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A "new" approach to the quantitative statistical dynamics of plasma turbulence: The optimum theory of rigorous bounds on steady-state transport

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ABSTRACT

The fundamental problem in the theory of turbulent transport is to find the flux Γ of a quantity such as heat. Methods based on statistical closures are mired in conceptual controversies and practical difficulties. However, it is possible to *bound* Γ by employing constraints derived rigorously from the equations of motion. Brief reviews of the general theory and its application to passive advection are given. Then, a detailed application is made to anomalous resistivity generated by *self-consistent* turbulence in a reversed-field pinch. A nonlinear variational principle for an upper bound on the turbulent electromotive force for fixed current is formulated from the magnetohydrodynamic equations in cylindrical geometry. Numerical solution of a case constrained solely by energy balance leads to a reasonable bound and nonlinear eigenfunctions that share intriguing features with experimental data: the dominant mode numbers appear to be correct, and field reversal is predicted at reasonable values of the pinch parameter. Although open questions remain, upon considering all bounding calculations to date one can conclude, remarkably, that global energy balance constrains transport sufficiently so that bounds derived therefrom are not unreasonable and that bounding calculations are feasible even for involved practical problems. The potential of the method has hardly been tapped; it provides a fertile area for future research.

I. INTRODUCTION

In this article we discuss applications of the theory of rigorous bounds on transport[1–5] due to steady-state turbulence. The method provides an intriguing alternative to the more familiar approaches based on statistical closure approximations.[6] For a closely related but less mathematical discussion, see Ref. 7.

A. The generic transport problem

Consider for definiteness the prototypical dissipative nonlinear equation

$$\frac{\partial}{\partial t}T(\boldsymbol{x},t) + \boldsymbol{\nabla}\cdot[\mathbf{u}(\boldsymbol{x},t)T] - \kappa\nabla^2 T = 0. \tag{1}$$

We distinguish two possibilities for the advecting field $\boldsymbol{u}$, which for definiteness we assume to be solenoidal ($\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$). If $\boldsymbol{u}$ is a functional of T, the problem is called "self-consistent"; otherwise, $\boldsymbol{u}$ must be specified externally and the problem is called "passive." In both cases one assumes that Eq. (1) exhibits statistical behavior, so it is relevant to introduce ensemble and/or spatial averaging operations. Statistical behavior may arise because of random initial conditions, intrinsic stochasticity, or, for passive advection, because $\boldsymbol{u}$ is assumed by fiat to be a random variable.

We assume that the statistics are homogeneous in all directions but x. In the x direction we assume that *inhomogeneous*, statistically sharp boundary conditions such as $T(x{=}0,t) = T_0$, $T(x{=}L,t) = T_1$ ($T_1 \neq T_0$) are imposed such that a nontrivial steady-state profile $\langle T\rangle(x)$ is assured. We allow no sources on the right-hand side of Eq. (1). This precludes the possibility that $\partial\langle T\rangle/\partial x$ is constant; the fluctuations are intrinsically inhomogeneous, and the fundamental nonlinearity in the theory is the self-consistent adjustment of the profile to the fluctuations. Although extensions to homogeneous turbulence are possible, they are quite difficult and have not yet been developed; we shall not discuss them here.

For a complete statistical description of Eq. (1) one requires the joint probability distribution functional $P\{T,\boldsymbol{u},t\}$. However, besides being very difficult to obtain, P provides an overabundance of detailed information; one is generally content with much less. For example, the *transport problem* can be defined from the mean continuity equation:

$$\frac{\partial}{\partial t}\langle T\rangle - \boldsymbol{\nabla}\cdot\boldsymbol{\Gamma}_{\mathrm{tot}} = 0,$$

where we have assumed $\langle\boldsymbol{u}\rangle = 0$. [The angular brackets denote a spatial average over the homogeneous directions (for self-consistent problems) or an ensemble average (for passive problems).] The total flux $\boldsymbol{\Gamma}_{\mathrm{tot}}$ is x-directed because of the symmetry assumptions: $\boldsymbol{\Gamma}_{\mathrm{tot}} =$

$\hat{x}\Gamma_{\rm tot}$. Let us normalize x to L, u to $u_{\rm rms} \doteq \langle \tilde{u}^2 \rangle^{1/2}$ [where $\tilde{u} \doteq u - \langle u \rangle$], and T to $\Delta T \doteq T_0 - T_1$. Then $\Gamma_{\rm tot}$ is the sum of a classical part $\Gamma_{\rm cl}$ and an advective part Γ, where

$$\Gamma_{\rm cl} \doteq -R^{-1}\frac{\partial \langle T \rangle}{\partial x}, \tag{2a}$$

$$\Gamma \doteq \langle \tilde{u}_x \tilde{T} \rangle, \tag{2b}$$

and $R \doteq u_{\rm rms} L/\kappa$ is a generic Reynolds-like number.

In steady state Eq. (1) reduces to $\partial \Gamma_{\rm tot}/\partial x = 0$, or $\Gamma_{\rm tot} =$ constant. This constant is a functional of the fluctuations. To see this, observe that because of the special form (2a) it is useful to define the barring operation $\overline{A} \doteq \int_0^1 dx\, A(x)$, so that $\overline{\Gamma}_{\rm cl} = -R^{-1}\int_0^1 dx\, \langle T \rangle'(x) = R^{-1}$ is entirely known in terms of the (statistically sharp) boundary conditions. Then, upon barring

$$\Gamma_{\rm tot} = \overline{\Gamma_{\rm cl}(x) + \Gamma(x)} = \text{constant}, \tag{3}$$

we find

$$\Gamma_{\rm tot} = R^{-1} + \overline{\Gamma}, \tag{4}$$

in which the only unknown is $\overline{\Gamma}$. Determining $\overline{\Gamma}$ is the principal goal of a transport theory.

For future use, note that Eqs. (3) and (4) can be combined to express the mean profile in terms of the fluctuations:

$$-\frac{\partial \langle T \rangle}{\partial x} = 1 - R\,\Delta\Gamma, \tag{5}$$

where $\Delta\Gamma(x) \doteq \Gamma(x) - \overline{\Gamma}$.

B. Difficulties with conventional approaches

One obvious way of determining $\overline{\Gamma}$ is to numerically integrate (many realizations of) Eq. (1), then to compute explicitly the ensemble and spatial averages. This procedure is nontrivial because of statistical noise and because the realizations need not be smooth. These difficulties are demonstrated explicitly in Ref. 8, where a one-dimensional version of Eq. (1) is studied in detail. A more subtle procedure is to determine analytically an approximate equation for $\Gamma(x)$, then solve that equation (probably numerically) and finally spatially average the solution to compute $\overline{\Gamma}$. A variety of techniques exist for constructing such reduced descriptions; their difficulties have been discussed extensively. See Refs. 9 and 6 for reviews with many references. For a discussion oriented specifically to the transport problem, see Ref. 7.

C. The "optimum" variational method

An alternative approach to the transport problem is suggested by the familiar observation that Eq. (1) determines an infinite hierarchy relating statistical moments of different orders. Thus, an infinity of constraints[10] must be satisfied in order to uniquely determine $P\{T, \boldsymbol{u}, t\}$ and, ultimately, the transport. Note, however, that most of the relations constrain extremely subtle details of P and may be practically unimportant for the quantitative prediction of low-order moments such as $\overline{\Gamma}$. Optimistically, one could hope that just one or a small number of judiciously chosen constraints would suffice to predict $\overline{\Gamma}$ reasonably well.

For the specific problem of thermal convection, Malkus speculated[11] that the steady-state flux was the *maximum* of all possible solutions of Eq. (1) subject to the boundary conditions. Such a criterion is difficult to formulate analytically and furthermore turns out to be false except in certain limiting cases. However, it captures the right intuitive idea and suggests a possible way of proceeding. Indeed, by somewhat inverting the logic, Howard[1] arrived at a nonlinear variational principle that was both rigorous and useful in practice. He posed the question "What is the maximum $\overline{\Gamma}$ subject to a finite subset of the infinity of constraints?" Let us denote such a bound by $\overline{\gamma}$. Of course, in the absence of any constraint at all the function space of all possible fields $\widetilde{\boldsymbol{u}}(\boldsymbol{x})$ and $\widetilde{T}(\boldsymbol{x})$ subject only to the boundary conditions is too vast; since such fields can be scaled arbitrarily, the unconstrained bound is infinity. However, Howard showed that one can demonstrate constrained variational problems that lead to finite $\overline{\gamma}$, and the rigorous nature of the formulation guarantees that $\overline{\gamma} \geq \overline{\Gamma}$.

In principle, further useful information would follow by bounding $\overline{\Gamma}$ from below as well as from above. Unfortunately, this problem is quite difficult. A lower bound for $\overline{\Gamma}$ is 0, and any moment constraint is satisfied by the zero solution. Kraichnan has proposed sophisticated techniques to overcome this problem,[12] but they remain to be fully developed. In the rest of this discussion we shall consider only bounds from above.

Conventionally, a fundamental constraint is derived from the steady-state balance of *production* $\mathcal{P}$, *transfer* $\mathcal{T}$, and *dissipation* $\mathcal{D}$. By definition, production is related to the interaction of the mean fields with the fluctuations, transfer describes conservative advection of the fluctuations, and dissipation is positive-definite when averaged over space. Although these quantities can be defined at various orders in the moment hierarchy, it is simplest and conventional to define them from the evolution equation of the variance of the fluctuations. One begins by multiplying the exact equation for the fluctuations,

$$\frac{\partial}{\partial t}\widetilde{T}(\boldsymbol{x},t) + \boldsymbol{\nabla}\cdot(\widetilde{\boldsymbol{u}}\langle T\rangle + \widetilde{\boldsymbol{u}}\,\widetilde{T} - \langle\widetilde{\boldsymbol{u}}\,\widetilde{T}\rangle) - R^{-1}\nabla^2\widetilde{T} = 0.$$

by $\widetilde{T}(\boldsymbol{x},t)$ (at the same point in space and time); upon ensemble-averaging, one obtains

$$\frac{\partial}{\partial t}\mathcal{E}(x,t) = \mathcal{P}(x,t) + \mathcal{T}(x,t) - \mathcal{D}(x,t), \tag{6}$$

where we have defined $\mathcal{E} \doteq \langle \widetilde{T}^2 \rangle /2$, $\mathcal{P} \doteq -\Gamma(x)\partial_x \langle T \rangle$, $\mathcal{T} \doteq -\partial_x \langle \widetilde{u}_x(\widetilde{T}^2/2)\rangle$, and $\mathcal{D} \doteq -R^{-1}\langle \widetilde{T}\,\nabla^2 \widetilde{T}\rangle$. The troublesome term here is $\mathcal{T}$, a triplet correlation function that describes mode-mode coupling and advection of fluctuations from point to point. In statistical closure theory one attempts to approximate $\mathcal{T}(x)$. By contrast, in the present method one appeals to the conservative nature of $\mathcal{T}$ and the sharp boundary conditions to obtain rigorously $\overline{\mathcal{T}} = 0$. Therefore, by barring Eq. (6) one obtains in steady state the constraint that global dissipation balances global production:

$$\overline{\mathcal{P}} = \overline{\mathcal{D}}. \tag{7}$$

Krommes and Smith[5] have called this the "basic" constraint. (It is often thought of as an energy balance, but $\mathcal{E}$ need not be the physical fluctuation energy; it might, for example, be more closely related to entropy production.) Note that the unknown, spatially varying profile can be eliminated from $\mathcal{P}$ in favor of the fluctuations by using Eq. (5), so

$$\overline{\mathcal{P}} = \overline{\Gamma(1 - R\Delta\Gamma)} = \overline{\Gamma} - R\,\overline{\Delta\Gamma^2}; \tag{8}$$

also, upon integrating by parts and using the boundary conditions one obtains the positive-definite form

$$\overline{\mathcal{D}} = R^{-1}\overline{\left\langle |\nabla\widetilde{T}|^2 \right\rangle}. \tag{9}$$

With the aid of Eqs. (8) and (9), one can rewrite Eq. (7) in the form

$$\overline{\Gamma} = R^{-1}\overline{\left\langle |\nabla\widetilde{T}|^2 \right\rangle} + R\,\overline{\Delta\Gamma^2}, \tag{10}$$

demonstrating that $\overline{\Gamma}$ is non-negative. This requirement is essential for successful use of the bounding method.

The simplest variational principle is, therefore, to maximize $\overline{\Gamma}$ subject to the basic constraint (7). If one employs the method of Lagrange multipliers, this amounts to maximizing the functional $\overline{\gamma}\{\widetilde{T}, \widetilde{u}\} \doteq \overline{\Gamma} + \lambda(\overline{\mathcal{P}} - \overline{\mathcal{D}})$. In this form the result is more general than its derivation from the specific equation (1).

One must inquire whether this single constraint is sufficient. First, note that the crucial term in Eq. (10) is $\overline{\Delta\Gamma^2}$, which unlike the other two terms is of fourth order in the fluctuations. This is the fundamental nonlinearity mentioned above. It turns out that for self-consistent problems the competition between second- and fourth-order terms is essential for obtaining a finite bound; when $\langle T \rangle'$ is taken to be constant ($= -1$), $\Delta\Gamma$ vanishes and there is sufficient freedom that the bound is infinity. Kraichnan has pointed out[12] that for this case, which includes homogeneous turbulence, one must consider higher-order moments, but this is very difficult and has not yet been carried out. For passive problems, one can still obtain a finite bound[5] for constant $\langle T \rangle'$, but it is not very useful.

The basic constraint is not necessarily sufficient, even for inhomogeneous problems. In general, when u is passive the basic constraint is enough to produce a well-posed variational problem with a finite bound.[5] Self-consistent problems are more involved. Depending on the particular form of the dynamical equations and the chosen definition of $\mathcal{E}$, often a second constraint is necessary in order to adequately pin down the relation between $\tilde{u}$ and $\tilde{T}$. However, in Sec. III we describe an example for which the basic constraint is sufficient.

Note that the basic theory produces time-*independent* Euler-Lagrange equations. This is a qualitative difference from the true physics. It means that the solutions of the Euler-Lagrange equations are at best suggestive of reality; however, one expects that they correctly capture aspects of the spatial dependence of an instantaneous snapshot of the true solution. One might find it remarkable that a time-independent theory can make rigorous predictions about equations that fluctuate randomly in time.

Since Howard's original work, the optimum approach has been applied to a variety of problems of interest in fluid dynamics; these were reviewed by Busse.[4] All of the calculations described by Busse were self-consistent and led to rather difficult problems in applied mathematics. (We shall see an example of such a calculation in Sec. III below.) However, the problem of passive advection is also of interest. It arises in a number of physical contexts, including the important problem of transport in specified stochastic magnetic fields, and it illustrates the method with a minimum of mathematical complexity. We shall briefly review results on the passive problem.

II. BOUNDS FOR PASSIVE ADVECTION

The general theory of bounds for passive advection was given by Krommes and Smith[5] (KS). Although passive problems are in general simpler than self-consistent ones, in one sense they are more difficult: Whereas for self-consistent problems the autocorrelation time and/or length of $\tilde{u}$ can be computed self-consistently in terms of solely Reynolds' number, for passive problems the correlation function of $\tilde{u}$ can be specified, thereby introducing extra parameters. As an example, KS considered a *reference model*, a version of Eq. (1) in which the velocity field is taken to be a one-dimensional, centered Gaussian random variable $\tilde{u}(t)$ with (in suitable dimensionless units) unit variance and autocorrelation time K (Kubo number). This model can be solved exactly, either analytically[5] for $K = \infty$ or numerically.[8] For $K = \infty$, the basic bound $\overline{\gamma}_\infty$ is about 25% too high in the worst case $R \to \infty$. This is considered to be excellent agreement in view of the simplicity of the calculation. (The direct-interaction approximation leads to a somewhat more precise result,[5] but at the cost of considerably more labor.) For finite K the basic bound is qualitatively deficient: since it is constructed from equal-time and equal-space moments, it cannot contain information about the dynamical effects of finite correlation times and/or lengths related to the advecting velocity. Krommes and Smith pointed out that in this situation it was essential to invoke a *two-time* constraint; they implemented a simple one. Upon solving an unusual integral equation, they found $\overline{\gamma}^{-1} = \overline{\gamma}_q^{-1} + \overline{\gamma}_\infty^{-1}$, where $\overline{\gamma}_q$ is the quasilinear flux that is the true answer for $K \to 0$. This result is intuitively reasonable; it reduces naturally to both limits $K \to \infty$ and $K \to 0$. However, it must be stressed that this is much more than just a plausible interpolation formula; it is the *rigorous* bound on the flux predicted by Eq. (1) (under the particular two-time constraint that was employed).

As a practical example, when the one-point basic bound is applied to the generic problem of particle transport in specified stochastic magnetic fields[13] one obtains[5] the scaling correct for the strong turbulence regime. To do better and recover the familiar quasilinear result[14,13] in the usual approximation of static fields one must employ a two-space-point constraint. Kim and Krommes have shown how to do this.[8]

III. SELF-CONSISTENT BOUND FOR THE TURBULENT ELECTROMOTIVE FORCE IN REVERSED-FIELD PINCHES

We would now like to gain experience with a self-consistent calculation relevant to plasma physics. As we have emphasized, the version of the bounding theory we describe here applies to problems in which the mean profile is imposed by inhomogeneous boundary conditions and is self-consistent with the fluctuations. This precludes its direct use for studies of turbulence in a region whose dimensions are small compared to macroscopic system size.[5] For example, models of homogeneous turbulence with constant background gradient and periodic boundary conditions cannot be treated with the present method. Rather, what is needed is a macroscopic system with statistically sharp boundary conditions at the walls and describable uniformly in space by one "simple" set of partial differential equations. Such a model is furnished by the reversed-field pinch[15] (RFP) in the resistive magnetohydrodynamic (MHD) description. This problem is of considerable theoretical and practical interest. Taylor's theory of relaxation in the presence of global helicity conservation[16] has had considerable success in predicting features of the RFP such as field reversal. Montgomery and Phillips have explored related principles.[17] However, it must be noted that none of these principles is rigorous, and they have little to say about the details of the underlying turbulence. The optimum theory offers intriguing possibilities for going beyond such work. First, it is entirely rigorous. Second, to the extent that the optimum variational principle reflects physical reality it makes definite statements about the nonlinear fluctuations. Unfortunately, the extent to which this is true is difficult to predict *a priori*, so experience and, ultimately, much deeper insights are desirable. We shall now explore the predictions of the basic bound. However, since the calculation is quite involved we can only present the highlights here. An initial account of this work can be found in Ref. 18. For more details, see Ref. 19.

A. The MHD description and turbulent electromotive force

Consider a cylindrical pinch described by resistive, viscous, incompressible MHD. Ohm's law is

$$\boldsymbol{E}_{\text{ext}} - \boldsymbol{E} - c^{-1}\boldsymbol{u}\times\boldsymbol{B} = \eta \boldsymbol{j}, \tag{11}$$

where $\boldsymbol{E}_{\text{ext}} \doteq \hat{\boldsymbol{z}} E_0$ is an external driving field and $\boldsymbol{E}$ is the internal field produced by the plasma. The resistivity η is (unrealistically) assumed to be constant. We shall assume that $\nabla\cdot\boldsymbol{u} = 0$, so the mass density ρ is constant. We also model the dissipative stress tensor as isotropic, assuming the kinematic viscosity ν to be constant. The momentum equation is then

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\boldsymbol{u} = -\nabla(P/\rho) + (\rho c)^{-1}\boldsymbol{j}\times\boldsymbol{B} + \nu\nabla^2\boldsymbol{u}. \tag{12a}$$

(Although it has been argued that compressibility may be important for a precise description of field reversal,[20] reversal is still possible in the incompressible model, which leads to major analytic simplifications.) We consider cylindrical geometry (r, θ, z) with constant toroidal field B_0. It is convenient to introduce as units of time, space, and magnetic field the resistive-diffusion time $\tau_\eta \doteq 4\pi a^2/\eta c^2$, the minor radius a, and B_0. If we also introduce the viscous-diffusion time $\tau_\nu \doteq a^2/\nu$ and an Alfvén transit time $\tau_A \doteq a/c_A$, where $c_A^2 \doteq B_0^2/4\pi\rho$, then it is natural to introduce as dimensionless parameters the magnetic Prandtl number $P_m \doteq \tau_\eta/\tau_\nu = 4\pi\nu/\eta c^2$ and the Hartmann number $H \doteq (\tau_\eta\tau_\nu/\tau_A^2)^{1/2} = (a^2B_0^2/c^2\eta\nu\rho)^{1/2}$. Then in terms of the vorticity $\boldsymbol{\omega} \doteq \nabla\times\boldsymbol{u}$ one can write Eq. (12a) as

$$P_m^{-1}\frac{\partial\boldsymbol{u}}{\partial t} = -\nabla P' + P_m^{-1}\boldsymbol{u}\times\boldsymbol{\omega} + H^2\boldsymbol{j}\times\boldsymbol{B} - \nabla\times\boldsymbol{\omega}, \tag{12b}$$

where $P' \doteq P/\rho + P_m^{-1}u^2/2$. We also have the (pre-)Maxwell equations

$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}, \tag{13a}$$

$$\nabla\times\boldsymbol{B} = \boldsymbol{j}, \tag{13b}$$

$$\nabla\cdot\boldsymbol{B} = 0. \tag{13c}$$

In addition to the assumption of steady state, we assume statistical homogeneity in the θ and z directions. This, together with regularity at $r = 0$, significantly constrains the form of the mean fields. For example, since $\partial\langle\ldots\rangle/\partial\theta = \partial\langle\ldots\rangle/\partial z = 0$, Eq. (13c) reduces to $r^{-1}\partial(r\langle B_r\rangle)/\partial r = 0$, the regular solution of which is $\langle B_r\rangle \equiv 0$. Similarly, we find $\langle u_r\rangle = \langle j_r\rangle = \langle\omega_r\rangle = 0$. Since in steady state $\nabla\times\langle\boldsymbol{E}\rangle = 0$ from Eq. (13a), we must have $\langle\boldsymbol{E}\rangle = -\nabla\langle\varphi\rangle \propto \hat{\boldsymbol{r}}$. From these results we deduce that $\langle\boldsymbol{E}\rangle + \langle\boldsymbol{u}\rangle\times\langle\boldsymbol{B}\rangle \propto \hat{\boldsymbol{r}}$. Because of the presence of viscosity, we assume that $\langle\boldsymbol{u}\rangle$ vanishes at the wall.

As boundary conditions on the fluctuations we shall assume that all fields are regular at $r = 0$ and that at the wall $\tilde{\boldsymbol{u}} = \tilde{B}_r = \hat{\boldsymbol{r}}\times\tilde{\boldsymbol{j}} = 0$. Other choices are possible and the issue is a difficult one.[19,18] The present choice allows us to demonstrate the use of the optimum principle with a minimum of complications.

We are now prepared to recognize the analogies to the generic transport problem sketched in Sec. I.A. Indeed, the curl of Eq. (11),

$$\frac{\partial}{\partial t}\boldsymbol{B} - \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) - \eta\nabla^2\boldsymbol{B} = 0,$$

is analogous to Eq. (1). We shall assume that the total axial current is fixed and treat E_0 as a derived quantity. In this way one may treat the current as the driving force, analogous to ΔT in the generic problem; E_0 is analogous to the total flux Γ_{tot}. More quantitatively, consider the mean Ohms' law

$$\boldsymbol{E}_{\text{ext}} + \langle\boldsymbol{E}\rangle + \langle\boldsymbol{u}\rangle\times\langle\boldsymbol{B}\rangle = \langle\boldsymbol{j}\rangle + \boldsymbol{Q}, \tag{14}$$

where $\boldsymbol{Q}(r) \doteq \langle\widetilde{\boldsymbol{B}}\times\widetilde{\boldsymbol{u}}\rangle(r)$. The z component of Eq. (14) leads to

$$E_0 = \langle j_z\rangle(r) + \varepsilon(r), \tag{15}$$

where $\varepsilon(r) \doteq Q_z(r)$ [*cf.* Eq. (2b)]; Eq. (15) should be compared with Eqs. (3) and (2). In cylindrical geometry the natural barring operation is $\overline{A} \doteq 2\int_0^1 dr\, r\, A(r)$. The driving force is then the mean axial current density $\mathcal{J} \doteq \overline{\langle j_z\rangle}$. Upon barring Eq. (15), we obtain the alternative form of Ohm's law

$$E_0 = \mathcal{J} + \overline{\varepsilon}; \tag{16}$$

cf. Eq. (4). Upon combining Eqs. (15) and (16), we obtain

$$\langle j_z\rangle = \mathcal{J} - \Delta\varepsilon;$$

cf. Eq. (5). For future use, we can use this result and $\langle j_r\rangle = 0$ to deduce

$$\overline{\langle\boldsymbol{j}\rangle\cdot\boldsymbol{Q}} = \mathcal{J}\overline{\varepsilon} - \left(\overline{\Delta\varepsilon^2} + \overline{Q_\theta^2}\right). \tag{17}$$

B. Energy constraint

For the basic constraint we use the energy functional $\langle(\widetilde{u}^2 + \widetilde{B}^2)/2\rangle$ and seek an upper bound on $\overline{\varepsilon}$ constrained by global energy balance. To determine the energy constraint we begin with the fluctuating parts of the curls of Eqs. (11) and (12):

$$\frac{\partial}{\partial t}\widetilde{\boldsymbol{B}} = \nabla\times[\widetilde{\boldsymbol{u}}\times\langle\boldsymbol{B}\rangle - \langle\boldsymbol{u}\rangle\times\widetilde{\boldsymbol{B}} - (\widetilde{\boldsymbol{B}}\times\widetilde{\boldsymbol{u}} - \boldsymbol{Q}) - \widetilde{\boldsymbol{j}}], \tag{18}$$

$$\begin{aligned} P_m^{-1}\frac{\partial}{\partial t}\widetilde{\boldsymbol{u}} = -\nabla\widetilde{P'} - P_m^{-1}[\widetilde{\boldsymbol{u}}\times\langle\boldsymbol{\omega}\rangle - \langle\boldsymbol{u}\rangle\times\widetilde{\boldsymbol{\omega}} + (\widetilde{\boldsymbol{u}}\times\widetilde{\boldsymbol{\omega}} - \langle\widetilde{\boldsymbol{u}}\times\widetilde{\boldsymbol{\omega}}\rangle)] \\ + H^2[\widetilde{\boldsymbol{j}}\times\langle\boldsymbol{B}\rangle + \langle\boldsymbol{j}\rangle\times\widetilde{\boldsymbol{B}} + (\widetilde{\boldsymbol{j}}\times\widetilde{\boldsymbol{B}} - \langle\widetilde{\boldsymbol{j}}\times\widetilde{\boldsymbol{B}}\rangle)] - \nabla\times\widetilde{\boldsymbol{\omega}}. \end{aligned} \tag{19}$$

Upon adding the scalar products of Eq. (18) with $\widetilde{\boldsymbol{B}}$ and Eq. (19) with $\widetilde{\boldsymbol{u}}$ and barring the result, we are led to the steady-state condition

$$\frac{\partial}{\partial t}\left[\overline{\tfrac{1}{2}\left(H^2\left\langle|\widetilde{\boldsymbol{B}}|^2\right\rangle + P_{\rm m}^{-1}\left\langle|\widetilde{\mathbf{u}}|^2\right\rangle\right)}\right] = 0 = \mathcal{C}_{\rm E} \doteq \overline{\langle\boldsymbol{u}\rangle\cdot\boldsymbol{G}} + H^2\overline{\langle\boldsymbol{j}\rangle\cdot\boldsymbol{Q}} - \left(\overline{H^2\left\langle|\widetilde{\boldsymbol{j}}|^2\right\rangle} + \overline{\left\langle|\widetilde{\boldsymbol{\omega}}|^2\right\rangle}\right), \tag{20}$$

where $\boldsymbol{G} \doteq H^2\langle\widetilde{\boldsymbol{B}}\times\widetilde{\boldsymbol{j}}\rangle + P_{\rm m}^{-1}\langle\widetilde{\boldsymbol{\omega}}\times\widetilde{\mathbf{u}}\rangle$. The mean of Eq. (12) can be used to show that (in steady state) $-\overline{\langle\boldsymbol{u}\rangle\cdot\boldsymbol{G}} = \overline{|\langle\boldsymbol{\omega}\rangle|^2} \doteq 2\boldsymbol{\Omega}$. This result and Eq. (17) can then be used to demonstrate that $\bar{\varepsilon} \geq 0$, analogous to Eq. (10). After considerable algebra an explicit formula for $\boldsymbol{\Omega}$ in terms of the fluctuations can be obtained.[19] Analogous manipulations lead one to a helicity balance constraint $\mathcal{C}_{\rm H}$. Although we shall not deal with the helicity constraint in this work, it may be helpful to show in the equations to follow where the effects of that constraint would enter [terms of $\mathcal{O}(\zeta_{\rm h})$]. For more discussion, see Ref. 19.

C. Variational principle

We consider then the variational problem: Maximize

$$\overline{\mathcal{A}}(\widetilde{\mathbf{u}}, \widetilde{\boldsymbol{B}}) \doteq \bar{\varepsilon} + \lambda_{\rm E}\mathcal{C}_{\rm E} + \lambda_{\rm H}\mathcal{C}_{\rm H} + \overline{\lambda_{\mathbf{u}}(\boldsymbol{x})(\boldsymbol{\nabla}\cdot\widetilde{\mathbf{u}})} + \overline{\lambda_{\boldsymbol{B}}(\boldsymbol{x})(\boldsymbol{\nabla}\cdot\widetilde{\boldsymbol{B}})}$$

for fixed $\mathcal{J}$. After some algebra, the Euler-Lagrange equations can be cast into the form

$$0 = \boldsymbol{\nabla}\times\widetilde{\boldsymbol{j}} + \zeta_{\rm h}\widetilde{\boldsymbol{j}} - \widetilde{\mathbf{u}}\times\boldsymbol{L} - H^{-2}\boldsymbol{\Omega}_{\rm B} + \boldsymbol{\nabla}\widehat{\lambda}_{\rm B}, \tag{21a}$$

$$0 = H^{-2}\boldsymbol{\nabla}\times\widetilde{\boldsymbol{\omega}} + \widetilde{\boldsymbol{B}}\times\boldsymbol{L} - H^{-2}\boldsymbol{\Omega}_{u} + \boldsymbol{\nabla}\widehat{\lambda}_{\mathbf{u}}, \tag{21b}$$

where $\widehat{\lambda} \propto \lambda$,

$$\boldsymbol{L} \doteq [-Q_\theta + \mathcal{O}(\zeta_{\rm h})]\hat{\boldsymbol{\theta}} - \tfrac{1}{2}[\zeta\mathcal{J} - 2\Delta\varepsilon + \mathcal{O}(\zeta_{\rm h})]\hat{\boldsymbol{z}},$$

$\boldsymbol{\Omega}_{\boldsymbol{v}} \doteq \delta\boldsymbol{\Omega}/\delta\boldsymbol{v}$ for $\boldsymbol{v} \in \{\widetilde{\boldsymbol{B}}, \widetilde{\boldsymbol{u}}\}$, $\zeta_{\rm h} \doteq \mathcal{J}(\lambda_{\rm H}/\lambda_{\rm E})$ (in the absence of the helicity constraint, $\zeta_{\rm h} \equiv 0$), and $\zeta \doteq 1 + \lambda_{\rm E}^{-1} + \mathcal{O}(\zeta_{\rm h})$. A relation between ζ and the fluctuations can be obtained[19] by manipulating the expressions for $\overline{\widetilde{\boldsymbol{B}}\cdot\delta\overline{\mathcal{A}}/\delta\widetilde{\boldsymbol{B}}}$ and $\overline{\widetilde{\mathbf{u}}\cdot\delta\overline{\mathcal{A}}/\delta\widetilde{\mathbf{u}}}$. For $\zeta_{\rm h} = 0$, one is eventually led to

$$\zeta = 2 - (\bar{\varepsilon}\mathcal{J})^{-1}\left[\overline{\left\langle|\widetilde{\boldsymbol{j}}|^2\right\rangle} + H^{-2}\overline{\left\langle|\widetilde{\boldsymbol{\omega}}|^2\right\rangle}\right].$$

It is convenient to eliminate $\lambda_{\boldsymbol{B}}$ and $\lambda_{\boldsymbol{u}}$ by applying the operators $\hat{\boldsymbol{r}}\cdot(\boldsymbol{\nabla}\times)^n$ ($n = 1$ or 2) to Eqs. (21). However, we must still ensure that the associated solenoidal constraints $\boldsymbol{\nabla}\cdot\boldsymbol{B}$ and $\boldsymbol{\nabla}\cdot\boldsymbol{u}$ remain satisfied. Therefore, we adopt the representation

$$\widetilde{\boldsymbol{B}} = \boldsymbol{\nabla}\times(\hat{\boldsymbol{r}}\psi_{\boldsymbol{B}}) - \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times(\hat{\boldsymbol{r}}\chi_{\boldsymbol{B}})$$

and similarly for $\tilde{\mathbf{u}}$. (For ψ and χ, the subscripts B or u are just labels; they do not denote differentiation.) We decompose $\boldsymbol{L}$ into linear and nonlinear parts, $\boldsymbol{L} = \boldsymbol{L}_l + \boldsymbol{L}_n$, where

$$\boldsymbol{L}_l \doteq \tfrac{1}{2}\mathcal{J}\zeta\hat{\boldsymbol{z}} + \mathcal{O}(\zeta_h)\hat{\boldsymbol{\theta}},$$
$$\boldsymbol{L}_n \doteq [-Q_\theta + \mathcal{O}(\zeta_h)]\hat{\boldsymbol{\theta}} + [-\Delta\varepsilon + \mathcal{O}(\zeta_h)]\hat{\boldsymbol{z}}.$$

Then we obtain the nonlinear, time-independent boundary-value problem

$$\hat{\boldsymbol{r}}\cdot\nabla\times\nabla\times\tilde{\boldsymbol{j}} + \zeta_h\hat{\boldsymbol{r}}\cdot\nabla\times\tilde{\boldsymbol{j}} - \boldsymbol{L}_l\cdot\nabla\tilde{u}_r = C_{\psi_B}, \tag{22a}$$
$$H^{-2}\hat{\boldsymbol{r}}\cdot\nabla\times\nabla\times\tilde{\boldsymbol{\omega}} + \boldsymbol{L}_l\cdot\nabla\tilde{B}_r = C_{\psi_u}, \tag{22b}$$
$$\hat{\boldsymbol{r}}\cdot\nabla\times\nabla\times\nabla\times\tilde{\boldsymbol{j}} + \zeta_h\hat{\boldsymbol{r}}\cdot\nabla\times\nabla\times\tilde{\boldsymbol{j}} - \boldsymbol{L}_l\cdot\nabla\tilde{\omega}_r = C_{\chi_B}, \tag{22c}$$
$$H^{-2}\hat{\boldsymbol{r}}\cdot\nabla\times\nabla\times\nabla\times\tilde{\boldsymbol{\omega}} + \boldsymbol{L}_l\cdot\nabla\tilde{j}_r = C_{\chi_u}, \tag{22d}$$

where

$$C_{\psi_B} \doteq \boldsymbol{L}_n\cdot\nabla\tilde{u}_r - H^{-2}\hat{\boldsymbol{r}}\cdot\nabla\times\boldsymbol{\Omega}_B,$$
$$C_{\psi_u} \doteq -\boldsymbol{L}_n\cdot\nabla\tilde{B}_r - H^{-2}\hat{\boldsymbol{r}}\cdot\nabla\times\boldsymbol{\Omega}_u,$$
$$C_{\chi_B} \doteq \hat{\boldsymbol{r}}\cdot\nabla\times\nabla\times(\tilde{\mathbf{u}}\times\boldsymbol{L}_n) - H^{-2}\hat{\boldsymbol{r}}\cdot\nabla\times\nabla\times\boldsymbol{\Omega}_B,$$
$$C_{\chi_u} \doteq -\hat{\boldsymbol{r}}\nabla\times\nabla\times(\tilde{\boldsymbol{B}}\times\boldsymbol{L}_n) - H^{-2}\hat{\boldsymbol{r}}\cdot\nabla\times\nabla\times\boldsymbol{\Omega}_u.$$

The highest order of the derivatives in each of Eqs. (22c) and (22d) is 6 [since, e.g., $\boldsymbol{j} = \nabla\times\boldsymbol{B} \sim \nabla\times\nabla\times\nabla\times(\hat{\boldsymbol{r}}\chi_B)$]; therefore, the system is of twelfth order. Determining the explicit expressions of the various curls requires tedious algebra; the results may be found in Ref. 19.

D. Energy stability and the critical current

The solution of the Euler-Lagrange problem determines the bounding curve $\bar{\varepsilon}(\mathcal{J})$ or, equivalently, $\mathcal{J}(\bar{\varepsilon})$. The latter form is often more convenient because the point $\bar{\varepsilon} = 0$ has a special status. Namely, since $\bar{\varepsilon} \geq 0$, the quantity $\mathcal{J}_c \doteq \mathcal{J}(0)$ determines the *critical current* below which no solution exists to the Euler-Lagrange equations. It can be shown[19] that $\mathcal{J}_c$ is identical to the energy stability criterion[3,21]: for currents below $\mathcal{J}_c$, perturbations of arbitrary size decay monotonically, so finite-amplitude steady-state turbulence cannot exist for $\mathcal{J} < \mathcal{J}_c$. Since energy stability is a very strong requirement, it is not surprising that $\mathcal{J}_c$ is very small. Nevertheless, because $\mathcal{J}_c$ is the first point on the bounding curve, it is important to compute it: The shape of the critical eigenfunctions provides the natural first guess for iterative numerical solution for small but finite $\bar{\varepsilon}$.

For $\bar{\varepsilon} = 0$ the right-hand sides of Eqs. (22) vanish and we are led to a linear eigenvalue problem. To solve this, we first Fourier transform in the homogeneous directions θ and z: $\varphi = \sum_{\boldsymbol{k}} \varphi_{\boldsymbol{k}}(r)\exp(i(m\theta+k_z z))$ where $\varphi \in \{\psi, \chi\}$, $\boldsymbol{k} \doteq (m, k_z)$, and $k_z \doteq 2\pi n/L$. The operator $\Lambda_{\mathbf{k}} \doteq -iL_1\cdot\nabla = k_z\mathcal{J}/2 + O(\zeta_h)$ plays the role of eigenvalue for fixed $\mathbf{k}$. From $\Lambda_{\mathbf{k}}$ we then obtain $\mathcal{J}_{\mathbf{k}}$; the desired critical current is $\mathcal{J}_c = \inf_{\mathbf{k}} \mathcal{J}_{\mathbf{k}}$.

For $\zeta_h = 0$ the Hartmann number can be removed by the rescaling $\tilde{\boldsymbol{B}} = H^{-1}\tilde{\boldsymbol{B}}'$, $\Lambda_{\mathbf{k}} = H^{-1}\Lambda'_{\mathbf{k}}$. The resulting system is somewhat analogous to Bessel's equation and must be solved numerically. Looking ahead to the nonlinear problem we choose the procedure of Lentini and Pereyra,[22] a variable-order, variable-step-size finite-difference method with deferred corrections.[23] To ensure that the variables remain regular at the origin, we follow the analysis of Lentini and Keller,[24] Keller,[25] and de Hoog and Weiss.[26] In this procedure the system is written in the form $\boldsymbol{y}' = r^{-1}\mathbf{A}(r)\boldsymbol{y} + \mathbf{B}(r,\boldsymbol{y})\boldsymbol{y}$, where $\mathbf{A}$ and $\mathbf{B}$ are regular as $r \to 0$. To determine the behavior near the origin we may ignore $\mathbf{B}$; then, in the special case that $\mathbf{A}$ can be diagonalized the variables $\boldsymbol{z}$ in the diagonal representation obey $z_i \propto r^{\Lambda_i}$, where Λ_i is an eigenvalue of $\mathbf{A}$. Regularity requires that all z's with negative Λ be set to 0. These conditions constrain linear combinations of the y's to vanish. In practice, we apply these conditions at $r = \epsilon$, where $\epsilon \ll 1$. In the present problem $\mathbf{A}$ cannot be diagonalized for most m's; however, it can be brought to Jordan canonical form and an analogous procedure applied. The details are quite involved.[19]

The results of the critical current calculation are[19] that the maximizing modes are $m = \pm 1$, $n = \pm 2$ for unit aspect ratio A, and that $\mathcal{J}_c \approx 40H^{-1}$. (To date, we have studied only $A = 1$, which is an interesting exemplary case.) Since H may be very large $[H \sim (\eta\nu)^{-1/2}]$, the critical current is far below the actual value observed in the RFP experiments $[\mathcal{J} = O(1)$ in the present units$]$. However, the dominance of $m = 1$ modes is in agreement with both numerical simulations and experiments. Also, it is in accord with the speculation of Caramana[27] that $|n| \sim 2R/a$.

E. Single-mode bounding curve

We now turn to the solution of the nonlinear system (22). Again, for arbitrary H this must be done numerically. For very large H (as in many experiments), an analytic calculation using singular perturbation theory is suggested[4]; however, this has not been attempted yet.

It is characteristic that the Euler-Lagrange equations derived from the optimum principle have nonlinearities of special form.[4] Specifically, the coefficients of $\boldsymbol{L}_n$ depend only on r, as do the coefficients of $\tilde{\boldsymbol{u}}$ or $\tilde{\boldsymbol{B}}$ in $\boldsymbol{\Omega}_u$ or $\boldsymbol{\Omega}_B$. Thus, Fourier transformation in θ and z is still appropriate. Unlike the linear problem, however, the Fourier modes are now coupled through the *values* of the nonlinear terms, which are determined by sums over all modes, and it is not guaranteed that solutions with single $\boldsymbol{k}$ maximize $\bar{\varepsilon}$. Presumably such solutions are appropriate for $\bar{\varepsilon}$ sufficiently close to 0 ($\mathcal{J}$ close to $\mathcal{J}_c$). (This does not necessarily imply that perturbation theory is adequate.) Motivated by experience with simpler problems,[4] it is presumed that as $\mathcal{J}$ is increased bifurcations eventually occur such that at $\mathcal{J} = \mathcal{J}_n$ n-mode solutions replace $(n-1)$-mode solutions as the maximizing ones. For our initial analysis of this extremely complicated problem we shall consider only single-mode solutions. In particular, we retain only the $(m,n) = (1,-2)$ mode, for which the rational surface falls inside the plasma. Furthermore, for reasons of computational simplicity we shall neglect in our preliminary calculations the terms describing the effects of the mean velocity. The resulting system is undetermined in the sense that $\mathrm{Re}\,\boldsymbol{B}_{\boldsymbol{k}} = C_{\boldsymbol{k}}\,\mathrm{Im}\,\boldsymbol{B}_{\boldsymbol{k}}$ for arbitrary real $C_{\boldsymbol{k}}$. However, for the single-mode solution it can be shown that $\bar{\varepsilon}$ is independent of $C_{\boldsymbol{k}}$. It is expected that this approximation affects the final bound by an amount of $\mathcal{O}(1)$.

In the absence of the helicity constraint the bounding curve satisfies the similarity law $H\mathcal{J} = f(H\bar{\varepsilon})$. The function f has been determined numerically for $H\mathcal{J} \lesssim 2 \times 10^3$; see Ref. 18 for a graph. At the largest values of $H\bar{\varepsilon}$ boundary layers are evident; this is consistent with the observation that the bounding curve appears to have entered its asymptotic regime, being essentially linear on a log-log plot for $H\mathcal{J} \gtrsim 5 \times 10^2$. In that regime we obtain

$$\bar{\varepsilon} \approx 0.4H^{0.1}\mathcal{J}^{1.1}.$$

Near the critical point $\bar{\varepsilon} \propto \mathcal{J} - \mathcal{J}_c$, as can be demonstrated analytically.

Examples of the nonlinear eigenfunctions are displayed in Ref. 19. The optimum profiles display field reversal for sufficiently large currents[18]; extrapolating a few data points obtained at modest H leads to a prediction of field reversal at a pinch parameter of ≈ 1.5 for $H \sim 10^5$, which is quite reasonable. The optimum states are not Taylor-like both near the wall (as would be expected) and near the center.

Further work must be done to explore in more detail the predictions and properties of the optimum states and to understand their relation to physical reality. However, we conclude that the energy upper bound on the turbulent emf is not unreasonable. This is in accord with previous experience[4] with self-consistent fluid problems. An obvious but technically quite difficult[19] extension of this calculation would be to include the helicity constraint.

IV. DISCUSSION

In summary, the theory of bounds is a "new" approach to the theory of plasma turbulence; though its history spans more than thirty years, to our knowledge the first discussion of it in the plasma physics literature is given in Ref. 21. Our calculations merely scratch the surface of possible plasma applications. Furthermore, there is renewed interest in the bounding technique in general because of its possible marriage with other approaches such as constrained decimation.[28,29] Although space does not permit a description of the techniques that have been proposed[12] for obtaining nontrivial lower bounds and extending the method to homogeneous turbulence, it is clear that such calculations would be highly desirable. In general, the theory of bounds on turbulent transport presents a fertile and challenging area for further research.

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REFERENCES

[1] L. N. Howard, *J. Fluid Mech.* **17**, 405 (1963).

[2] L. N. Howard, Ann. Rev. Fluid Mech. **4**, 473 (1972).

[3] D. D. Joseph, *Stability of Fluid Motions (Springer-Verlag, New York, 1976).*

[4] F. H. Busse, *Adv. Appl. Mech.* **18**, 77 (1978), and references therein.

[5] J. A. Krommes and R. A. Smith, *Ann. Phys.* **177**, 246 (1987).

[6] J. A. Krommes, in *Basic Plasma Physics II*, edited by A. A. Galeev and R. N. Sudan (North-Holland, Amsterdam, 1984), Chap. 5.5.

[7] J. A. Krommes, in *International Conference on Plasma Physics* (Pramāna, Indian Academy of Sciences, Bangalore, India, in press).

[8] C.-B. Kim and J. A. Krommes, *J. Stat. Phys.* **53**, 1103 (1988).

[9] R. H. Kraichnan, *J. Stat. Phys.* **51**, 949 (1988).

[10] R. H. Kraichnan, in *Nonlinear Dynamics*, edited by H.G. Helleman (New York Academy of Sciences, New York, 1980), p. 37.

[11] W. Malkus, *Proc. Roy. Soc. A* **225**, 196 (1954).

[12] R. H. Kraichnan, in Institute for Fusion Studies Report No. IFSR 318 (1987).

[13] J. A. Krommes, C. Oberman, and R. H. Kleva, *J. Plasma Phys.* **30**, 11 (1983).

[14] A. B. Rechester and M. N. Rosenbluth, *Phys. Rev. Lett.* **40**, 38 (1978).

[15] H. A. Bodin and A. A. Newton, *Nucl. Fusion* **20**, 1255 (1980).

[16] J. B. Taylor, *Rev. Mod. Phys.* **58**, 741 (1986).

[17] D. Montgomery and L. Phillips, *Phys. Rev. A* **38**, 2953 (1988).

[18] C.-B. Kim and J. A. Krommes, submitted to *Phys. Rev. Lett.* (1989).

[19] C.-B. Kim, Ph.D. thesis, Princeton University, 1989.

[20] D. Y. Aydemir, D. C. Barnes, E. J. Caramana, A. A. Mirin, R. A. Nebel, D. D. Schnack, and A. G. Sgro, *Phys. Fluids* **28**, 898 (1985).

[21] R. A. Smith, Ph.D. thesis, Princeton University, 1986.

[22] V. Pereyra, in *Lecture Notes in Computer Science* (Springer-Verlag, Berlin, 1978), Vol. 76, p. 67.

[23] This algorithm is implemented in the International Mathematical Subroutine Library (IMSL) [see, *e.g.*, *Sources and Development of Mathematical Software*, edited by Wayne R. Cowell (Prentice-Hall, Englewood Cliffs, NJ, 1984)] as the routine DVCPR.

[24] M. Lentini and H. B. Keller, *SIAM J. Numer. Anal.* **17**, 577 (1980).

[25] H. B. Keller, *Numerical Solution of Two Point Boundary Value Problems (SIAM, Philadelphia, 1976).*

[26] F. R. de Hoog and R. Weiss, *SIAM J. Numer. Anal.* **13**, 775 (1976).

[27] E. J. Caramana, *Phys. Fluids B* **1**, 2186 (1989).

[28] R. H. Kraichnan, in *Theoretical Approaches to Turbulence*, edited by D. L. Dwoyer, M. Y. Hussaini, and R. G. Voight (Springer, New York, 1985), p. 91.

[29] R. H. Kraichnan and S. Chen, *Physica D* **37**, 160 (1989).