \ ; \ \ C_{AB}L^{-B}{}_C L^{-A}{}_D = C_{DC}\varepsilon \ , \tag{4.4.8}$$

The costructures are the ones given in (4.3.13)-(4.3.15) with L^+ replaced by $\mathcal{L}^+$.
□□□

Note 4.4.2 We can consider the extension $\hat{IU} \subset \hat{U}_{q,r}(s(N+2))$ obtained by including the elements $\ell^{\pm A}{}_A$ ($\ell^{\pm A}{}_A = \ln L^{\pm A}{}_A$, see the previous section). Then $\hat{IU}$ is generated by the symbols $L^{-A}{}_B$, $\mathcal{L}^{+A}{}_B$, $\ell^{\pm A}{}_A$ modulo the relations (4.4.4)-(4.4.8) and (4.3.21) [(4.3.17) in the uniparametric case]. Equivalently, from (4.3.22)-(4.3.23), we have that $\hat{IU}$ is generated by $\hat{U}_{q,r}(s(N))$, the dilatation $\ell^{-\circ}{}_\circ$ and the N elements $L^{-a}{}_\circ$ (satisfying, in the orthogonal case, the quantum plane relations). All the relations are then given by those between the generators of $\hat{U}_{q,r}(s(N))$ –listed in (4.3.5)-(4.3.8), (4.3.21) with lower case indices– and by the following ones

$$L^{-\circ}{}_\circ L^{-a}{}_\circ = q_{\circ a}^{-1} L^{-a}{}_\circ L^{-\circ}{}_\circ \tag{4.4.9}$$

$$P_A{}^{ab}{}_{fe} L^{-e}{}_\circ L^{-f}{}_\circ = 0 \tag{4.4.10}$$

$$L^{-\circ}{}_\circ L^{\pm b}{}_d = \frac{q_{b\circ}}{q_{d\circ}} L^{\pm b}{}_d L^{-\circ}{}_\circ \tag{4.4.11}$$

$$L^{-a}{}_\circ L^{\pm b}{}_d = \frac{r}{q_{d\circ}} (R^{\pm})^{ba}{}_{ef} L^{\pm e}{}_d L^{-f}{}_\circ \tag{4.4.12}$$

where $R^{\pm}$ is defined in (4.3.3). The number of generators is $N(N-1)/2 + N + 1$ in the orthogonal case and $N(N+1)/2 + N + 2$ in the symplectic case because we consider also the element $L^{-\bullet}{}_\circ$ so that the $L^{-a}{}_\circ$ elements can be ordered (cf. last footnote).

Note 4.4.3 When $q_{a\bullet} = r \ \forall a$, then $L^{-\circ}{}_\circ = L^{+\bullet}{}_\bullet$, $L^{-\bullet}{}_\bullet = L^{+\circ}{}_\circ$ and, in complete analogy to (4.2.25), IU is generated by $U_{q,r}(s(N))$, $L^{-a}{}_\circ$, $L^{-\circ}{}_\circ$ and $L^{-\bullet}{}_\bullet = (L^{-\circ}{}_\circ)^{-1}$. With abuse of notations we will consider IU generated by these elements also for arbitrary values of the parameters $q_{a\bullet}$; this is what actually happens in $\hat{IU}$.

Note 4.4.4 From the second equation in (4.4.4) applied to t we obtain the quantum Yang-Baxter equation for the matrix $\mathcal{R}$.

The results of Note 4.2.5 holds also for $U_{q,r}(is(N))$ with the obvious changes in notation. The projection π [well defined only if $q_{a\bullet} = r\ \forall a$ see (4.4.11)] is given by:

$$\pi(L^{-a}{}_{\circ}) = 0 \ , \quad \pi(L^{-\circ}{}_{\circ}) = I \ , \quad \pi(L^{\pm a}{}_{b}) = L^{\pm a}{}_{b} \ , \quad \pi(\varepsilon) = \varepsilon \ .$$

The semidirect product structure is:

$$U_{q,r}(is(N)) \cong B' \rtimes U_{q,r}(s(N)) \tag{4.4.13}$$

where B' is the subalgebra of $U_{q,r}(is(N))$ generated by $L^{-\circ}{}_{\circ}$ and $L^{-a}{}_{\circ}$. Moreover $B' = IU_{\text{inv}}$, the space of all right invariant elements of the $U_{q,r}(s(N))$–bicovariant algebra $U_{q,r}(is(N))$. The ordered monomials that form a basis of B' and that freely generate IU as a right module, in the orthogonal case are:

$$\eta^i = (L^{-\circ}{}_{\circ})^{i_\circ}(L^{-1}{}_{\circ})^{i_1} \ldots (L^{-N}{}_{\circ})^{i_N} \quad \text{with } i_\circ \in \boldsymbol{Z} \ , \quad i_1, \ldots i_N \in \boldsymbol{N} \cup \{0\} \ .$$

In the symplectic case, if we also consider the element z, a linear basis of B' is given by the ordered monomials

$$\eta^i = (L^{-\circ}{}_{\circ})^{i_\circ}(L^{-1}{}_{\circ})^{i_1} \ldots (L^{-N}{}_{\circ})^{i_N}(L^{-\bullet}{}_{\circ}L^{-\circ}{}_{\circ})^{i_{N+1}} \text{ with } i_\circ \in \boldsymbol{Z} \ , \ i_1, \ldots i_N, i_{N+1} \in \boldsymbol{N} \cup \{0\}$$

[$L^{-\bullet}{}_{\circ}L^{-\circ}{}_{\circ}$ commutes with the elements $L^{-a}{}_{\circ}$. Use (4.4.9), (4.4.10) and then (4.4.11) and (4.4.12), to exlicitly write a generic element of IU as $\eta^i a_i$ where $a_i \in U_{q,r}(s(N))$].

The $(\mathbf{Z},\mathbf{N})$ grading is: *grade*$(T^a{}_b) = (0,0)$, *grade*$(L^{-a}{}_{\circ}) = (0,1)$, *grade*$(L^{-\circ}{}_{\circ}) = (1,0)$, so that:

$$U_{q,r}(is(N)) \ = \sum_{(h,k)\in(\mathbf{Z},\mathbf{N})}^{\oplus} \Gamma'^{(h,k)} \tag{4.4.14}$$

where $\Gamma'^{(0,0)} = U_{q,r}(s(N))$

$$\Gamma'^{(0,1)} = \{L^{-a}{}_{\circ}\varphi^a \ / \ \varphi^a \in U_{q,r}(s(N))\} \ , \qquad \Gamma'^{(\pm 1,0)} = \{(L^{-\circ}{}_{\circ})^{\pm 1}\varphi \ / \ \varphi \in U_{q,r}(s(N))\}$$

$$\Gamma'^{(h,k)} = \{(L^{-\circ}{}_{\circ})^h L^{-a_1}{}_{\circ} L^{-\circ}{}_{a_2} \ldots L^{-\circ}{}_{a_k}\varphi^{a_1 a_2 \ldots a_k} \ / \ \varphi^{a_1 a_2 \ldots a_k} \in U_{q,r}(s(N))\}$$

Any submodule $\Gamma'^{(h,k)}$ is a $U_{q,r}(s(N))$–bicovariant bimodule freely generated by the elements η^i with degree $(h,k) \in (\mathbf{Z},\mathbf{N})$. Also the analogue of Note 3.3.5 and Note 3.3.6 still holds for IU.

Duality $U_{q,r}(iso(N)) \leftrightarrow IS_{q,r}(N)$

We now show that IU is dually paired to $S_{q,r}(N+2)$. This is the fundamental step allowing to interpret IU as the algebra of regular functionals on $IS_{q,r}(N)$.

Theorem 4.4.4 IU annihilates H.

Proof : Let $\mathcal{L}$ and $\mathcal{T}$ be generic generators of IU and H respectively. As discussed in Note 4.4.1, $\mathcal{L}(\mathcal{T}) = 0$. A generic element of the ideal is given by $a\mathcal{T}b$ where sum of polynomials is understood; we have (using Sweedler's notation for the coproduct): $\mathcal{L}(a\mathcal{T}b) = \mathcal{L}_{(1)}(a)\mathcal{L}_{(2)}(\mathcal{T})\mathcal{L}_{(3)}(b) = 0$ because $\mathcal{L}_{(2)}(\mathcal{T}) = 0$. Indeed $\mathcal{L}_{(2)}$ is still a generator of IU since IU is a sub-coalgebra of $U_{q,r}(s(N+2))$. Thus $\mathcal{L}(H) = 0$. Recalling that a product of functionals annihilating H still annihilates the co-ideal H, we also have $IU(H) = 0$. □□□

In virtue of Theorem 4.4.4 the following bracket is well defined:

Definition $\langle\ ,\ \rangle : \quad IU \otimes IS_{q,r}(N) \longrightarrow \boldsymbol{C}$

$$\langle a', P(a)\rangle \equiv a'(a) \quad \forall a' \in IU\ ,\ \forall a \in S_{q,r}(N+2) \tag{4.4.15}$$

where $P\ :\ S_{q,r}(N+2) \to S_{q,r}(N+2)/H \equiv IS_{q,r}(N)$ is the canonical projection, which is surjective. The bracket is well defined because two generic counterimages of $P(a)$ differ by an addend belonging to H.

Note that when we use the bracket $\langle\ ,\ \rangle$, a' is seen as an element of IU , while in the expression $a'(a)$, a' is seen as an element of $U_{q,r}(s(N+2))$ (vanishing on H).

Theorem 4.4.5 The bracket (4.4.15) defines a pairing between IU and $IS_{q,r}(N)$: $\forall a', b' \in IU\ ,\ \forall P(a), P(b) \in IS_{q,r}(N)$

$$\langle a'b', P(a)\rangle = \langle a' \otimes b', \Delta(P(a))\rangle \tag{4.4.16}$$

$$\langle a', P(a)P(b)\rangle = \langle \Delta'(a'), P(a) \otimes P(b)\rangle \tag{4.4.17}$$

$$\langle \kappa'(a'), P(a)\rangle = \langle a', \kappa(P(a))\rangle \tag{4.4.18}$$

$$\langle I, P(a)\rangle = \varepsilon(P(a))\ \ ;\quad \langle a', P(I)\rangle = \varepsilon'(a') \tag{4.4.19}$$

Proof : The proof is easy since IU is a Hopf subalgebra of $U_{q,r}(s(N+2))$ and P is compatible with the structures and costructures of $S_{q,r}(N+2)$ and $IS_{q,r}(N)$. Indeed we have

$$\langle a', P(a)P(b)\rangle = \langle a', P(ab)\rangle = a'(ab) = \Delta'(a')(a \otimes b) = \langle \Delta'(a'), P(a) \otimes P(b)\rangle$$

$$\langle a'b', P(a)\rangle = a'b'(a) = (a'\otimes b')\Delta_{N+2}(a) = \langle a'\otimes b', (P\otimes P)\Delta_{N+2}(a)\rangle = \langle a'\otimes b', \Delta(P(a))\rangle$$

$$\langle \kappa'(a'), P(a)\rangle = \kappa'(a')(a) = a'(\kappa_{N+2}(a)) = \langle a', P(\kappa_{N+2}(a))\rangle = \langle a', \kappa(P(a))\rangle$$

□□□

We now recall that IU and $IS_{q,r}(N)$, besides being dually paired, are bicovariant algebras with the same graded structure (4.2.34) and (4.4.14), and can both be obtained as a cross-product cross-coproduct construction: $IS_{q,r}(N) \cong B \rtimes\!\!\!\lhd S_{q,r}(N)$, $IU \cong B' \rtimes\!\!\!\lhd U_{q,r}(s(N))$. In particular $IS_{q,r}(N)$ and IU are freely generated (as modules) by B and B' i.e. by the two isomorphic sets of the monomials in the q-plane plus dilatation coordinates $L^{-\circ}{}_{\circ}$, $L^{-a}{}_{\circ}$ and u, x^a respectively. We then conclude that IU is the universal enveloping algebra of $IS_{q,r}(N)$:

$$U_{q,r}(is(N)) \equiv IU\ . \tag{4.4.20}$$

Note 4.4.5 Given a $*$-structure on $IS_{q,r}(N)$, the duality $IS_{q,r}(N) \leftrightarrow U_{q,r}(is(N))$ induces a $*$-structure on $U_{q,r}(is(N))$. If in particular the $*$-conjugation on $IS_{q,r}(N)$ is found by projecting a $*$-conjugation on $S_{q,r}(N+2)$, then the induced $*$ on $U_{q,r}(iso(N))$ is simply the restriction to $U_{q,r}(is(N))$ of the $*$ on $U_{q,r}(s(N+2))$. This is the case for the $*$-structures that lead to the real forms $IS_{q,r}(N, \boldsymbol{R})$ and $ISO_{q,r}(n+1, n-1)$ and in particular to the quantum Poincaré group.

4.5 Bicovariant calculus on $SO_{q,r}(N+2)$ and $Sp_{q,r}(N+2)$

The bicovariant differential calculus on the multiparametric q-groups of the B, C, D series can be formulated following Section 2.2. We list here some formulae and comments that do not appear in that section.

The commutations between the generators $T^R{}_S$ and the 1-forms $\omega_{A_1}{}^{A_2}$ are explicitly given by

$$\omega_{A_1}{}^{A_2} T^R{}_S = (R^{-1})^{TB_1}{}_{CA_1} (R^{-1})^{A_2C}{}_{B_2S} T^R{}_T \omega_{B_1}{}^{B_2} \tag{4.5.1}$$

Using (2.2.45) we compute the exterior derivative on the basis elements of $S_{q,r}(N+2)$:

$$d\,T^A{}_B = \frac{1}{r-r^{-1}}[(R^{-1})^{CR}{}_{ET}(R^{-1})^{TE}{}_{SB}T^A{}_C - \delta^R_S T^A{}_B]\,\omega_R{}^S \equiv T^A{}_C X^{CR}{}_{BS}\omega_R{}^S \tag{4.5.2}$$

where we have

$$X^{A_1B_1}{}_{A_2B_2} \equiv \frac{1}{r-r^{-1}}[(R^{-1})^{A_1B_1}{}_{ET}(R^{-1})^{TE}{}_{B_2A_2} - \delta^{B_1}_{B_2}\delta^{A_1}_{A_2}] = zK^{A_1B_1}{}_{A_2B_2} - (\hat{R}^{-1})^{A_1B_1}{}_{A_2B_2} \tag{4.5.3}$$

with $z \equiv \epsilon r^{N-\epsilon}, K^{A_1B_1}{}_{A_2B_2} = C^{A_1B_1}C_{A_2B_2}$. [From (4.1.14) and (4.1.21), the second equality in (4.5.3) is easily proven.] Notice also that from (4.5.2) and (2.1.85), $X^{A_1B_1}{}_{A_2B_2}$ is the fundamental representation of the q-Lie algebra generators $\chi^{B_1}{}_{B_2}$:

$$X^{A_1B_1}{}_{A_2B_2} = \chi^{B_1}{}_{B_2}(T^{A_1}{}_{A_2})$$

Every element ρ of Γ, which by definition is written in a unique way as $\rho = a^{A_1}{}_{A_2}\omega_{A_1}{}^{A_2}$, can also be written as

$$\rho = \sum_k a_k db_k \tag{4.5.4}$$

for some a_k, b_k belonging to A. This can be proven directly by inverting the relation (4.5.2). The result is an expression of the ω in terms of a linear combination of $\kappa(T)dT$, as in the classical case:

$$\omega_{A_1}{}^{A_2} = Y_{A_1B_1}{}^{A_2B_2}\kappa(T^{B_1}{}_C)dT^C{}_{B_2} \tag{4.5.5}$$

where Y satisfies $X^{A_1B_1}{}_{A_2B_2}Y_{B_1C_1}{}^{B_2C_2} = \delta^{A_1}_{C_1}\delta^{C_2}_{A_2}$, $Y_{A_1B_1}{}^{A_2B_2}X^{B_1C_1}{}_{B_2C_2} = \delta^{C_1}_{A_1}\delta^{A_2}_{C_2}$ and is given explicitly by

$$Y_{A_1B_1}{}^{A_2B_2} = \alpha[(z-\lambda)C_{A_1B_1}C^{A_2B_2} + C_{A_1D}R^{DA_2}{}_{CB_1}C^{CB_2} - \frac{\lambda}{z(z-z^{-1})}D^{A_2}_{A_1}(D^{-1})^{B_2}_{B_1}] \tag{4.5.6}$$

with $\alpha = \frac{1}{z(z-z^{-1}-\lambda)}$ and $D^E_C \equiv C^{EF}C_{CF}$. The $r=1$ limit of (4.5.2) is discussed in the next section.

The braiding matrix Λ that defines the exterior product of forms is given by (2.2.35). It satisfies the characteristic equation:

$$\begin{array}{c}(\Lambda + r^2 I)\ (\Lambda + r^{-2}I)\ (\Lambda + \epsilon r^{\epsilon+1-N}I)(\Lambda + \epsilon r^{-\epsilon-1+N}I)\times \\ (\Lambda - \epsilon r^{-\epsilon+1+N}I)\ (\Lambda - \epsilon r^{\epsilon-1-N}I)\ (\Lambda - I) = 0\end{array} \tag{4.5.7}$$

due to the characteristic equation (4.1.13). For simplicity we will at times use the adjoint indices $i, j, k, ...$ with ${}^i = {}_A{}^B$, ${}_i = {}^A{}_B$. Define

$$(P_I, P_J)_{a_1}{}^{a_2}{}_{d_1}{}^{d_2}|^{c_1}{}_{c_2}{}^{b_1}{}_{b_2} \equiv d^{f_2}d^{-1}_{c_2}\hat{R}^{b_1f_2}{}_{c_2g_1}(P_I)^{c_1g_1}{}_{a_1e_1}(\hat{R}^{-1})^{a_2e_1}{}_{d_1g_2}(P_J)^{d_2g_2}{}_{b_2f_2} \tag{4.5.8}$$

where $P_I = P_S, P_A, P_0$ are given in (4.1.16) and $d^{f_2} \equiv D^{f_2}_{f_2}$, $d^{-1}_{c_2} \equiv (D^{-1})^{c_2}_{c_2}$. The (P_I, P_J) are themselves projectors, i.e.:

$$(P_I, P_J)(P_K, P_L) = \delta_{IK}\delta_{JL}(P_I, P_J) \tag{4.5.9}$$

Moreover

$$(I, I) = I \tag{4.5.10}$$

From $\omega^i \wedge \omega^j \equiv \omega^i \otimes \omega^j - \Lambda^{ij}{}_{kl}\omega^k \otimes \omega^l$ we find

$$\omega^i \wedge \omega^j = -Z^{ij}{}_{kl}\omega^k \wedge \omega^l \tag{4.5.11}$$

with

$$Z = (P_S, P_S) + (P_A, P_A) + (P_0, P_0) - I \tag{4.5.12}$$

see ref. [30]. The inverse of Λ always exists, and is given by

$$\begin{aligned}(\Lambda^{-1})^{A_2}_{A_1}{}^{D_2}_{D_1}|^{B_1}_{B_2}{}^{C_1}_{C_2} &= f_{D_1}{}^{D_2B_1}{}_{B_2}(T^{A_2}{}_{C_2}\kappa^{-1}(T^{C_1}{}_{A_1})) = \\ &= R^{F_1B_1}{}_{A_1G_1}(R^{-1})^{A_2G_1}{}_{E_2D_1}(R^{-1})^{D_2E_2}{}_{G_2C_2}R^{G_2C_1}{}_{B_2F_1}(d^{-1})^{C_1}d_{F_1}\end{aligned} \tag{4.5.13}$$

Note that for $r=1$, $\Lambda^2 = I$ and $(\Lambda + I)(\Lambda - I) = 0$ replaces the seventh-order spectral equation (4.5.7). In this special case, one finds the simple formula:

$$\omega^i \wedge \omega^j = -\Lambda^{ij}{}_{kl}\omega^k \wedge \omega^l \qquad \text{i.e. } Z = \Lambda\ . \tag{4.5.14}$$

The ***q*** **-Cartan-Maurer equations** are given by:

$$d\omega_{C_1}^{C_2} = \frac{1}{r-r^{-1}}(\omega_B{}^B \wedge \omega_{C_1}^{C_2} + \omega_{C_1}^{C_2} \wedge \omega_B{}^B) \equiv -\frac{1}{2}C^{A_1}{}_{A_2}{}^{B_1}{}_{B_2}|^{C_2}_{C_1}\ \omega_{A_1}^{A_2} \wedge \omega_{B_1}^{B_2} \tag{4.5.15}$$

with:

$$C^{A_1\ B_1}_{A_2\ B_2}|^{C_2}_{C_1} = -\frac{2}{(r-r^{-1})}[Z_B{}^B{}_{C_1}{}^{C_2}|^{A_1\ B_1}_{A_2\ B_2} + \delta^{A_1}_{C_1}\delta^{C_2}_{A_2}\delta^{B_1}_{B_2}] \tag{4.5.16}$$

To derive this formula we have used the flip operator Z on $\omega_B{}^B \wedge \omega_{C_1}{}^{C_2}$.

Finally, we recall that the χ operators close on the **q-Lie algebra** :

$$\chi_i\chi_j - \Lambda^{kl}{}_{ij}\chi_k\chi_l = \mathbf{C}_{ij}{}^k\chi_k \tag{4.5.17}$$

where the q-structure constants are given by

$$\mathbf{C}_{jk}{}^i = \chi_k(M_j{}^i) \quad \text{explicitly :} \quad \mathbf{C}^{A_1\ B_1}_{A_2\ B_2}|^{C_2}_{C_1} = \frac{1}{r-r^{-1}}[-\delta^{B_1}_{B_2}\delta^{A_1}_{C_1}\delta^{C_2}_{A_2} + \Lambda_B{}^B{}_{C_1}{}^{C_2}|^{A_1\ B_1}_{A_2\ B_2}]. \tag{4.5.18}$$

The C structure constants appearing in the Cartan-Maurer equations are related to the $\mathbf{C}$ constants of the q-Lie algebra by:

$$\mathbf{C}_{jk}{}^i = \frac{1}{2}[C_{jk}{}^i - \Lambda^{rs}{}_{jk}C_{rs}{}^i] \,. \tag{4.5.19}$$

In the particular case $\Lambda^2 = I$ (i.e. for $r = 1$) it is not difficult to see that in fact $C = \mathbf{C}$, and that the q-structure constants are Λ-antisymmetric:

$$\mathbf{C}_{jk}{}^i = -\Lambda^{rs}{}_{jk}\mathbf{C}_{rs}{}^i. \tag{4.5.20}$$

Note 4.5.1 The formulae characterizing the bicovariant calculus have been written in the basis $\{\chi^A{}_B\}$, $\{\omega_C{}^D\}$ because of the particularly simple expression of the $f_A{}^{BC}{}_D$ and $\chi^A{}_B$ functionals in terms of $L^{\pm A}{}_B$, see (2.2.32) and (2.2.52). Obviously the calculus is independent from the basis chosen. If we consider the linear transformation

$$\omega^i \to \omega'^i = X^i{}_j\omega^j$$

(where we use adjoint indices ${}^i = {}_{A_1}{}^{A_2}, {}_j = {}^{B_1}{}_{B_2}$), from the exterior differential

$$da = (\chi_i * a)\omega^i = (\chi'_i * a)\omega'^i \tag{4.5.21}$$

we find

$$\chi_i \to \chi'_i = \chi_j\,(X^{-1})^j{}_i\,,$$

and from the coproduct rule (2.1.35) of the χ_i we find $f^i{}_j \to f'^i{}_j = X^i{}_l f^l{}_m\,(X^{-1})^m{}_j$; while from (2.1.38) we have $M_i{}^j \to M_i'^j = (X^{-1})^l{}_i M_l{}^m X^j{}_m$.

A useful change of basis is obtained via the following transformation:

$$\begin{array}{l} \omega_{A_1}{}^{A_2} \to \vartheta^{A_1}{}_{A_2} = X^{A_1B_1}{}_{A_2B_2}\omega_{B_1}{}^{B_2} \\ \chi^{A_1}{}_{A_2} \to \psi_{A_1}{}^{A_2} = \chi^{B_1}{}_{B_2}Y_{B_1A_1}{}^{B_2A_2} \end{array} \tag{4.5.22}$$

where X and its (second) inverse Y are defined in (4.5.3) and (4.5.6). Using (4.5.5) it is immediate to see that

$$\vartheta^{A_1}_{\ A_2} = \kappa(T^{A_1}_{\ \ C})dT^C_{\ \ A_2} \ . \tag{4.5.23}$$

We also have:

$$\psi_{A_1}^{\ A_2}(T^{B_1}_{\ \ B_2}) = \psi_{A_1}^{\ A_2}(\tilde{T}^{B_1}_{\ \ B_2}) = \delta^{B_1}_{A_1}\delta^{A_2}_{B_2} \quad \text{where} \quad \tilde{T}^{B_1}_{\ \ B_2} \equiv T^{B_1}_{\ \ B_2} - \delta^{B_1}_{B_2} I \ . \tag{4.5.24}$$

Formula (4.5.24) follows from $\psi_{A_1}^{\ A_2}(I) = 0$ and:

$$\begin{aligned} \vartheta^{A_1}_{\ A_2} &= \kappa(T^{A_1}_{\ \ C})dT^C_{\ \ A_2} = \kappa(T^{A_1}_{\ \ C})(\psi_{B_1}^{\ B_2} * T^C_{\ \ A_2})\vartheta^{B_1}_{\ B_2} \\ &= \kappa(T^{A_1}_{\ \ C})T^C_{\ \ D}\psi_{B_1}^{\ B_2}(T^D_{\ \ A_2})\vartheta^{B_1}_{\ B_2} = \psi_{B_1}^{\ B_2}(T^{A_1}_{\ \ A_2})\vartheta^{B_1}_{\ B_2} \ . \end{aligned} \tag{4.5.25}$$

The analogue of the coordinates $\tilde{T}^{B_1}_{\ \ B_2}$ in the old basis is given by

$$x_{B_1}^{\ B_2} \equiv Y_{B_1 C_1}^{\quad B_2 C_2}\tilde{T}^{C_1}_{\ \ C_2} \quad , \quad \chi^{A_1}_{\ \ A_2}(x_{B_1}^{\ B_2}) = \delta^{A_1}_{B_1}\delta^{B_2}_{A_2} \ . \tag{4.5.26}$$

The set of coordinates $x_{B_1}^{\ B_2}$ and $\tilde{T}^{C_1}_{\ \ C_2}$ span the space X described in (2.1.89) and dual to the q-Lie algebra of $S_{q,r}(N+2)$.

Conjugation

From the $*$-structures (4.3.29), (4.3.30) and the definition (2.2.52) it is straightforward to find how the $*$-conjugation acts on the tangent vectors χ. Both conjugations (4.3.29) and (4.3.30) are compatibile with the differential calculus. Indeed they respectively yield [use (2.2.52), (4.3.15), (4.3.5), (4.1.19), (4.1.20) and (4.1.21) with $N \to N+2$ since we have capital indices]:

$$\begin{aligned} (\chi^A_{\ \ B})^* &= -r^{-N-1}\chi^C_{\ \ D}\mathcal{D}^F_{\ \ B}\mathcal{D}^A_{\ \ G}R^{EG}_{\ \ \ FC}D^D_{\ \ E} \quad \text{for } SO_{q,r}(n+2,n;\mathbf{R}) \ ; \quad 2n+2 = N+2 \\ (\chi^A_{\ \ B})^* &= -\epsilon r^{\epsilon-(N+2)}\chi^C_{\ \ D}R^{EA}_{\ \ \ BC}D^D_{\ \ E} \quad \text{for } SO_{q,r}(n+1,n+1;\mathbf{R}) \text{ or } Sp_{q,r}(n+1,n+1;\mathbf{R}) \end{aligned} \tag{4.5.27}$$

with $D^E_{\ \ C} \equiv C^{EF}C_{CF}$. As for the L matrices (and similarly to the T matrices) we have $\kappa^2(\chi^A_{\ \ B}) = D^{-1}{}^A_{\ E}\chi^E_{\ \ F}D^F_{\ \ B}$.

In a basis $\{\chi^A_{\ \ B}\}$ relation (2.3.52) reads

$$(\chi^A_{\ \ B})^* = V^{A\ \ Q}_{\ BP}\chi^P_{\ \ Q} \quad \text{if and only if} \quad \omega_C^{\ \ D} = -\omega_E^{\ \ F}\,\overline{V^{E\ \ D}_{\ FC}} \tag{4.5.28}$$

where V is a matrix with complex entries and $\overline{V}$ is its complex conjugate . Using this espression [or the inversion formulae (4.5.5)] one finds the induced conjugation on the left invariant 1-forms (use $\overline{D^A_{\ \ B}} = D^{-1}{}^A_{\ B}$):

$$\begin{aligned} (\omega_A^{\ \ B})^* &= r^{N+1}\mathcal{D}^F_{\ \ D}\mathcal{D}^C_{\ \ G}D^{-1}{}^B_{\ E}R^{-1}{}^{EG}_{\ \ \ FA}\omega_C^{\ \ D} \quad \text{for } SO_{q,r}(n+2,n;\mathbf{R}) \ ; \quad 2n+2 = N+2 \\ (\omega_A^{\ \ B})^* &= \epsilon r^{N+2-\epsilon}D^{-1}{}^B_{\ E}R^{-1}{}^{EC}_{\ \ \ DA}\omega_C^{\ \ D} \quad \text{for } SO_{q,r}(n+1,n+1;\mathbf{R}) \text{ or } Sp_{q,r}(n+1,n+1;\mathbf{R}). \end{aligned} \tag{4.5.29}$$

4.6 Differential calculus on $SO_{q,r=1}(N+2)$ and $Sp_{q,r=1}(N+2)$

As discussed in Section 4.2, we have obtained the quantum inhomogeneous groups $IS_{q,r}(N)$ via the projection

$$P \ : \quad S_{q,r}(N+2) \longrightarrow \frac{S_{q,r}(N+2)}{H} = IS_{q,r}(N) \tag{4.6.1}$$

with H=Hopf ideal in $S_{q,r}(N+2)$ defined after (4.2.1). As a consequence, the universal enveloping algebra $U(is_{q,r}(N))$ is a Hopf subalgebra of $U(s_{q,r}(N+2))$ and contains all the functionals that annihilate $H = Ker(P)$.

Let us consider now the χ functionals in the differential calculus on $S_{q,r}(N+2)$. Decomposing the indices we find:

$$\chi^a{}_b = \frac{1}{r-r^{-1}}[f_c{}^{ca}{}_b - \delta^a_b\varepsilon] \qquad + \frac{1}{r-r^{-1}} f_\bullet{}^{\bullet a}{}_b \tag{4.6.2}$$

$$\chi^a{}_\circ = \frac{1}{r-r^{-1}} f_c{}^{ca}{}_\circ \qquad + \frac{1}{r-r^{-1}} f_\bullet{}^{\bullet a}{}_\circ \tag{4.6.3}$$

$$\chi^\circ{}_b = \qquad + \frac{1}{r-r^{-1}}[f_c{}^{c\circ}{}_b + f_\bullet{}^{\bullet\circ}{}_b] \tag{4.6.4}$$

$$\chi^a{}_\bullet = \qquad + \frac{1}{r-r^{-1}} f_\bullet{}^{\bullet a}{}_\bullet \tag{4.6.5}$$

$$\chi^\bullet{}_b = \frac{1}{r-r^{-1}} f_\bullet{}^{\bullet\bullet}{}_b \tag{4.6.6}$$

$$\chi^\circ{}_\circ = \frac{1}{r-r^{-1}}[f_\circ{}^{\circ\circ}{}_\circ - \varepsilon] \qquad + \frac{1}{r-r^{-1}}[f_c{}^{c\circ}{}_\circ + f_\bullet{}^{\bullet\circ}{}_\circ] \tag{4.6.7}$$

$$\chi^\circ{}_\bullet = \qquad + \frac{1}{r-r^{-1}} f_\bullet{}^{\bullet\circ}{}_\bullet \tag{4.6.8}$$

$$\chi^\bullet{}_\circ = \frac{1}{r-r^{-1}} f_\bullet{}^{\bullet\bullet}{}_\circ \tag{4.6.9}$$

$$\chi^\bullet{}_\bullet = \underbrace{\frac{1}{r-r^{-1}}[f_\bullet{}^{\bullet\bullet}{}_\bullet - \varepsilon]}_{\text{terms annihilating } H} \tag{4.6.10}$$

where we have indicated the terms that do and do not annihilate the Hopf ideal H, i.e. that belong or do not belong to $U_{q,r}(is(N))$. We see that only the functionals $\chi^\bullet{}_b$, $\chi^\bullet{}_\circ$ and $\chi^\bullet{}_\bullet$ do annihilate H, and therefore belong to $U(is_{q,r}(N))$. The resulting bicovariant differential calculus, see Chapter 5, contains dilatations and translations, but does not contain the tangent vectors of $S_{q,r}(N)$, i.e. the functionals $\chi^a{}_b$. Indeed these contain $f_\bullet{}^{\bullet a}{}_b$, in general not vanishing on H. We can, however, try to find restrictions on the parameters q, r such that $f_\bullet{}^{\bullet a}{}_b(H) = 0$. As we will see, this happens for $r = 1$. For this reason we consider in the following the particular multiparametric deformations called "minimal deformations" or twistings, corresponding to $r = 1$.

We first examine what happens to the bicovariant calculus on $S_{q,r}(N+2)$ in the $r=1$ limit[3]. The R matrix is given by, cf. (4.1.8):

$$R^{AB}{}_{AB} = q_{AB}^{-1} + O(\lambda) \qquad (4.6.11)$$
$$R^{AB}{}_{BA} = \lambda \qquad A > B, A' \neq B \qquad (4.6.12)$$
$$R^{AA'}{}_{A'A} = \lambda\ (1 - \epsilon r^{\rho_A - \rho_{A'}}) \qquad A > A' \qquad (4.6.13)$$
$$R^{AA'}{}_{BB'} = -\lambda \epsilon_A \epsilon_B + O(\lambda^2) \qquad A > B, A' \neq B \qquad (4.6.14)$$

where $O(\lambda^n)$ indicates an infinitesimal of order $\geq \lambda^n$; the q_{AB} parameters satisfy:

$$q_{AB} = q_{AB'}^{-1} = q_{A'B}^{-1} = q_{BA}^{-1} \ ; \qquad q_{AA} = q_{AA'} = 1 \qquad (4.6.15)$$

up to order $O(\lambda)$. Note that the components $R^{AA'}{}_{A'A}$ are of order $O(\lambda^2)$ for the orthogonal case ($\epsilon = 1$) and of order $O(\lambda)$ for the symplectic case ($\epsilon = -1$). The RTT relations simply become:

$$T^{B_1}{}_{A_1} T^{B_2}{}_{A_2} = \frac{q_{B_1B_2}}{q_{A_1A_2}} T^{B_2}{}_{A_2} T^{B_1}{}_{A_1} \ . \qquad (4.6.16)$$

For $r=1$ the metric is $C_{AB} = \epsilon_A \delta_{AB'}$ and therefore we have $C_{AB} = \epsilon C_{BA}$. Using the definition (4.3.1), it is easy to see that

$$L^{\pm A}{}_A(T^C{}_D) = \delta^C_D q_{AC} + O(\lambda) \qquad (4.6.17)$$
$$L^{\pm A}{}_B(T^B{}_A) = \pm\lambda \qquad A \neq B, A' \neq B;\ A < B \text{ for } L^+,\ A > B \text{ for } L^- \qquad (4.6.18)$$
$$L^{\pm A}{}_{A'}(T^{A'}{}_A) = \pm\lambda\ [1 - \epsilon r^{\pm(\rho_A - \rho_{A'})}] \quad A < A' \text{ for } L^+,\ A > A' \text{ for } L^- \qquad (4.6.19)$$
$$L^{\pm A}{}_B(T^{A'}{}_{B'}) = \mp\lambda\epsilon_A\epsilon_B + O(\lambda^2) \quad A \neq B, A' \neq B;\ \ A < B \text{ for } L^+,\ A > B \text{ for } L^- \qquad (4.6.20)$$

all other $L^{\pm}(T)$ vanishing. Relations (4.6.18) and (4.6.20) imply that for any generator $T^C{}_D$ we have $L^{\pm A}{}_B(T^C{}_D) = -\epsilon_A\epsilon_B L^{\pm B'}{}_{A'}(T^C{}_D) + O(\lambda^2)$ with $A \neq B,\ A \neq B'$. In general, since

$$\Delta(L^{\pm A}{}_A) = L^{\pm A}{}_A \otimes L^{\pm A}{}_A \ ; \ \ \Delta(L^{\pm A}{}_B) = L^{\pm A}{}_A \otimes L^{\pm A}{}_B + L^{\pm A}{}_B \otimes L^{\pm B}{}_B + O(\lambda^2), \quad A \neq B$$

we find that

$$L^{\pm A}{}_A = O(1) \qquad (4.6.21)$$
$$L^{\pm A}{}_B = O(\lambda), \qquad A \neq B,\ A \neq B' \qquad (4.6.22)$$
$$L^{\pm A}{}_{A'} = O(\lambda^2) \text{ for } SO_q, \qquad O(\lambda) \text{ for } Sp_q \qquad (4.6.23)$$

[3]By $\lim_{r\to 1} a$, where the generic element $a \in S_{q,r}(N+2)$ is a polynomial in the matrix elements $T^A{}_B$ with complex coefficients $f(r)$ depending on r, we understand the element of $S_{q,r=1}(N+2)$ with coefficients given by $\lim_{r\to 1} f(r)$. The expression $\lim_{r\to 1} \phi = \varphi$, where $\phi \in U(s_{q,r}(N+2))$ and $\varphi \in U(S_{q,r=1}(N+2))$ means that $\lim_{r\to 1} \phi(a) = \varphi(\lim_{r\to 1} a)$ for any $a \in S_{q,r}(N+2)$ such that $\lim_{r\to 1} a$ exists. Finally, the left invariant 1-forms ω^i are symbols, and therefore $\lim_{r\to 1} a_i\omega^i \equiv (\lim_{r\to 1} a_i)\omega^i$.

where, by definition, $\phi = O(\lambda^n)$ (ϕ being a functional) means that for any element $a \in S_{q,r}(N+2)$ with well-defined classical limit, we have $\phi(a) = O(\lambda^n)$.
Moreover the following relations hold:

$$L^{\pm A}{}_A = L^{\mp A}{}_A + O(\lambda) \;, \tag{4.6.24}$$

$$\kappa(L^{\pm A}{}_B) = \epsilon_A \epsilon_B L^{\pm B'}{}_{A'} + O(\lambda) \quad \text{and therefore, } \kappa^2 = id + O(\lambda) \;. \tag{4.6.25}$$

Similarly one can prove the relations involving the f functionals (no sum on repeated indices):

$$f_A{}^{AA}{}_A = \varepsilon + O(\lambda) \tag{4.6.26}$$

$$f_A{}^{BA}{}_B = O(1) \qquad \text{and} \quad f_A{}^{BA}{}_B = f_{B'}{}^{A'B'}{}_{A'} + O(\lambda) \tag{4.6.27}$$

$$f_C{}^{CA}{}_A = O(\lambda^2) \qquad C \neq A \tag{4.6.28}$$

$$f_C{}^{CA}{}_B = O(\lambda^2) \qquad [A < B, C \neq B] \text{ or } [A > B, C \neq A] \tag{4.6.29}$$

[hint: check (4.6.27)-(4.6.29) first on the generators, then use the coproduct in (2.1.35)]. From the last relation we deduce

$$\chi^A{}_B = \frac{1}{\lambda} f_B{}^{BA}{}_B + O(\lambda), \qquad A < B \tag{4.6.30}$$

$$\chi^A{}_B = \frac{1}{\lambda} f_A{}^{AA}{}_B + O(\lambda), \qquad A > B \tag{4.6.31}$$

and from (4.6.26) and (4.6.28) one has

$$\chi^A{}_A = \frac{1}{\lambda}[f_A{}^{AA}{}_A - \varepsilon] \tag{4.6.32}$$

Next one can verify that

$$\left.\begin{array}{l} \chi^A{}_B(T^B{}_A) = -q_{BA} + O(\lambda) \\ \chi^A{}_B(T^{A'}{}_{B'}) = \epsilon_A \epsilon_B + O(\lambda) \\ \chi^A{}_B(T^C{}_D) = 0 \;\; \text{otherwise} \end{array}\right\} \; A \neq B,\, A \neq B' \tag{4.6.33}$$

$$\forall\, T^C{}_D \,, \quad \chi^A{}_A(T^C{}_D) = -\chi^{A'}{}_{A'}(T^C{}_D) + O(\lambda). \tag{4.6.34}$$

Eq.s (4.6.33) yield the relation between χ functionals:

$$\forall\, T^C{}_D \,, \quad \chi^{B'}{}_{A'}(T^C{}_D) = -\frac{\epsilon_A \epsilon_B}{q_{BA}} \chi^A{}_B(T^C{}_D) + O(\lambda), \;\; A \neq B,\, A \neq B'. \tag{4.6.35}$$

It is not difficult to prove that the coproduct rule in (2.1.35) is compatible with (4.6.35) and (4.6.34) making them valid on arbitrary polynomials in the $T^A{}_B$ elements:

$$\chi^{B'}{}_{A'} = -\frac{\epsilon_A \epsilon_B}{q_{BA}} \chi^A{}_B + O(\lambda), \;\; A \neq B,\, A \neq B' \quad ; \quad \chi^A{}_A = -\chi^{A'}{}_{A'} + O(\lambda) \;. \tag{4.6.36}$$

Finally:

$$\chi^A{}_{A'} = O(\lambda) \text{ for } SO_q\ , \quad O(1) \text{ for } Sp_q\ , \quad A \neq A'. \tag{4.6.37}$$

Summarizing, in the $r \to 1$ limit, only the following χ functionals survive:

$$\chi^A{}_A \equiv \lim_{r\to 1} \frac{1}{\lambda}\ [f_A{}^{AA}{}_A - \varepsilon] \tag{4.6.38}$$

$$\chi^A{}_B \equiv \lim_{r\to 1} \frac{1}{\lambda}\ f_A{}^{AA}{}_B, \quad A > B, A \neq B' \tag{4.6.39}$$

$$\chi^A{}_B \equiv \lim_{r\to 1} \frac{1}{\lambda}\ f_B{}^{BA}{}_B, \quad A < B, A \neq B' \tag{4.6.40}$$

$$\chi^A{}_{A'} \equiv \lim_{r\to 1} \frac{1}{\lambda}\ \sum_C f_C{}^{CA}{}_{A'} = 0 \text{ for } SO_q, \quad \neq 0 \text{ for } Sp_q \tag{4.6.41}$$

Notice that (4.6.36) and (4.6.37) are all contained in the formula:

$$\chi^{B'}{}_{A'} = -\frac{\epsilon_A \epsilon_B}{q_{BA}} \chi^A{}_B + O(\lambda) \tag{4.6.42}$$

thus in the $r \to 1$ limit there are $(N+2)(N+1)/2$ tangent vectors for $SO_q(N+2)$ and $(N+2)(N+3)/2$ tangent vectors for $Sp_q(N+2)$, exactly as in the classical case.

The $r = 1$ limit of (4.5.2) reads:

$$dT^A{}_B = -\sum_C T^A{}_C q_{CB} (\omega_B{}^C - \epsilon_B \epsilon_C q_{BC} \omega_{C'}{}^{B'})\ , \tag{4.6.43}$$

and therefore, for $r = 1$, ω appears only in the combination

$$\Omega_A{}^B \equiv \omega_A{}^B - \epsilon_A \epsilon_B q_{AB} \omega_{B'}{}^{A'}\ , \tag{4.6.44}$$

Only $(N+2)(N+1)/2$ $[(N+2)(N+3)/2$ for $Sp_q(N+2)]$ of the $(N+2)^2$ one forms $\Omega_A{}^B$ are linearly independent because [compare with (4.6.42)]:

$$\Omega_{B'}{}^{A'} = -\frac{\epsilon_A \epsilon_B}{q_{AB}} \Omega_A{}^B\ . \tag{4.6.45}$$

In the sequel, instead of considering the left module of 1-forms freely generated by $\omega_A{}^B$, we consider the submodule Γ freely generated by $\Omega_A{}^B$ with $A' < B$ for SO_q and $A' \leq B$ for Sp_q. In fact only this submodule will be relevant for the $r = 1$ differential calculus. As in the classical case [4] , in order to simplify notations in sums we often

[4] To make closer contact with the classical case one may define:

$$\Omega^{AB} \equiv \Omega_C{}^B C^{CA} = \epsilon_A \Omega_{A'}{}^B \ ; \quad \chi_{AB} \equiv C_{AC} \chi^C{}_B = \epsilon_A \chi^{A'}_B\ ,$$

and retrieve the more familiar q-antisymmetry:

$$\Omega^{AB} = -\epsilon q_{BA} \Omega^{BA} \ ; \quad \chi_{AB} = -\epsilon q_{AB} \chi_{BA}\ .$$

use $\chi^A{}_B$ and $\Omega_A{}^B$ without the restriction $A' \leq B$ see for ex. (4.6.50) below. The bimodule structure on Γ, see Theorem 4.5.1, is given by the $r \to 1$ limit of the $f^i{}_j$ functionals. These are diagonal in the i, j indices [i.e. they vanish for $i \neq j$, see (4.6.26)-(4.6.29)] and still satisfy the property (2.1.32). We have:

$$\begin{aligned} \Omega_A{}^B a &= (\omega_A{}^B - \epsilon_A \epsilon_B q_{AB} \omega_{B'}{}^{A'}) a \\ &= (f_A{}^{BC}{}_D * a)\omega_C{}^D - \epsilon_A \epsilon_B q_{AB} (f_{B'}{}^{A'D'}{}_{C'} * a)\omega_{D'}{}^{C'} \\ &= (f_A{}^{BA}{}_B * a)\Omega_A{}^B \end{aligned} \tag{4.6.46}$$

where in the last equality we have used (4.6.27) and no sum is understood. We see that the bimodule structure is very simple since it does not mix different Ω's. Moreover, relation (4.6.43) is invertible and yields:

$$\Omega_A{}^B = -q_{AB} \kappa(T^B{}_C) dT^C{}_A \ ; \tag{4.6.47}$$

in the limit $q_{AB} = 1$, the $\Omega_A{}^B$ are to be identified with the classical 1-forms, and indeed for $q_{AB} = 1$ eq. (4.6.47) reproduces the correct classical limit $\Omega = -g^{-1}dg$ for the left-invariant 1-forms on the group manifold.

The bimodule commutation rule (4.6.46) yields a formula similar to (4.5.1), replacing the values of the R matrix for $r = 1$ we find the commutations:

$$\Omega_{A_1}{}^{A_2} T^R{}_S = \frac{q_{A_2 S}}{q_{A_1 S}} T^R{}_S \Omega_{A_1}{}^{A_2} \tag{4.6.48}$$

For $r = 1$ the coproduct on the χ functionals reads

$$\Delta'(\chi^A{}_B) = \chi^A{}_B \otimes f_A{}^{BA}{}_B + \varepsilon \otimes \chi^A{}_B \qquad \text{no sum on repeated indices.} \tag{4.6.49}$$

We then consider the $r = 1$ limit of (2.3.30): $da = (\chi_i * a)\omega^i$ and therefore obtain the following definition of the exterior differential:

$$da \equiv \frac{1}{2}(\chi^A{}_B * a)\Omega_A{}^B = \sum_{A' \leq B} (\chi^A{}_B * a)\Omega_A{}^B, \qquad \forall a \in A \ , \tag{4.6.50}$$

where in the second expression we have used the basis of linear independent tangent vectors $\{\chi^A{}_B\}_{A' \leq B}$ and dual 1-forms $\{\Omega_A{}^B\}_{A' \leq B}$ (notice that in the SO_q case we have $A' < B$ because $\chi^A{}_A = \Omega_A{}^A = 0$). The Leibniz rule is satisfied for d defined in (4.6.50) because of (4.6.49) and (4.6.46). Moreover any $\rho = a^A{}_B \Omega_A{}^B \in \Gamma$ can be written as $\rho = \sum_k a_k db_k$, [use (4.6.47)].

We now introduce a left and a right action on the bimodule Γ of 1-forms:

$$\Delta_L(a\Omega_A{}^B) \equiv \Delta(a) I \otimes \Omega_A{}^B \ , \tag{4.6.51}$$

$$\Delta_R(a\Omega_A{}^B) \equiv \Delta(a)(\Omega_C{}^D \otimes M^C_{DA}{}^B) \ . \tag{4.6.52}$$

where $M^C_{DA}{}^B = T^C{}_A \kappa(T^B{}_D)$. [Using (4.6.44) one can check that this is the $r = 1$ limit of $\Delta_L(a\omega_{A_1}{}^{A_2}) = \Delta(a)\,(I \otimes \omega_{A_1}{}^{A_2})$ and $\Delta_R(a\omega_{A_1}{}^{A_2}) = \Delta(a)\,(\omega_{B_1}{}^{B_2} \otimes M^{B_1}_{B_2 A_1}{}^{A_2})$]. Relation (4.6.52) is well defined i.e. $\Delta_R(\Omega_{B'}{}^{A'}) = \Delta_R(-\frac{\epsilon_A \epsilon_B}{q_{AB}} \Omega_A{}^B)$ because $\epsilon_F \epsilon_E q_{FE} M^{F'}_{E'A}{}^B$

$= \epsilon_A \epsilon_B q_{AB} M^{E}_{FB'}{}^{A'}$. Since in the $r = 1$ case the bicovariant bimodule conditions (2.1.32), (2.1.44) and (2.1.51) are still satisfied, it is easy to deduce that Δ_L and Δ_R give a bicovariant bimodule structure to Γ.

The differential (4.6.50) gives a bicovariant differential calculus if it is compatible with Δ_L and Δ_R, i.e. if:

$$\Delta_L(adb) = \Delta(a)(id \otimes d)\Delta(b) \ , \tag{4.6.53}$$

$$\Delta_R(adb) = \Delta(a)(d \otimes id)\Delta(b) \ . \tag{4.6.54}$$

The proof of the compatibility of d with Δ_L is straightforward, just use (4.6.50) and the coassociativity of the coproduct Δ. In order to prove (4.6.54) we recall, from Proposition 2.3.1, that in the case $r \neq 1$ property (4.6.54) holds if and only if

$$b_1\omega^j \otimes b_2\chi_i(b_3)M_j{}^i = b_1\omega^j \otimes \chi_j(b_2)b_3 \tag{4.6.55}$$

and this last relation is equivalent to

$$b_1\chi_i(b_2)M_j{}^i = \chi_j(b_1)b_2 \ , \quad \text{i.e.} \quad \chi_i * b = (b * \chi_j)\kappa(M_i{}^j) \tag{4.6.56}$$

as one can verify by applying $m(\kappa \otimes id)\Delta_L \otimes id$ (m denotes multiplication) to (4.6.55), and using the linear independence of the ω^i. Now formula (4.6.56) holds also in the limit $r = 1$. Indeed if we consider b to be a polynomial in the $T^A{}_B$ with well behaved coefficents in the $r \to 1$ limit, then $\lim_{r\to 1}[b_1\chi_i(b_2)M_j{}^i] = \lim_{r\to 1}[\chi_j(b_1)b_2]$ i.e. $b_1[\lim_{r\to 1}\chi_i(b_2)]M_j{}^i = [\lim_{r\to 1}\chi_j(b_1)]b_2$ so that relation (4.6.56) remains valid for $r = 1$, cf .(4.6.38)-(4.6.41). At this point one can prove (4.6.54) in the $r = 1$ case simply by substituting Ω to ω in (2.3.37), (2.3.38) and (4.6.55). Since (4.6.56) holds for $r = 1$, then also (4.6.55) holds in this limit and the theorem is proved.
□□□

We conclude that (4.6.50) defines a bicovariant differential calculus on $S_q(N+2)$.

Note 4.6.1 We have found the $S_{q,r=1}(N)$ differential calculus studying the $r = 1$ limit of the χ functionals and of the bicovariant bimodule of 1-forms [see (4.6.51), (4.6.52), (4.6.53), (4.6.54)]. This has given a comprehensive analysis of the $r \to 1$ limit. The classical limit $r \to 1$, $q \to 1$ and the classical differential calculus are now easily recovered. From a slightly different perspective, since the calculus can be defined from the q-Lie algebra alone, we could just have studied only the limit of the q-Lie algebra. It is immediate to see that (2.3.3), (2.3.4), (2.3.5) or (2.3.18) still hold. This is another proof of the bicovariance of the $S_{q,r=1}(N)$ calculus.

We have chosen to study the $r \to 1$ limit of the quantum Lie algebra in the χ basis because this gives the classical Lie algebra. Another possiblity is to perform the limit in the ψ basis. In this case the ψ are linearly independent also when $r = 1$, see (4.5.24) and the classical differential calculus is contained in this calculus.

Note 4.6.2 In (4.6.56) the sum on the indices $j = (C, D)$ can be restricted to $C' \le D$, thus using the basis $\{\chi^C{}_D\}_{C' \le D}$, provided one replaces M by

$$\begin{array}{lll} M_-{}^C{}_{DA}{}^B \equiv M^C{}_{DA}{}^B - \epsilon_A \epsilon_B q_{AB} M^C{}_{DB'}{}^{A'} & \text{for } C' \ne D,\ A' \ne B \\ M_-{}^C{}_{C'A}{}^B \equiv 0 \quad , \quad M_-{}^C{}_{DA}{}^{A'} \equiv 0 & \text{for } SO_q \\ M_-{}^C{}_{C'A}{}^B \equiv M^C{}_{C'A}{}^B, \quad M_-{}^C{}_{DA}{}^{A'} \equiv M^C{}_{DA}{}^{A'} & \text{for } Sp_q \end{array} \tag{4.6.57}$$

This is easily seen from (4.6.42). We have thus obtained the fundamental relation (2.3.23): $(\chi^{B_1}{}_{B_2} * b) M_-{}^{A_1}{}_{A_2 B_1}{}^{B_2} = (b * \chi^{A_1}{}_{A_2})$, with $A_1' \le A_2$, $B_1' \le B_2$ for the $ISq_{,r=1}(N)$ differential calculus. Notice that $M_-{}^C{}_{DA}{}^B = M^C{}_{DA}{}^B - \epsilon_C \epsilon_D q_{DC} M^{D'}_{C'A}{}^B$, this equality is due to the RTT relations (4.6.16). We can also write $\Delta_R(a\Omega_A{}^B) = \sum_{C' \le D} \Delta(a)(\Omega_C{}^D \otimes M_-{}^C{}_{DA}{}^B)$ cf.(4.6.52), thus using the basis $\{\Omega_C{}^D\}_{C' \le D}$. According to the general theory the elements $M_-{}^C{}_{DA}{}^B$ with $C' \le D$, $A' \le B$ are the adjoint representation for the differential calculus on $S_{q,r=1}(N+2)$. Since the calculus is bicovariant [cf.(4.6.53), (4.6.54)] we know a priori that the $M_-{}^C{}_{DA}{}^B$ with $C' \le D$, $A' \le B$ satisfy the properties (2.1.44) and (2.1.51).[5]

It is useful to express the bicovariant algebra (4.5.17), (2.1.112)-(2.1.114) in the $r \to 1$ limit. Due to the R matrix being diagonal for $r = 1$, the Λ tensor $\Lambda^{A_2}{}_{A_1}{}^{D_2}{}_{D_1}|^{C_1}{}_{C_2}{}^{B_1}{}_{B_2} \equiv f_{A_1}{}^{A_2 B_1}{}_{B_2}(M^{C_1}{}_{C_2 D_1}{}^{D_2})$ takes the simple form:

$$\Lambda^{A_2}{}_{A_1}{}^{B_2}{}_{B_1}|^{B_1}{}_{B_2}{}^{A_1}{}_{A_2} = q_{A_1 B_2} q_{A_2 B_1} q_{B_1 A_1} q_{B_2 A_2}\ , \qquad 0 \text{ otherwise} \tag{4.6.58}$$

Therefore (2.1.112)-(2.1.114) read (no sum on repeated indices):

$$f^i{}_i f^j{}_j = f^j{}_j f^i{}_i \tag{4.6.59}$$

$$\mathbf{C}_{jk}{}^i f^j{}_j f^k{}_k + f^i{}_j \chi_k = \Lambda^{kj}{}_{jk} \chi_k f^i{}_j + \mathbf{C}_{jk}{}^i f^i{}_i \tag{4.6.60}$$

$$\chi_k f^i{}_i = \Lambda^{ik}{}_{ki} f^i{}_i \chi_k\ . \tag{4.6.61}$$

[5] A direct proof in the SO_q case is also instructive. We call P_- the "q-antisymmetric" projector defined by:

$$P_-{}^A{}_{BC}{}^D \equiv \frac{1}{2}(\delta^A_C \delta^D_B - q_{BA} \delta^{B'}_C \delta^D_{A'}) = \frac{1}{2}(\delta^A_C \delta^D_B - q_{CD} \delta^{B'}_C \delta^D_{A'})\ .$$

Then one easily shows that $P_-{}^A{}_{BC}{}^D = -q_{BA} P_-{}^{B'}{}_{A'D}{}^C$, $P_-{}^A{}_{BC}{}^D = -q_{CD} P_-{}^A{}_{BD'}{}^{C'}$ and

$$\begin{array}{c} \Omega^j P_-{}_j{}^i = \Omega^i\ , \quad P_-{}_i{}^j \chi_j = \chi_i\ , \quad P_-{}_k{}^i f^k{}_j = f^i{}_k P_-{}_j{}^k = P_-{}_k{}^i f^k{}_n P_-{}_j{}^n\ , \\ M_-{}_i{}^j = 2P_-{}_i{}^l M_l{}^j = 2M_i{}^l P_-{}_l{}^j\ , \quad M_-{}_i{}^j = P_-{}_i{}^l M_-{}_l{}^j = M_-{}_i{}^l P_-{}_l{}^j = 2P_-{}_i{}^\alpha M_-{}_\alpha{}^j = 2M_-{}_i{}^\beta P_-{}_\beta{}^j\ , \end{array}$$

where greek letters α, β represent adjoint indices (A_1, A_2), (B_1, B_2) with the restriction $A_1' < A_2$, $B_1' < B_2$. It is then straightforward to show that $\Delta(M_-{}_i{}^j) = M_-{}_i{}^\alpha \otimes M_-{}_\alpha{}^j$ and $\varepsilon(M_-{}_\alpha{}^\beta) = \delta^\beta_\alpha$. Applying P_- to (2.1.51) and using $f^i{}_j = 0$ unless $i = j$ cf. (4.6.26)-(4.6.29) one also proves $M_-{}_\alpha{}^j(a * f^\alpha{}_k) = (f^j{}_\beta * a) M_-{}_k{}^\beta$. These formulae hold in particular if all indices are greek, thus proving (2.1.44) and (2.1.51) for $SO_{q,r=1}(N+2)$.

Explicitly the q-Lie algebra (4.5.17) reads:

$$\chi^{C_1}_{\ C_2}\chi^{B_1}_{\ B_2} - q_{B_1C_2}q_{C_1B_1}q_{B_2C_1}q_{C_2B_2}\ \chi^{B_1}_{\ B_2}\chi^{C_1}_{\ C_2} =$$
$$- q_{B_1C_2}q_{C_2B_2}q_{B_2B_1}\delta^{C_1}_{B_2}\ \chi^{B_1}_{\ C_2} + q_{C_1B_1}q_{B_2B_1}C_{B_2C_2}\ \chi^{B_1}_{\ C_1'} +$$
$$+ q_{C_2B_2}q_{B_1C_2}C^{C_1B_1}\ \chi^{B_2'}_{\ C_2} - q_{B_2C_1}\delta^{B_1}_{C_2}\ \chi^{B_2'}_{\ C_1'}\ . \tag{4.6.62}$$

The Cartan-Maurer equations are obtained by differentiating (4.6.47):

$$d\Omega_A^{\ B} = q_{AB}q_{BC}q_{CA}C_{CD}\ \Omega_C^{\ B}\wedge\Omega_A^{\ D} \tag{4.6.63}$$

The commutations between Ω 's are easy to find using (4.5.14):

$$\Omega_{A_1}^{\ A_2}\wedge\Omega_{D_1}^{\ D_2} = -q_{A_1D_2}q_{D_1A_1}q_{A_2D_1}q_{D_2A_2}\Omega_{D_1}^{\ D_2}\wedge\Omega_{A_1}^{\ A_2} \tag{4.6.64}$$

Finally, we turn to the $*$-conjugations given by equations (4.5.27) and (4.5.29). Their $r\to 1$ limit yields, for (4.3.30)

$$(\Omega_A^{\ B})^* = q_{BA}\mathcal{D}^C_{\ A}\Omega_C^{\ D}\mathcal{D}^B_{\ D}\quad ;\quad (\chi^A_{\ B})^* = -q_{CD}\mathcal{D}^A_{\ C}\chi^C_{\ D}\mathcal{D}^D_{\ B} = -\overline{q_{BA}}\mathcal{D}^A_{\ E}\chi^E_{\ F}\mathcal{D}^F_{\ B}\ , \tag{4.6.65}$$

while for the conjugation (4.3.29)

$$(\Omega_A^{\ B})^* = \epsilon q_{BA}\Omega_A^{\ B}\quad ;\quad (\chi^A_{\ B})^* = -\epsilon q_{AB}\chi^A_{\ B}\ . \tag{4.6.66}$$

This shows that we have a bicovariant $*$-differential calculus.

4.7 Differential calculus on $ISO_{q,r=1}(N)$ and $ISp_{q,r=1}(N)$

We have found the inhomogeneous quantum group $IS_{q,r}(N)$ by means of a projection from $S_{q,r}(N+2)$; dually, its universal enveloping algebra is a given Hopf subalgebra of $U_{q,r}(s(N+2))$. Using the same techniques and the results of Section 2.3 we here derive the differential calculus on $IS_{q,r=1}(N)$.

From (2.3.4), (2.3.5) and (2.3.18) it is immediate to see that $T'\equiv T\cap U_{q,r=1}(is(N))$ satisfies

$$\Delta(T')\subset T'\otimes\varepsilon + U_{q,r=1}(is(N))\otimes T' \tag{4.7.1}$$

$$[T',T']\subseteq T\cap U_{q,r=1}(is(N)) = T' \tag{4.7.2}$$

$$\forall\psi\in U_{q,r}(is(N))\ ,\qquad ad_\psi T'\subseteq T' \tag{4.7.3}$$

indeed $U_{q,r}(is(N))$ is a Hopf subalgebra of $U_{q,r}(s(N+2))$. Also condition (2.3.3) is fulfilled since T' generates $U_{q,r}(is(N))$ in the same way T generates $U_{q,r}(s(N+2))$ [78], this is a consequence of the upper and lower triangularity of the L^+ and L^-

matrices and of the dependence of the diagonal elements of L^+ from the diagonal elements of L^-; this is true for $r \neq 1$ and therefore also for $r = 1$. From this last statement, (4.7.1) and (4.7.2)–or just from (4.7.1) and (4.7.3)– we obtain that T' generates an $IS_{q,r=1}(N)$ bicovariant differential calculus.

We reconsider now, in the $r \to 1$ limit, the functionals given in eq.s (4.6.2)-(4.6.10). We list below the functionals among these that belong to T':

$$\begin{aligned}
\chi^{a}{}_{b} &= \frac{1}{r-r^{-1}}[f_c{}^{ca}{}_b - \delta^a_b \varepsilon] \\
\chi^{a}{}_{\circ} &= \frac{1}{r-r^{-1}} f_c{}^{ca}{}_{\circ} \\
\chi^{\bullet}{}_{b} &= \frac{1}{r-r^{-1}} f_{\bullet}{}^{\bullet\bullet}{}_b \\
\chi^{\circ}{}_{\circ} &= \frac{1}{r-r^{-1}}[f_{\circ}{}^{\circ\circ}{}_{\circ} - \varepsilon] \\
\chi^{\bullet}{}_{\bullet} &= \frac{1}{r-r^{-1}}[f_{\bullet}{}^{\bullet\bullet}{}_{\bullet} - \varepsilon] \\
\chi^{\bullet}{}_{\circ} &= \frac{1}{r-r^{-1}} f_{\bullet}{}^{\bullet\bullet}{}_{\circ}
\end{aligned} \tag{4.7.4}$$

Note that in the $r \to 1$ limit $\chi^{\bullet}{}_{\circ}$ vanishes for $SO_{q,r=1}(N+2)$, and does not vanish in the case $Sp_{q,r=1}(N+2)$.

For $r = 1$ the χ's in (4.7.4) are not linearly independent, cf. relation (4.6.42) of previous section, and we have:

$$\chi^{b'}{}_{a'} = -q_{ab}\chi^{a}{}_{b}, \quad \chi^{b'}{}_{\circ} = -\frac{1}{q_{b\bullet}}\chi^{\bullet}{}_{b}, \quad \chi^{\circ}{}_{\circ} = -\chi^{\bullet}{}_{\bullet} \tag{4.7.5}$$

A basis of tangent vectors for T', in the orthogonal case, is therefore given by

$$\chi^{a}{}_{b} = \lim_{r\to 1} \frac{1}{\lambda}\,[f_c{}^{ca}{}_b - \delta^a_b\varepsilon]\ , \quad \text{with } a+b > N+1 \quad \text{i.e.} \quad a' < b; \tag{4.7.6}$$

$$\chi^{\bullet}{}_{b} = \lim_{r\to 1}\frac{1}{\lambda} f_{\bullet}{}^{\bullet\bullet}{}_b \ ; \quad \chi^{\bullet}{}_{\bullet} = \lim_{r\to 1}\frac{1}{\lambda}\,[f_{\bullet}{}^{\bullet\bullet}{}_{\bullet} - \varepsilon]\ , \tag{4.7.7}$$

The q-Lie algebra commutations are a subset of (4.6.62) obtained specializing the capital indices of (4.6.62) to the indices ${}^a{}_b$, ${}^{\bullet}{}_b$ and ${}^{\bullet}{}_{\bullet}$. We have the $SO_{q,r=1}(N)$ q-Lie algebra that reads as in eq. (4.6.62) with lower case indices; the remaining commutations are:

$$\chi^{c_1}{}_{c_2}\chi_{b_2} - \frac{q_{c_1\bullet}}{q_{c_2\bullet}} q_{b_2c_1}q_{c_2b_2}\chi_{b_2}\chi^{c_1}{}_{c_2} = \frac{q_{c_1\bullet}}{q_{c_2\bullet}}[C_{b_2c_2}\chi_{c_1'} - \delta^{c_1}_{b_2} q_{c_2c_1}\chi_{c_2}]\ , \tag{4.7.8}$$

$$\chi_{c_2}\chi_{b_2} - \frac{q_{b_2\bullet}}{q_{c_2\bullet}} q_{c_2b_2}\chi_{b_2}\chi_{c_2} = 0\ , \tag{4.7.9}$$

$$\chi^{c_1}{}_{c_2}\chi^{\bullet}{}_{\bullet} - \chi^{\bullet}{}_{\bullet}\chi^{c_1}{}_{c_2} = 0 \quad , \quad \chi_{c_2}\chi^{\bullet}{}_{\bullet} - \chi^{\bullet}{}_{\bullet}\chi_{c_2} = -\chi_{c_2} \tag{4.7.10}$$

where we have defined $\chi_a \equiv \chi^{\bullet}{}_a$. The exterior differential reads, $\forall a \in ISO_{q,r=1}(N)$

$$da = \sum_{a'<b} (\chi^a{}_b * a)\Omega_a{}^b + (\chi^{\bullet}{}_b * a)\Omega_{\bullet}{}^b + (\chi^{\bullet}{}_{\bullet} * a)\Omega_{\bullet}{}^{\bullet} \tag{4.7.11}$$

where $\Omega_a{}^b$, $\Omega_{\bullet}{}^b$, and $\Omega_{\bullet}{}^{\bullet}$ are the 1-forms dual to the tangent vectors (4.7.6) and (4.7.7). As discussed in [69], these 1-forms can be seen as the projection of the $S_{q,r=1}(N+2)$ 1-forms : $P(\Omega_A{}^B) = -q_{AB} P[\kappa(T^B{}_C)] dP(T^C{}_A)$.

The adjoint representation, defined by (2.3.18): $ad_\psi = M_i{}^j(\psi)\chi_j$, is given by the elements $P(M\_{}^C{}_{DA}{}^B) \in ISO_{q,r=1}(N)$ with $C' \leq D,\ A' \leq B$ obtained by projecting with P those of $SO_{q,r=1}(N+2)$.
Proof : In $SO_{q,r}(N+2)$ we have $ad_\psi \chi^C{}_D = M\_{}^C{}_{DA}{}^B(\psi)\chi^C{}_D$ with $C' \leq D,\ A' \leq B$, since $M\_{}^C{}_{DA}{}^B$ is the adjoint representation of the $SO_{q,r}(N+2)$ calculus (see Note 4.6.2). Now $M\_{}^C{}_{DA}{}^B(\psi) = \psi(M\_{}^C{}_{DA}{}^B) = \langle \psi\, ,\, P(M\_{}^C{}_{DA}{}^B)\rangle$ where the last bracket is the duality bracket between $ISO_{q,r}(N)$ and $U_{q,r}(iso(N))$ [cf. (4.4.15)]. We then obtain:

$$ad_\psi \chi^C{}_D = \langle \psi\, ,\, P(M\_{}^C{}_{DA}{}^B)\rangle\, \chi^C{}_D \quad \text{with } C' \leq D,\ A' \leq B\ ,$$

this is the defining formula for the adjoint representation associated to the quantum Lie algebra T'. The nonvanishing elements are:

$$\begin{aligned}
&P(M_{}^{b_1}{}_{b_2 a_1}{}^{a_2}) = T^{b_1}{}_{a_1}\kappa(T^{a_2}{}_{b_2}) - q_{b_2 b_1} T^{b_2'}{}_{a_1}\kappa(T^{a_2}{}_{b_1'})\\
&P(M_{}^{b_1}{}_{b_2 \bullet}{}^{a_2}) = x^{b_1}\kappa(T^{a_2}{}_{b_2}) - q_{b_2 b_1} x^{b_2'}\kappa(T^{a_2}{}_{b_1'})\\
&P(M_{}^{\bullet}{}_{b_2 \bullet}{}^{a_2}) = v\kappa(T^{a_2}{}_{b_2})\\
&P(M_{}^{\bullet}{}_{\bullet\bullet}{}^{a_2}) = v\kappa(x^{a_2})\\
&P(M_{}^{\bullet}{}_{\bullet\bullet}{}^{\bullet}) = I
\end{aligned} \tag{4.7.12}$$

We will later use the relation between left invariant and right invariant vectorfields; in our case (2.3.23) reads:

$$(\chi^{A_1}{}_{A_2} * b)\, P(M\_{}^{B_1}{}_{B_2 A_1}{}^{A_2}) = b * \chi^{B_1}{}_{B_2} \quad \text{with } A_1' < A_2,\ B_1' < B_2\ . \tag{4.7.13}$$

The $ISp_{q,r=1}(N)$ differential calculus has the same structure as the $ISO_{q,r=1}(N)$ one, provided one considers $a' \leq b$ in (4.7.6) and (4.7.11), and includes the extra generator $\chi^{\bullet}{}_{\circ}$ in (4.7.7) and his dual form $\Omega_{\circ}{}^{\bullet}$ in the definition of the exterior differential. The adjoint representation is obtained by projecting with P the adjoint representation of the $Sp_{q,r}(N+2)$ differential calculus.

We now show that it is possible to exclude the generator $\chi^{\bullet}{}_{\bullet}$ (and $\chi^{\circ}{}_{\circ}$) and obtain a dilatation-free bicovariant differential calculus on $ISO_{q,r=1}(N)$.

We study the $ISO_{q,r=1}(N)$ subspace g linearly spanned by the functionals $\chi^a{}_b$, χ_b:

$$g \equiv span\{\chi^a{}_b\, ,\ \chi_b\}\ . \tag{4.7.14}$$

The space g is our candidate q-Lie algebra. A basis of g is $\{\chi_\alpha\} = \{\chi^a{}_b(a' < b), \chi^\bullet{}_b\}$. In the sequel greek letters will denote adjoint indices $\alpha = (a_1, a_2)$ with $a_1' < a_2$, and $\alpha = (\bullet, a_2)$. The coproduct on the elements χ_α reads $\Delta' \chi_\alpha = \chi_\alpha \otimes f^\alpha{}_\alpha + \varepsilon \otimes \chi_\alpha$; this shows that g satisfies condition (2.3.4). We also have $\Delta' f^\alpha{}_\alpha = f^\alpha{}_\alpha \otimes f^\alpha{}_\alpha$. To prove that g defines a bicovariant differential calculus we can proceed as in Section 3.5. We here give an alternative proof based on the results of Section 2.3. Recalling Theorem 2.3.1, g defines a bicovariant differential calculus if there exists a set of elements $M_i{}^j \in ISO_{q,r=1}(N)$ that satysfy (2.3.4) and(2.3.23): $(\chi_j * b) M_i{}^j = b * \chi_i$. It is immediate to verify that the subset of (4.7.12) given by

$$
\begin{aligned}
&P(M\_{}^{b_1}{}_{b_2 a_1}{}^{a_2}) = T^{b_1}{}_{a_1} \kappa(T^{a_2}{}_{b_2}) - q_{b_2 b_1} T^{b_2'}{}_{a_1} \kappa(T^{a_2}{}_{b_1'}) \\
&P(M\_{}^{b_1}{}_{b_2 \bullet}{}^{a_2}) = x^{b_1} \kappa(T^{a_2}{}_{b_2}) - q_{b_2 b_1} x^{b_2'} \kappa(T^{a_2}{}_{b_1'}) \\
&P(M\_{}^{\bullet}{}_{b_2 \bullet}{}^{a_2}) = v \kappa(T^{a_2}{}_{b_2})
\end{aligned}
\tag{4.7.15}
$$

satisfies

$$
(\chi_\beta * a) M_\alpha{}^\beta = a * \chi_\alpha \tag{4.7.16}
$$

indeed $P(M\_{}^{b_1}{}_{b_2 \bullet}{}^{\bullet}) = P(M\_{}^{\bullet}{}_{b_2 \bullet}{}^{\bullet}) = 0$ and therefore (4.7.13) closes also on the subset of χ and M_- with greek indices. We have therefore shown:

Theorem 4.7.1 g is a quantum Lie algebra and defines a bicovariant differential calculus on $ISO_{q,r=1}(N)$ that has the same dimension as in the commutative case. □□□

We now analize this differential calculus. The exterior derivative is

$$
da = (\chi_\alpha * a) \Omega^\alpha \tag{4.7.17}
$$

The left $ISO_{q,r=1}(N)$–module Γ freely generated by the 1-forms Ω^α dual to the tangent vectors χ_α is a bicovariant bimodule over $ISO_{q,r=1}(N)$ with the right multiplication (no sum on repeated indices):

$$
\Omega^\alpha a = (f^\alpha{}_\alpha * a) \Omega^\alpha , \qquad a \in ISO_{q,r=1}(N) \tag{4.7.18}
$$

and with the left and right actions of $ISO_{q,r=1}(N)$ on Γ given by:

$$
\Delta_L(a_\alpha \Omega^\alpha) \equiv \Delta(a_\alpha) I \otimes \Omega^\alpha \tag{4.7.19}
$$

$$
\Delta_R(a_\alpha \Omega^\alpha) \equiv \Delta(a_\alpha) \Omega^\beta \otimes P(M\_{}_\beta{}^\alpha) \; . \tag{4.7.20}
$$

Using the general formula (4.7.18) we can deduce the Ω, T commutations:

$$
\Omega_{a_1}{}^{a_2} T^r{}_s = \frac{q_{a_2 s}}{q_{a_1 s}} T^r{}_s \Omega_{a_1}{}^{a_2} \tag{4.7.21}
$$

$$
\Omega_{a_1}{}^{a_2} x^r = \frac{q_{a_2 \bullet}}{q_{a_1 \bullet}} x^r \Omega_{a_1}{}^{a_2} \tag{4.7.22}
$$

$$\Omega_{a_1}^{\ a_2} u = \frac{q_{a_1\bullet}}{q_{a_2\bullet}} u \ \Omega_{a_1}^{\ a_2} \tag{4.7.23}$$

$$\Omega_{\bullet}^{\ a_2} T^r{}_s = q_{s\bullet} q_{a_2 s} T^r{}_s \Omega_{\bullet}^{\ a_2} \tag{4.7.24}$$

$$\Omega_{\bullet}^{\ a_2} x^r = q_{a_2\bullet} x^r \Omega_{\bullet}^{\ a_2} \tag{4.7.25}$$

$$\Omega_{\bullet}^{\ a_2} u = \frac{1}{q_{a_2\bullet}} u \ \Omega_{\bullet}^{\ a_2} \tag{4.7.26}$$

Note 4.7.1 u commutes with all Ω 's only if $q_{a\bullet} = 1$ (cf. Note 4.2.2). This means that $u = I$ is consistent with the differential calculus on $ISO_{q_{ab}, r=1, q_{a\bullet}=1}(N)$.

The exterior derivative on the generators $T^A{}_B$ is given by:

$$\begin{aligned} dT^a{}_b &= -\sum_c T^a{}_c q_{cb} \Omega_b^{\ c} \\ dx^a &= -\sum_c T^a{}_c q_{c\bullet} V^c \\ du &= dv = 0 \end{aligned} \tag{4.7.27}$$

where we have defined $V^a \equiv \Omega_{\bullet}^{\ a}$. Again, for $q_{a\bullet} = 1$, $u = v = I$ is a consistent choice.

Inverting (4.7.27) yields:

$$\Omega_a^{\ b} = -q_{ab} \kappa(T^b{}_c) \ dT^c{}_a \tag{4.7.28}$$

$$V^b = -\frac{1}{q_{b\bullet}} \kappa(T^b{}_c) \ dx^c \tag{4.7.29}$$

The exterior product of the left-invariant 1-forms is defined as

$$\Omega^\alpha \wedge \Omega^\beta \equiv \Omega^\alpha \otimes \Omega^\beta - \Lambda^{\alpha\beta}{}_{\gamma\delta} \Omega^\gamma \otimes \Omega^\delta \tag{4.7.30}$$

where

$$\Lambda^{\alpha\beta}{}_{\gamma\delta} \equiv \langle f^\alpha{}_\delta \,, \, P(M_{-\gamma}^{\ \ \beta}) \rangle = f^\alpha{}_\delta (M_{-\gamma}^{\ \ \beta}) \tag{4.7.31}$$

[cf. (4.4.15)]; so that this Λ tensor is obtained from the one of $SO_{q,r=1}(N+2)$ by restricting its indices to the subset $ab, \bullet b$. We therefore just specialize the indices in equation (4.6.64) to deduce the q-commutations for the 1-forms Ω and V:

$$\Omega_{a_1}^{\ a_2} \wedge \Omega_{d_1}^{\ d_2} = -q_{a_1 d_2} q_{d_1 a_1} q_{a_2 d_1} q_{d_2 a_2} \Omega_{d_1}^{\ d_2} \wedge \Omega_{a_1}^{\ a_2} \tag{4.7.32}$$

$$\Omega_{a_1}^{\ a_2} \wedge V^{d_2} = -\frac{q_{a_2\bullet}}{q_{a_1\bullet}} q_{a_1 d_2} q_{d_2 a_2} V^{d_2} \wedge \Omega_{a_1}^{\ a_2} \tag{4.7.33}$$

$$V^{a_2} \wedge V^{d_2} = -\frac{q_{a_2\bullet}}{q_{d_2\bullet}} q_{d_2 a_2} V^{d_2} \wedge V^{a_2} \tag{4.7.34}$$

The Cartan-Maurer equations

$$d\Omega^{\alpha} = -\frac{1}{2} C_{\beta\gamma}{}^{\alpha} \Omega^{\beta} \wedge \Omega^{\gamma} \tag{4.7.35}$$

can be explicitly written for the Ω and V by differentiating eq.s (4.7.28) and (4.7.29) [or again secializing the indices in (4.6.63)]:

$$d\Omega_a{}^b = q_{ab} q_{bc} q_{ca} \; \Omega_c{}^b \wedge \Omega_a{}^c \tag{4.7.36}$$

$$dV^b = \frac{q_{a\bullet}}{q_{b\bullet}} q_{ba} \; \Omega_a{}^b \wedge V^a \tag{4.7.37}$$

where the 1-forms $\Omega_a{}^b$ with $a' > b$ are given by $\Omega_a{}^b = -q_{ab} \Omega_{b'}{}^{a'}$; i.e. we consider (as it is usually done in the classical limit), the 1-forms $\Omega_a{}^b$ to be "q-antisymmetric" $\Omega_a{}^b = -q_{ab} \Omega_{b'}{}^{a'}$, cf. eq. (4.6.45).

The $*$-conjugation on the χ functionals and on the 1-forms Ω can be deduced from (4.6.65):

$$(\chi^a{}_b)^* = -q_{cd} \mathcal{D}^a{}_c \chi^c{}_d \mathcal{D}^d{}_b, \quad (\chi_b)^* = -(q_{d\bullet})^{-1} \chi_d \mathcal{D}^d{}_b = -\overline{q_{b\bullet}} \chi_d \mathcal{D}^d{}_b \tag{4.7.38}$$

$$(\Omega_a{}^b)^* = q_{ba} \mathcal{D}^c{}_a \Omega_c{}^d \mathcal{D}^b{}_d, \quad (V^b)^* = q_{b\bullet} V^d \mathcal{D}^b{}_d \tag{4.7.39}$$

Note 4.7.2 As discussed at the end of Section 4.2, a q-Poincaré group without dilatations (i.e. $u = I$) has only one free real parameter q_{12}, which is the real parameter related to the q-Lorentz subalgebra. Then the formulas of this section can be specialized to describe a bicovariant calculus on the dilatation-free $ISO_{q,r=1}(3,1)$ provided $q_{a\bullet} = 1$ and $q_{12} \in \mathbf{R}$. It is however possible to have a bicovariant calculus without the dilatation generator $\chi^{\bullet}{}_{\bullet}$ even on $ISO_{q,r=1}(3,1)$ with $u \neq I$. The q-Poincaré algebra presented in [14] corresponds to the case $q \equiv q_{1\bullet}$, $q_{2\bullet} = q_{12} = 1$, for which the Lorentz subalgebra is undeformed and the q-Poincaré group contains $u \neq I$. The possibility of having a dilatation-free q-Lie algebra describing a bicovariant calculus on a q-group containing dilatations u was already observed in the case of IGL q-groups (see Section 3.5).

Note 4.7.3 We here study a differential calculus on $ISp_{q,r=1}(N)$ that has the same number of tangent vectors as in the classical case. Following the same arguments given after (4.7.14) we have an $ISp_{q,r=1}(N)$ differential calculus with quantum Lie algebra generators $\chi^a{}_b$ with $a' \leq b$, χ_b and $\chi^{\bullet}{}_{\circ}$. To further restrict the quantum Lie algebra to the one spanned by the basis $\{\chi^a{}_b \, (a' \leq b) \, , \chi_b\}$, observe that from (4.6.19),

$$\chi^{\bullet}{}_{\circ} = \lim_{r \to 1} \frac{1}{\lambda} \; \kappa'(L^{+\bullet}{}_{\bullet}) L^{-\bullet}{}_{\circ} \tag{4.7.40}$$

is different from zero only on monomials that contain the element z. However in the $q, r \to 1$ limit, as noticed after (4.2.30), z can be set to zero since there is no more any constraint between z and the generators $T^a{}_b, x^a, u, v = u^{-1}$. Then $\chi^{\bullet}{}_{\circ}$ is zero as well and we have an $[N(N+1)/2+N]$-dimensional bicovariant differential calculus on the twisted inhomogeneous symplectic group generated by $T^a{}_b, x^a, u, v = u^{-1}$. The adjoint representation is given by the elements $P(M_{\alpha}{}^{\beta}) \in ISp_{q,r}(N)$ obtained by projecting with P those of $Sp_{q,r=1}(N+2)$. The explicit formulae carachterizing this differential calculus are as in (4.7.17)–(4.7.35), where now greek letters denote adjoint indices $\alpha = (a_1, a_2)$ with $a_1' \le a_2$, and $\alpha = (\bullet, a_2)$.

Chapter 5

Geometry of the quantum orthogonal plane

We present here a bicovariant calculus on the full multiparametric $ISO_{q,r}(N)$ *without the restriction* $r = 1$. This calculus, however, is trivial on the $SO_{q,r}(N)$ quantum subgroup: it can really be seen as a non-trivial calculus only on the coset $Fun_{q,r}[ISO(N)/SO(N)]$, i.e. on the quantum orthogonal plane. We therefore call this calculus on the quantum plane $ISO_{q,r}(N)$-bicovariant. We find that in the $r \neq 1$ case this $ISO_{q,r}(N)$-bicovariant calculus necessarily contains dilatations.

If we break $ISO_{q,r}(N)$ bicovariance and require right covariance under $ISO_{q,r}(N)$ and left covariance only under $SO_{q,r}(N)$, i.e. compatibility of the exterior differential on the quantum plane with the right $ISO_{q,r}(N)$-coaction and the left $SO_{q,r}(N)$-coaction, the calculus can be expressed in terms of coordinates x, differentials dx and partial derivatives ∂, *without the need of dilatations.* In this case q-commutations between x, dx and ∂ close by themselves, and in fact generalize to the multiparametric case the known results of ref.s [51, 53, 54]. Here these results emerge from the broader setting of the bicovariant calculus on $ISO_{q,r}(N)$.

The two $*$-conjugations of the previous sections, consistent with the q-group structure, lead to a $ISO_{q,r}(n+1, n-1)$, and a $ISO_{q,r}(n,n)$ or $ISO_{q,r}(n, n+1)$ bicovariant calculus on the quantum orthogonal plane respectively with $(n+1, n-1)$, (n, n) or $(n, n+1)$ signature. We will be concerned with the conjugation that gives the $ISO_{q,r}(n-1, n+1)$ calculus. [To retrieve the other conjugations, both for N=even and N=odd, just take $\mathcal{D}^A{}_B = \delta^A_B$ in the formulae where $\mathcal{D}^A{}_B$ appears].

Using this conjugation one can define real coordinates X and hermitian partial derivative operators P i.e. momenta. This is achieved by a canonical procedure, using the compatibility of the $*$-structure with the bicovariant calculus on $ISO_{q,r}(N)$, i.e. the property that $*$ is a linear operation on the q-Lie algebra. The q-commutations of the momenta P with the coordinates X define a deformed version of the Heisenberg X, P commutation relations (with no extra operator as in ref.s [9]). In the same spirit as in ref.s [9] it would be interesting to investigate the Hilbert space representations of this deformed phase-space algebra.

In Section 5.1 we present the $ISO_{q,r}(N)$ bicovariant differential calculus with $r \neq 1$, then, in Section 5.2 we restrict this calculus to the quantum orthogonal plane. We find that in order to obtain a space of 1-forms that has the same dimension as in classical case we have to break $ISO_{q,r}(N)$-bicovariance. This naturally leads to a right $ISO_{q,r}(N)$-covariant and $SO_{q,r}(N)$-bicovariant calculus. The commutation relations caracterizing this calculus are explicitly given in the tables at the end of the chapter.

5.1 Bicovariant calculus on $ISO_{q,r}(N)$ with $r \neq 1$

In this section we study, with projection techniques, a differential calculus on $ISO_{q,r}(N)$ with $r \neq 1$, a similar calculus exists also for $ISp_{q,r}(N)$; for physical reasons we here treat in detail the orthogonal case.

As discussed at the beginning of Section 4.6, in the $r \neq 1$ case, the quantum tangent space $T' \equiv T \cup U_{q,r}(is(N))$ contains dilatations and translations, but does not contain the tangent vectors of $S_{q,r}(N)$, i.e. the functionals $\chi^a{}_b$. However T' defines a bicovariant differential calculus on $ISO_{q,r}(N)$ or $ISp_{q,r}(N)$ because conditions (2.3.4) and (2.3.18) are satisfied. The proof is as in (4.7.1) and (4.7.3).

The q-Lie algebra in the orthogonal case is explicitly given by

$$\chi^\bullet{}_\circ \chi^\bullet{}_b - (q_{\bullet b})^{-2} \chi^\bullet{}_b \chi^\bullet{}_\circ = 0 \tag{5.1.1}$$

$$\chi^\bullet{}_c \chi^\bullet{}_\bullet - r^{-2} \chi^\bullet{}_\bullet \chi^\bullet{}_c = -r^{-1} \chi^\bullet{}_c \tag{5.1.2}$$

$$\chi^\bullet{}_\circ \chi^\bullet{}_\bullet - r^{-4} \chi^\bullet{}_\bullet \chi^\bullet{}_\circ = \frac{-(1+r^2)}{r^3} \chi^\bullet{}_\circ \tag{5.1.3}$$

$$q_{\bullet a} P_A^{ab}{}_{cd} \chi^\bullet{}_b \chi^\bullet{}_a = 0 \tag{5.1.4}$$

Relation (5.1.4) is equivalent to $q_{b\bullet} P_A^{ab}{}_{cd} \chi^\bullet{}_b \chi^\bullet{}_a = 0$ and $q_{a\bullet} [(P_A)_{q^{-1},r^{-1}}]^{ab}_{cd} \chi^\bullet{}_a \chi^\bullet{}_b = 0$. A combination of (5.1.1)-(5.1.4) yields:

$$\chi^\bullet{}_\circ + \lambda \chi^\bullet{}_\circ \chi^\bullet{}_\bullet = \lambda \frac{-r^{\frac{N}{2}}}{r^2 + r^N} \frac{1}{q_{d\bullet}} \chi^\bullet{}_b C^{db} \chi^\bullet{}_d \tag{5.1.5}$$

Notice the similar structure of eq.s (4.2.23), (4.3.23) and (5.1.5).

Following the same arguments as in (4.7.12), the adjoint representation is given by the elements

$$P(M^\bullet{}_{B\bullet}{}^D) = P(T^\bullet{}_\bullet \kappa_{N+2}(T^D{}_B)) = v P(\kappa_{N+2}(T^D{}_B)) \tag{5.1.6}$$

that explicitly read

$$\begin{array}{lll} P(M^\bullet{}_{\circ\bullet}{}^\circ) = v^2 & P(M^\bullet{}_{\circ\bullet}{}^d) = 0 & P(M^\bullet{}_{\circ\bullet}{}^\bullet) = 0 \\ P(M^\bullet{}_{b\bullet}{}^\circ) = v r^{-\frac{N}{2}} x^e C_{eb} & P(M^\bullet{}_{b\bullet}{}^d) = v\kappa(T^d{}_b) & P(M^\bullet{}_{b\bullet}{}^\bullet) = 0 \\ P(M^\bullet{}_{\bullet\bullet}{}^\circ) = -\frac{1}{r^N(r^{\frac{N}{2}} + r^{-\frac{N}{2}+2})} x^e C_{ef} x^f & P(M^\bullet{}_{\bullet\bullet}{}^d) = v\kappa(x^d) & P(M^\bullet{}_{\bullet\bullet}{}^\bullet) = I \end{array} \tag{5.1.7}$$

The differential related to this calculus is given by

$$\forall a \in ISO_{q,r}(N) \qquad da = (\chi^{\bullet}{}_{b} * a)\omega_{\bullet}{}^{b} + (\chi^{\bullet}{}_{\bullet} * a)\omega_{\bullet}{}^{\bullet} + (\chi^{\bullet}{}_{\circ} * a)\omega_{\bullet}{}^{\circ} \tag{5.1.8}$$

where $\omega_{\bullet}{}^{b}$, $\omega_{\bullet}{}^{\bullet}$ and $\omega_{\bullet}{}^{\circ}$ are the 1-forms dual to the tangent vectors $\chi^{\bullet}{}_{b}$, $\chi^{\bullet}{}_{\circ}$ and $\chi^{\bullet}{}_{\bullet}$. The left and right actions $\Delta_L : \Gamma \to ISO_{q,r}(N) \otimes \Gamma$ and $\Delta_R : \Gamma \to \Gamma \otimes ISO_{q,r}(N)$ are defined by:

$$\Delta_L(\omega_{\bullet}{}^{A}) = I \otimes \omega_{\bullet}{}^{A} \quad , \qquad \Delta_R(\omega_{\bullet}{}^{A}) = \omega_{\bullet}{}^{B} \otimes P(M^{\bullet}{}_{B\bullet}{}^{A}) \tag{5.1.9}$$

We now explicitly give the relation characterizing this differential calculus. These formulae will be needed in the next chapter.

To simplify notations, we write the composite indices as follows:

$${}_{\bullet}{}^{a} \to {}^{a}, \; {}_{\bullet}{}^{\bullet} \to {}^{\bullet}, \; {}_{\bullet}{}^{\circ} \to {}^{\circ}; \quad {}^{\bullet}{}_{a} \to {}_{a}, \; {}^{\bullet}{}_{\bullet} \to {}_{\bullet}, \; {}^{\bullet}{}_{\circ} \to {}_{\circ} \tag{5.1.10}$$

Similarly we'll write q_b instead of $q_{b\bullet}$. The explicit expression for the tangent vectors then reads:

$$\begin{aligned} \chi_b &= \frac{1}{r - r^{-1}} f^{\bullet}{}_{b} \\ \chi_\circ &= \frac{1}{r - r^{-1}} f^{\bullet}{}_{\circ} \\ \chi_\bullet &= \frac{1}{r - r^{-1}} [f^{\bullet}{}_{\bullet} - \varepsilon] \end{aligned} \tag{5.1.11}$$

and their coproduct is given by

$$\Delta(\chi_b) = \chi_\bullet \otimes f^{\bullet}{}_{b} + \chi_c \otimes f^{c}{}_{b} + \varepsilon \otimes \chi_b \tag{5.1.12}$$

$$\Delta(\chi_\bullet) = \chi_\bullet \otimes f^{\bullet}{}_{\bullet} + \varepsilon \otimes \chi_\bullet \tag{5.1.13}$$

$$\Delta(\chi_\circ) = \chi_\circ \otimes f^{\circ}{}_{\circ} + \chi_\bullet \otimes f^{\bullet}{}_{\circ} + \chi_c \otimes f^{c}{}_{\circ} + \varepsilon \otimes \chi_\circ \tag{5.1.14}$$

Using the general formula (4.7.18) we can deduce the ω, T commutations for $ISO_{q,r}(N)$:

$$\omega^b T^c{}_d = \frac{q_f}{r} (R^{-1})^{bf}{}_{ed} T^c{}_f \omega^e \tag{5.1.15}$$

$$\omega^b x^c = \frac{q_b}{r^2} x^c \omega^b + \lambda r^{\frac{N}{2}-1} q_d C^{bd} T^c{}_d \omega^\circ \tag{5.1.16}$$

$$\omega^b u = \frac{r^2}{q_b} u \; \omega^b \tag{5.1.17}$$

$$\omega^b v = \frac{q_b}{r^2} v \; \omega^b \tag{5.1.18}$$

$$\omega^\bullet T^c{}_d = T^c{}_d \omega^\bullet \tag{5.1.19}$$

$$\omega^\bullet x^c = \frac{1}{r^2} x^c \omega^\bullet - \lambda \frac{q_b}{r} T^c{}_b \omega^b \tag{5.1.20}$$

$$\omega^\bullet u = r^2 u\omega^\bullet \quad (5.1.21)$$
$$\omega^\bullet v = r^{-2} v\omega^\bullet \quad (5.1.22)$$
$$\omega^\circ T^c{}_d = q_d^2 r^{-2} T^c{}_d \omega^\circ \quad (5.1.23)$$
$$\omega^\circ x^c = x^c \omega^\circ \quad (5.1.24)$$
$$\omega^\circ u = u\omega^\circ \quad (5.1.25)$$
$$\omega^\circ v = v\omega^\circ \quad (5.1.26)$$

The 1-form $\tau \equiv \omega^\bullet \equiv \omega_\bullet{}^\bullet$ is bi-invariant, and one can check that $\forall\, a \in A$, $da = \frac{1}{\lambda}[\tau a - a\tau]$. The exterior derivative on the generators of $ISO_{q,r}(N)$ reads:

$$dT^c{}_d = 0 \quad (5.1.27)$$
$$dx^c = -q_b r^{-1} T^c{}_b \omega^b - r^{-1} x^c \omega^\bullet \quad (5.1.28)$$
$$du = ru\omega^\bullet \quad (5.1.29)$$
$$dv = -r^{-1} v\omega^\bullet \quad (5.1.30)$$
$$dz = -q_b r^{-1} y_b \omega^b - r(1-r^N)u\omega^\circ - r^{-1} z\omega^\bullet \quad (5.1.31)$$

where we have included the exterior derivative on z for convenience. Note that the calculus is trivial on the $SO_{q,r}(N)$ subgroup of $ISO_{q,r}(N)$, as is evident from (5.1.27). Thus effectively we are discussing a bicovariant calculus on the orthogonal q-plane generated by the coordinates x^a and the "dilatations" u, v.

Every element ρ of Γ can be written as $\rho = \sum_k a_k db_k$ for some a_k, b_k belonging to $ISO_{q,r}(N)$. Indeed inverting the relations (5.1.28)-(5.1.31) yields:

$$\omega^a = -\frac{r}{q_a}\kappa(T^a{}_c)[dx^c - x^c u dv] = \frac{r}{q_a}[d\kappa(x^a)]v = r^{-1} v d\kappa(x^a) \quad (5.1.32)$$
$$\omega^\bullet = -rudv = r^{-1} v du \quad (5.1.33)$$
$$\omega^\circ = -\frac{vdz + r^{-N} z dv + r^{-\frac{N}{2}} C_{ab} x^a dx^b}{r(1-r^N)} \quad (5.1.34)$$

The exterior product of left-invariant 1-forms is as usual defined by

$$\omega^i \wedge \omega^j \equiv \omega^i \otimes \omega^j - \Lambda^{ij}{}_{kl} \omega^k \otimes \omega^l \quad (5.1.35)$$

where

$$\Lambda^{ij}{}_{kl} = f^i{}_l(M_k{}^j) \quad (5.1.36)$$

As in (4.7.31) this Λ tensor is obtained from the one of $SO_{q,r}(N+2)$ by restricting its indices to the subset $\bullet b, \bullet\bullet, \bullet\circ$. The non-vanishing components of Λ read:

$$\begin{array}{lll}
\Lambda^{ad}{}_{cb} = \frac{q_a}{q_c} r^{-1} R^{ad}{}_{bc} & \Lambda^{\bullet\circ}{}_{cb} = -\frac{r^{-\frac{N}{2}-1}}{q_c}\lambda C_{bc} & \Lambda^{\bullet d}{}_{c\bullet} = r^{-2}\delta^d_c \\
\Lambda^{a\circ}{}_{c\circ} = r^{-1}\lambda\delta^a_c & \Lambda^{\circ d}{}_{c\circ} = (\frac{r}{q_c})^2\delta^d_c & \Lambda^{a\bullet}{}_{\bullet b} = \delta^a_b \\
\Lambda^{\bullet d}{}_{\bullet b} = r^{-1}\lambda\delta^d_b & \Lambda^{a\circ}{}_{\circ b} = r^{-4}(q_a)^2\delta^a_b & \Lambda^{\bullet\bullet}{}_{\bullet\bullet} = 1 \\
\Lambda^{ad}{}_{\bullet\circ} = -q_a r^{-\frac{N}{2}-1}\lambda C^{da} & \Lambda^{\bullet\circ}{}_{\bullet\circ} = \lambda r^{-1}(1-r^{-N}) & \Lambda^{\circ\bullet}{}_{\bullet\circ} = 1 \\
\Lambda^{\bullet\circ}{}_{\circ\bullet} = r^{-4} & \Lambda^{\circ\circ}{}_{\circ\circ} = 1 &
\end{array}$$

From (5.1.35) it is not difficult to deduce the commutations between the ω's:

$$\frac{1}{q_c} P_S^{ab}{}_{cd}\, \omega^d \wedge \omega^c = 0 \tag{5.1.37}$$

$$\omega^a \wedge \omega^\bullet = -r^2 \omega^\bullet \wedge \omega^a \tag{5.1.38}$$

$$\omega^a \wedge \omega^\circ = -r^{-4}(q_a)^2 \omega^\circ \wedge \omega^a \tag{5.1.39}$$

$$\omega^\bullet \wedge \omega^\bullet = \omega^\circ \wedge \omega^\circ = 0 \tag{5.1.40}$$

$$\omega^\bullet \wedge \omega^\circ = -r^{-4} \omega^\circ \wedge \omega^\bullet + \frac{\lambda r^{-\frac{N}{2}-1}}{q_a(1-r^{-N})} C_{ba}\, \omega^a \wedge \omega^b \tag{5.1.41}$$

Notice that the dimension of the space of 2-forms generated by $\omega^a \wedge \omega^b$ is larger than in the commutative case since P_S project into an $N(N+1)/2-1$ (and not into an $N(N+1)/2$) dimensional space. This is not surprising since the exterior algebra of homogeneous orthogonal quantum groups is known to be larger than its classical counterpart.

The Cartan-Maurer equations

$$d\omega^i = \frac{1}{r - r^{-1}} (\tau \wedge \omega^i + \omega^i \wedge \tau) \tag{5.1.42}$$

can be explicitly found after use of the commutations (5.1.37)- (5.1.41):

$$d\omega^a = r^{-1} \omega^a \wedge \omega^\bullet \tag{5.1.43}$$

$$d\omega^\bullet = 0 \tag{5.1.44}$$

$$d\omega^\circ = -r(1+r^2) \omega^\bullet \wedge \omega^\circ + \frac{r^3}{r^{\frac{N}{2}} - r^{-\frac{N}{2}}} \frac{C_{ba}}{q_a} \omega^a \wedge \omega^b \tag{5.1.45}$$

Finally, the nonvanishing structure constants $\boldsymbol{C}$, given by $\boldsymbol{C}_{jk}{}^i = \chi_k(M_j{}^i)$, read:

$$\boldsymbol{C}_{ab}{}^\circ = -q_a^{-1} r^{-\frac{N}{2}-1} C_{ba} \quad \boldsymbol{C}_{a\bullet}{}^c = -r^{-1}\delta_a^c \quad \boldsymbol{C}_{\bullet b}{}^c = r^{-1}\delta_b^c$$
$$\boldsymbol{C}_{\circ\bullet}{}^\circ = -r^{-3}(1+r^2) \quad \boldsymbol{C}_{\bullet\circ}{}^\circ = r^{-1}(1-r^{-N})$$

These structure constants can be obtained from those of $SO_{q,r}(N+2)$ by specializing indices, for the same reason as for the Λ components.

The $*$-conjugation on the χ functionals can be deduced from (4.5.27) [use $(q_f)^{-1}\mathcal{D}^f{}_b = \overline{q}_b \mathcal{D}^f{}_b$]

$$(\chi_b)^* = -r^{-N} \mathcal{D}^f{}_b \frac{1}{q_f} D^d_f \chi_d = -r^{-N} \overline{q}_b \mathcal{D}^f{}_b D^d_f \chi_d = -r^{-N} \overline{q}_b D^f_b \mathcal{D}^d{}_f \chi_d \tag{5.1.46}$$

$$(\chi_\bullet)^* = -\chi_\bullet \tag{5.1.47}$$

$$(\chi_\circ)^* = -r^{-2N-2} \chi_\circ \tag{5.1.48}$$

whereas the conjugation on the ω 1-forms can be deduced from (2.3.52) and (5.1.46)-(5.1.48) or directly from their expression in terms of dx, du, dv differentials (5.1.32)-(5.1.34) remembering that $(da)^* = d(a^*)$:

$$(\omega^a)^* = \overline{q}_a{}^{-1} r^N (D^{-1})^a{}_b \mathcal{D}^b{}_c = \overline{q}_a{}^{-1} r^N \mathcal{D}^a{}_b (D^{-1})^b{}_c \omega^c \tag{5.1.49}$$

$$(\omega^\bullet)^* = \omega^\bullet \tag{5.1.50}$$

$$(\omega^\circ)^* = r^{2N+2}\omega^\circ \tag{5.1.51}$$

5.2 Calculus on the multiparametric orthogonal quantum plane

In this section we concentrate on the orthogonal quantum plane

$$M \equiv Fun_{q,r}\left(\frac{ISO(N)}{SO(N)}\right) , \tag{5.2.52}$$

i.e. the $ISO_{q,r}(N)$ subalgebra generated by the coordinates x^a and the dilatations u, v. This is the algebra we called B in the study of the cross-product cross-coproduct construction $ISO_{q,r}(N) \cong B \rtimes SO_{q,r}(N)$ of Section 4.2.

We study the action of the exterior differential d on M and the corresponding space Γ_M of 1-forms. Γ_M is the sub-bimodule of Γ formed by all the elements adb or $(da')b'$ where a, b, a', b' are polynomials in x^a, u and v [of course $adb = d(ab) - (da)b$].

We will see that a generic element ρ of Γ_M cannot be generated, as a left module, only by the differentials dx, dv, i.e. it cannot be written as $\rho = a_i dx^i + adv$. We need also to introduce the differential dz (or equivalently $dL \equiv d(x^a C_{ab} x^b)$). Thus the basis of differentials is given by dx^a, dv, dz and corresponds to the intrinsic basis of *independent* 1-forms $\omega^a, \omega^\bullet$ and ω°. Note that du can be expressed in terms of dv since $du = -u(dv)u = -r^2u^2dv = -r^{-2}(dv)u^2$ [see (5.3.124) below].

In Subsection 5.2.1 we consistently impose an extra conditon in order to relate dz to dx and dv. This is done in two different ways: checking explicilty the consistency of the extra condition as in [48] [53] and also deriving it using $ISO_{q,r}(N)$ symmetry principles.

Commutations

The commutations between the coordinates x^a, u and v have been given in Section 4.2. The commutations between coordinates and differentials are found by expressing the differentials in terms of the 1-forms ω as in (5.1.28)-(5.1.31), and using then the x, u, v commutations with the ω's given in (5.1.15)-(5.1.26). The resulting q-commutations between x and dx are found to be:

$$(r^{-2}P_S - P_A)(x \otimes dx) = (P_S + P_A)(dx \otimes x) \tag{5.2.53}$$

where we have used the tensor notation $A^{ab}{}_{cd}x^c dx^d \equiv A(x\otimes dx)$ etc. The remaining commutations are given in formulae (5.3.121)–(5.3.132) in Table 1.

Let us consider the above formula, giving the x, dx commutations. If we multiply it by P_0 we find $0 = 0$. Thus from this equation we have no information on $P_0(x\otimes dx)$. Applying instead the projectors P_S and P_A yields

$$P_S(x\otimes dx) = r^2 P_S(dx\otimes x)\;;\quad P_A(x\otimes dx) = -P_A(dx\otimes x) \tag{5.2.54}$$

which does not allow to express $x^a dx^b$ only in terms of linear combinations of $(dx)x$ since no linear combination of P_S and P_A is invertible. The space of 1-forms has therefore one more dimension than his classical analogue because we are missing a condition involving the one dimensional projector $P_0{}^{ab}{}_{ef} = (C^{lm}C_{\ lm})^{-1}C^{ab}C_{\ ef}$, see (4.1.16).

However, if we consider the 1-form $dL \equiv d(x^e C_{ef} x^f)$ – an exterior derivative of *polynomials* in the basic elements – we can write the commutations between the x and dx elements as follows:

$$\begin{aligned} dx\otimes x &= -(P_S+P_A+P_0)x\otimes dx + (P_S+P_A)d(x\otimes x) + P_0 d(x\otimes x)\\ &= P_S dx\otimes x + P_A dx\otimes x - P_0 x\otimes dx + P_0 d(x\otimes x)\\ &= (r^{-2}P_S - P_A - P_0)x\otimes dx + P_0 d(x\otimes x) \end{aligned} \tag{5.2.55}$$

where we have used the Leibniz rule, the commutations (5.2.54) and $P_S + P_A + P_0 = I$. Equivalently we have

$$dx\otimes x = (r^{-2}P_S - P_A - P_0)x\otimes dx - C\frac{r^{\frac{N}{2}-2}(1-r^2)}{1-r^N}(vdz + zdv) \tag{5.2.56}$$

involving the dv and dz differentials.

The presence of dz can also be explained within the general theory by recalling that Γ is a free right module [see paragraph following (2.1.47)]. A basis of right invariant 1-forms is given by (2.1.47): $\eta^A \equiv \kappa^{-1}(M_B{}^A)\omega^B$, we explicitly have:

$$\begin{aligned} \eta^a &= -r^{-1}dx^a\, u = -r^{-1}dT^a{}_\bullet\,\kappa(T^\bullet{}_\bullet) && (5.2.57)\\ \eta^\bullet &= -r^{-1}dv\, u = -r^{-1}dT^\bullet{}_\bullet\,\kappa(T^\bullet{}_\bullet) && (5.2.58)\\ \eta^\circ &= \frac{r^{\frac{N}{2}-1}}{(1-r^N)(1+r^{N-2})}[dx^e C_{ef}x^f - r^{N-2}x^e C_{ef}dx^f]u^2 && (5.2.59)\\ &= \frac{-r^{N-1}}{r^N-1}[dz\, u + dy_b\,\kappa(x^b) + du\,\kappa(z)] = \frac{-r^{N-1}}{r^N-1}dT^\circ{}_B\,\kappa(T^B{}_\bullet) && (5.2.60) \end{aligned}$$

To derive the expressions for η° use: $y_b = -r^{-\frac{N}{2}}ux^e C_{ef}T^f{}_b$; $dy_b\,\kappa(x^b) = r^{-\frac{N}{2}}du\, xxu + r^{-\frac{N}{2}}dx\, xu^2$; $dz = d(\frac{-r^{-\frac{N}{2}}}{1+r^{2-N}}uxx) = \frac{-r^{-\frac{N}{2}}}{1+r^{2-N}}(du\, xx + dx\, xu + xdx\, u)$; $\kappa(z) = \kappa(T^\circ{}_\bullet) = r^{-N}z$; $uxx = r^2 xxu$; $udx\, x = dx\, xu$, where $xx \equiv L \equiv x^e C_{ef}x^f$.

The 1-forms (5.2.58)-(5.2.59) in Γ do not contain any $T^a{}_b$ element and therefore belong to Γ_M as well; they are linearly independent and freely generate Γ_M as a right module because they freely generate the full Γ as a right module. The extra 1-form η° (or dz) is therefore a natural consequence of the right module structure of Γ.

In summary: either dL or dz or η° are necessary in order to close the commutation algebra between coordinates and differentials. Thus the commutations involving z and dz appear in Table 1.

We have seen that $dv\, u$; $dx^a\, u$ and η° freely generate Γ_M as a right module; recalling that Γ is also a free left module, we have the :

Proposition 5.1 The M-bimodule Γ_M, as a left module (or as a right module), is freely generated by the differentials dx, dL (or dz) and dv. *Proof:* to show that $a_i dx^i + a dL + a_\bullet dv = 0 \Rightarrow a_i = 0, a = 0, a_\bullet = 0$ express dx^i, dL, dv in terms of $\omega^a, \omega^\circ \omega^\bullet$, see (5.1.28)-(5.1.31), and recall that Γ is a free left module.

Note 5.2.1 From (5.3.121), (5.3.122) and the commutations of L with x and u we have $x^c dL = dL\, x^c$, $u dL = dL\, u$ and $v dL = dL\, v$. These relations and (5.3.120) show that inside Γ_M there is the smaller bimodule generated by the differentials dx^a and dL.

We now examine the space of 2-forms. By simply applying the exterior derivative d to the relations (5.3.120)-(5.3.132) we deduce the commutations between the differentials given in Table 1. As with the ω^a's in eq. (5.1.37), the relations in (5.3.133) are not sufficient to order the differentials dx^a.

$ISO_{q,r}(N)$ **- coactions**

All the relations we have been deriving have many symmetry properties because they are covariant, under the actions on M and Γ, of the full $ISO_{q,r}(N)$ q-group. In fact we have the following three $ISO_{q,r}(N)$ actions:

1) the coproduct of $ISO_{q,r}(N)$ can be seen as a left-coaction $\Delta : M \rightarrow ISO_{q,r}(N) \otimes M$:

$$\Delta x^a = T^a{}_b \otimes x^b + x^a \otimes v \ , \qquad \Delta(u) = u \otimes u \ , \qquad \Delta(v) = v \otimes v \tag{5.2.61}$$

2) the left coaction $\Delta_L : \Gamma \rightarrow ISO_{q,r}(N) \otimes \Gamma$, when restricted to Γ_M gives

$$\Delta_L|_{\Gamma_M} : \Gamma_M \rightarrow ISO_{q,r}(N) \otimes \Gamma_M \tag{5.2.62}$$

and defines a left coaction of $ISO_{q,r}(N)$ on Γ_M compatible with the bimodule structure of Γ_M and the exterior differential: $\Delta_L|_{\Gamma_M}(adb) = \Delta(a)(id \otimes d)\Delta(b)$.

3) the right coaction $\Delta_R : \Gamma \to \Gamma \otimes ISO_{q,r}(N)$, does not become a right coaction of $ISO_{q,r}(N)$ on Γ_M; however we have

$$\Delta_R|_{\Gamma_M} : \Gamma_M \to \Gamma \otimes M \subset \Gamma \otimes ISO_{q,r}(N) \tag{5.2.63}$$

this map is obviously well defined and satisfies $\Delta_R|_{\Gamma_M}(adb) = \Delta(a)(d \otimes id)\Delta(b)$ $\forall a, b \in M$ since $M \subset ISO_{q,r}(N)$.

We call this calculus $ISO_{q,r}(N)$-bicovariant because $\Delta_L|_{\Gamma_M}$ and $\Delta_R|_{\Gamma_M}$ are compatible with the bimodule structure of Γ_M and with the exterior differential.

5.2.1 $ISO_{q,r}(N)$-covariant and $SO_{q,r}(N)$-bicovariant calculus

Commutations

Since the $P_A^{ab}{}_{cd} x^c x^d = 0$ commutation relations allow for an ordering of the coordinates (moreover the Poincaré series of the polynomials on the quantum orthogonal plane is the same as the classical one), it is tempting to impose extra conditions on the differential algebra of the q-Minkowski plane, so that the space of 1-forms has the same dimension as in the classical case. We require that the commutation relations between x and dx close on the algebra generated by x and dx:

$$dx^a x^b = \alpha^{ab}{}_{ef} x^e dx^f \tag{5.2.64}$$

where α is an unknown matrix whose entries are complex numbers. Any matrix can be expanded as $\alpha = aP_S + bP_A + cP_0$ with $a, b, c = const.$ From (5.2.54) we have $\alpha = r^{-2}P_S - P_A + cP_0$; therefore condition (5.2.64) is equivalent to

$$P_0(dx \otimes x) = cP_0(x \otimes dx) \tag{5.2.65}$$

and supplements eq.s (5.2.54). Taking its exterior derivative leads to a supplementary condition on the dx, dx products (for $c \neq -1$):

$$P_0(dx \wedge dx) = 0 \ . \tag{5.2.66}$$

From (5.3.133) and (5.2.66) it follows that $dx \wedge dx = (P_S + P_A + P_0)(dx \wedge dx) = P_A(dx \wedge dx)$, or [see the definition of P_A in (4.1.16)] :

$$dx \wedge dx = -r\hat{R}\ dx \wedge dx \ . \tag{5.2.67}$$

which allows the ordering of dx, dx products.

Using (5.2.55), (5.2.65) and (4.1.15), we find

$$\begin{aligned}
dx \otimes x &= (r^{-2}P_S - P_A)(x \otimes dx) + P_0(dx \otimes x) && (5.2.68)\\
&= (r^{-2}P_S - P_A)(x \otimes dx) + cP_0(x \otimes dx) && (5.2.69)\\
&= (r^{-2}P_S - P_A + r^{N-2}P_0)(x \otimes dx) + (c - r^{N-2})P_0(x \otimes dx) && (5.2.70)\\
&= r^{-1}\hat{R}^{-1}(x \otimes dx) + (c - r^{N-2})P_0(x \otimes dx) \ . && (5.2.71)
\end{aligned}$$

The consistency of the commutation relations (5.2.67) and (5.2.71) with the associativity condition on the triple $dx^i\, dx^j\, x^k$ fixes $c = r^{N-2}$ i.e.:

$$P_0(dx \otimes x) = r^{N-2} P_0(x \otimes dx) \; ; \tag{5.2.72}$$

the x, dx commutations (5.2.71) then become:

$$x \otimes dx = r\hat{R}(dx \otimes x) \tag{5.2.73}$$

and reproduce (in the uniparametric case) the known x, dx commutations of the quantum orthogonal plane [54].

Coactions

This calculus is no more bicovariant under the $ISO_{q,r}(N)$ action,

$$x^a \longrightarrow T^a{}_b \otimes x^b \; + x^a \otimes v \; , \quad u \longrightarrow u \otimes u \; , \quad v \longrightarrow v \otimes v \tag{5.2.74}$$

but we are left with bicovariance under the $SO_{q,r}(N)$ action

$$x^a \longrightarrow T^a{}_b \otimes x^b \; . \tag{5.2.75}$$

In other words, $\delta_L : \Gamma'_M \to SO_{q,r} \otimes \Gamma'_M$ defined by $\delta_L(adb) = \delta(c)(id \otimes d)\delta(b)$ with $\delta(x^a) = T^a{}_b \otimes x^b$ is a left coaction of $SO_{q,r}(N)$ on the bimodule Γ'_M where Γ'_M is Γ_M with the extra condition (5.2.65) [cf. (5.2.62)]. Similarly, the map $\delta_R(adb) = \delta(a)(d \otimes id)\delta(b)$ is well defined [cf. (5.2.63)].

Left covariance under (5.2.74) is broken only by (5.2.65). Indeed, while relations (5.2.54) are left and right $ISO_{q,r}(N)$-covariant, the extra condition (5.2.65) is *not* left $ISO_{q,r}(N)$-covariant : $\Delta_L[P_0(dx \otimes x) - cP_0(x \otimes dx)] \neq 0, \forall c$. It is right $ISO_{q,r}(N)$-covariant, $\Delta_R[P_0(dx \otimes x) - cP_0(x \otimes dx)] = 0$, only for $c = r^{N-2}$, as can be seen using $T^b{}_d dx^a = d(T^b{}_d x^a) = \frac{r}{q_d} R^{ab}{}_{ef} dx^e\, T^f{}_d$ and (4.1.21). Therefore the choice $c = r^{N-2}$ preserves the right coaction Δ_R i.e. the right $ISO_{q,r}(N)$-covariance.

Note 5.2.2 We can reformulate the quotient procedure $\Gamma_M \to \Gamma'_M$ in a more abstact setting by considering that Γ_M is a subbimodule of the *bicovariant* bimodule Γ. In (5.2.59) we have expressed the $x^e C_{ef} dx^f \leftrightarrow dx^e C_{ef} x^f$ commutation via the right invariant 1-form η°. A condition on Γ (and therefore on Γ_M) that preserves *right $ISO_{q,r}(N)$ covariance*, i.e. compatible with Δ_R, is: η° linearly dependent from the remaining right invariant 1-forms: $dv\, u$ and $dx^a\, u$. It is easily seen that since η° is quadratic in the basis elements x^a the only possible linear condition is $\eta^\circ = 0$, and this gives exactly (5.2.72). The M-bimodule Γ'_M is therefore generated by the differentials dx^b and dv. Since left $ISO_{q,r}(N)$ covariance, contrary to right $ISO_{q,r}(N)$ covariance, is broken, the relations between the left invariants 1-forms is nonlinear. Explicitly we have

$$\omega^\circ = -\frac{q_a}{r^2} v y_a\, \omega^a + \frac{r^{\frac{N}{2}-2}}{r^N + r^2} C_{ab} x^a x^b \omega^\bullet \tag{5.2.76}$$

[express dz in terms of dx^i,dv in (5.1.34) and use the expansion of dx^b and dv on ω^a and $\omega^\bullet$ as given in (5.1.28),(5.1.30)].

Partial derivatives

The tangent vectors χ in (5.1.11) and the corresponding vector fields $\chi*$ have "flat" indices. To compare $\chi*$ with partial derivative operators with "curved" indices, we need to define the operators $\overleftarrow{\partial}$ such that

$$d\,a = \overleftarrow{\partial}_c(a)\;dx^c + \overleftarrow{\partial}_\bullet(a)dv \equiv \overleftarrow{\partial}_C(a)dx^C \tag{5.2.77}$$

$[C = (c, \bullet),\; dx^C = (dx^c, dv)]$. The action of $\overleftarrow{\partial}_C$ on the coordinates $x^C = (x^c, v)$ is given by

$$\overleftarrow{\partial}_C(x^A) = \delta^A_C I\;, \tag{5.2.78}$$

From the Leibniz rule $d(ab) = (da)b + a(db)$, using (5.2.77) and the fact that $dx^C = (dx^c, dv)$ is a basis for 1-forms, we find

$$\overleftarrow{\partial}_c(ax^b) = a\delta^b_c + \overleftarrow{\partial}_d(a)r^{-1}(\hat{R}^{-1})^{db}{}_{ec}x^e - (1-r^2)\overleftarrow{\partial}_\bullet\delta^b_c v \tag{5.2.79}$$

$$\overleftarrow{\partial}_\bullet(ax^b) = q_b^{-1}\overleftarrow{\partial}_\bullet(a)x^b \tag{5.2.80}$$

$$\overleftarrow{\partial}_c(av) = r^{-2}q_c\overleftarrow{\partial}_c(a)v \tag{5.2.81}$$

$$\overleftarrow{\partial}_\bullet(av) = r^{-2}\overleftarrow{\partial}_\bullet(a)v + a \tag{5.2.82}$$

Note the dilatation operator $\overleftarrow{\partial}_\bullet$ appearing on the right-hand side of (5.2.79).

From $d^2(a) = 0 = d(\overleftarrow{\partial}_C(a)dx^C) = \overleftarrow{\partial}_B(\overleftarrow{\partial}_C(a))dx^B \wedge dx^C$ and the q-commutations of the differentials (5.3.133)-(5.3.139) one finds the commutations between the "curved" partial derivatives:

$$(P_A)^{ab}{}_{cd}\overleftarrow{\partial}_a\overleftarrow{\partial}_b = 0 \tag{5.2.83}$$

$$\overleftarrow{\partial}_b\overleftarrow{\partial}_\bullet - \frac{q_b}{r^2}\overleftarrow{\partial}_\bullet\overleftarrow{\partial}_b = 0 \tag{5.2.84}$$

We can also define the partial derivatives ∂_C so that [48], [53],

$$d\,a = dx^C\;\partial_C(a)\;; \tag{5.2.85}$$

again the action of ∂_C on the coordinates is

$$\partial_C(x^A) = \delta^A_C I\;. \tag{5.2.86}$$

We now give an explicit relation between the ∂_C and the q-Lie algebra generators χ_C (a similar expression holds also for the $\overleftarrow{\partial}$ derivatives). From (2.3.32) we have:

$$d\,a = -\eta^C(a * \kappa'(\chi_C)) \tag{5.2.87}$$

where $C = (c, \bullet)$ because we have set $\eta^\circ = 0$. Relations (5.2.85) and (5.2.87) give, $\forall a \in ISO_{q,r}(N)$:

$$\partial_c(a) = r^{-1}u(a * \kappa'(\chi_c)) \quad , \qquad \partial_\bullet(a) = r^{-1}u(a * \kappa'(\chi_\bullet)) \tag{5.2.88}$$

The commutations between the partial derivatives can be derived as done above for $\overleftarrow{\partial}$, or via (5.2.88) and the q-Lie algebra (5.1.1)-(5.1.4). They are given in Table 2. Similarly we can introduce the right invariant vectorfields

$$\tilde{h}_C \equiv \tilde{h}_{\kappa'(\chi_C)} \equiv [\kappa'(\chi_C) \otimes id]\Delta \tag{5.2.89}$$

and use their Leibniz rule [it follows from $\Delta(\kappa'(\chi_C)) = \kappa'(\chi_C) \otimes \varepsilon + \kappa'(f^D{}_C) \otimes \kappa'(\chi_D)$]:

$$\tilde{h}_C(ab) = \tilde{h}_C(a)b + \kappa'(f^D{}_C)(a_1)\, a_2 \tilde{h}_D(b) \tag{5.2.90}$$

to derive the ∂, x, u commutations. For example we have $\tilde{h}_a x^b = rv\delta^b_a + (r/q_b)R^{lb}{}_{ac}x^c\tilde{h}_l$ $+ r\lambda\tilde{h}_\bullet$ that togheter with $\partial_C = r^{-1}u\tilde{h}_C$ gives

$$\partial_a x^b = \delta^i_j[I + (r^2 - 1)v\partial_\bullet] + rR^{lb}{}_{ac}x^c\partial_l \ .$$

Similarly for the the other relations, see Table 2.

Conjugation

The commutations in Table 2 are consistent under the conjugation (already defined for x^a and dx^a)

$$(x^a)^* = \mathcal{D}^a{}_b x^b,\ (dx^a)^* = \mathcal{D}^a{}_b dx^b,\ (\partial_a)^* = -r^N d_b^{-1}\mathcal{D}^b{}_a\partial_b \tag{5.2.91}$$

$$v^* = v,\ (dv)^* = dv,\ (\partial_\bullet)^* = u - \partial_\bullet \tag{5.2.92}$$

where we have used the notation $D^a_a = d^a$, $D^{-1}{}^a_a = d_a^{-1}$ ($D^a_b = C^{ae}C_{be}$ is diagonal). This consistency can be checked directly by taking the $*$-conjugates of the relations in Table 2, and by using the identity (4.1.34) and:

$$\bar{C} = C^T;\ [Q_N(r)]^* = Q_N(r); \tag{5.2.93}$$

$$\begin{array}{ll} d^c d_h^{-1} R^{cg}{}_{ha}(R^{-1})^{ea}{}_{cd} = \delta^e_h \delta^g_d; & R^{ab}{}_{cd} d^a d^b = R^{ab}{}_{cd} d^c d^d \\ d^c R^{cg}{}_{hc} = r^{N-1}\delta^g_h; & R^{ab}{}_{cd} d_a^{-1} d_b^{-1} = R^{ab}{}_{cd} d_c^{-1} d_d^{-1} \end{array} \tag{5.2.94}$$

$$\overline{q_a} = \frac{1}{q_a}\text{for } a \neq n, n+1, \quad \overline{q_n} = \frac{1}{q_{n+1}} \tag{5.2.95}$$

We now *derive* the conjugation on the partial derivatives from the differential calculus on $ISO_{q,r}(N)$. This is achieved by studying the conjugation on the right invariant vectorfields $\tilde{h}$.

For a general Hopf algebra, with tangent vectors χ_i, we deduce the conjugation on $\tilde{h}$ from the commutation relations between $\tilde{h}$ and a generic element of the Hopf algebra:

$$\tilde{h}_j b = \tilde{h}_j(b) + \langle \kappa'(f^s{}_j)\,,\, b_1\rangle\, b_2 \tilde{h}_s \;= \langle \kappa'(\chi_j)\,,\, b_1\rangle\, b_2 + \langle \kappa'(f^s{}_j)\,,\, b_1\rangle\, b_2 \tilde{h}_s \qquad (5.2.96)$$

We multiply this expression by $\langle \kappa'^2(f^j{}_i)\,,\, b_0\rangle$ [where we have used the notation $(id \otimes \Delta)\Delta(b) = b_0 \otimes b_1 \otimes b_2$] to obtain

$$\langle \kappa'^2(f^j{}_i)\,,\, b_1\rangle\, \tilde{h}_j b_2 + \langle \kappa'^2(\chi_i)\,,\, b_1\rangle b_2 = b\tilde{h}_i \qquad (5.2.97)$$

Now, using $\langle \psi\,,\, b\rangle = \overline{\langle [\kappa'(\psi)]^*, b^*\rangle}$ and then applying $*$ we obtain (here $a = b^*$)

$$\tilde{h}^*_j a = \langle [\kappa'^3(\chi_j)]^*\,,\, a_1\rangle\, a_2 + \langle [\kappa'^3(f^s{}_j)]^*\,,\, a_1\rangle\, a_2 \tilde{h}_s\,. \qquad (5.2.98)$$

This last relation implies

$$\tilde{h}^*_i \equiv [\tilde{h}_{\kappa'(\chi_i)}]^* = \tilde{h}_{[\kappa'^3(\chi_i)]^*} \qquad (5.2.99)$$

notice that $*\circ\kappa'^2$ is a well defined conjugation since $(*\circ\kappa'^2)^2 = id$.

We now apply formula (5.2.99), valid for a generic Hopf algebra, to the $*$-conjugation and the right invariant vectorfields of this section; we have:

$$[\tilde{h}_{\kappa'(\chi_a)}]^* = -r^N \overline{q}_a d_a^{-1} \mathcal{D}^b{}_a \tilde{h}_{\kappa'(\chi_b)} \qquad (5.2.100)$$

$$[\tilde{h}_{\kappa'(\chi_\bullet)}]^* = -\tilde{h}_{\kappa'(\chi_\bullet)}\,. \qquad (5.2.101)$$

From these last relations and (5.2.88) we then finally deduce $(\partial_a)^* = -d_b^{-1}\mathcal{D}^b{}_a r^N \partial_b$ and $(\partial_\bullet)^* = u - \partial_\bullet$.

5.2.2 The reduced x^a, dx^a, ∂_a algebra and the quantum Minkowski phase-space.

Note that the algebra in Table 2 actually contains a subalgebra generated only by x^a, dx^a, ∂_a, indeed $\partial_\bullet$ vanishes when acting on monomials containing only the coordinates x^b, as can be seen from (5.4.149). This calculus is $ISO_{q,r}(N)$-right covariant because it can also be obtained imposing the conditions $\eta^\bullet = 0$ and $\chi_\bullet = 0$ that are compatible with the right coaction Δ_R and the bimodule structure given by the f^i_j functionals.

Table 3 contains the multiparametric orthogonal quantum plane algebra of coordinates, differentials and partial derivatives, together with a consistent conjugation for any even dimension. We emphasize here that this conjugation *does not* require an additional scaling operator as in ref. [9]. Thus the algebra in Table 3 can be taken as starting point for a deformed Heisenberg algebra (i.e. a deformed phase-space) .

Real coordinates and hermitean momenta

We note that the transformation

$$X^a = \frac{1}{\sqrt{2}}(x^a + x^{a'}), \quad a \leq n \tag{5.2.102}$$

$$X^{n+1} = \frac{i}{\sqrt{2}}(x^n - x^{n+1}) \tag{5.2.103}$$

$$X^a = \frac{1}{\sqrt{2}}(x^a - x^{a'}), \quad a > n+1 \tag{5.2.104}$$

defines real coordinates X^a. On this basis the metric becomes $C' = (M^{-1})^T C M^{-1}$ (where M is the transformation matrix $X = Mx$):

$$C' = \frac{1}{2}\begin{pmatrix} r^{\frac{N}{2}-1} + r^{-\frac{N}{2}+1} & 0 & & & 0 & -(r^{\frac{N}{2}-1} - r^{-\frac{N}{2}+1}) \\ 0 & r^{\frac{N}{2}-2} + r^{-\frac{N}{2}+2} & & & -(r^{\frac{N}{2}-2} - r^{-\frac{N}{2}+2}) & 0 \\ \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ \vdots & \vdots & & & \vdots & \vdots \\ 0 & r^{\frac{N}{2}-2} - r^{-\frac{N}{2}+2} & & & -(r^{\frac{N}{2}-2} + r^{-\frac{N}{2}+2}) & 0 \\ r^{\frac{N}{2}-1} - r^{-\frac{N}{2}+1} & 0 & & & 0 & -(r^{\frac{N}{2}-1} + r^{-\frac{N}{2}+1}) \end{pmatrix} \tag{5.2.105}$$

and reduces for $r \to 1$ to the usual $SO(n+1, n-1)$ diagonal metric with $n+1$ plus signs and $n-1$ minus signs. Notice that the diagonal elements of C' are real while the off diagonal ones are pure imaginary, moreover C' is hermitian (and can therefore be diagonalized via a unitary matrix).

As for the coordinates X, it is possible to define antihermitian χ and ∂, and real ω and dx. To define hermitian momenta we first notice that the partial derivatives

$$\tilde{\partial}_a \equiv r^{\frac{N}{2}} d_a^{-\frac{1}{2}} \partial_a \tag{5.2.106}$$

behave, under the hermitian conjugation $*$, similarly to the coordinates x^a:

$$(\tilde{\partial}_a)^* = -\tilde{\partial}_a \qquad a \neq n,\, n+1 \tag{5.2.107}$$

$$(\tilde{\partial}_n)^* = -\tilde{\partial}_{n+1}\ . \tag{5.2.108}$$

As in (5.2.102)–(5.2.104) we then define:

$$P_a = \frac{-i\hbar}{\sqrt{2}}(\tilde{\partial}_a + \tilde{\partial}_{a'}), \quad a \leq n \tag{5.2.109}$$

$$P_{n+1} = \frac{-\hbar}{\sqrt{2}}(\tilde{\partial}_n - \tilde{\partial}_{n+1}) \tag{5.2.110}$$

$$P_a = \frac{-i\hbar}{\sqrt{2}}(\tilde{\partial}_a - \tilde{\partial}_{a'}), \quad a > n+1 \tag{5.2.111}$$

It is easy to see that the P_a are hermitian: $P_a{}^* = P_a$ and that in the classical limit are the momenta conjugated to the coordinates X^a: $P_a(X^b) = -i\hbar\delta^b_a$. In the $r \neq 1$ case we explicitly have (use $d_{a'} = d_a^{-1}$, $d_n = d_{n+1} = 1$):

$$P_a(X^a) = \tfrac{-i}{2} r^{\frac{N}{2}} \hbar (d_a^{\frac{1}{2}} + d_a^{-\frac{1}{2}})$$

$$P_a(X^{a'}) = \epsilon_a \tfrac{-i}{2} r^{\frac{N}{2}} \hbar (d_a^{\frac{1}{2}} - d_a^{-\frac{1}{2}}) \quad \text{where } \epsilon_a = 1 \text{ if } a < n \text{ and } \epsilon_a = -1 \text{ if } a > n+1$$

while the other entries of the $P_a(X^b)$ matrix are zero.

By defining the transformation matrix $N_a{}^b$ as:

$$P_a \equiv -i\hbar N_a{}^b \partial_b \tag{5.2.112}$$

we find the deformed canonical commutation relations:

$$P_a X^b - r S^{bc}{}_{ad} X^d P_c = -i\hbar E^b_a I \tag{5.2.113}$$

where

$$S^{bc}{}_{ad} = N_a{}^e M^b{}_f \hat{R}^{fh}{}_{eg} (M^{-1})^g{}_d (N^{-1})_h{}^c, \quad E^b_a \equiv \frac{i}{\hbar} P_a(X^b) = N_a{}^c M^b{}_c \tag{5.2.114}$$

Similarly one finds all the remaining commutations of the P, X and dX algebra. Notice that no unitary operator appears on the right-hand side of (5.2.113). Our conjugation is consistent with (5.2.113) without the need of the extra operator of ref. [9] .

For $n = 2$ the results of this section immediately yield the bicovariant calculus on the quantum Minkowski space, i.e. on the multiparametric orthogonal quantum plane $Fun_{q,r}(ISO(3,1)/SO(3,1))$.

Note 5.2.3 In the $r \to 1$ limit the reduced differential calculus on the coordinates x^a coincides with the $r = 1$ bicovariant differential calculus on $ISO_{q,r=1}(N)$ of Section 4.7 [see (4.7.17)]. This is so because the $ISO_{q,r=1}(N)$ bicovariant differential can be written $da = (\chi^a{}_b * a)\omega_a{}^b + (\chi^\bullet{}_c * a)\omega_\bullet{}^c = -\eta_a{}^b(a * \kappa'(\chi^a{}_b)) - \eta_\bullet{}^c(a * \kappa'(\chi^\bullet{}_c))$. Similarly to Theorem 3.7.1, we have that $(a * \kappa'(\chi^a{}_b)) = 0$ when a is a polynomial in x^a. Then the exterior differential on such polynomials reads $da = -\eta_\bullet{}^c(a * \kappa'(\chi^\bullet{}_c))$ as in the reduced differential calculus on the coordinates x^a.

5.3 Table 1: the $ISO_{q,r}(N)$-bicovariant algebra

$$P_A{}^{ab}{}_{cd}x^c x^d = 0 \tag{5.3.115}$$

$$x^b v = q_b v x^b \; ; \quad x^b u = q_b^{-1} u x^b \tag{5.3.116}$$

$$z = -\frac{1}{(r^{-\frac{N}{2}} + r^{\frac{N}{2}-2})} x^b C_{ba} x^a u \tag{5.3.117}$$

$$zv = r^2 vz \; ; \quad zu = r^{-2} uz \tag{5.3.118}$$

$$q_a x^a z = z x^a \tag{5.3.119}$$

$$(x \otimes dx) = (r^2 P_S - P_A - P_0)(dx \otimes x) + P_0 d(x \otimes x) \tag{5.3.120}$$

$$x^c du = \frac{1}{q_c}(du)x^c - \frac{\lambda}{r}(dx^c)u; \quad x^c dv = q_c(dv)x^c + \lambda r(dx^c)v \tag{5.3.121}$$

$$x^c dz = \frac{1}{q_c}(dz)x^c \tag{5.3.122}$$

$$u dx^c = \frac{q_c}{r^2}(dx^c)u \tag{5.3.123}$$

$$u du = r^{-2}(du)u; \quad u dv = r^{-2}(dv)u \tag{5.3.124}$$

$$u dz = (dz)u \tag{5.3.125}$$

$$v dx^c = \frac{r^2}{q_c}(dx^c)v \tag{5.3.126}$$

$$v du = r^2(du)v; \quad v dv = r^2(dv)v \tag{5.3.127}$$

$$v dz = (dz)v \tag{5.3.128}$$

$$z dx^c = q_c(dx^c)z \tag{5.3.129}$$

$$z du = r^{-2}(du)z + (r^{-2} - 1)(dz)u \tag{5.3.130}$$

$$z dv = r^2(dv)z + (r^2 - 1)(dz)v \tag{5.3.131}$$

$$z dz = r^{-2}(dz)z \tag{5.3.132}$$

$$P_S(dx \wedge dx) = 0 \tag{5.3.133}$$

$$dx^c \wedge du = -\frac{r^2}{q_c} du \wedge dx^c; \quad dx^c \wedge dv = -\frac{q_c}{r^2} dv \wedge dx^c \tag{5.3.134}$$

$$dx^c \wedge dz = -\frac{1}{q_c} dz \wedge dx^c \tag{5.3.135}$$

$$du \wedge du = dv \wedge dv = 0 \tag{5.3.136}$$

$$du \wedge dv = -r^{-2} dv \wedge du \quad (5.3.137)$$
$$dz \wedge du = -du \wedge dz; \quad dz \wedge dv = -dv \wedge dz \quad (5.3.138)$$
$$dz \wedge dz = 0 \quad (5.3.139)$$

5.4 Table 2: the $ISO_{q,r}(N)$-covariant $x^a, v, \partial_a, \partial_\bullet, dx^a, dv$ algebra

$$P_A^{ab}{}_{cd} x^c x^d = 0 \quad (5.4.140)$$
$$x^b v = q_b v x^b \quad (5.4.141)$$

$$x \otimes dx = r\hat{R}(dx \otimes x) \quad (5.4.142)$$
$$x^c dv = q_c (dv) x^c + \lambda r (dx^c) v \quad (5.4.143)$$
$$v dx^c = \frac{r^2}{q_c}(dx^c) v \quad (5.4.144)$$

$$dx \wedge dx = -r\hat{R} dx \wedge dx \quad (5.4.145)$$
$$dx^c \wedge dv = -\frac{q_c}{r^2} dv \wedge dx^c \quad (5.4.146)$$
$$dv \wedge dv = 0 \quad (5.4.147)$$

$$\partial_c x^b = r\hat{R}^{be}{}_{cd} x^d \partial_e + \delta^b_c [I + (r^2 - 1) v \partial_\bullet] \quad (5.4.148)$$
$$\partial_\bullet x^b = q_b x^b \partial_\bullet \quad (5.4.149)$$
$$\partial_\bullet v = r^2 v \partial_\bullet + I \quad (5.4.150)$$

$$(P_A)^{ab}{}_{cd} \partial_b \partial_a = 0 \quad (5.4.151)$$
$$\partial_b \partial_\bullet - \frac{q_b}{r^2} \partial_\bullet \partial_b = 0 \quad (5.4.152)$$

Conjugation:

$$(x^a)^* = \mathcal{D}^a{}_b x^b, \ (dx^a)^* = \mathcal{D}^a{}_b dx^b, \ (\partial_a)^* = -r^N d_a^{-1} \mathcal{D}^b{}_a \partial_b \quad (5.4.153)$$
$$v^* = v, \ (dv)^* = dv, \ (\partial_\bullet)^* = u - \partial_\bullet \quad (5.4.154)$$
$$r^* = r^{-1}, \quad q_a^* = \frac{1}{q_a} \text{for } a \neq n, n+1, \quad q_n^* = \frac{1}{q_{n+1}} \quad (5.4.155)$$

5.5 Table 3: the reduced $ISO_{q,r}(N)$-covariant x^a, ∂_a, dx^a algebra

$$P_A{}^{ab}{}_{cd} x^c x^d = 0 \tag{5.5.156}$$

$$x \otimes dx = r\hat{R}(dx \otimes x) \tag{5.5.157}$$

$$dx \wedge dx = -r\hat{R}(dx \wedge dx) \tag{5.5.158}$$

$$\partial_c x^b = r\hat{R}^{be}{}_{cd} x^d \partial_e + \delta^b_c I \tag{5.5.159}$$

$$P_A{}^{ab}{}_{cd} \partial_b \partial_a = 0 \tag{5.5.160}$$

Conjugation:

$$(x^a)^* = \mathcal{D}^a{}_b x^b,\ (dx^a)^* = \mathcal{D}^a{}_b dx^b,\ (\partial_a)^* = -r^N d_a^{-1} \mathcal{D}^b{}_a \partial_b \tag{5.5.161}$$

Chapter 6

Appendix

A The Hopf algebra axioms

A Hopf algebra over the field K is a unital algebra over K endowed with the linear maps:

$$\Delta \;:\; A \to A \otimes A, \quad \varepsilon \;:\; A \to K, \quad \kappa \;:\; A \to A \tag{A.1}$$

satisfying the following properties $\forall a, b \in A$:

$$(\Delta \otimes id)\Delta(a) = (id \otimes \Delta)\Delta(a) \tag{A.2}$$

$$(\varepsilon \otimes id)\Delta(a) = (id \otimes \varepsilon)\Delta(a) = a \tag{A.3}$$

$$m(\kappa \otimes id)\Delta(a) = m(id \otimes \kappa)\Delta(a) = \varepsilon(a)I \tag{A.4}$$

$$\Delta(ab) = \Delta(a)\Delta(b) \;; \quad \Delta(I) = I \otimes I \tag{A.5}$$

$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b) \;; \quad \varepsilon(I) = 1 \tag{A.6}$$

where m is the multiplication map $m(a \otimes b) = ab$. From these axioms we deduce:

$$\kappa(ab) = \kappa(b)\kappa(a) \;; \quad \Delta[\kappa(a)] = \tau(\kappa \otimes \kappa)\Delta(a) \;; \quad \varepsilon[\kappa(a)] = \varepsilon(a) \;; \quad \kappa(I) = I \tag{A.7}$$

where $\tau(a \otimes b) = b \otimes a$ is the twist map.

B The derivation of two equations

In this Appendix we derive the two equations (2.1.108) and (2.1.110). Consider the exterior derivative of eq. (2.1.30):

$$d(\omega^i a) = d[(f^i{}_j * a)\omega^j]. \tag{B.1}$$

The left-hand side is equal to:

$$
\begin{aligned}
& d(\omega^i a) = \\
& \quad = d\omega^i \wedge a - \omega^i \wedge da = -\boldsymbol{C}_{jk}{}^i \omega^j \otimes \omega^k a - \omega^i \wedge (\chi_j * a)\omega^j = \\
& \quad = -\boldsymbol{C}_{jk}{}^i \omega^j \otimes \omega^k a - (f^i{}_s * \chi_j * a)\omega^s \wedge \omega^j = \\
& \quad = -\boldsymbol{C}_{jk}{}^i (f^j{}_p * f^k{}_q * a)\omega^p \otimes \omega^q - (f^i{}_s * \chi_j * a)(\omega^s \otimes \omega^j - \Lambda^{sj}_{pq}\omega^p \otimes \omega^q) = \\
& \quad = [(-\boldsymbol{C}_{jk}{}^i f^j{}_p f^k{}_q - f^i{}_p \chi_q + \Lambda^{sj}_{pq} f^i{}_s \chi_j) * a](\omega^p \otimes \omega^q) \qquad \text{(B.2)}
\end{aligned}
$$

The right-hand side reads:

$$
\begin{aligned}
& d[(f^i{}_j * a)\omega^j] = \\
& \quad = d(f^i{}_j * a)\omega^j + (f^i{}_j * a)d\omega^j = \\
& \quad = (\chi_k * f^i{}_j * a)\omega^k \wedge \omega^j - (f^i{}_j * a)\boldsymbol{C}_{pq}{}^j \omega^p \otimes \omega^q = \\
& \quad = (\chi_k * f^i{}_j * a)(\omega^k \otimes \omega^j - \Lambda^{kj}_{pq}\omega^p \otimes \omega^q) - (f^i{}_j * a)\boldsymbol{C}_{pq}{}^j \omega^p \otimes \omega^q = \\
& \quad = [(\chi_p f^i{}_q - \Lambda^{kj}_{pq}\chi_k f^i{}_j - \boldsymbol{C}_{pq}{}^j f^i{}_j) * a](\omega^p \otimes \omega^q), \qquad \text{(B.3)}
\end{aligned}
$$

so that we deduce the equation

$$
\begin{aligned}
& -\boldsymbol{C}_{jk}{}^i f^j{}_p f^k{}_q - f^i{}_p \chi_q + \Lambda^{sj}_{pq} f^i{}_s \chi_j = \\
& = \chi_p f^i{}_q - \Lambda^{kj}\ pq \chi_k f^i{}_j - \boldsymbol{C}_{pq}{}^j f^i{}_j. \qquad \text{(B.4)}
\end{aligned}
$$

We now need two lemmas.

Lemma 1

$$
f^n{}_l * a\theta = (f^n{}_r * a)(f^r{}_l * \theta), \quad a \in A,\ \theta \in \Gamma^{\otimes n}. \qquad \text{(B.5)}
$$

Proof

$$
\begin{aligned}
& f^n{}_l * a\theta = \\
& \quad (id \otimes f^n{}_l)\Delta(a)\Delta_R(\theta) = a_1\theta_1 f^n{}_l(a_2\theta_2) = \\
& \quad a_1\theta_1 f^n{}_r(a_2) f^r{}_l(\theta_2) = a_1 f^n{}_r(a_2)\theta_1 f^r{}_l(\theta_2) = \\
& \quad (f^n{}_r * a)\theta_1 f^r{}_l(\theta_2) = (f^n{}_r * a)(f^r{}_l * \theta). \qquad \text{(B.6)}
\end{aligned}
$$

Lemma 2

$$
f^r{}_l * \omega^j = \Lambda^{rj}_{kl}\omega^k. \qquad \text{(B.7)}
$$

Proof

$$
\begin{aligned}
& f^r{}_l * \omega^j = \\
& \quad (id \otimes f^r{}_l)\Delta_R(\omega^j) = (id \otimes f^r{}_l)[\omega^k \otimes M_k{}^j] = \\
& \quad = \omega^k f^r{}_l(M_k{}^j) = \Lambda^{rj}_{kl}. \qquad \text{(B.8)}
\end{aligned}
$$

Consider now eq. (2.1.107) with $h = f^n{}_l$:

$$d(f^n{}_l * a) = f^n{}_l * da. \qquad \text{(B.9)}$$

The first member is equal to $(\chi_k * f^n{}_l * a)\omega^k$, while the second member is:

$$\begin{aligned} f^n{}_l * da &= f^n{}_l * [(\chi_j * a)\omega^j] = (f^n{}_r * \chi_j * a)(f^r{}_l * \omega^j) \\ &= (f^n{}_r * \chi_j * a)(\Lambda^{rj}_{kl}\omega^k) \end{aligned} \qquad \text{(B.10)}$$

We have used here the two lemmas (B.5) and (B.7). Therefore the following equation holds:

$$\chi_k * f^n{}_l = \Lambda^{rj}_{kl}\, f^n{}_r * \chi_j, \qquad \text{(B.11)}$$

which is just eq. (2.1.108). Equation (2.1.110) is obtained simply by subtracting (B.11) from eq. (B.4).

C Two theorems on i_t and ℓ_t

Theorem 2.4.11

$$\ell_{t_i} = i_{t_i} d + d i_{t_i}$$

that is

$$\begin{aligned} &\forall\, a_{i_1 \dots i_n}\omega^{i_1} \wedge \dots \omega^{i_n} \in \Gamma^{\wedge n}, \\ &\ell_{t_i}(a_{i_1 \dots i_n}\omega^{i_1} \wedge \dots \omega^{i_n}) = (i_{t_i} d + d i_{t_i})(a_{i_1 \dots i_n}\omega^{i_1} \wedge \dots \omega^{i_n}). \end{aligned} \qquad \text{(C.1)}$$

We will show this theorem by induction on the integer n. To do this, we need the following:

Lemma

If $n = 1$, the theorem is true, i.e.

$$\ell_{t_i}(b_k\omega^k) = (i_{t_i} d + d i_{t_i})(b_k\omega^k). \qquad \text{(C.2)}$$

First we show that:

$$\ell_{t_i}(\omega^k) = (i_{t_i} d + d i_{t_i})\omega^k. \qquad \text{(C.3)}$$

We already know that $\ell_{t_i}(\omega^k) = \omega^j \boldsymbol{C}_{ji}{}^k$. The right-hand side of (C.3) yields:

$$\begin{aligned} (i_{t_i} d + d i_{t_i})(\omega^k) &= i_{t_i} d\omega^k + d(i_{t_i}\omega^k) = \\ &= -\tfrac{1}{2} C_{nj}{}^k i_{t_i}(\omega^n \wedge \omega^j) = \\ &= -\tfrac{1}{2} C_{nj}{}^k \left(f^n{}_i * \omega^j - \omega^n \delta^j_i\right) = \\ &= -\tfrac{1}{2} C_{nj}{}^k \left[(id \otimes f^n{}_i)\left(\omega^\ell \otimes M_\ell{}^j\right) - \delta^j_i \omega^n\right] = \\ &= -\tfrac{1}{2} C_{nj}{}^k \left[\omega^\ell \Lambda^{nj}_{\ell i} - \delta^j_i \omega^n\right] = \\ &= +\tfrac{1}{2} C_{nj}{}^k \left[\delta^n_\ell \delta^j_i - \Lambda^{nj}_{\ell i}\right] \omega^\ell = \\ &= \boldsymbol{C}_{\ell i}{}^k \omega^\ell \end{aligned}$$

and (C.3) is thus proved.

The right-hand side of (C.2) gives:

$$\begin{aligned}(i_{t_i}d + di_{t_i})(b_k\omega^k) &= i_{t_i}\left(db_k \wedge \omega^k + b_k d\omega^k\right) + d\left(b_k i_{t_i}(\omega^k)\right) = \\ &= i_{t_j}(db_k)f^j{}_i * \omega^k - (db_k)i_{t_i}(\omega^k) + \\ &\quad + b_k i_{t_i}(d\omega^k) + (db_k)i_{t_i}(\omega^k) = \\ &= i_{t_j}((\chi_n * b_k)\omega^n)f^j{}_i * \omega^k + b_k i_{t_i}(d\omega^k) = \\ &= (\chi_n * b_k)\delta^n_j f^j{}_i * \omega^k + b_k(i_{t_i}d + di_{t_i})\omega^k = \\ &= (\chi_n * b_k)f^n{}_i * \omega^k + b_k \ell_{t_i}(\omega^k) = \\ &= \ell_{t_n}(b_k)f^n{}_i * \omega^k + b_k\ell_{t_i}(\omega^k) = \\ &= \ell_{t_i}(b_k\omega^k),\end{aligned}$$

and the lemma is proved. We now finally prove the theorem.
Let us suppose it to be true for a $(n-1)$-form:

$$\ell_{t_a}(a_{i_2\ldots i_n}\omega^{i_2} \wedge \ldots \omega^{i_n}) = (i_{t_i}d + di_{t_i})(a_{i_2\ldots i_n}\omega^{i_2} \wedge \ldots \omega^{i_n}). \tag{C.4}$$

Then it holds also for an n-form. Indeed, the left-hand side of (C.1) yields

$$\begin{aligned}&\ell_{t_i}(a_{i_1\ldots i_n}\omega^{i_1} \wedge \ldots \omega^{i_n}) = \\ &= \ell_{t_j}(a_{i_1\ldots i_n}\omega^{i_1}) \wedge f^j{}_i * (\omega^{i_2} \wedge \ldots \omega^{i_n}) + a_{i_1\ldots i_n}\omega^{i_1} \wedge \ell_{t_i}(\omega^{i_2} \wedge \ldots \omega^{i_n})\end{aligned}$$

while the right-hand side of (C.1) is given by :

$$\begin{aligned}&(i_{t_i}d + di_{t_i})(a_{i_1\ldots i_n}\omega^{i_1} \wedge \ldots \omega^{i_n}) = \\ = \quad & i_{t_i}[d(a_{i_1\ldots i_n}\omega^{i_1}) \wedge \omega^{i_2} \wedge \ldots \omega^{i_n} - (a_{i_1\ldots i_n}\omega^{i_1}) \wedge d(\omega^{i_2} \wedge \ldots \omega^{i_n})] + \\ & d[i_{t_j}(a_{i_1\ldots i_n}\omega^{i_1})f^j{}_i * (\omega^{i_2} \wedge \ldots \omega^{i_n}) - (a_{i_1\ldots i_n}\omega^{i_1}) \wedge i_{t_i}(\omega^{i_2} \wedge \ldots \omega^{i_n})] = \\ = \quad & i_{t_j}(da_{i_1\ldots i_n}\omega^{i_1}) \wedge f^j{}_i * (\omega^{i_2} \wedge \ldots \omega^{i_n}) + d(a_{i_1\ldots i_n}\omega^{i_1}) \wedge i_{t_i}(\omega^{i_2} \wedge \ldots \omega^{i_n}) + \\ & -i_{t_j}(a_{i_1\ldots i_n}\omega^{i_1})f^j{}_i * d(\omega^{i_2} \wedge \ldots \omega^{i_n}) + a_{i_1\ldots i_n}\omega^{i_1} \wedge i_{t_i}d(\omega^{i_2} \wedge \ldots \omega^{i_n}) + \\ & +di_{t_j}(a_{i_1\ldots i_n}\omega^{i_1})f^j{}_i * (\omega^{i_2} \wedge \ldots \omega^{i_n}) + i_{t_j}(a_{i_1\ldots i_n}\omega^{i_1}) \wedge f^j{}_i * d(\omega^{i_2} \wedge \ldots \omega^{i_n}) + \\ & -d(a_{i_1\ldots i_n}\omega^{i_1}) \wedge i_{t_i}(\omega^{i_2} \wedge \ldots \omega^{i_n}) + a_{i_1\ldots i_n}\omega^{i_1} \wedge di_{t_i}(\omega^{i_2} \wedge \ldots \omega^{i_n}) = \\ = \quad & [(i_{t_j}d + di_{t_j})(a_{i_1\ldots i_n}\omega^{i_1})] \wedge f^j{}_i * (\omega^{i_2} \wedge \ldots \omega^{i_n}) + \\ & +a_{i_1\ldots i_n}\omega^{i_1}(i_{t_i}d + di_{t_i})(\omega^{i_2} \wedge \ldots \omega^{i_n}) = \\ = \quad & \ell_{t_j}(a_{i_1\ldots i_n}\omega^{i_1}) \wedge f^j{}_i * (\omega^{i_2} \wedge \ldots \omega^{i_n}) + a_{i_1\ldots i_n}\omega^{i_1} \wedge \ell_{t_i}(\omega^{i_2} \wedge \ldots \omega^{i_n})\end{aligned}$$

and the theorem is proved. □□□

Lemma $[i_{t_i}, \ell_{t_k}] = i_{t_i}\circ\ell_{t_k} - \Lambda^{ef}_{ik}\ell_{t_e}\circ i_{t_f}$

Proof By definition we have

$$[i_{t_i}, \ell_{t_k},](\vartheta) \equiv \ell_{\kappa'(\chi_{k1})*}\circ i_{t_i}\circ\ell_{\chi_{k2}*}(\vartheta) = \kappa'(\chi_j) * (i_{t_i}(f^j{}_k * \vartheta)) + i_{t_i}(\ell_{t_k}(\vartheta)) \tag{C.5}$$

we therefore have to prove that

$$\kappa'(\chi_j) * (i_{t_i}(f^j{}_k * \vartheta)) = -\Lambda^{ef}_{ik}\ell_{t_e} \circ i_{t_f} \tag{C.6}$$

First notice that on a generic covariant tensor $\tau \in \Gamma^{\otimes}$

$$i_{t_i}(f^j{}_k * \tau) = \Lambda^{ef}_{ik} f^j{}_e * i_{t_f}(\tau) \tag{C.7}$$

as can be easily proved by induction with $\tau = \tau' \otimes \omega^\ell$, $\tau' \in \Gamma^{\otimes}$. To complete the proof recall that $\kappa(\chi_j) f^j{}_e = -\chi_e$ [apply $m(\kappa' \otimes id)$ to $\Delta'(\chi_e)$]. □□□

Theorem $[i_{t_i}, \ell_{t_k}] = i_{[t_i, t_k]} = \boldsymbol{C}_{ik}{}^l i_{t_l}$

Proof
The proof is by induction, it holds on 1-forms, let assume it holds for a generic ϑ form of order n, we prove it holds also for the generic $n+1$ form $\omega^a \wedge \vartheta$. Use the previous lemma to rewrite $[i_{t_i}, \ell_{t_k}]$ as (1)+(2):

$$\begin{aligned}
(1) \quad & i_{t_i}\ell_{t_j}(\omega^a \wedge \vartheta) = i_{t_i}[(\omega^b \boldsymbol{C}_{bp}{}^a \wedge f^p{}_j * \vartheta + \omega^a \wedge \ell_{t_j}(\vartheta)] \\
& = [\boldsymbol{C}_{bp}{}^a f^b{}_i f^p{}_j + f^a{}_i \chi_j] * \vartheta - \omega^b \wedge \boldsymbol{C}_{bp}{}^a i_{t_i}(f^p{}_j * \vartheta) - \omega^a \wedge i_{t_i}\ell_{t_j}(\vartheta) \\
(2) \quad & -\Lambda^{kl}_{ij}\ell_{t_k} i_{t_l}(\omega^a \wedge \vartheta) = -\Lambda^{kl}_{ij}\ell_{t_k}[f^a{}_l * \vartheta - \omega^a i_{t_l}(\vartheta)] \\
& = [-\Lambda^{kl}_{ij}\chi_k f^a{}_l] * \vartheta + \Lambda^{kl}_{ij}\omega^b \wedge \boldsymbol{C}_{bp}{}^a f^p{}_k * i_{t_l}(\vartheta) - \omega^a \wedge (-1)\Lambda^{kl}_{ij}\ell_{t_k} i_{t_l}(\vartheta) \\
(3) \quad & \boldsymbol{C}_{ij}{}^k i_{t_k}(\omega^a \wedge \vartheta) = \boldsymbol{C}_{ij}{}^k f^a{}_k * \vartheta - \boldsymbol{C}_{ij}{}^k \omega^a \wedge i_{t_k}(\vartheta)
\end{aligned} \tag{C.8}$$

We then have

$$\begin{aligned}
(1)+(2)-(3) \quad &= \quad [\boldsymbol{C}_{bp}{}^a f^b{}_i f^p{}_j + f^a{}_i \chi_j - \Lambda^{kl}_{ij}\chi_k f^a{}_l - \boldsymbol{C}_{ij}{}^k f^a{}_k] * \vartheta && \text{(C.9)} \\
& \quad -\omega^b \wedge \boldsymbol{C}_{bp}{}^a [i_{t_i}(f^p{}_j * \vartheta) - \Lambda^{kl}_{ij} f^p{}_k * i_{t_l}(\vartheta)] && \text{(C.10)} \\
& \quad -\omega^a \wedge [i_{t_i}\ell_{t_j} - \Lambda^{kl}_{ij}\ell_{t_k} i_{t_l} - \boldsymbol{C}_{ij}{}^k i_{t_k}](\vartheta) && \text{(C.11)} \\
&= \quad 0 && \text{(C.12)}
\end{aligned}$$

Where the first addend is zero because of (2.1.113), the second is zero because of (C.7), the third because of the inductive hypothesis. □□□

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