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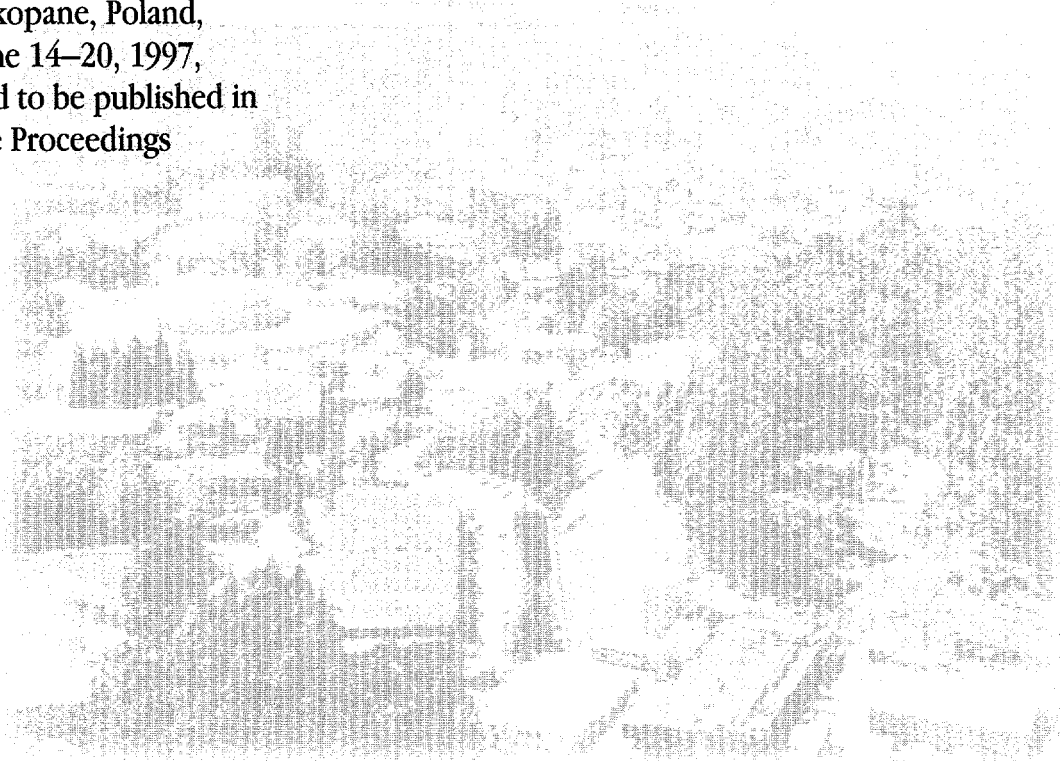
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UNIFICATION OF THE GENERAL NON-LINEAR SIGMA MODEL AND THE VIRASORO MASTER EQUATION *

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abstract

The Virasoro master equation describes a large set of conformal field theories known as the affine-Virasoro constructions, in the operator algebra (affine Lie algebra) of the WZW model, while the Einstein equations of the general non-linear sigma model describe another large set of conformal field theories. This talk summarizes recent work which unifies these two sets of conformal field theories, together with a presumable large class of new conformal field theories. The basic idea is to consider spin-two operators of the form $L_{ij}\partial x^i\partial x^j$ in the background of a general sigma model. The requirement that these operators satisfy the Virasoro algebra leads to a set of equations called the unified Einstein-Virasoro master equation, in which the spin-two spacetime field L_{ij} couples to the usual spacetime fields of the sigma model. The one-loop form of this unified system is presented, and some of its algebraic and geometric properties are discussed.

1. INTRODUCTION

There have been two broadly successful approaches to the construction of conformal field theories,

- The general affine-Virasoro construction¹⁻⁷
- The general non-linear sigma model⁸⁻¹³ (1)

but, although both approaches have been formulated as Einstein-like systems^{12, 2}, the relation between the two has remained unclear.

This talk summarizes recent work¹⁴ which unifies these two approaches, following the organization of Fig. 1. The figure shows the two developments (1) with the left column (the general affine-Virasoro construction) as a special case of the right column (the general non-linear sigma model). Our goal here is to explain the unification shown in the lower right of the figure.

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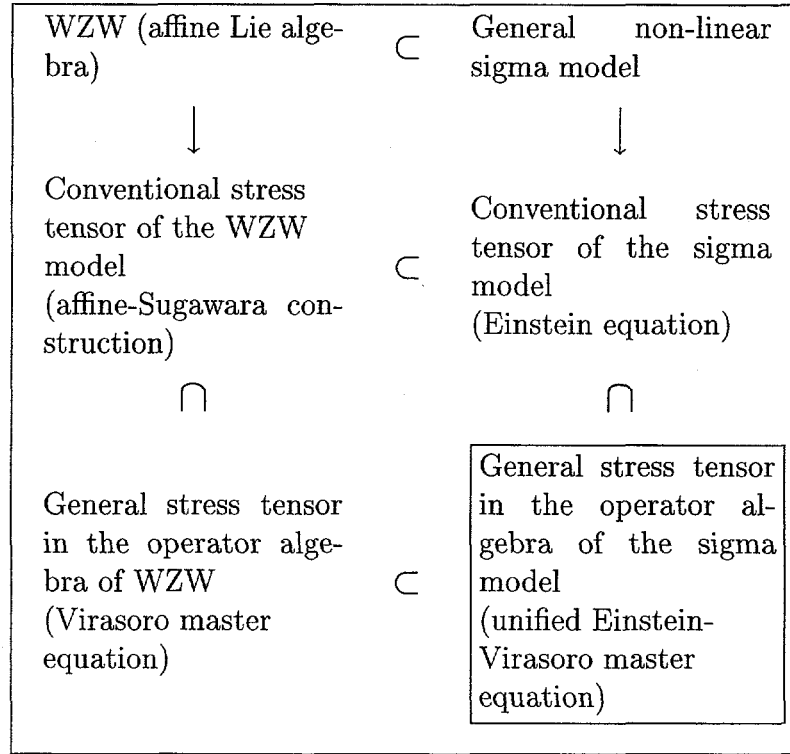


Figure 1. Conformal Field Theory

In the general affine-Virasoro construction, a large class of exact Virasoro operators^{1, 3}

$$T = L^{ab} * J_a J_b * + i D^a \partial J_a, \quad a, b = 1 \dots \dim(g) \quad (2)$$

are constructed as quadratic forms in the currents J of the general affine Lie algebra^{15, 16}. The coefficients $L^{ab} = L^{ba}$ and D^a are called the inverse inertia tensor and the improvement vector respectively. The general construction is summarized^{1, 3} by the (improved) Virasoro master equation (VME) for L and D , and this approach is the basis of irrational conformal field theory⁷ which includes the affine-Sugawara¹⁶⁻¹⁹ and coset constructions^{16, 17, 20} as a small subspace. The construction (2) can also be considered as the general Virasoro construction in the operator algebra of the WZW model^{21, 22}, which is the field-theoretic realization of the affine algebras. See Ref. 7 for a more detailed history of affine Lie algebra and the affine-Virasoro constructions.

For each non-linear sigma model, a Virasoro operator²³

$$T = -\frac{1}{2\alpha'} G_{ij} \partial x^i \partial x^j + \mathcal{O}(\alpha'^0) = -\frac{1}{2\alpha'} G^{ab} \Pi_a \Pi_b + \mathcal{O}(\alpha'^0) \quad (3a)$$

$$G^{ab} = e_i^a G^{ij} e_j^b, \quad \Pi_a = G_{ab} e_i^b \partial x^i, \quad i, j, a, b = 1, \dots, \dim(M) \quad (3b)$$

is constructed in a semiclassical expansion on an arbitrary manifold M , where G_{ij} is the metric on M and G^{ab} is the inverse of the tangent space metric. This is the canonical or conventional stress tensor of the sigma model and this construction is summarized^{12, 23} by the Einstein equations of the sigma model, which couple the metric G , the antisymmetric tensor field B and the dilaton Φ . In what follows we refer to these equations as the conventional Einstein equations of the sigma model, to distinguish them from the generalized Einstein equations obtained below.

In this paper, we unify these two approaches, using the fact that the WZW action is a special case of the general sigma model. More precisely, we study the general Virasoro construction

$$T = -\frac{1}{\alpha'} L_{ij} \partial x^i \partial x^j + \mathcal{O}(\alpha'^0) = -\frac{1}{\alpha'} L^{ab} \Pi_a \Pi_b + \mathcal{O}(\alpha'^0) \quad (4a)$$

$$i, j, a, b = 1, \dots, \dim(M) \quad (4b)$$

at one loop in the operator algebra of the general sigma model, where L is a symmetric second-rank spacetime tensor field, the inverse inertia tensor, which is to be determined. The unified construction is described by a system of equations which we call

- the Einstein-Virasoro master equation

of the general sigma model. This geometric system, which resides schematically in the lower right of Fig. 1, describes the covariant coupling of the spacetime fields L , G , B and Φ_a , where the vector field Φ_a generalizes the derivative $\nabla_a \Phi$ of the dilaton Φ .

The unified system contains as special cases the two constructions in (1): For the particular solution

$$L^{ab} = L_G^{ab} = \frac{G^{ab}}{2} + \mathcal{O}(\alpha'), \quad \Phi_a = \Phi_a^G = \nabla_a \Phi \quad (5)$$

the general stress tensors (4) reduce to the conventional stress tensors (3) and the Einstein-Virasoro master equation reduces to the conventional Einstein equations of the sigma model. Moreover, the unified system reduces to the general affine-Virasoro construction and the VME when the sigma model is taken to be the WZW action. In this case we find that the contribution of Φ_a to the unified system is precisely the known improvement term of the VME.

More generally, the unified system describes a space of conformal field theories which is presumably much larger than the sum of the general affine-Virasoro construction and the sigma model with its canonical stress tensors.

2. BACKGROUND

To settle notation and fix concepts which will be important below, we begin with a brief review of the two known constructions in (1), which are the two columns of Fig. 1.

2.1. The General Affine-Virasoro Construction

The improved VME

The general affine-Virasoro construction, which is the left column of Figure 1, begins with the currents of a general affine Lie algebra^{15, 16}

$$J_a(z) J_b(w) = \frac{G_{ab}}{(z-w)^2} + \frac{if_{ab}^c J_c(w)}{z-w} + \text{reg.} \quad (6)$$

where $a, b = 1 \dots \dim g$ and f_{ab}^c are the structure constants of g . For simple g , the central term in (6) has the form $G_{ab} = k\eta_{ab}$ where η_{ab} is the Killing metric of g and k is the level of the affine algebra. Then the general affine-Virasoro construction is¹

$$T = L^{ab} {}^* J_a J_b {}^* + iD^a \partial J_a \quad (7)$$

where the coefficients $L^{ab} = L^{ba}$ and D^a are the inverse inertia tensor and the improvement vector respectively. The stress tensor T is a Virasoro operator

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \text{reg.} \quad (8)$$

iff the improved Virasoro master equation¹

$$L^{ab} = 2L^{ac}G_{cd}L^{db} - L^{cd}L^{ef}f_{ce}^a f_{df}^b - L^{cd}f_{ce}^f f_{df}^{(a}L^{b)e} - f_{cd}^{(a}L^{b)c}D^d \quad (9a)$$

$$D^a(2G_{ab}L^{be} + f_{ab}^d L^{bc}f_{cd}^e) = D^e \quad (9b)$$

$$c = 2G_{ab}(L^{ab} + 6D^a D^b) \quad (9c)$$

is satisfied[§] by L and D , and the central charge of the construction is given in (9c). The unimproved VME^{1, 3} is obtained by setting the improvement vector D to zero.

K-conjugation covariance

A central property of the VME at zero improvement is K -conjugation covariance^{16, 17, 20, 1} which says that all solutions come in K -conjugate pairs L and $\tilde{L}$,

$$L^{ab} + \tilde{L}^{ab} = L_g^{ab}, \quad T + \tilde{T} = T_g, \quad c + \tilde{c} = c_g \quad (10a)$$

$$T(z)\tilde{T}(w) = \text{reg.} \quad (10b)$$

whose K -conjugate stress tensors $T, \tilde{T}$ commute and add to the affine-Sugawara construction [15–18] on g

$$T_g = L_g^{ab} {}^* J_a J_b {}^*. \quad (11)$$

For simple g , the inverse inertia tensor of the affine-Sugawara construction is

$$L_g^{ab} = \frac{\eta^{ab}}{2k + Q_g} = \frac{\eta^{ab}}{2k} + \mathcal{O}(k^{-2}) = \frac{G^{ab}}{2} + \mathcal{O}(k^{-2}) \quad (12)$$

where η^{ab} is the inverse Killing metric of g and Q_g is the quadratic Casimir of the adjoint. K -conjugation covariance can be used to generate new solutions $\tilde{L} = L_g - L$ from old solutions L and the simplest application of the covariance generates the coset constructions^{16, 17, 20} as $\tilde{L} = L_g - L_h = L_{g/h}$.

Semiclassical expansion

At zero improvement, the high-level or semiclassical expansion^{24, 7} of the VME has been studied in some detail. On simple g , the leading term in the expansion has the form

$$L^{ab} = \frac{P^{ab}}{2k} + \mathcal{O}(k^{-2}), \quad c = \text{rank}(P) + \mathcal{O}(k^{-1}) \quad (13a)$$

[§]Our convention is $A^{(a}B^{b)} = A^a B^b + A^b B^a$, $A^{[a}B^{b]} = A^a B^b - A^b B^a$.

$$P^{ac}\eta_{cd}P^{db} = P^{ab} \quad (13b)$$

where P is the high-level projector of the L theory. These are the solutions of the classical limit of the VME,

$$L^{ab} = 2L^{ac}G_{cd}L^{db} + \mathcal{O}(k^{-2}) \quad (14)$$

but a semiclassical quantization condition²⁴ provides a restriction on the allowed projectors. In the partial classification of the space of solutions by graph theory^{5, 25, 7}, the projectors P are closely related to the adjacency matrices of the graphs.

Irrational conformal field theory

Given also a set of antiholomorphic currents $\bar{J}_a$, $a = 1 \dots \dim(g)$, there is a corresponding antiholomorphic Virasoro construction

$$\bar{T} = L^{ab} * \bar{J}_a \bar{J}_b * + iD^a \partial \bar{J}_a \quad (15)$$

with $\bar{c} = c$. Each pair of stress tensors T and $\bar{T}$ then defines a conformal field theory (CFT) labelled by L and D . Starting from the modules of affine $g \times g$, the Hilbert space of a particular CFT is obtained^{26, 27, 7} by modding out by the local symmetry of the Hamiltonian.

It is known that the CFTs of the master equation have generically irrational central charge, even when attention is restricted to the space of unitary theories, and the study of all the CFTs of the master equation is called irrational conformal field theory (ICFT), which contains the affine-Sugawara and coset constructions as a small subspace.

In ICFT at zero improvement, world-sheet actions are known for the following cases: the affine-Sugawara constructions (WZW models^{21, 22}), the coset constructions (spin-one gauged WZW models²⁸) and the generic ICFT (spin-two gauged WZW models^{26, 29, 30}). The spin-two gauge symmetry of the generic ICFT is a consequence of K -conjugation covariance.

See Ref. 7 for a comprehensive review of ICFT, and Ref. 31 for a recent construction of a set of semiclassical blocks and correlators in ICFT.

In this talk, we restrict ourselves to holomorphic stress tensors, and the reader is referred to Ref. 14 for the antiholomorphic version.

WZW model

The left column of Fig. 1 can be considered as the set of constructions in the operator algebra of the WZW model, which is affine Lie algebra.

The WZW action is a special case of the general nonlinear sigma model, where the target space is a group manifold G and g is the algebra of G .

2.2. The General Non-Linear Sigma Model

The general non-linear sigma model (the right column of Fig. 1) has been extensively studied^{32, 33, 8, 34, 9, 10, 35, 36, 11, 12, 37, 38, 23, 13}.

The Euclidean action of the general non-linear sigma model is

$$S = \frac{1}{2\alpha'} \int d^2z (G_{ij} + B_{ij}) \partial x^i \bar{\partial} x^j \quad (16a)$$

$$d^2z = \frac{dx dy}{\pi}, \quad z = x + iy, \quad H_{ijk} = \partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij}. \quad (16b)$$

Here x^i , $i = 1 \dots \dim(M)$ are coordinates with the dimension of length on a general manifold M and α' , with dimension length squared, is the string tension or Regge slope. The fields G_{ij} and B_{ij} are the (covariantly constant) metric and antisymmetric tensor field on M .

We also introduce a covariantly constant vielbein e_i^a , $a = 1 \dots \dim(M)$ on M and use it to translate between Einstein and tangent-space indices, e.g. $G_{ij} = e_i^a G_{ab} e_j^b$, where G_{ab} is the covariantly constant metric on tangent space. Covariant derivatives are defined as usual in terms of the spin connection, $R_{ija}{}^b$ is the Riemann tensor and $R_{ab} = R_{acb}{}^c$ is the Ricci tensor. It will also be convenient to define the generalized connections and covariant derivatives with torsion,

$$\hat{\nabla}_i^\pm v_a = \partial_i v_a - \hat{\omega}_{ia}^{\pm b} v_b \quad (17a)$$

$$\hat{\omega}_{ia}^{\pm b} = \omega_{ia}^b \pm \frac{1}{2} H_{ia}^b \quad (17b)$$

$$\hat{R}_{ija}{}^b = (\partial_i \hat{\omega}_j^\pm - \partial_j \hat{\omega}_i^\pm - [\hat{\omega}_i^\pm, \hat{\omega}_j^\pm])_a{}^b \quad (17c)$$

where $\hat{\omega}_{iab}^\pm$ is antisymmetric under (a, b) interchange and $\hat{R}_{ijab}^\pm$ is pairwise antisymmetric in (i, j) and (a, b) .

Following Banks, Nemeschansky and Sen²³, the canonical or conventional stress tensors of the general sigma model have the form

$$T_G = -\frac{G_{ij}}{2\alpha'} \partial x^i \partial x^j + \partial^2 \Phi + T_1 + \mathcal{O}(\alpha') \quad (18a)$$

$$= -\frac{G^{ab}}{2\alpha'} \Pi_a \Pi_b + \partial^2 \Phi + T_1 + \mathcal{O}(\alpha') \quad (18b)$$

$$\Pi_a = G_{ab} e_i^b \partial x^i, \quad \bar{\Pi}_a = G_{ab} e_i^b \bar{\partial} x^i \quad (18c)$$

where Φ is the dilaton and T_1 is a finite one-loop counterterm which depends on the renormalization scheme. The condition that T_G is one-loop conformal reads¹²

$$R_{ij} + \frac{1}{4} (H^2)_{ij} - 2 \nabla_i \nabla_j \Phi = \mathcal{O}(\alpha') \quad (19a)$$

$$\nabla^k H_{kij} - 2 \nabla^k \Phi H_{kij} = \mathcal{O}(\alpha') \quad (19b)$$

$$c_G = \bar{c}_G = \dim(M) + 3\alpha' (4|\nabla \Phi|^2 - 4\nabla^2 \Phi + R + \frac{1}{12} H^2) + \mathcal{O}(\alpha'^2) \quad (19c)$$

where (19a) and (19b) are the conventional Einstein equations of the sigma model and (19c) is the central charge of the construction. The result for the central charge includes two-loop information, but covariant constancy of the field-dependent part of the central charge follows by Bianchi identities from the Einstein equations, so all three relations in (19) can be obtained with a little thought from the one-loop calculation. It will also be useful to note that the conventional Einstein equations (19a),(19b) can be written in either of two equivalent forms

$$\hat{R}_{ij}^\pm - 2 \hat{\nabla}_i^\pm \hat{\nabla}_j^\pm \Phi = \mathcal{O}(\alpha') \quad (20)$$

by using the generalized quantities (17) with torsion.

WZW data

The WZW action is a special case of the general sigma model (16a) on a group manifold G . Identifying the vielbein e on M with the left-invariant vielbein e on G , we find that $J_a = \frac{i}{\sqrt{\alpha'}} \Pi_a$ are the classical currents of WZW and

$$G_{ab} = k\eta_{ab}, \quad H_{ab}{}^c = \frac{1}{\sqrt{\alpha'}} f_{ab}{}^c. \quad (21)$$

Here $f_{ab}{}^c$ and η_{ab} are the structure constants and the Killing metric of g and k is the level of the affine algebra. From this data, one also computes

$$\omega_{ab}{}^c = -\frac{1}{2\sqrt{\alpha'}} f_{ab}{}^c \quad (22a)$$

$$\hat{\omega}_{ab}{}^c = 0, \quad \hat{\omega}_{ab}{}^c = -\frac{1}{\sqrt{\alpha'}} f_{ab}{}^c \quad (22b)$$

$$\hat{R}_{ija}{}^b = 0. \quad (22c)$$

Manifolds with vanishing generalized Riemann tensors are called parallelizable^{35, 37}.

2.3. Strategy

As seen in Fig. 1, our strategy here is a straightforward generalization of the VME to the sigma model, following the relation of the general affine-Virasoro construction to the WZW model. In the operator algebra of the general sigma model, we use the technique of Banks et al.²³ to study the general Virasoro construction

$$T = -\frac{L_{ij}}{\alpha'} \partial x^i \partial x^j + \mathcal{O}(\alpha'^0) = -\frac{L^{ab}}{\alpha'} \Pi_a \Pi_b + \mathcal{O}(\alpha'^0) \quad (23a)$$

$$\bar{\partial} T = 0 \quad (23b)$$

$$\langle T(z)T(w) \rangle = \frac{c/2}{(z-w)^4} + \frac{2\langle T(w) \rangle}{(z-w)^2} + \frac{\langle \partial T(w) \rangle}{(z-w)} + \text{reg.} \quad (23c)$$

where the dilatonic contribution is included at $\mathcal{O}(\alpha'^0)$ and L is a symmetric second-rank spacetime tensor field (the inverse inertia tensor) to be determined.

It is clear that this one-loop construction includes the conventional stress tensor T_G of the general sigma model, as well as the general affine-Virasoro construction when the sigma model is chosen to be WZW.

3. CLASSICAL PREVIEW OF THE CONSTRUCTION

The classical limit of the general construction (23a) can be studied with the classical equations of motion of the general sigma model

$$\bar{\partial} \Pi_a + \bar{\Pi}_b \Pi_c \hat{\omega}^{+bc}{}_a = 0 \quad (24)$$

where $\Pi, \bar{\Pi}$ are defined in (18c) and $\hat{\omega}^\pm$ are the generalized connections (17b) with torsion.

One then finds that the classical stress tensor is holomorphic

$$T = -\frac{L^{ab}}{\alpha'} \Pi_a \Pi_b, \quad \bar{\partial} T = 0 \quad (25)$$

iff the inverse inertia tensor is covariantly constant

$$\hat{\nabla}_i^+ L^{ab} = 0 \quad (26)$$

where $\hat{\nabla}^\pm$ are the generalized covariant derivatives (17a) with torsion. Further discussion of this covariant-constancy condition is found in Sections 5.2 and especially 5.5, which places the relation in a more geometric context.

To study the classical Virasoro conditions, we introduce Poisson brackets in Minkowski space, and study the classical chiral stress tensor

$$T_{++} = \frac{1}{8\pi\alpha'} L^{ab} J_a^+ J_b^+ \quad (27)$$

where J_a^+ is the Minkowski-space version of Π_a . This stress tensor satisfies the equal-time Virasoro algebra iff

$$L^{ab} = 2L^{ac} G_{cd} L^{db}, \quad (28)$$

which is the analogue on general manifolds of the high-level or classical limit (14) of the VME on group manifolds.

4. THE UNIFIED EINSTEIN-VIRASORO MASTER EQUATION

We summarize here the results obtained by enforcing the Virasoro condition (23c) at one loop. Details of the relevant background field expansions, Feynman diagrams and dimensional regularization can be found in Ref. 14.

Including the one-loop dilatonic and counterterm contributions, the holomorphic stress tensor T is

$$T = -L^{ab} \left(\frac{\Pi_a \Pi_b}{\alpha'} + \frac{1}{2} \Pi_c \Pi_d H_{ae}^c H_b^{ed} \right) + \partial(\Pi_a \Phi^a) + \mathcal{O}(\alpha') \quad (29a)$$

$$a, b = 1, \dots, \dim(M) \quad (29b)$$

where $L^{ab} = L^{ba}$ is the inverse inertia tensor and Π_a is defined in (18c). The second term in T is a finite one-loop counterterm which characterizes our renormalization scheme. The quantity Φ^a in (29a) is called the dilaton vector, and we will see below that the dilaton vector includes the conventional dilaton as a special case.

The necessary and sufficient condition that T satisfies the Virasoro algebra is the unified Einstein-Virasoro master equation

$$L^{cd} \hat{R}_{acdb}^+ + \hat{\nabla}_a^+ \Phi_b = \mathcal{O}(\alpha') \quad (30a)$$

$$\Phi_a = 2L_a^b \Phi_b + \mathcal{O}(\alpha') \quad (30b)$$

$$\hat{\nabla}_i^+ L^{ab} = \mathcal{O}(\alpha') \quad (30c)$$

$$\begin{aligned} L^{ab} = & 2L^{ac} G_{cd} L^{db} \\ & - \alpha' (L^{cd} L^{ef} H_{ce}^a H_{df}^b + L^{cd} H_{ce}^f H_{df}^{(a} L^{b)e}) \\ & - \alpha' (L^{c(a} G^{b)d} \nabla_{[c} \Phi_{d]}) + \mathcal{O}(\alpha'^2) \end{aligned} \quad (30d)$$

$$c = 2G_{ab} L^{ab} + 6\alpha' (2\Phi_a \Phi^a - \nabla_a \Phi^a) + \mathcal{O}(\alpha'^2) \quad (30e)$$

where the first line of (30d) is the classical master equation in (28).

In what follows, we refer to (30a) as the generalized Einstein equation of the sigma model, and equation (30b) is called the eigenvalue relation of the dilaton vector. Equation (30d) is called the generalized Virasoro master equation (VME) of the sigma model. The central charge (30e) is consistent¹⁴ by Bianchi identities with the rest of the unified system. The $\mathcal{O}(\alpha')$ corrections to the covariant-constancy condition (30c) can be computed in principle from the solutions of the generalized VME.

Some simple observations

1. Algebraic form of the generalized VME. In parallel with the VME, the generalized VME (30d) is an algebraic equation for L . This follows because any derivative of L can be removed by using the covariant-constancy condition (30c).
2. Semiclassical solutions of the generalized VME. The solutions of (30c) and (30d) have the form

$$L_a{}^b = \frac{1}{2} P_a{}^b + \mathcal{O}(\alpha') \quad (31a)$$

$$\hat{\nabla}_i^+ P_a{}^b = 0 \quad (31b)$$

where P is a covariantly constant projector, in parallel with the form (13) of the high-level solutions of the VME. The solutions of (31b) are further discussed in Section 5.5.

3. Correspondence with the VME. The non-dilatonic terms of the generalized VME (30d) have exactly the form of the unimproved VME (see eq. (9a)), after the covariant substitution

$$f_{ab}{}^c \rightarrow \sqrt{\alpha'} H_{ab}{}^c \quad (32)$$

for the general sigma model. This correspondence is the inverse of the WZW datum in (21),

$$H_{ab}{}^c = \frac{1}{\sqrt{\alpha'}} f_{ab}{}^c \quad (33)$$

which means that, for the special case of WZW, the non-dilatonic terms of the generalized VME will reduce correctly to those of the unimproved VME. We return to complete the WZW reduction in Section 5.2.

4. Dilaton solution for the dilaton vector. According to the classical limit (28) of the generalized VME, one solution of the eigenvalue relation (30b) for the dilaton vectors is

$$\Phi^a(\Phi) \equiv 2L^{ab} \nabla_b \Phi \quad (34)$$

In what follows, this solution is called the dilaton solution, and we shall see in the following section that the scalar field Φ is in fact the conventional dilaton of the sigma model.

5. PROPERTIES OF THE UNIFIED SYSTEM

5.1. The Conventional Stress Tensors of the Sigma Model

In this section, we check that the conventional stress tensors of the sigma model are correctly included in the unified system.

In the full system, the conventional stress tensor T_G of the sigma model corresponds to the particular solution of the generalized VME whose classical limit is

$$L^{ab} = L_G^{ab} = \frac{G^{ab}}{2} + \mathcal{O}(\alpha') \quad (35)$$

where G^{ab} is the inverse of the metric in the sigma model action. The covariant-constancy condition (30c) is trivially solved to this order because $\hat{\nabla}_i^\pm G_{ab} = 0$.

To obtain the form of T_G through one loop, we must also take the dilaton solution (34) for the dilaton vector, so that the dilaton contributes to the system as

$$\Phi_a = \Phi_a^G = \nabla_a \Phi + \mathcal{O}(\alpha'), \quad \nabla_{[a} \Phi_{b]}^G = \mathcal{O}(\alpha'). \quad (36)$$

The relations (35) and (36) then tell us that the generalized Einstein equation (30a) simplifies to the conventional Einstein equation

$$\hat{R}_{ab}^\pm - 2\hat{\nabla}_a^\pm \hat{\nabla}_b^\pm \Phi = \mathcal{O}(\alpha'). \quad (37)$$

Moreover, eq. (36) tells us that the dilaton terms do not contribute to the generalized VME in this case, and we may easily obtain

$$L^{ab} = L_G^{ab} = \frac{G^{ab}}{2} - \frac{\alpha'}{4}(H^2)^{ab} + \mathcal{O}(\alpha'^2) \quad (38a)$$

$$\hat{\nabla}_i^\pm L_G^{ab} = -\frac{\alpha'}{4}\hat{\nabla}_i^\pm (H^2)^{ab} + \mathcal{O}(\alpha'^2) \quad (38b)$$

$$T_G(\Phi) = -\frac{G^{ab}}{2\alpha'}\Pi_a\Pi_b + \partial^2\Phi + \mathcal{O}(\alpha') \quad (38c)$$

by solving the generalized VME through the indicated order. In this case, the stress tensor counterterm in (29a) cancels against the $\mathcal{O}(\alpha')$ correction to L_G , and (38c) are consistent with (18). In what follows, the stress tensor $T_G(\Phi)$ is called the conventional stress tensor of the sigma model.

To complete the check, we evaluate the central charge $c = c_G(\Phi)$ in this case,

$$c_G(\Phi) = 2G_{ab}\left(\frac{G^{ab}}{2} - \frac{\alpha'}{4}(H^2)^{ab}\right) + 6\alpha'(2|\nabla\Phi|^2 - \nabla^2\Phi) + \mathcal{O}(\alpha'^2) \quad (39a)$$

$$= \dim(M) + 3\alpha'(4|\nabla\Phi|^2 - 2\nabla^2\Phi - \frac{1}{6}H^2) + \mathcal{O}(\alpha'^2) \quad (39b)$$

$$= \dim(M) + 3\alpha'(4|\nabla\Phi|^2 - 4\nabla^2\Phi + R + \frac{1}{12}H^2) + \mathcal{O}(\alpha'^2) \quad (39c)$$

which agrees with the conventional central charge in (19c). To obtain the usual form in (39c), we used the conventional Einstein equations (19a) in the form $R = 2\nabla^2\Phi - \frac{1}{4}H^2$.

We also note the form of the system for $L = L_G$ with general dilaton vector Φ_a^G ,

$$T_G(\Phi_a) = -\frac{G^{ab}}{2\alpha'}\Pi_a\Pi_b + \partial(\Pi^a\Phi_a^G) + \mathcal{O}(\alpha') \quad (40a)$$

$$c_G(\Phi_a) = \dim(M) + 3\alpha'(4\Phi_a^G\Phi_G^a - 4\nabla_a\Phi_G^a + R + \frac{1}{12}H^2) + \mathcal{O}(\alpha'^2) \quad (40b)$$

$$\hat{R}_{ab}^+ - 2\hat{\nabla}_a^+\Phi_b^G = \mathcal{O}(\alpha') \quad (40c)$$

where Φ_a^G is unrestricted because its eigenvalue equation is trivial.

5.2. WZW and the Improved VME

In this section we check that, for the special case of WZW, the unified system reduces to the improved VME (9a), where the improvement vector D is constructed from the general dilaton vector.

Using the WZW datum above we find that the generalized VME (30d) has the form

$$L^{ab} = (\text{usual } L^2 \text{ and } L^2 f^2 \text{ terms}) + \sqrt{\alpha'} f_{cd}^{(a} L^{b)c} \Phi^d + \mathcal{O}(\alpha'^2) \quad (41)$$

when the sigma model is taken as WZW. The terms in parentheses are the usual terms (see eq. (9a)) of the unimproved VME. Next, we solve the generalized Einstein equation (30a) to find (using $\hat{R}^\pm = 0$) that the dilaton vector is a constant

$$\partial_i \Phi^a = 0. \quad (42)$$

It follows that the dilaton vector can be identified with the improvement vector of the VME in (9a)

$$D^a \equiv -\sqrt{\alpha'} \Phi^a = \text{constant}. \quad (43)$$

Moreover, the solution of the covariant-constancy condition (30c) is

$$\partial_i L^{ab} = 0, \quad L^{ab} = \text{constant} \quad (44)$$

because $\hat{\omega}^+ = 0$. This completes the recovery of the improved VME in (9a).

The central charge reduces in this case to

$$c = 2G_{ab}(L^{ab} + 6D^a D^b) + \mathcal{O}(k^{-2}) \quad (45)$$

in agreement with the central charge (9c) of the improved VME. We finally note that the eigenvalue relation (30b) of the dilaton vector can be rewritten with (43) as

$$2L^{ab} G_{bc} D^c = D^a + \mathcal{O}(k^{-2}) \quad (46)$$

which is recognized as the leading term of the exact eigenvalue relation (9b) of the improved VME. This completes the one-loop check of the unified Einstein-Virasoro master equation against the improved VME.

5.3. Alternate Forms of the Central Charge

Using the generalized Einstein equation and the generalized VME, the central charge (30e) can be written in a variety of forms,

$$c = 2L_a^a + 6\alpha'(2\Phi_a \Phi^a - \hat{\nabla}_a^+ \Phi^a) + \mathcal{O}(\alpha'^2) \quad (47a)$$

$$= 4L_a^b L_b^a + 2\alpha' [L_b^e L_d^f H^{bda} H_{efa} + 3(2\Phi_a \Phi^a - \hat{\nabla}_a^+ \Phi^a)] + \mathcal{O}(\alpha'^2) \quad (47b)$$

$$= \text{rank}(P) + 2\alpha' [L_a^b L_c^e (4L_d^f - 3\delta_d^f) H^{acd} H_{bef} + 3(2\Phi_a \Phi^a - \hat{\nabla}_a^+ \Phi^a)] + \mathcal{O}(\alpha'^2) \quad (47c)$$

$$= \text{rank}(P) + 2\alpha' [3L^{ab} \hat{R}_{ab}^+ + L_a^b L_c^e (4L_d^f - 3\delta_d^f) H^{acd} H_{bef} + 6(\Phi_a \Phi^a - \hat{\nabla}_a^+ \Phi^a)] + \mathcal{O}(\alpha'^2) \quad (47d)$$

$$= 4L_a^b L_b^a + 2\alpha' [3L^{ab} \hat{R}_{ab}^+ + L_a^b L_c^e H^{acd} H_{bed} + 6(\Phi_a \Phi^a - \hat{\nabla}_a^+ \Phi^a)] + \mathcal{O}(\alpha'^2). \quad (47e)$$

The first form in (47a) is the 'affine-Virasoro form' of the central charge. The form

in (47d), with the first occurrence of the generalized Ricci tensor, is called the 'conventional form' of the central charge because it reduces easily to the central charge of the conventional stress tensor

$$c_G(\Phi) = \dim(M) + 3\alpha'(4|\nabla\Phi|^2 - 4\nabla^2\Phi + R + \frac{1}{12}H^2) + \mathcal{O}(\alpha'^2) \quad (48)$$

when $P = G$ and $\Phi_a^G = \nabla_a\Phi$. The conventional form is also the form in which we found¹⁴ it most convenient to prove the constancy of c

$$\partial_i c = \hat{\nabla}_i^+ c = \mathcal{O}(\alpha'^2) \quad (49)$$

using the Bianchi identities and the rest of the unified system. The final form of c in (47e) is the form which we believe comes out directly from the two-loop computation.

5.4. Solution Classes and a Simplification

Class I and Class II solutions

The solutions of the unified system (30) can be divided into two classes:

Class I. T conformal but $T_G(\Phi_a)$ not conformal

Class II. T and $T_G(\Phi_a)$ both conformal.

The distinction here is based on whether or not (in addition to the generalized Einstein equation) the dilaton-vector Einstein equation in (40a) is *also* satisfied. In the case when the dilaton solution $\Phi_a(\Phi)$ in (34) is taken for the dilaton vector, the question is whether or not the background sigma model is itself conformal in the conventional sense.

In Class I, we are constructing a conformal stress tensor T in the operator algebra of a sigma model whose conventional stress tensor $T_G(\Phi_a)$ with general dilaton vector is not conformal. This is a situation not encountered in the general affine-Virasoro construction because the conventional stress-tensor T_g of the WZW model is the affine-Sugawara construction, which is conformal. It is expected that Class I solutions are generic in the unified system, since there are so many non-conformal sigma models, but there are so far no non-trivial[†] examples (see however Ref. 40, which proposes a large set of candidates).

In Class II, we are constructing a conformal stress tensor T in the operator algebra of a sigma model whose conventional stress tensor $T_G(\Phi_a)$ with general dilaton vector is conformal. This class includes the case where the conventional stress tensors $T_G(\Phi)$ are conformal so that the sigma model is conformal in the conventional sense. The general affine-Virasoro construction provides a large set of non-trivial examples in Class II when the sigma model is the WZW action. Other examples are known from the general affine-Virasoro construction which are based on coset constructions, instead of WZW. In particular, Halpern et al.⁴¹ construct exact Virasoro operators in the Hilbert space of a certain class of g/h coset constructions, and we are presently studying these conformal field theories as Class II solutions in the sigma model description of the coset constructions (see also the Conclusions).

[†]Trivial examples in Class I are easily constructed as tensor products of conformal and non-conformal theories.

It is also useful to subdivide Class II solutions into Class IIa and IIb. In Class IIb, we require the natural identification

$$\Phi_a = 2L_a{}^b \Phi_b^G + \mathcal{O}(\alpha') \quad (50)$$

which solves (30b), and Class IIa is the set of solutions in Class II without this identification. Note in particular that Class IIb contains all solutions in Class II with the dilaton solution $\Phi_a(\Phi)$ in (34).

Simplification for Class IIb with the dilaton solution

A simplification in Class IIb follows for the dilaton solution $\Phi_a(\Phi)$. In this case the unified system reads

$$R_{ij} + \frac{1}{4}(H^2)_{ij} - 2\nabla_i \nabla_j \Phi = \mathcal{O}(\alpha') \quad (51a)$$

$$\nabla^k H_{kij} - 2\nabla^k \Phi H_{kij} = \mathcal{O}(\alpha') \quad (51b)$$

$$\hat{\nabla}_i^+ L^{ab} = \mathcal{O}(\alpha'), \quad \hat{\nabla}_i^- \bar{L}^{ab} = \mathcal{O}(\alpha') \quad (51c)$$

$$\begin{aligned} L^{ab} = & 2L^{ac} G_{cd} L^{db} \\ & - \alpha' (L^{cd} L^{ef} H_{ce}{}^a H_{df}{}^b + L^{cd} H_{ce}{}^f H_{df}{}^{(a} L^{b)e)} \\ & - \alpha' (L^{c(a} G^{b)d} \nabla_{[c} \Phi_{d]}) + \mathcal{O}(\alpha'^2) \end{aligned} \quad (51d)$$

$$c = 2G_{ab} L^{ab} + 6\alpha' (2\Phi_a \Phi^a - \nabla_a \Phi^a) + \mathcal{O}(\alpha'^2) \quad (51e)$$

$$\Phi_a = \Phi_a(\Phi) = 2L_a{}^b \nabla_b \Phi. \quad (51f)$$

This simplified system is close in spirit to the VME of the general affine-Virasoro construction: The solution of the conventional Einstein equation in (51a), (51b) provides a conformal background, in which we need only look for solutions of the generalized VME in the form

$$L_a{}^b = \frac{P_a{}^b}{2} + \mathcal{O}(\alpha') \quad (52)$$

where P is a covariantly constant projector. Moreover, as in the VME, it has been shown¹⁴ that all solutions of the simplified system (51) exhibit K -conjugation covariance, so that

$$\tilde{T} \equiv T_G - T, \quad \tilde{c} = c_G - c \quad (53)$$

is also a conformal stress tensor when T is conformal.

5.5. Integrability Conditions

The inverse inertia tensor L^{ab} is a second-rank symmetric spacetime tensor, and we know that its associated projector P is covariantly constant

$$\hat{\nabla}_i^+ P_a{}^b = 0 \quad (54)$$

Operating with a second covariant derivative, we find that the integrability conditions

$$\hat{R}_{cd}^{+ae} P_e{}^b + \hat{R}_{cd}^{+be} P_e{}^a = 0 \quad (55)$$

follow as necessary conditions for the existence of solutions to (54).

On any manifold, there is always at least one solution to the covariant-constancy condition (54) and its integrability conditions (55), namely

$$P^{ab} = G^{ab} \quad (56a)$$

$$\hat{R}_{cd}^{\pm ab} + \hat{R}_{cd}^{\pm ba} = 0 \quad (56b)$$

$$L^{ab} = L_G^{ab} = \frac{G^{ab}}{2} + \mathcal{O}(\alpha') \quad (56c)$$

where G_{ab} is the metric of the sigma model action. This solution corresponds to the classical limit of the conventional sigma model stress tensor, as discussed in Section 5.1. For WZW, the integrability conditions (55) are also trivially satisfied (because $\hat{R}_{abcd}^{\pm} = 0$) and the general solutions of the covariant-constancy conditions were obtained for this case in Section 5.2.

In general we are interested in the classification of manifolds with at least one more solution P^{ab} , beyond G^{ab} . In what follows, we outline the sufficient and necessary condition for this phenomenon.

In a suitable basis, any projector P can be written as

$$P = \left(\begin{array}{c|c} \mathbf{1} & 0 \\ \hline 0 & 0 \end{array} \right). \quad (57)$$

Inserting this form in (54) and (55) shows then that $\hat{R}^+$ and $\hat{\omega}^+$ must be 'block diagonal' in the same basis, i.e. they can be written as

$$(\hat{R}_{cd}^+)_a^b = \left(\begin{array}{c|c} A_{cd} & 0 \\ \hline 0 & D_{cd} \end{array} \right), \quad (\hat{\omega}_i^+)_a^b = \left(\begin{array}{c|c} D_i & 0 \\ \hline 0 & E_i \end{array} \right) \quad (58)$$

for some matrices A_{cd}, B_{cd}, D_i, E_i . Thus, a necessary condition for new solutions to the covariant-constancy condition to exist is that $\hat{R}$ and $\hat{\omega}$ should be block diagonal.

Conversely, given a block diagonal $\hat{\omega}^+$, we can construct a new solution to the covariant-constancy condition with P given in (57). In fact, with $\hat{\omega}^+$ written in terms of the smallest possible blocks we can classify all possible solutions to the covariant constancy condition. If we denote the smallest diagonal blocks of $\hat{\omega}^+$ by $D_1, \dots, D_k$, then the most general covariantly constant projector is

$$P = p_1 \mathbf{1}_1 + \dots + p_k \mathbf{1}_k \quad (59)$$

where $p_i \in \{0, 1\}$ and $\mathbf{1}_j$ is the matrix which consists of the identity matrix in the j^{th} block and zeroes everywhere else. In the case when one of the blocks in $\hat{\omega}^+$ is zero, say D_j , then $p_j \mathbf{1}_j$ can be replaced by an arbitrary projector P_j in the j^{th} subspace.

New solutions obtained following this procedure are discussed in the Conclusions.

Mathematically, the problem of finding block-diagonal curvatures is the problem of finding manifolds with reducible holonomy. In the absence of torsion, it is known that block-diagonal curvatures exist only on product manifolds, but in the presence of torsion the question of manifolds with a block-diagonal curvature is an unsolved problem, except for the group manifolds discussed above (where $\hat{R}^+ = 0$), and the new examples given in the Conclusions.

6. CONCLUSIONS

We have studied the general Virasoro construction

$$T = -\frac{L_{ij}}{\alpha'} \partial x^j \partial x^i + \mathcal{O}(\alpha'^0) \quad (60)$$

at one loop in the operator algebra of the general non-linear sigma model, where L is a spin-two spacetime tensor field called the inverse inertia tensor. The construction is summarized by a unified Einstein-Virasoro master equation which describes the covariant coupling of L to the spacetime fields G , B and Φ_a , where G and B are the metric and antisymmetric tensor of the sigma model and Φ_a is the dilaton vector, which generalizes the derivative $\nabla_a \Phi$ of the dilaton Φ . As special cases, the unified system contains the Virasoro master equation of the general affine-Virasoro construction and the conventional Einstein equations of the canonical sigma model stress tensors. More generally, the unified system describes a space of conformal field theories which is presumably much larger than the sum of these two special cases.

In addition to questions posed in the text, we list here a number of other important directions.

1. New solutions. It is important to find new solutions of the unified system, beyond the canonical stress tensors of the sigma model and the general affine-Virasoro construction.

Although it is not in the original paper¹⁴, we have recently discovered a large class of new solutions of the covariant-constancy condition: It is not hard to see that the spin connection in the sigma model description of the g/h coset constructions has the form

$$(\hat{\omega}_i^+)_a{}^b = N_i^A f_{Aa}{}^b \quad (61)$$

where A is an h -index and a, b are g/h -indices, and $f_{Aa}{}^b$ are the structure constants of g . The structure constants and hence the spin connection can be taken block diagonal, where the blocks correspond to irreducible representations of h . As discussed in Section 5.5, this allows us to classify all possible covariantly-constant projectors on these manifolds. More work remains to be done in this case, including the solution of the generalized VME, but there are indications that the resulting conformal field theories may be identified as the set of local Lie h -invariant conformal field theories⁴¹ on g , which have in fact been studied in the Virasoro master equation itself.

2. Duality. The unified system contains the coset constructions in two distinct ways, that is, both as $G_{ab} = k\eta_{ab}$, $L^{ab} = L_{g/h}^{ab}$ in the general affine-Virasoro construction and among the canonical stress tensors of the sigma model with the sigma model metric that corresponds to the coset construction. This is an indicator of new duality transformations in the system, possibly exchanging L and G , which may go beyond the coset constructions. Indeed, if the conjecture of the previous paragraph holds, this duality would extend over all local Lie h -invariant conformal field theory, and perhaps beyond.

In this connection, we remind the reader that the VME has been identified² as an Einstein-Maxwell system with torsion on the group manifold, where the inverse inertia tensor is the inverse metric on tangent space. Following this hint, it may be possible to cast the unified system on group manifolds as two coupled Einstein systems, with exact covariant constancy of both G and L .

3. Non-renormalization theorems. The unified Einstein-Virasoro master equation is at present a one-loop result, while the Virasoro master equation is exact to all orders.

This suggests a number of possibly exact relations¹⁴ to all orders in the WZW model and in the general non-linear sigma model.

4. Spacetime action and/or C -function. These have not yet been found for the unified system, but we remark that they are known for the special cases unified here: The spacetime action^{12, 42} is known for the conventional Einstein equations of the sigma model, and, for this case, the C -function is known¹³ for constant dilaton. Moreover, an exact C -function is known⁴³ for the special case of the unimproved VME.

5. World-sheet actions. We have studied here only the Virasoro operators constructible in the operator algebra of the general sigma model, but we have not yet worked out the world-sheet actions of the corresponding new conformal field theories, whose beta functions should be the unified Einstein-Virasoro master equation. This is a familiar situation in the general affine-Virasoro construction, whose Virasoro operators are constructed in the operator algebra of the WZW model, while the world-sheet actions of the corresponding new conformal field theories include spin-one²⁸ gauged WZW models for the coset constructions and spin-two^{26, 29, 30} gauged WZW models for the generic construction.

As a consequence of this development in the general affine-Virasoro construction, more or less standard Hamiltonian methods now exist for the systematic construction of the new world-sheet actions from the new stress tensors, and we know for example that K -conjugation covariance is the source of the spin-two gauge invariance in the generic case. At least at one loop, a large subset of Class IIb solutions of the unified system exhibit K -conjugation covariance, so we may reasonably expect that the world-sheet actions for generic constructions in this subset are spin-two gauged sigma models. For solutions with no K -conjugation covariance, the possibility remains open that these constructions are dual descriptions of other conformal sigma models.

6. Superconformal extensions. The method of Ref. 23 has been extended⁴⁴⁻⁴⁶ to the canonical stress tensors of the supersymmetric sigma model. The path is therefore open to study general superconformal constructions in the operator algebra of the general sigma model with fermions. Such superconformal extensions should then include and generalize the known $N = 1$ and $N = 2$ superconformal master equations⁴⁷ of the general affine-Virasoro construction.

In this connection, we should mention that that the Virasoro master equation is the true master equation, because it includes as a small subspace all the solutions of the superconformal master equations. It is reasonable to expect therefore that, in the same way, the unified system of this paper will include the superconformal extensions.

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REFERENCES

1. M.B. Halpern and E. Kiritsis, *Mod. Phys. Lett.* A4:1373 (1989); Erratum *ibid.* A4:1797 (1989).
2. M.B. Halpern and J.P. Yamron, *Nucl. Phys.* B332:411 (1990).
3. A.Yu Morozov, A.M. Perelomov, A.A. Rosly, M.A. Shifman and A.V. Turbiner, *Int. J. Mod. Phys.* A5:803 (1990).
4. M.B. Halpern, E. Kiritsis, N.A. Obers, M. Porrati and J.P. Yamron, *Int. J. Mod. Phys.* A5:2275 (1990).
5. M.B. Halpern and N.A. Obers, *Comm. Math. Phys.* 138:63 (1991).
6. M.B. Halpern and N.A. Obers, *J. Math. Phys.* 36:1080 (1995).
7. M.B. Halpern, E. Kiritsis, N.A. Obers and K. Clubok, "Irrational Conformal Field Theory", *Physics Reports* 265:1 (1996).
8. D. Friedan, "Nonlinear Models in $2 + \epsilon$ Dimensions", PhD thesis, LBL-11517 (1980); *Phys. Rev. Lett.* 45:1057 (1980).
9. L. Alvarez-Gaumé, D.Z. Freedman and S. Mukhi, *Ann. of Phys.* 134:85 (1981).
10. C. Lovelace, *Phys. Lett.* 135B:75 (1984).
11. E.S. Fradkin and A.A. Tseytlin, *Nucl. Phys.* B261:1 (1985).
12. C.G. Callan, D. Friedan, E.J. Martinec and M.J. Perry, *Nucl. Phys.* B262:593 (1985).
13. A.A. Tseytlin, *Int. J. Mod. Phys.* A4:1257 (1989).
14. J. de Boer and M.B. Halpern, *Int. J. Mod. Phys.* A12:1551 (1997).
15. V.G. Kac, *Funct. Anal. App.* 1:328 (1967); R.V. Moody, *Bull. Am. Math. Soc.* 73:217 (1967).
16. K. Bardakçi and M.B. Halpern, *Phys. Rev.* D3:2493 (1971).
17. M.B. Halpern, *Phys. Rev.* D4:2398 (1971).
18. R. Dashen and Y. Frishman, *Phys. Rev.* D11:2781 (1975).
19. V.G. Knizhnik and A.B. Zamolodchikov, *Nucl. Phys.* B247:83 (1984).
20. P. Goddard, A. Kent and D. Olive, *Phys. Lett.* B152:88 (1985).
21. S.P. Novikov, *Usp. Math. Nauk.* 37:3 (1982).
22. E. Witten, *Comm. Math. Phys.* 92:455 (1984).
23. T. Banks, D. Nemeschansky and A. Sen, *Nucl. Phys.* B277:67 (1986).
24. M.B. Halpern and N.A. Obers, *Nucl. Phys.* B345:607 (1990).
25. M.B. Halpern and N.A. Obers, *J. Math. Phys.* 33:3274 (1992).
26. M.B. Halpern and J.P. Yamron, *Nucl. Phys.* B351:333 (1991).
27. M.B. Halpern, Recent progress in irrational conformal field theory, in: "Strings 1993", M.B. Halpern et al., eds., World Scientific, Singapore (1995).
28. K. Bardakçi, E. Rabinovici and B. Säring, *Nucl. Phys.* B299:151 (1988); D. Altschuler, K. Bardakçi and E. Rabinovici, *Comm. Math. Phys.* 118:241 (1988).
29. A.A. Tseytlin, *Nucl. Phys.* B418:173 (1994).
30. J. de Boer, K. Clubok and M.B. Halpern, *Int. J. Mod. Phys.* A9:2451 (1994).
31. M.B. Halpern and N.A. Obers, *Int. J. Mod. Phys.* A11:4837 (1996).
32. G. Ecker and J. Honerkamp, *Nucl. Phys.* B35:481 (1971).
33. J. Honerkamp, *Nucl. Phys.* B36:130 (1972).
34. E.S. Fradkin and A.A. Tseytlin, *Phys. Lett.* 106B:63 (1981).
35. T.L. Curtright and C.K. Zachos, *Phys. Rev. Lett.* 53:1799 (1984).
36. S. Gates, C. Hull and M. Roček, *Nucl. Phys.* B248:157 (1984).
37. E. Braaten, T.L. Curtright and C.K. Zachos, *Nucl. Phys.* B260:630 (1985).
38. S. Mukhi, *Nucl. Phys.* B264:640 (1986).
39. M.B. Green, J.H. Schwarz and E. Witten, "Superstring Theory", Cambridge University Press, Cambridge (1987).
40. K. Clubok and M.B. Halpern, The generic world-sheet action of irrational conformal field theory, in: "Strings'95", I. Bars et al., eds., World Scientific, Singapore (1996).
41. M.B. Halpern, E. Kiritsis and N.A. Obers, in: "Infinite Analysis"; *Int. J. Mod. Phys.* A7, [Suppl. 1A]:339 (1992).
42. G. Horowitz, The dark side of string theory, in: "String Theory and Quantum Gravity 1992", hep-th/9210119.
43. A. Giveon, M.B. Halpern, E. Kiritsis and N.A. Obers, *Nucl. Phys.* B357:655 (1991).
44. G. Aldazabal, F. Hussain and R. Zhang, *Phys. Lett.* 185B:89 (1987).
45. T. Itoh and M. Takao, *Int. J. Mod. Phys.* A5:2265 (1990).
46. G. Aldazabal and J.M. Maldacena, *Int. J. Mod. Phys.* A8:3359 (1993).

47. A. Givon, M.B. Halpern, E. Kiritsis and N.A. Obers, *Int. J. Mod. Phys. A*7:947 (1992).