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"INTRODUCTION TO GAUGE THEORIES AND UNIFICATION"

By

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INTRODUCTION TO GAUGE THEORIES AND UNIFICATION

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Lecture I

Basic Notations:

The theories which we will study in these lectures are supposed to describe fundamental processes at extremely high energies. Consequently, these theories will be relativistic theories invariant under Lorentz transformations. Let me, therefore, begin by establishing some notation which I will be using throughout the lectures.

Let us recall that if we have two vectors $\vec{x}$ and $\vec{y}$ in the three dimensional Euclidean space, their product invariant under rotations is defined to be (we will assume repeated indices to be summed unless otherwise specified.)

$$\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x} = x_1 y_1 + x_2 y_2 + x_3 y_3 = x_i y_i \qquad i = 1, 2, 3 \tag{1.1}$$

This is, of course, the scalar product and from this we obtain the length squared of a given vector $\vec{x}$ as

$$\vec{x}^2 = \vec{x} \cdot \vec{x} = x_1^2 + x_2^2 + x_3^2 = x_i x_i \tag{1.2}$$

which is also invariant under rotations. (Rotations define the isometry group of the Euclidean space.)

In constrast, in the four dimensional Minkowski space, one can define two kinds of vectors, namely, the covariant and the contravariant vectors denoted respectively by A_μ and A^μ. These are four component objects (also known as four-vectors) with μ taking the values, $\mu = 0, 1, 2, 3$. Furthermore, the covariant and the contravariant vectors are related through the metric of the Minkowski space as

$$\begin{aligned} A^\mu &= \eta^{\mu\nu} A_\nu \\ A_\mu &= \eta_{\mu\nu} A^\nu \end{aligned} \tag{1.3}$$

where I will assume the second rank metric tensors $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$ to take the diagonal matrix form

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \eta_{\mu\nu} \tag{1.4}$$

The metric tensors can be used to raise or lower tensor indices and the choice of the metric in Eq. (1.4) is commonly known as the Bjorken-Drell convention.

It is clear now that if we write the components of A^μ as

$$A^\mu = \left(A^0, \vec{A}\right) \tag{1.5}$$

then the components of A_μ would take the form

$$A_\mu = \eta_{\mu\nu} A^\nu = \left(A^0, -\vec{A}\right) \tag{1.6}$$

In a sense, the covariant and the contravariant vectors have opposite transformation properties under a Lorentz transformation so that given two vectors A_μ and B^μ, we can define a scalar product

$$A \cdot B = A_\mu B^\mu = A^\mu B_\mu = \eta^{\mu\nu} A_\mu B_\nu = \eta_{\mu\nu} A^\mu B^\nu = A^0 B^0 - \vec{A} \cdot \vec{B} \tag{1.7}$$

which will be invariant under Lorentz transformations. The length squared of a vector A_μ in Minkowski space now follows to be

$$A^2 = A_\mu A^\mu = \left(A^0\right)^2 - \vec{A}^2 \tag{1.8}$$

This is Lorentz invariant but is no longer positive definite as would be true in the Euclidean space.

Let us also recall that space and time coordinates define a four vector in Minkowski space. Thus writing

$$x^\mu = (t, \vec{x}) \tag{1.9}$$

we obtain

$$x_\mu = (t, -\vec{x}) \tag{1.10}$$

and

$$x^2 = x_\mu x^\mu = t^2 - \vec{x}^2 \tag{1.11}$$

which is, of course, the invariant length (we will set $\hbar = 1 = c$ throughout). It is clear now that Minkowski space can be divided into four cones and the physical processes are assumed to take place in the forward light cone (so that causality holds) defined by

$$x^2 = t^2 - \vec{x}^2 \geq 0 \qquad\qquad t \geq 0 \tag{1.12}$$

Just as space and time coordinates define a four-vector, derivatives with respect to these coordinates also define a four-vector. Thus the contragradient is defined to be

$$\frac{\partial}{\partial x_\mu} = \partial^\mu = \left(\frac{\partial}{\partial t}\,, -\vec{\nabla}\right) \tag{1.13}$$

from which we obtain

$$\frac{\partial}{\partial x^\mu} = \partial_\mu = \eta_{\mu\nu}\partial^\nu = \left(\frac{\partial}{\partial t}\,, \vec{\nabla}\right) \tag{1.14}$$

The generalization of the Laplacian to the Minkowski space is known as the D'Alembertian and is given by

$$\Box = \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 \tag{1.15}$$

There is one other kind of four-vectors that we will need for our discussions. These are known as the Dirac matrices and are denoted by γ^μ and γ_μ. They satisfy the anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} I \tag{1.16}$$

where I is the identity matrix. It follows, therefore, that

$$(\gamma^0)^2 = I \qquad (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -I \tag{1.17}$$

I would choose the Hermiticity properties of these matrices to be

$$(\gamma^0)^\dagger = \gamma^0 \qquad (\gamma^i)^\dagger = -\gamma^i \tag{1.18}$$

A particular representation for these 4×4 matrices can be written in terms of 2×2 blocks as

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

where σ_i's represent the Pauli matrices. From the four Dirac matrices we can construct a nontrival scalar matrix

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{1.20}$$

which satisfies

$$\gamma_5^\dagger = \gamma_5 \qquad (\gamma_5)^2 = I \tag{1.21}$$

As we will see later, γ_5 describes the chirality or the handedness of a massless Dirac spinor.

Scalar Field Theory:

With this introduction, let us look at the simplest of the field theories, namely, the free, massive, real scalar field theory. The Lagrangian (or more appropriately, the Lagrangian density) has the form

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\,\phi^2 \tag{1.22}$$

where $\phi(x) = \phi(t, \vec{x})$ is Hermitian and is known as a spin zero field or a scalar field because it transforms like a scalar under a Lorentz transformation. Most of the physical theories are at most quadratic in the derivatives. In this case, the Euler-Lagrange equations take the form

$$\partial_\mu\,\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \tag{1.23}$$

From Eq. (1.22), therefore, we see the dynamical equations to have the form

$$\left(\Box + m^2\right)\phi = 0 \tag{1.24}$$

This is a generalization of the wave equation, known as the Klein-Gordon equation, whose solutions are plane waves of the form

$$\phi(x) \sim e^{\pm ik\cdot x}$$

with (1.25)

$$k^2 = k_\mu k^\mu = m^2$$

The field $\phi(x)$ can describe neutral spin zero particles. In physical processes, however, particles are not completely free - rather they are interacting. Thus a more realistic theory would be one which describes a scalar field interacting with an external source. The Lagrangian, in this case, has the form

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\,\phi^2 + j\phi \tag{1.26}$$

where $j(x)$ represents an external source and the Euler-Lagrange equation, in this case, takes the form

$$\left(\Box + m^2\right)\phi(x) = j(x) \tag{1.27}$$

The solution to this equation can be obtained from the Greens function for the problem which satisfies

$$\left(\Box_x + m^2\right)G(x-y) = -\delta^{(4)}(x-y) \tag{1.28}$$

In terms of $G(x-y)$ then, we can write

$$\phi(x) = \phi^{(0)}(x) - \int d^4y\, G(x-y)j(y) \tag{1.29}$$

where $\phi^{(0)}(x)$ is any solution of the homogeneous equation (1.24). The formal solution of Eq. (1.29) is useful only if we know the explicit form of $G(x-y)$. Note that in momentum space Eq. (1.28) has the form

$$\left(-k^2+m^2\right)G(k)=-1$$

$$\text{or,}\qquad G(k)=\frac{1}{k^2-m^2} \tag{1.30}$$

so that we can write

$$\begin{aligned} G(x-y) &= \frac{1}{(2\pi)^4}\int d^4k\, e^{-ik\cdot(x-y)}G(k) \\ &= \frac{1}{(2\pi)^4}\int d^4k\, \frac{e^{-ik\cdot(x-y)}}{k^2-m^2-i\epsilon} \end{aligned} \tag{1.31}$$

Here the infinitesimal parameter ϵ is added to the denominator in order to obtain the retarded Greens function. Thus we see that a crucial ingredient in studying any physical system is the Greens function which is also known as the propagator. Note that so far in our discussion we have not brought in the quantum nature of the theory. This can be done simply by noting that from the Lagrangian in Eq. (1.22) or (1.26), we can define a momentum canonically conjugate to the $\phi(x)$ as

$$\Pi(x)=\frac{\partial\mathcal{L}}{\partial\dot{\phi}(x)}=\dot{\phi}(x) \tag{1.32}$$

The quantization rules now follow to be

$$\begin{aligned} \left[\phi(x),\phi(x')\right]_{t=t'} &= 0 = \left[\Pi(x),\Pi(x')\right]_{t=t'} \\ \left[\phi(x),\Pi(x')\right]_{t=t'} &= i\delta^{(3)}(x-x') \end{aligned} \tag{1.33}$$

The fields $\phi(x)$ and $\Pi(x)$ can now be expanded in terms of creation and annihilation operators and we can build up a Hilbert space for the quantum system.

<u>Self-Interacting Scalar Field Theory:</u>

Just as a scalar field can interact with an external source, it can also interact with itself. Thus let us choose the following Lagrangian as a model of a self-interacting scalar field theory (also known as the ϕ^4 theory).

$$\mathcal{L}=\frac{1}{2}\,\partial_\mu\phi\partial^\mu\phi-\frac{m^2}{2}\,\phi^2-\frac{\lambda}{4!}\,\phi^4 \qquad \lambda>0 \tag{1.34}$$

where λ represents the strength of self interaction or the coupling constant. From this Lagrangian, we can construct the Hamiltonian as

$$\begin{aligned} H &= \Pi\dot{\phi} - \mathcal{L} \\ &= \frac{1}{2}\,\Pi^2 + \frac{1}{2}\,\vec{\nabla}\phi \cdot \vec{\nabla}\phi + \frac{m^2}{2}\,\phi^2 + \frac{\lambda}{4!}\,\phi^4 \end{aligned} \tag{1.35}$$

It is clear now that classically the field configuration for which the energy would be a minimum has the form

$$\phi_c(x) = \text{constant} \; = 0 \tag{1.36}$$

Quantum mechanically, we say that the ground state or the vacuum state is one where

$$< 0|\phi(x)|0 >= 0 \tag{1.37}$$

For a constant field configuration, the minimum of the energy can be simply obtained by noting that in such a case

$$H = V(\phi) = -\mathcal{L}_{\text{int}} = \frac{m^2}{2}\,\phi^2 + \frac{\lambda}{4!}\,\phi^4 \tag{1.38}$$

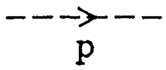
from which we obtain

$$\frac{\partial V(\phi)}{\partial \phi} = 0 \qquad \text{for} \qquad \phi = \phi_c = 0 \tag{1.39}$$

As we will see later, these observations will be useful in studying the phenomenon of spontaneous symmetry breaking.

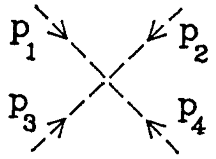
In a laboratory experiment, we would like to study the scattering involving particles. The scattering amplitudes can be calculated using the Feynman rules following from the theory, in Eq. (1.34). The theory, as we have seen, has a propagator and a set of interaction vertices. In the present case, we have

$$iG(p) = \frac{i}{p^2 - m^2}$$

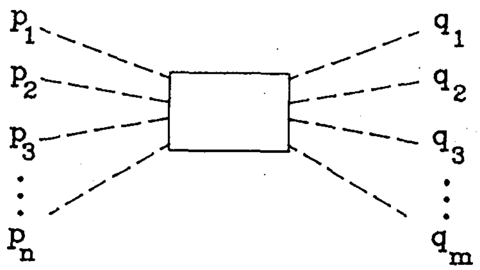


$$-i\Gamma^{(4)}(p_1, p_2, p_3, p_4) = -i \left. \frac{\partial^4 V}{\partial \phi^4} \right|_{\phi=0}$$

$$= -i\lambda\delta^{(4)}(p_1 + p_2 + p_3 + p_4) \tag{1.40}$$

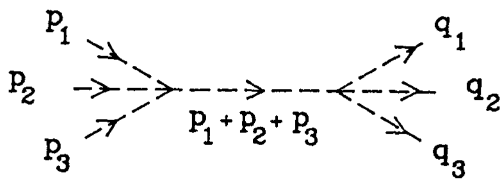


Any physical scattering process such as



n particles $\longrightarrow$ m particles

can now be constructed and computed using the propagator and the interaction vertices. Thus for example, 3 particles $\longrightarrow$ 3 particles, in this theory has the lowest order graph given by

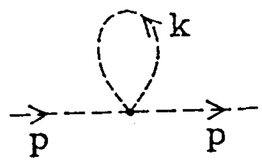


$$p_1 + p_2 + p_3 = q_1 + q_2 + q_3$$

and has the value

$$(-i\lambda)\, \frac{i}{(p_1 + p_2 + p_3)^2 - m^2}\, (-i\lambda) = -\frac{i\lambda^2}{(p_1 + p_2 + p_3)^2 - m^2} \tag{1.41}$$

The diagram describing the scattering process above is a simple one and such diagrams are known as tree diagrams. However, scattering can take place through complicated diagrams also. For example, in the lowest order in the ϕ^4 theory a particle can scatter by emitting a pair of particles which would annihilate each other. The Feynman diagram corresponding to this would look like



Such a diagram involves an internal loop representing the creation and annihilation of a pair of particles and is known as a loop diagram. In fact, it is called a one-loop diagram since the number of loops involved is one. Use of the Feynman rules now gives this scattering amplitude to be

$$-i\lambda \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} = \frac{\lambda}{(2\pi)^4} \int d^4k \, \frac{1}{k^2 - m^2} \tag{1.42}$$

The difference from the tree diagram is now obvious in that we have an integration over a momentum variable. This merely reflects the fact that the process involving the pair creation and annihilation is a virtual process and can occur with any momentum. This integral can be evaluated in many ways. The simplest is to go to the Euclidean space by letting

$$\begin{aligned} k_0 &\longrightarrow ik_4 \\ k^2 = k_0^2 - \vec{k}^2 &\longrightarrow -k_4^2 - \vec{k}^2 = -k_E^2 \end{aligned} \tag{1.43}$$

so that the integral takes the form

$$\begin{aligned} \frac{i\lambda}{(2\pi)^4} \int d^4k_E \, \frac{1}{-(k_E^2 + m^2)} &= -\frac{i\lambda}{(2\pi)^4} \int k_E^3 dk_E d\Omega \, \frac{1}{k_E^2 + m^2} \\ &= -\frac{i\lambda}{(2\pi)^4} \cdot 2\pi^2 \int_0^\infty \frac{1}{2} \, dk_E^2 \, \frac{k_E^2}{k_E^2 + m^2} \end{aligned} \tag{1.44}$$

Clearly, the integral in Eq. (1.44) diverges and one way to define the integral is to cut off the integral at some large value of k_E^2. Thus defined this way, Eq. (1.44) becomes

$$-\frac{i\lambda}{16\pi^2}\int_0^{\Lambda^2} dk_E^2\, \frac{k_E^2+m^2-m^2}{k_E^2+m^2}$$

$$= -\frac{i\lambda}{16\pi^2}\int_0^{\Lambda^2} dk_E^2 \left(1-\frac{m^2}{k_E^2+m^2}\right) \tag{1.45}$$

$$= -\frac{i\lambda}{16\pi^2}\left(\Lambda^2 - m^2 \ln\,(\frac{\Lambda^2+m^2}{m^2})\right)$$

The true value of the integral (1.44) is, of course, obtained in the limit $\Lambda \to \infty$ and it diverges. But doing it this way brings out the nature of the divergence. This example also brings home another difference between the tree and the loop diagrams, namely, the loop diagrams diverge and, consequently, need to be regularized.

Lecture II

Dimensional Regularization:

As we saw in the last lecture, there are inherent divergences in a quantum field theory which need to be regularized. There are many possible ways of regularizing a theory. For example, in the earlier calculation, we used a cut off to regularize the amplitude. But we could have chosen one of many other available regularization schemes such as the Pauli-Villars regularization or the point splitting regularization or the dimensional regularization or the higher derivative regularization and so on. Given a system, one chooses a regularization scheme which respects all the symmetry properties of the theory. In the case of gauge theories, the regularization that works well (it respects gauge invariance) and has become the standard regularization is dimensional regularization which I will describe next.

Let us now study the ϕ^4 theory, which we have analyzed in some detail, not in four dimensions but rather in n dimensions where $n = 4 - \epsilon$ with ϵ an infinitesimal parameter. The action defined as

$$S = \int d^n x\, \mathcal{L} \tag{2.1}$$

is a scalar in units of $\hbar = c = 1$ so that the canonical dimension of $\mathcal{L}$ follows to be

$$[\mathcal{L}] = n \tag{2.2}$$

Note that

$$\mathcal{L} = \frac{1}{2}\, \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\, \phi^2 - \frac{\lambda}{4!}\, \phi^4 \tag{2.3}$$

and since

$$[x^\mu] = -1 \qquad\qquad [\partial_\mu] = 1 \tag{2.4}$$

the canonical dimension of ϕ now follows to be

$$[\phi] = \frac{n-2}{2} \tag{2.5}$$

We also obtain

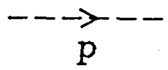
$$[m] = 1 \qquad\qquad [\lambda] = 4 - n = \epsilon \tag{2.6}$$

We would, however, like the coupling constant λ to be dimensionless and this can be achieved if we introduce an arbitrary mass scale μ and write the Lagrangian as

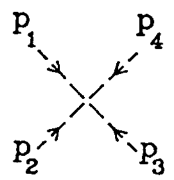
$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\,\phi^2 - \frac{\mu^\epsilon}{4!}\,\lambda\phi^4 \tag{2.7}$$

The coupling constant, λ, now will be dimensionless. The Feynman rules for this theory in n-dimensions take the form

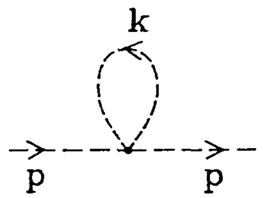
$$iG(p) = \frac{i}{p^2 - m^2}$$



$$\begin{aligned} -i\Gamma^{(4)}(p_1,p_2,p_3,p_4) &= -i\,\frac{\partial^4 V}{\partial\phi^4} \\ &= -i\mu^\epsilon\lambda\delta^{(4)}(p_1+p_2+p_3+p_4) \end{aligned} \tag{2.8}$$



Let us next go on and calculate all the one loop diagrams in this theory. Remembering that we are in n-dimensions, we obtain



$$\begin{aligned} &= -i\mu^\epsilon\lambda \int \frac{d^n k}{(2\pi)^n}\,\frac{i}{k^2 - m^2} \\ &= \mu^\epsilon\lambda \int \frac{d^n k}{(2\pi)^n}\,\frac{1}{k^2 - m^2} \end{aligned} \tag{2.9}$$

The fundamental formula for n-dimensional integrals that is of use to us is

$$\begin{aligned} I_\alpha &= \int \frac{d^n k}{(2\pi)^n}\,\frac{1}{(k^2 + 2k\cdot p - M^2)^\alpha} \\ &= (-1)^\alpha\,\frac{i\pi^{n/2}}{(2\pi)^n}\,\frac{\Gamma(\alpha - n/2)}{\Gamma(\alpha)}\,\frac{1}{(p^2 + M^2)^{\alpha - n/2}} \end{aligned} \tag{2.10}$$

where

$$\Gamma(\alpha + 1) = \alpha!$$

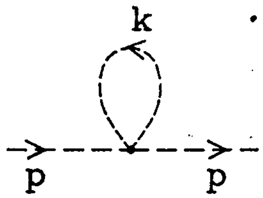
Differentiating $I_{\alpha-1}$ with respect to p^μ, we can obtain other useful formulae such as

$$
\begin{aligned}
-\frac{1}{2(\alpha-1)}\,\frac{\partial I_{\alpha-1}}{\partial p^\mu} &= \int \frac{d^n k}{(2\pi)^n}\,\frac{k_\mu}{(k^2+2k\cdot p-M^2)^\alpha} \\
&= (-1)^{\alpha-1}\,\frac{i\pi^{n/2}}{(2\pi)^n}\,\frac{\Gamma(\alpha-n/2)}{\Gamma(\alpha)}\,\frac{p_\mu}{(p^2+M^2)^{\alpha-n/2}}
\end{aligned}
\tag{2.11}
$$

and similarly

$$
\begin{aligned}
&\int \frac{d^n k}{(2\pi)^n}\,\frac{k_\mu k_\nu}{(k^2+2k\cdot p-M^2)^\alpha} \\
&\qquad = (-1)^\alpha\,\frac{i\pi^{n/2}}{(2\pi)^n}\,\frac{1}{\Gamma(\alpha)}\,\frac{1}{(p^2+M^2)^{\alpha-n/2}}\Big[p_\mu p_\nu \Gamma(\alpha-n/2) \\
&\qquad\qquad -\frac{1}{2}\,\eta_{\mu\nu}(p^2+M^2)\Gamma(\alpha-1-n/2)\Big]
\end{aligned}
\tag{2.12}
$$

Using Eq. (2.10), we can now evaluate the expression in Eq. (2.9) which takes the form



$$
\begin{aligned}
&= \mu^\epsilon \lambda(-1)\,\frac{i\pi^{n/2}}{(2\pi)^n}\,\frac{\Gamma(1-n/2)}{\Gamma(1)}\,\frac{1}{(m^2)^{1-n/2}} \\
&= -\,i\mu^\epsilon\lambda\,\frac{\pi^{n/2}}{(2\pi)^n}\,\Gamma\Big(-1+\frac{\epsilon}{2}\Big)\big(m^2\big)^{1-\epsilon/2}
\end{aligned}
\tag{2.13}
$$

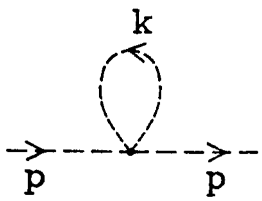
We can now use the gamma function identities

$$
\Gamma(n) = \frac{\Gamma(n+1)}{n}
$$

and (2.14)

$$
\Gamma\Big(\frac{\epsilon}{2}\Big) = \frac{2}{\epsilon} + \text{finite}
$$

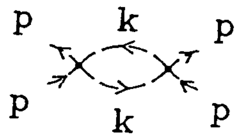
to simplify the expression in Eq. (2.13). Thus



$$
= m^2 \cdot \frac{i\lambda}{16\pi^2}\left(\frac{2}{\epsilon} - \ln\frac{m^2}{\mu^2} + \text{finite}\right)
\tag{2.15}
$$

Thus we see that in $n = 4 - \epsilon$ dimensions, this Feynman amplitude is well defined. However, as we approach $\epsilon \to 0$, the divergence appears as a pole.

There is one other one loop diagram that we can construct in this theory. Let us calculate a simplified version of this.



$$= \frac{1}{2}\left(-i\mu^{\epsilon}\lambda\right)^2 \int \frac{d^n k}{(2\pi)^n} \left(\frac{i}{k^2 - m^2}\right)^2$$

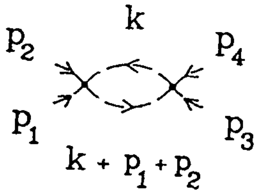
$$= \frac{1}{2}\,\mu^{2\epsilon}\lambda^2 \int \frac{d^n k}{(2\pi)^n}\,\frac{1}{(k^2 - m^2)^2}$$

$$= \frac{1}{2}\,\mu^{2\epsilon}\lambda^2(-1)^2\,\frac{i\pi^{n/2}}{(2\pi)^n}\,\frac{\Gamma(2 - n/2)}{\Gamma(2)}\,\frac{1}{(m^2)^{2-n/2}}$$

$$= \frac{1}{2}\,\mu^{2\epsilon}\lambda^2\,\frac{i\pi^{n/2}}{(2\pi)^n}\,\Gamma\!\left(\frac{\epsilon}{2}\right)\left(m^2\right)^{-\epsilon/2}$$

$$= i\mu^{\epsilon}\lambda \cdot \frac{\lambda}{32\pi^2}\left(\frac{2}{\epsilon} - \ln\frac{m^2}{\mu^2} + \text{finite}\right) \tag{2.16}$$

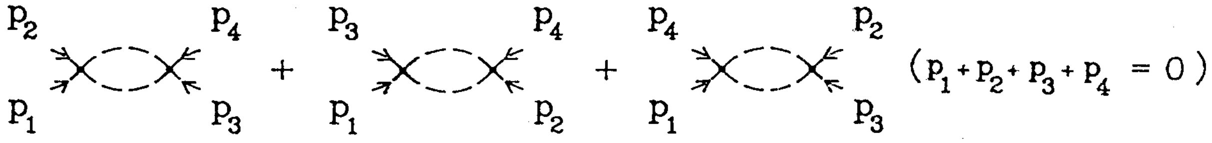
Here the factor $\frac{1}{2}$ is known as the symmetry factor and arises because the amplitude is symmetric under the interchange of the two internal lines. Once again we see that the divergence appears as a pole and is independent of the external momentum. We can now explicitly work out and show that even for arbitrary external momenta, the Feynman amplitude will have the form



$$= i\mu^{\epsilon}\lambda \cdot \frac{\lambda}{32\pi^2}\left(\frac{2}{\epsilon} - \ln\frac{m^2}{\mu^2} + \text{finite}\right) \tag{2.17}$$

$p_1 + p_2 + p_3 + p_4 = 0$

Adding in the other two channels, we see that at one loop the total four particle scattering amplitude will have the form



$$= i\mu^{\epsilon}\lambda \cdot \frac{3\lambda}{32\pi^2}\left(\frac{2}{\epsilon} - \ln\frac{m^2}{\mu^2} + \text{finite}\right) \tag{2.18}$$

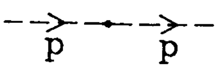
The divergence structure of the theory is now completely determined at the one loop level and we see that if we start from the theory in Eq. (2.7), then at one loop, the two point function as well as the four point function develop divergences. On the other hand, let us note that if we had started from the theory

$$\mathcal{L}_0 = \frac{1}{2}\,\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\,\phi^2 - \frac{\mu^\epsilon\lambda}{4!}\,\phi^4 - \frac{m^2}{2}\,A\phi^2 - \frac{\mu^\epsilon\lambda}{4!}\,B\phi^4 \tag{2.19}$$

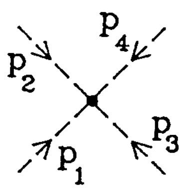
with

$$A = \frac{\lambda}{8\pi^2\epsilon}$$
$$B = \frac{3\lambda}{16\pi^2\epsilon} \tag{2.20}$$

then we would have additional vertices in the theory given by



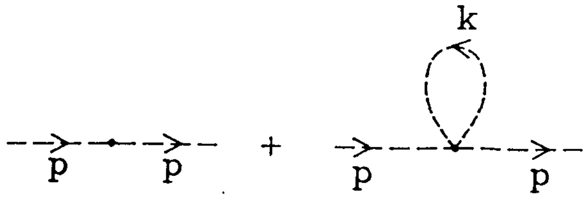
$$= -\,im^2 A = -im^2\,\frac{\lambda}{8\pi^2\epsilon}$$



$$= -\,i\mu^\epsilon\lambda B\delta^4(p_1 + p_2 + p_3 + p_4) \tag{2.21}$$

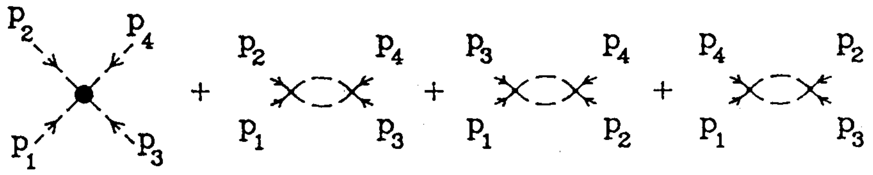
$$= -\,i\mu^\epsilon\lambda \cdot \frac{3\lambda}{16\pi^2\epsilon}\,\delta^4(p_1 + p_2 + p_3 + p_4)$$

Note that the constants A and B are really one loop quantities. (this would have been easier to see if we had kept the $\hbar$ terms.) Thus in this new theory, at one loop we will have



$$= -im^2 \cdot \frac{\lambda}{8\pi^2\epsilon} + im^2 \cdot \frac{\lambda}{16\pi^2}\left(\frac{2}{\epsilon} - \ln\frac{m^2}{\mu^2} + \text{finite}\right)$$

$$= -im^2 \cdot \frac{\lambda}{16\pi^2}\left(\ln\frac{m^2}{\mu^2} + \text{finite}\right) \tag{2.22}$$



$(p_1 + p_2 + p_3 + p_4 = 0)$

$$= -i\mu^\epsilon\lambda \cdot \frac{3\lambda}{16\pi^2\epsilon} + i\mu^\epsilon\lambda \cdot \frac{3\lambda}{32\pi^2}\left(\frac{2}{\epsilon} - \ln\frac{m^2}{\mu^2} + \text{finite}\right)$$

$$= -i\mu^\epsilon\lambda \cdot \frac{3\lambda}{32\pi^2}\left(\ln\frac{m^2}{\mu^2} + \text{finite}\right) \tag{2.23}$$

Thus we see that had we started with the theory $\mathcal{L}_0$, there would be no divergence at least at the one loop level. However, the Lagrangian $\mathcal{L}_0$ would appear to be different from $\mathcal{L}$. But on closer inspection we find that $\mathcal{L}_0$ really has the same form as $\mathcal{L}$ with redefined parameters, namely,

$$\mathcal{L}_0 = \frac{1}{2}\,\partial_\mu\phi_0\partial^\mu\phi_0 - \frac{m_0^2}{2}\,\phi_0^2 - \frac{\lambda_0}{4!}\,\phi_0^4 \tag{2.24}$$

where

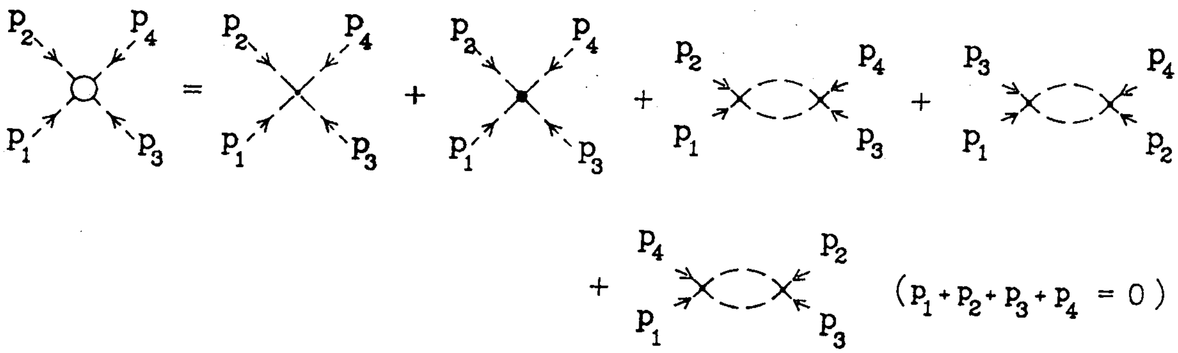
$$\phi_0 = \phi$$

$$m_0^2 = m^2(1+A) = m^2\left(1 + \frac{\lambda}{8\pi^2\epsilon}\right) \tag{2.25}$$

$$\lambda_0 = \mu^\epsilon\lambda(1+B) = \mu^\epsilon\lambda\left(1 + \frac{3\lambda}{16\pi^2\epsilon}\right)$$

The terms, depending on the constants A and B, which are added to the original Lagrangian to render the amplitudes finite are known as counterterms and this process of removing divergences is known as renormalization. Although we have explicitly shown this only for one-loop, in any given theory one can add counterterms order by order so that the amplitudes are finite at every loop. Furthermore, a renormalizable theory is one (physical theories are renormalizable.) where the counterterms can be completely absorbed into a redefinition of the fields and the various parameters in the theory.

Let me say this again in a different way. Let us call ϕ_0 to be the bare field and m_0, λ_0 to be the bare parameters. Similarly, let us call ϕ to be the renormalized field and m, λ to be the renormalized parameters. Then if we start from the bare Lagrangian $\mathcal{L}_0$ and calculate the amplitudes in terms of bare parameters m_0 and λ_0 then the process of renormalization guarantees that, in a renormalizable theory, when the bare parameters are expressed in terms of a set of renormalized parameters as in Eq. (2.25), the amplitudes would be finite. Thus if we calculated the four point function up to one-loop from $\mathcal{L}_0$ in terms of m and λ, we would obtain



$$= -i\mu^\epsilon\lambda - i\mu^\epsilon\lambda \cdot \frac{3\lambda}{32\pi^2}\left(\ln\frac{m^2}{\mu^2} + \text{finite}\right)$$

$$= -i\mu^\epsilon\lambda\left(1 + \frac{3\lambda}{32\pi^2}\left(\ln\frac{m^2}{\mu^2} + \text{finite}\right)\right) = \text{finite} \qquad (2.26)$$

The choice of the renormalized parameters is, however, not unique. As is obvious, their definition depends on the mass scale μ. Thus for example, given a bare theory $\mathcal{L}_0$, two different people can choose two sets of renormalized parameters - depending on different mass scales - which would lead to finite scattering amplitudes. The mass scale μ, therefore, would correspond in some sense to the subtraction point or the energy scale at which a process is evaluated. Thus we see that in a renormalizable theory, the renormalized parameters become energy dependent. In the case of coupling constant one fondly says that the renormalized coupling runs with energy or that it becomes a running coupling. This dependence of the renormalized parameters on the mass scale μ leads to the renormalization group equation which basically gives how various quantities would change as μ is changed. For the coupling constant, this evolution up to one-loop can be obtained from Eq. (2.25). Note that whereas the renormalized parameters depend on μ, the bare parameters do not depend on the arbitrary mass scale. Thus from Eq. (2.25) we obtain

$$\mu \frac{\partial \lambda_0}{\partial \mu} = 0$$

$$\text{or,} \quad \mu \frac{\partial}{\partial \mu} \left(\mu^{\epsilon} \lambda \left(1 + \frac{3\lambda}{16\pi^2 \epsilon} \right) \right) = 0$$

$$\text{or,} \quad \mu \frac{\partial}{\partial \mu} \left(\left(\lambda + \frac{3\lambda^2}{16\pi^2 \epsilon} \right) \left(1 + \epsilon \ln \mu + O\left(\epsilon^2\right) \right) \right) = 0$$

$$\text{or,} \quad \mu \frac{\partial}{\partial \mu} \left(\lambda + \frac{3\lambda^2}{16\pi^2 \epsilon} + \left(\epsilon\lambda + \frac{3\lambda^2}{16\pi^2} \right) \ln \mu + O\left(\epsilon^2\right) \right) = 0$$

$$\text{or,} \quad \mu \frac{\partial \lambda}{\partial \mu} \left(1 + \frac{3\lambda}{8\pi^2 \epsilon} + \left(\epsilon + \frac{3\lambda}{8\pi^2} \right) \ln \mu \right) + \epsilon\lambda + \frac{3\lambda^2}{16\pi^2} + O\left(\epsilon^2\right) = 0$$

$$\text{or,} \quad \mu \frac{\partial \lambda}{\partial \mu} = -\left(\epsilon\lambda + \frac{3\lambda^2}{16\pi^2} \right) \left(1 - \frac{3\lambda}{8\pi^2 \epsilon} - \left(\epsilon + \frac{3\lambda}{8\pi^2} \right) \ln \mu \right) + O\left(\epsilon^2\right) \qquad (2.27)$$

Keeping terms up to order λ^2 which is the consistent one-loop case, we obtain in the limit $\epsilon \to 0$

$$\mu \frac{\partial \lambda}{\partial \mu} = -\frac{3\lambda^2}{16\pi^2} + \frac{3\lambda^2}{8\pi^2} = \frac{3\lambda^2}{16\pi^2} = \beta(\lambda) \tag{2.28}$$

This equation can be solved in a straightforward manner.

$$\mu \frac{\partial \lambda}{\partial \mu} = \frac{3\lambda^2}{16\pi^2}$$

$$\text{or,} \quad \frac{d\lambda}{\lambda^2} = \frac{3}{16\pi^2} \frac{d\mu}{\mu}$$

$$\text{or,} \quad \int_{\lambda(\bar{\mu})}^{\lambda(\mu)} \frac{d\lambda}{\lambda^2} = \frac{3}{16\pi^2} \int_{\bar{\mu}}^{\mu} \frac{d\mu}{\mu}$$

$$\text{or,} \quad -\frac{1}{\lambda(\mu)} + \frac{1}{\lambda(\bar{\mu})} = \frac{3}{16\pi^2} \ln \frac{\mu}{\bar{\mu}}$$

$$\text{or,} \quad \lambda(\mu) = \frac{\lambda(\bar{\mu})}{1 - \frac{3\lambda(\bar{\mu})}{16\pi^2} \ln \frac{\mu}{\bar{\mu}}} \tag{2.29}$$

This, indeed, shows how the coupling constant, $\lambda(\mu)$, changes with the scale or energy. As μ increases relative to $\bar{\mu}$, $\lambda(\mu)$ grows. The coupling becomes stronger and beyond a certain value perturbation theory breaks down. One says that the ϕ^4 theory is not asymptotically free. I would also like to emphasize here that although our discussion so far has been within the context of the real scalar field, it can be generalized to other theories in a straightforward manner.

Lecture III

Complex Scalar Field Theory:

Let us next consider a self-interacting scalar field theory where the field $\phi(x)$ is not real. Such a theory can describe processes involving charged spin zero particles. Thus let,

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi - \frac{\lambda}{4} \left(\phi^\dagger \phi\right)^2 \qquad \lambda > 0 \tag{3.1}$$

Since $\phi(x)$ is complex, we can express it in terms of two real fields $\sigma(x)$ and $\zeta(x)$ as

$$\begin{aligned} \phi(x) &= \frac{1}{\sqrt{2}} \left(\sigma(x) + i\zeta(x)\right) \\ \phi^\dagger(x) &= \frac{1}{\sqrt{2}} \left(\sigma(x) - i\zeta(x)\right) \end{aligned} \tag{3.2}$$

When expressed in terms of $\sigma(x)$ and $\zeta(x)$, the Lagrangian of Eq. (3.1) takes the form

$$\mathcal{L} = \frac{1}{2}\, \partial_\mu \sigma \partial^\mu \sigma + \frac{1}{2}\, \partial_\mu \zeta \partial^\mu \zeta - \frac{m^2}{2} \left(\sigma^2 + \zeta^2\right) - \frac{\lambda}{16} \left(\sigma^2 + \zeta^2\right)^2 \tag{3.3}$$

Thus we see that a self-interacting complex scalar field theory is equivalent to a theory of two coupled, self-interacting real scalar fields.

The fields $\phi(x)$ and $\phi^\dagger(x)$ in Eq. (3.1) can be taken to be independent variables. Correspondingly, the Euler-Lagrange equations for the system are

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^\dagger} - \frac{\partial \mathcal{L}}{\partial \phi^\dagger} = 0$$

$$\text{or,} \qquad \left(\Box + m^2\right) \phi = -\frac{\lambda}{2} \left(\phi^\dagger \phi\right) \phi \tag{3.4}$$

and

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\text{or,} \qquad \left(\Box + m^2\right) \phi^\dagger = -\frac{\lambda}{2} \left(\phi^\dagger \phi\right) \phi^\dagger \tag{3.5}$$

The canonical momenta conjugate to $\phi(x)$ and $\phi^\dagger(x)$ are given respectively by

$$\begin{aligned} \Pi_\phi(x) &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)} = \dot{\phi}^\dagger(x) \\ \Pi_{\phi^\dagger}(x) &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}^\dagger(x)} = \dot{\phi}(x) \end{aligned} \tag{3.6}$$

The quantization conditions, therefore, become

$$[\phi(x), \Pi_\phi(x')]_{t=t'} = i\delta^{(3)}(x - x') = \left[\phi^\dagger(x), \Pi_{\phi^\dagger}(x')\right]_{t=t'} \tag{3.7}$$

with all other equal time commutators vanishing.

Noether's Theorem:

Let us note that the Lagrangian of Eq. (3.1) is invariant under a phase transformation of the form

$$\begin{aligned} \phi(x) &\rightarrow \bar{\phi}(x) = e^{i\alpha}\phi(x) \\ \phi^\dagger(x) &\rightarrow \bar{\phi}^\dagger(x) = e^{-i\alpha}\phi^\dagger(x) \end{aligned} \tag{3.8}$$

where α is a space-time independent constant parameter. Thus the phase transformations of Eq. (3.8) are a symmetry of the system described by the Lagrangian in Eq. (3.1). The phase transformations do not involve a change in the space-time coordinates of the fields and hence do not correspond to a space-time symmetry transformation. Rather, such a transformation is known as an internal symmetry transformation. Furthermore, the parameter of transformation, α, is a constant - it is the same at all coordinate points. Thus such a transformation defines a global symmetry trnasformation. Often times, it is more convenient to study the infinitesimal form of a transformation. Thus when the parameter of transformation is infinitesimally small, the transformations in Eq. (3.8) can be written as

$$\begin{aligned} \delta_\epsilon\phi(x) &= \bar{\phi}(x) - \phi(x) = i\epsilon\phi(x) \\ \delta_\epsilon\phi^\dagger(x) &= \bar{\phi}^\dagger(x) - \phi^\dagger(x) = -i\epsilon\phi^\dagger(x) \end{aligned} \tag{3.9}$$

where ϵ is the infinitesimal parameter of transformation.

If the Lagrangian of a system is invariant under a continuous symmetry transformation, Noether's theorem guarantees the existence of a conserved current. In the case of an internal symmetry transformation, the conserved current can be obtained as

$$\epsilon\, j^\mu(x) = \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi(x)}\,\delta_\epsilon\psi(x) \tag{3.10}$$

where $\psi(x)$ generically represents all the field variables of the theory. Thus for the Lagrangian in Eq. (3.1), we can determine the current associated with the infinitesimal transformations of Eq. (3.9) to be

$$\epsilon\, j^\mu(x) = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi(x)}\, \delta_\epsilon \phi(x) + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^\dagger(x)}\, \delta_\epsilon \phi^\dagger(x)$$

$$= \partial^\mu \phi^\dagger(x)(i\epsilon\phi(x)) + \partial^\mu \phi(x)(-i\epsilon\phi^\dagger(x))$$

$$= -i\epsilon\big(\phi^\dagger(x)\partial^\mu\phi(x) - \partial^\mu\phi^\dagger(x)\phi(x)\big)$$

or, $$j^\mu(x) = -i\,\big(\phi^\dagger(x)\partial^\mu\phi(x) - \partial^\mu\phi^\dagger(x)\phi(x)\big) = -i\phi^\dagger(x)\overleftrightarrow{\partial^\mu}\,\phi(x) \tag{3.11}$$

Let us note that

$$\begin{aligned} \partial_\mu j^\mu(x) &= -i\partial_\mu\Big(\phi^\dagger(x)\partial^\mu\phi(x) - \partial^\mu\phi^\dagger(x)\phi(x)\Big) \\ &= -i\Big(\phi^\dagger(x)\,\Box\,\phi(x) - \Box\,\phi^\dagger(x)\phi(x)\Big) \end{aligned} \tag{3.12}$$

Using the equations of motion in (3.4 - 3.5), we see that

$$\begin{aligned} \partial_\mu j^\mu(x) = -i\Big(&\phi^\dagger(x)\big(-m^2\phi(x) - \frac{\lambda}{2}\,(\phi^\dagger\phi)\phi\big) \\ &- \big(-m^2\phi^\dagger(x) - \frac{\lambda}{2}\,(\phi^\dagger\phi)\phi^\dagger\big)\phi(x)\Big) \end{aligned} \tag{3.13}$$

or, $$\partial_\mu j^\mu(x) = 0$$

In other words, the current associated with the transformations in Eq. (3.9) is, indeed, conserved.

Given a conserved current, j^μ, we can construct a charge as

$$Q = \int d^3x\, j^0(\vec{x},t) \tag{3.14}$$

which can be shown to be independent of time. Thus, for the complex scalar field theory, we see that there exists a charge operator

$$\begin{aligned} Q = \int d^3x\, j^0(x) &= -i\int d^3x\big(\phi^\dagger(x)\dot\phi(x) - \dot\phi^\dagger(x)\phi(x)\big) \\ &= -i\int d^3x\,\big(\phi^\dagger(x)\Pi_{\phi^\dagger}(x) - \Pi_\phi(x)\phi(x)\big) \end{aligned} \tag{3.15}$$

Here we have used the relations in Eq. (3.6). The algebra of the charge can now be obtained using Eq. (3.7) to be

$$[Q,Q] = 0 \tag{3.16}$$

In other words, the conserved charge associated with the symmetry transformations of Eq. (3.9) is Abelian. This can be identified with the electric charge operator of the theory. Note also that using Eq. (3.7), we can show that

$$\begin{aligned}\delta_\epsilon \phi(x) &= i\epsilon\phi(x) = -i[\phi(x), \epsilon Q] \\ \delta_\epsilon \phi^\dagger(x) &= -i\epsilon\phi^\dagger(x) = -i[\phi^\dagger(x), \epsilon Q]\end{aligned} \tag{3.17}$$

This shows that the charge Q is the generator of the infinitesimal transformations in Eq. (3.9).

Spontaneous Symmetry Breaking:

Let us now consider the complex scalar field theory described by the Lagrangian

$$\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi + m^2\phi^\dagger\phi - \frac{\lambda}{4}\,\left(\phi^\dagger\phi\right)^2 \qquad\qquad \lambda > 0 \tag{3.18}$$

This Lagrangian differs from the one in Eq. (3.1) in the sign of the quadratic term. It is still invariant under the phase transformations of Eq. (3.8) or the infinitesimal transformations of Eq. (3.9). The conserved charge Q of Eq. (3.15) must, therefore, commute with the Hamiltonian of the system and one would naively expect that the charge operator Q would annihilate the vacuum of the theory. But as we will see next, this is not true.

Let us rewrite the Lagrangian in terms of the real fields $\sigma(x)$ and $\zeta(x)$ of Eq. (3.2).

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\,\partial_\mu\zeta\partial^\mu\zeta + \frac{m^2}{2}\,\left(\sigma^2+\zeta^2\right) - \frac{\lambda}{16}\,\left(\sigma^2+\zeta^2\right)^2 \tag{3.19}$$

In terms of these fields, the infinitesimal transformations of Eq. (3.9) would take the form

$$\begin{aligned}\delta_\epsilon\sigma(x) &= -\frac{1}{\sqrt{2}}\,\epsilon\zeta(x) = -i[\sigma(x), \epsilon Q] \\ \delta_\epsilon\zeta(x) &= \frac{1}{\sqrt{2}}\,\epsilon\sigma(x) = -i[\zeta(x), \epsilon Q]\end{aligned} \tag{3.20}$$

Let us analyze the ground state of the theory. For constant field configurations, we see that the minimum of the potential

$$V = -\frac{m^2}{2}\,\left(\sigma^2+\zeta^2\right) + \frac{\lambda}{16}\,\left(\sigma^2+\zeta^2\right)^2 \tag{3.21}$$

would give the ground state. Note that the solutions to the minimum equations

$$\frac{\partial V}{\partial \sigma} = -m^2\sigma + \frac{\lambda}{4}\,\sigma(\sigma^2+\zeta^2) = 0$$
$$\frac{\partial V}{\partial \zeta} = -m^2\zeta + \frac{\lambda}{4}\,\zeta(\sigma^2+\zeta^2) = 0 \tag{3.22}$$

are given by

$$\sigma = 0 = \zeta$$

or (3.23)

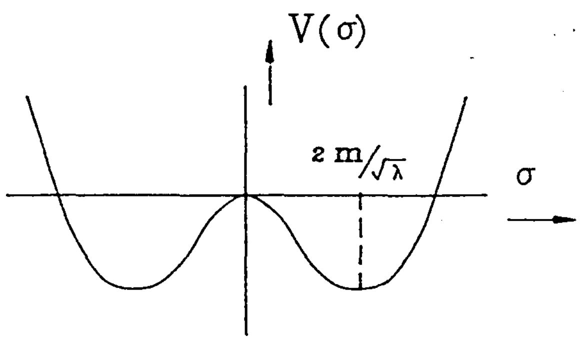
$$\sigma^2 + \zeta^2 = \frac{4m^2}{\lambda}$$

It can be verified readily that $\sigma = \zeta = 0$ defines a local maximum whereas the true minimum is given by

$$\sigma^2 + \zeta^2 = \frac{4m^2}{\lambda} \tag{3.24}$$

For simplicity, we choose the minimum to correspond to

$$\zeta = 0 \qquad\qquad \sigma = \frac{2m}{\sqrt{\lambda}} \tag{3.25}$$



As we have discussed before, in the quantum theory, it corresponds to the fact that the vacuum state satisfies

$$<0|\zeta|0> = 0$$
$$<0|\sigma|0> = \frac{2m}{\sqrt{\lambda}} \tag{3.26}$$

From Eq. (3.20), on the other hand, we see that this implies

$$<0|\delta_\epsilon\zeta|0> = \frac{1}{\sqrt{2}}\,\epsilon <0|\sigma|0> = \epsilon\sqrt{\frac{2}{\lambda}}\,m = -i <0|[\zeta,\epsilon Q]|0> \tag{3.27}$$

This cannot be satisfied if

$$Q|0>=0 \tag{3.28}$$

In other words, we see that in this case, a symmetry of the theory is not a symmetry of the vacuum. In such a case, we say that there is a spontaneous breakdown of the symmetry.

To obtain the consequences of spontaneous symmetry breaking, let us observe that any perturbtion of a system can only be stable around the ground state. Thus we should expand the theory around the stable minimum by redefining

$$\sigma \rightarrow \sigma + \frac{2m}{\sqrt{\lambda}} \tag{3.29}$$

The Lagrangian now becomes

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\,\partial_\mu \sigma \partial^\mu \sigma + \frac{1}{2}\,\partial_\mu \zeta \partial^\mu \zeta + \frac{m^2}{2}\left((\sigma + \frac{2m}{\sqrt{\lambda}})^2 + \zeta^2\right) \\ &\quad - \frac{\lambda}{16}\left((\sigma + \frac{2m}{\sqrt{\lambda}})^2 + \zeta^2\right)^2 \\ &= \frac{1}{2}\,\partial_\mu \sigma \partial^\mu \sigma + \frac{1}{2}\,\partial_\mu \zeta \partial^\mu \zeta - m^2\sigma^2 + \frac{m^4}{\lambda} \\ &\quad - \frac{m\sqrt{\lambda}}{2}\,\sigma(\sigma^2 + \zeta^2) - \frac{\lambda}{16}\,(\sigma^2 + \zeta^2)^2 \end{aligned} \tag{3.30}$$

The interesting point to note in the above Lagrangian is that there is no mass term for the ζ-field. That is, the ζ-field has become massless. This is known as the Goldstone theorem which roughly says that whenever a continuous global symmetry is spontaneously broken, there must arise massless particles in the theory. The massless field ζ is also known as the Goldstone field.

Dirac Field Theories:

Let us next discuss theories which describe particles obeying Fermi-Dirac statistics. The simplest theory is, of course, one which describes a massless spin $\frac{1}{2}$ particle. The Lagrangian has the form

$$\mathcal{L} = i\bar{\psi}(x)\gamma^\mu \partial_\mu \psi(x) \tag{3.31}$$

Here γ^μ's are the 4×4 Dirac matrices we discussed in Lecture I and $\psi(x)$ is a four component spinor field which has the form

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix} \tag{3.32}$$

and

$$\bar{\psi}(x) = \psi^\dagger(x)\gamma^0 \tag{3.33}$$

The Euler-Lagrange equations have the form

$$\begin{aligned} &i\gamma^\mu \partial_\mu \psi(x) = i\not{\partial}\psi(x) = 0 \\ & \qquad\qquad\qquad\qquad\qquad\qquad (\not{A} = \gamma^\mu A_\mu) \\ &i\bar{\psi}(x)\overleftarrow{\partial_\mu}\gamma^\mu = i\bar{\psi}(x)\overleftarrow{\not{\partial}} = 0 \end{aligned} \tag{3.34}$$

The momentum canonically conjugate to $\psi_\alpha(x)$ is

$$\Pi_\alpha(x) = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_\alpha(x)} = i\left(\bar{\psi}(x)\gamma^0\right)_\alpha = i\psi^\dagger_\alpha(x) \qquad \alpha = 1,2,3,4 \tag{3.35}$$

The theory can now be quantized in the following way.

$$\begin{aligned} &\left\{\psi_\alpha(x), \psi_\beta(x')\right\}_{t=t'} = \left(\psi_\alpha(x)\psi_\beta(x') + \psi_\beta(x')\psi_\alpha(x)\right)_{t=t'} = 0 \\ &\left\{\Pi_\alpha(x), \Pi_\beta(x')\right\}_{t=t'} = 0 \\ &\left\{\psi_\alpha(x), \Pi_\beta(x')\right\}_{t=t'} = i\delta_{\alpha\beta}\delta^{(3)}(x - x') \end{aligned} \tag{3.36}$$

We see the basic difference between the bosonic theories and the fermionic theories in that the fermionic theories are quantized with anticommutators. This is connected with the fact that the fermionic theories describe particles obeying Fermi-Dirac statistics.

The Lagrangian in Eq. (3.31) is invariant under the constant phase transformations

$$\begin{aligned} \psi(x) &\rightarrow \widetilde{\psi}(x) = e^{i\alpha}\psi(x) \\ \bar{\psi}(x) &\rightarrow \widetilde{\bar{\psi}}(x) = e^{-i\alpha}\bar{\psi}(x) \end{aligned} \tag{3.37}$$

as well as the chiral phase transformations

$$\psi(x) \to \overset{\approx}{\psi}(x) = e^{i\gamma_5\beta}\psi(x)$$
$$\bar{\psi}(x) \to \overset{\approx}{\bar{\psi}}(x) = \bar{\psi}(x)e^{i\gamma_5\beta} \tag{3.38}$$

where α and β are constant parameters independent of any space-time coordinates. Infinitesimally, the transformations take the form

$$\delta_\epsilon\psi(x) = \tilde{\psi}(x) - \psi(x) = i\epsilon\psi(x)$$
$$\delta_\epsilon\bar{\psi}(x) = \tilde{\bar{\psi}}(x) - \bar{\psi}(x) = -i\epsilon\bar{\psi}(x) \tag{3.39}$$

and

$$\delta_\eta\psi(x) = \overset{\approx}{\psi}(x) - \psi(x) = i\eta\gamma_5\psi(x)$$
$$\delta_\eta\bar{\psi}(x) = \overset{\approx}{\bar{\psi}}(x) - \bar{\psi}(x) = i\eta\bar{\psi}(x)\gamma_5 \tag{3.40}$$

The conserved currents associated with these symmetry transformations can be constructed from the Noether's theorem (see Eq. (3.10)) and take the form

$$\epsilon j^\mu(x) = \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi(x)}\,\delta_\epsilon\psi(x) = i\bar{\psi}(x)\gamma^\mu(i\epsilon\psi(x))$$

$$\text{or,} \qquad j^\mu(x) = -\bar{\psi}(x)\gamma^\mu\psi(x) \tag{3.41}$$

and

$$\eta j_5^\mu(x) = \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi(x)}\,\delta_\eta\psi(x) = i\bar{\psi}(x)\gamma^\mu\left(i\eta\gamma_5\psi(x)\right)$$

$$\text{or,} \qquad j_5^\mu(x) = -\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x) \tag{3.42}$$

The corresponding conserved charges are

$$Q = \int d^3x\, j^0(x) = -\int d^3x\, \bar{\psi}(x)\gamma^0\psi(x)$$
$$= -\int d^3x\, \psi^\dagger(x)\psi(x) = i\int d^3x\, \Pi_\alpha(x)\psi_\alpha(x) \tag{3.43}$$

and

$$Q_5 = \int d^3x\, j_5^0(x) = -\int d^3x\, \bar{\psi}(x)\gamma^0\gamma_5\psi(x)$$
$$= -\int d^3x\, \psi^\dagger(x)\gamma_5\psi(x)$$
$$= i\int d^3x\, \Pi_\alpha(x)(\gamma_5)_{\alpha\beta}\psi_\beta(x) \tag{3.44}$$

Using the quantization conditions of Eq. (3.36), the charge algebra can now be shown to satisfy

$$[Q,Q] = 0 = [Q_5,Q_5] = [Q,Q_5] \tag{3.45}$$

Let us note here that although the currents j^μ and j_5^μ in Eqs. (3.41) and (3.42) are conserved classically, quantum mechanical corrections may spoil this. When a current is conserved classically, but quantum mechanically

$$\partial_\mu j^\mu(x) \neq 0 \tag{3.46}$$

we say that there are anomalies in the theory and that the symmetry has become anomalous. Anomalies associated with global symmetries are harmless (in fact, sometimes useful as shown in $\pi^0 \to 2\gamma$) but as we will see, anomalies associated with local symmetries can render the theory inconsistent.

Local Symmetry:

Let us next analyze what would happen if we tried to make the phase parameter in Eq. (3.37) to be a local function. Under

$$\psi(x) \to \tilde{\psi}(x) = e^{i\alpha(x)}\psi(x)$$
$$\bar{\psi}(x) \to \tilde{\bar{\psi}}(x) = e^{-i\alpha(x)}\bar{\psi}(x) \tag{3.47}$$

we see that

$$\mathcal{L} \to \tilde{\mathcal{L}} = i\tilde{\bar{\psi}}(x)\gamma^\mu\partial_\mu(\tilde{\psi}(x))$$
$$= i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - \partial_\mu\alpha(x)\bar{\psi}(x)\gamma^\mu\psi(x) \tag{3.48}$$
$$= \mathcal{L} - \partial_\mu\alpha(x)\bar{\psi}(x)\gamma^\mu\psi(x)$$

In other words, the Lagrangian of Eq. (3.31) is not invariant under a local phase transformation. On the other hand, we note that if we had started from the Lagrangian

$$\bar{\mathcal{L}} = i\bar{\psi}\gamma^{\mu}\left(\partial_{\mu} - ieA_{\mu}\right)\psi \tag{3.49}$$

where "e" is a constant, then this would be invariant under

$$\begin{aligned} \psi(x) &\to e^{i\alpha(x)}\psi(x) \\ \bar{\psi}(x) &\to e^{-i\alpha(x)}\bar{\psi}(x) \\ A_{\mu}(x) &\to A_{\mu}(x) + \frac{1}{e}\,\partial_{\mu}\alpha(x) \end{aligned} \tag{3.50}$$

Infinitesimally, these take the form

$$\begin{aligned} \delta_{\epsilon}\psi(x) &= i\epsilon(x)\psi(x) \\ \delta_{\epsilon}\bar{\psi}(x) &= -i\epsilon(x)\bar{\psi}(x) \\ \delta_{\epsilon}A_{\mu}(x) &= \frac{1}{e}\,\partial_{\mu}\epsilon(x) \end{aligned} \tag{3.51}$$

Thus we see that the local symmetry of Eq. (3.31) requires an additional field $A_{\mu}(x)$. $A_{\mu}(x)$ is known as a gauge field and the transformations of Eqs. (3.50) and (3.51) are known as gauge transformations.

Let us also note here that the Lagrangian $\bar{\mathcal{L}}$ can be obtained from $\mathcal{L}$ with the replacement

$$\begin{aligned} &\partial_{\mu} \to \partial_{\mu} - ieA_{\mu} \\ \text{or,}\qquad &p_{\mu} = -i\partial_{\mu} \to p_{\mu} - eA_{\mu} \end{aligned} \tag{3.52}$$

This is, of course, the prescription of minimal coupling we are familiar with in trying to couple charged particles to electromagnetic fields. Thus we can identify the gauge field, $A_{\mu}(x)$, with the photon of the theory. This also suggests that a local symmetry must always be accompanied by physical forces. Conversely, we may try to describe physical forces in terms of theories with local symmetries (or gauge theories).

Lecture IV

Quantum Electrodynamics:

We have seen how to couple charged spin $\frac{1}{2}$ particles to the electromagnetic field or the photon field. If we now introduce the dynamics of the photon fields, we would have an interacting theory of say, electrons and photons - otherwise known as quantum electrodynamics. The Lagrangian, in this case, has the form (e can now be thought of as the electromagnetic coupling.)

$$\mathcal{L}_{\text{QED}} = i\bar{\psi}\gamma^{\mu}\left(\partial_{\mu} - ieA_{\mu}\right)\psi - \frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} \tag{4.1}$$

where

$$\begin{aligned} F_{\mu\nu} &= \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} = -F_{\nu\mu} \\ F_{0i} &= E_i \\ F_{ij} &= -\epsilon_{ijk}B_k \end{aligned} \tag{4.2}$$

The Euler-Lagrange equations for this theory are

$$i\gamma^{\mu}\left(\partial_{\mu} - ieA_{\mu}\right)\psi = \left(i\partial\!\!\!/ + eA\!\!\!/\right)\psi = 0 \tag{4.3}$$

and

$$\partial_{\mu}F^{\mu\nu} = -e\bar{\psi}\gamma^{\nu}\psi = ej^{\nu} \tag{4.4}$$

The $\nu = 0$ and $\nu = i$ components of Eq. (4.4) give respectively

$$\vec{\nabla}\cdot\vec{E} = ej^{0}$$

and (4.5)

$$\vec{\nabla}\times\vec{B} = \frac{\partial\vec{E}}{\partial t} + e\vec{j}$$

Similarly, from the definition of $F_{\mu\nu}$ in Eq. (4.2) we see that it satisfies

$$\partial_{\mu}F_{\nu\lambda} + \partial_{\nu}F_{\lambda\mu} + \partial_{\lambda}F_{\mu\nu} = 0 \tag{4.6}$$

These set of equations can be shown to be equivalent to the pair of equations

$$\vec{\nabla} \cdot \vec{B} = 0$$

and (4.7)

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

We recognize Eqs. (4.5) and (4.7) together as the set of Maxwell's equations and, therefore, we conclude that the additional term in the Lagrangian in Eq. (4.1), indeed, gives the dynamics of the photon fields.

The Lagrangian of Eq. (4.1) can be checked to be invariant under the gauge transformations

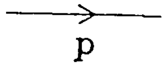
$$\begin{aligned} \delta_\epsilon \psi(x) &= i\epsilon(x)\psi(x) \\ \delta_\epsilon \bar{\psi}(x) &= -i\epsilon(x)\bar{\psi}(x) \\ \delta_\epsilon A_\mu(x) &= \frac{1}{e}\,\partial_\mu \epsilon(x) \end{aligned} \tag{4.8}$$

This is, of course, a local symmetry. Note that gauge invariance requires the photon to be massless since a mass term would break gauge invariance. From Eq. (4.4) we see that since $F^{\mu\nu}$ is antisymmetric, consistency of the equation would require

$$\partial_\nu \partial_\mu F^{\mu\nu} = 0 = e\partial_\nu j^\nu \tag{4.9}$$

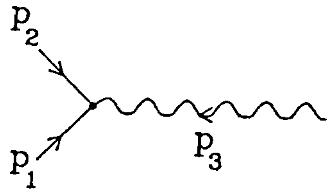
In such a case, therefore, the current must be conserved even quantum mechanically. Any violation of current conservation or any anomaly would render the dynamical equations inconsistent. This is, of course, what we have noted earlier, namely, whereas anomalous global symmetries are harmless, anomalies in local symmetries must be avoided.

The gauge invariances of the QED Lagrangian has both advantages as well as disadvantages. To see the advantages, let us write down the fermionic Feynman rules of the theory.



$$= iS(p) = \frac{i}{\not p}$$

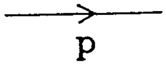
(4.10)



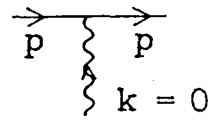
$$= -i\Gamma_\mu(p_1, p_2, p_3) = ie\gamma_\mu \delta^{(4)}(p_1 + p_2 + p_3)$$

From the structure of the propagator and the vertex, we see that

$$\frac{\partial}{\partial p^\mu}$$



$$= \frac{\partial}{\partial p^\mu} \frac{i}{\not p} = \frac{i}{\not p} i\gamma_\mu \frac{i}{\not p} = \frac{1}{e} \left(\quad \right)$$



$$\text{or,} \qquad \frac{\partial}{\partial p^\mu} \left(iS(p)\right) = \frac{1}{e} \left(iS(p)\right)\left(-i\Gamma_\mu(p, -p, 0)\right)\left(iS(p)\right)$$

$$\text{or,} \qquad \frac{\partial S^{-1}(p)}{\partial p^\mu} = -\frac{1}{e}\, \Gamma_\mu(p, -p, 0) \tag{4.11}$$

This relation is quite important in that it relates different scattering amplitudes. It is, in fact, a consequence of the gauge invariance of the system (although our simple derivation does not make it seem so). Although, we have derived this relation for the case when the electrons are massless, the same holds for massive electrons. Furthermore, this relation holds order by order in perturbation theory and plays a crucial role in the renormalization of the theory. Relation (4.11) is also known as the Ward identity of QED.

The difficulties of gauge invariance can be seen from Eq. (4.4).

$$\partial_\mu F^{\mu\nu} = ej^\nu$$

$$\text{or,} \qquad \partial_\mu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = ej^\nu$$

$$\text{or,} \qquad \left(\Box\, \eta^{\mu\nu} - \partial^\mu \partial^\nu\right) A_\mu = ej^\nu \tag{4.12}$$

The Greens function associated with this equation must satisfy

$$\left(\Box_x \eta^{\mu\nu} - \partial_x^\mu \partial_x^\nu\right) G_{\nu\lambda}(x - y) = -\delta^\mu_\lambda \delta^{(4)}(x - y) \tag{4.13}$$

But note that the operator $(\Box_x \eta^{\mu\nu} - \partial_x^\mu \partial_x^\nu)$ is a transverse projection operator in the sense that

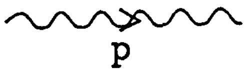
$$\partial_{x\mu} \left(\Box_x \eta^{\mu\nu} - \partial_x^\mu \partial_x^\nu\right) = \Box_x \partial_x^\nu - \Box_x \partial_x^\nu = 0 \tag{4.14}$$

Since projection operators do not have inverses, the Greens function of Eq. (4.13) does not exist. Consequently, the Cauchy initial value problem cannot be solved uniquely. Classically, we know that in such a case, we have to choose a gauge in which the problem can be solved. The rationale for this, of course, comes from the fact that any observable is gauge invariant and is, therefore, insensitive to a choice of gauge.

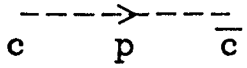
In the quantum theory, there is a well-defined procedure (known as the Faddeev-Popov procedure) for doing this. One adds a gauge fixing term to the Lagrangian (corresponding to the choice of a gauge) and a compensating ghost Lagrangian. Thus with a covariant gauge choice, the complete gauge fixed Lagrangian for QED takes the form

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= \mathcal{L}_{\text{QED}} - \frac{1}{2\alpha}\left(\partial_\mu A^\mu\right)^2 + \partial^\mu \bar{c} \partial_\mu c \\ &= i\bar{\psi}\gamma^\mu(\partial_\mu - ieA_\mu)\psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}\left(\partial_\mu A^\mu\right)^2 + \partial^\mu \bar{c} \partial_\mu c\end{aligned} \tag{4.15}$$

Here α is an arbitrary constant parameter known as the gauge fixing parameter. c and $\bar{c}$ are known as ghost fields and satisfy anticommutation relations like the fermions. (They are scalars with opposite statistics.) Physically, one can think of the ghost fields as subtracting out two degrees of freedom from the four component photon field to give effectively two physical degrees of freedom (namely, the transverse degrees). The complete theory of QED now has the additional Feynman rules given by



$$= iG_{\mu\nu}(p) = -\frac{i}{p^2}\left(\eta_{\mu\nu} - (1-\alpha)\frac{p_\mu p_\nu}{p^2}\right)$$



$$= iG_c(p) = \frac{i}{p^2} \tag{4.16}$$

The procedure of gauge fixing, while gives well defined calculational rules, has changed the theory also (at least appears to). For example, the theory is no longer gauge invariant and, consequently, it is not clear whether the Ward identities which we derived earlier and which characterize gauge invariance still hold in the full theory. A crucial observation which helps answer this question is that even though $\mathcal{L}_{\text{eff}}$ is not gauge invariant, it is invariant under a symmetry transformation involving the ghost field (also known as the BRST transformation). This can be appreciated by rewriting $\mathcal{L}_{\text{eff}}$ as

$$\mathcal{L}_{\text{eff}} = i\bar{\psi}\gamma^{\mu}\left(\partial_{\mu} - ieA_{\mu}\right)\psi - \frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} - F\partial_{\mu}A^{\mu} + \frac{\alpha}{2}\,F^{2} + \partial^{\mu}\bar{c}\partial_{\mu}c \tag{4.17}$$

Note that if we eliminate the auxiliary field F from Eq. (4.17), the Lagrangian of Eq. (4.15) is obtained. The Lagrangian (4.17) is invariant under the transformations

$$\begin{aligned}
\delta_{\beta}A_{\mu}(x) &= \frac{\beta}{e}\,\partial_{\mu}c(x)\\
\delta_{\beta}\psi(x) &= i\beta c(x)\psi(x)\\
\delta_{\beta}\bar{\psi}(x) &= -i\beta c(x)\bar{\psi}(x)\\
\delta_{\beta}c(x) &= 0\\
\delta_{\beta}\bar{c}(x) &= -\frac{\beta}{e}\,F(x)\\
\delta_{\beta}F(x) &= 0
\end{aligned} \tag{4.18}$$

Here β is a space-time independent anticommuting parameter. The invariance of the Lagrangian can be checked by noting that these transformations correspond to a gauge transformation if we identify

$$\alpha(x) = \beta c(x) \tag{4.19}$$

Since the original Lagrangian is gauge invariant, it follows now that

$$\begin{aligned}
\delta_\beta \mathcal{L}_{\text{eff}} &= \delta_\beta \Big[- F\partial_\mu A^\mu + \frac{\alpha}{2}\, F^2 + \partial^\mu \bar{c}\partial_\mu c \Big] \\
&= -F\partial_\mu \delta_\beta A^\mu + \partial^\mu \delta_\beta \bar{c}\partial_\mu c \\
&= -\frac{\beta}{e}\, F\partial_\mu\partial^\mu c - \frac{\beta}{e}\, \partial^\mu F\partial_\mu c \\
&= -\frac{\beta}{e}\, \partial_\mu (F\partial^\mu c)
\end{aligned} \tag{4.20}$$

Therefore, the action is invariant.

The BRST symmetry of the theory imposes relations between different scattering amplitudes which include the Ward identities we discussed earlier. But more importantly, the symmetry transformations of Eq. (4.18) lead to a conserved charge, Q_{BRST}, through the Noether procedure. This charge has the important property that it is nilpotent, that is,

$$Q^2_{\text{BRST}} = 0 \qquad\qquad (Q_{\text{BRST}} \text{ is fermionic.}) \tag{4.21}$$

This allows us to define the physical states in this theory as those states which are annihilated by Q_{BRST}, namely,

$$Q_{\text{BRST}} |\text{phys}> = 0 \tag{4.22}$$

Note that since Q_{BRST} is the generator of the BRST transformations, we can write (up to a total derivative)

$$\begin{aligned}
&-F\partial_\mu A^\mu + \frac{\alpha}{2}\, F^2 + \partial^\mu \bar{c}\partial_\mu c \\
&= \delta\Big(- e\big(A_\mu \partial^\mu \bar{c} + \frac{\alpha}{2}\, F\bar{c}\big)\Big) \\
&= \Big\{ Q_{\text{BRST}}, -e\Big(A_\mu \partial^\mu \bar{c} + \frac{\alpha}{2}\, F\bar{c}\Big)\Big\}
\end{aligned} \tag{4.23}$$

It follows now that

$$
\begin{aligned}
&< \text{phys}'|\mathcal{L}_{\text{eff}}|\text{phys} > \\
&\qquad =< \text{phys}'|\mathcal{L}_{\text{QED}} - F\partial_\mu A^\mu + \frac{\alpha}{2}\, F^2 + \partial^\mu \bar{c}\partial_\mu c|\text{phys} > \\
&\qquad =< \text{phys}'|\mathcal{L}_{\text{QED}} - e\left\{Q_{\text{BRST}}, A^\mu\partial_\mu \bar{c} + \frac{\alpha}{2}\, F\bar{c}\right\}|\text{phys} > \\
&\qquad =< \text{phys}'|\mathcal{L}_{\text{QED}}|\text{phys} >
\end{aligned}
\tag{4.24}
$$

Here in the last step we have used Eq. (4.22). This shows that even though the gauge fixing procedure may have changed the theory, the effect is not observable in the physical sector. The BRST symmetry also plays a crucial role in proving the perturbative unitarity of the theory.

Higgs Mechanism:

Let us reconsider the self-interacting theory of the complex scalar field which displays spontaneous symmetry breaking. However, let us also assume the complex (charged) scalar fields to be interacting with photons. As we have seen, interaction with photons can be introduced through the minimal coupling. Thus the Lagrangian for this theory is given by

$$
\mathcal{L} = -\frac{1}{4}\, F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi) + m^2\phi^\dagger\phi - \frac{\lambda}{4}\,(\phi^\dagger\phi)^2 \qquad \lambda > 0 \tag{4.25}
$$

where

$$
\begin{aligned}
D_\mu\phi &= (\partial_\mu - ieA_\mu)\phi \\
(D_\mu\phi)^\dagger &= (\partial_\mu + ieA_\mu)\phi^\dagger
\end{aligned}
\tag{4.26}
$$

In terms of the real fields σ and ζ, the Lagrangian of Eq. (4.25) takes the form

$$
\begin{aligned}
\mathcal{L} = &-\frac{1}{4}\, F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\,\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\,\partial_\mu\zeta\partial^\mu\zeta \\
&- e\sigma \overleftrightarrow{\partial_\mu}\zeta A^\mu + \frac{e^2}{2}\, A_\mu A^\mu(\sigma^2 + \zeta^2) \\
&+ \frac{m^2}{2}\,(\sigma^2 + \zeta^2) - \frac{\lambda}{16}\,(\sigma^2 + \zeta^2)^2
\end{aligned}
\tag{4.27}
$$

This Lagrangian is invariant under the gauge transformations

$$\delta_\epsilon \sigma(x) = -\frac{1}{\sqrt{2}}\,\epsilon(x)\zeta(x)$$

$$\delta_\epsilon \zeta(x) = \frac{1}{\sqrt{2}}\,\epsilon(x)\sigma(x) \tag{4.28}$$

$$\delta_\epsilon A_\mu(x) = \frac{1}{e}\,\partial_\mu \epsilon(x)$$

However, as we have seen earlier, the ground state of this theory occurs for

$$\begin{aligned} \zeta &= 0 \\ \sigma &= \frac{2m}{\sqrt{\lambda}} \end{aligned} \tag{4.29}$$

For a stable perturbation, the theory must be expanded around this ground state by letting

$$\sigma \to \sigma + \frac{2m}{\sqrt{\lambda}} \tag{4.30}$$

This leads the Lagrangian in Eq. (4.27) to take the form

$$\begin{aligned} \mathcal{L} = &-\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\,\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\,\partial_\mu\zeta\partial^\mu\zeta \\ &- \frac{2em}{\sqrt{\lambda}}\,A^\mu\partial_\mu\zeta - e\sigma\,\overleftrightarrow{\partial_\mu}\,\zeta A^\mu + \frac{2m^2e^2}{\lambda}\,A_\mu A^\mu \\ &+ \frac{2me^2}{\sqrt{\lambda}}\,\sigma A_\mu A^\mu + \frac{e^2}{2}\,A_\mu A^\mu(\sigma^2+\zeta^2) + \frac{m^4}{\lambda} - m^2\sigma^2 \\ &- \frac{m\sqrt{\lambda}}{2}\,\sigma(\sigma^2+\zeta^2) - \frac{\lambda}{16}\,(\sigma^2+\zeta^2)^2 \end{aligned} \tag{4.31}$$

As we have seen before, there is spontaneous breakdown of symmetry in this theory and in the absence of the photon fields, Goldstone's theorem guarantees the existence of massless particles. However, in the presence of the photon field, there is a gauge invariance which allows us to choose a gauge. In particular, if we choose the gauge (this is also known as the unitary gauge)

$$\zeta = 0 \tag{4.32}$$

Then the Lagrangian of Eq. (4.31) takes the form

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} \partial_\mu\sigma\partial^\mu\sigma + \frac{2m^2e^2}{\lambda} A_\mu A^\mu$$

$$- m^2\sigma^2 + \frac{m^4}{\lambda} + \frac{2me^2}{\sqrt{\lambda}} \sigma A_\mu A^\mu + \frac{e^2}{2} \sigma^2 A_\mu A^\mu \qquad (4.33)$$

$$- \frac{m\sqrt{\lambda}}{2} \sigma^3 - \frac{\lambda}{16} \sigma^4$$

We note that in such a case, the massless particle has disappeared and instead the photon field has become massive with a mass given by

$$m_{ph} = \frac{2me}{\sqrt{\lambda}} \qquad (4.34)$$

This is known as the Higgs mechanism and we say that the photon has become massive by eating the Goldstone boson ζ. Note that a massive photon has three helicity states as opposed to the two states a massless photon can have and, consequently, the total number of degrees of freedom is unchanged (otherwise unitarity would be violated). The Higgs mechanism is quite useful in generating masses for particles in a physical theory.

Lecture V

Non-Abelian Symmetries:

So far we have only considered symmetries where the generator of the symmetry, namely, the charge satisfied a trivial algebra - that is, it commuted with itself. Such symmetries are known as Abelian symmetries. Let us next consider some symmetries where the generators satisfy a nontrivial algebra. Such symmetries are known as non-Abelian symmetries and we are quite familiar with them also. For example, we know that the angular momentum operators generate rotations and satisfy the algebra

$$\left[J^a, J^b\right] = i\epsilon^{abc} J^c \qquad\qquad a,b,c = 1,2,3 \tag{5.1}$$

As we know, this is a non-Abelian algebra corresponding to the group SU(2). We also know that the quantum mechanical operator generating rotations is given by

$$U(\theta) = e^{iJ^a\theta^a} \tag{5.2}$$

where θ^a is the parameter of rotation. Thus, for example, we know that if ψ is a two component spinor corresponding to the $j = 1/2$ representation, then under a rotation

$$\psi \longrightarrow \widetilde{\psi} = U_{1/2}(\theta)\psi = e^{\frac{i}{2}\sigma^a\theta^a}\psi \tag{5.3}$$

where σ^a are the Pauli matrices and correspond to the angular momentum operators for this representation (actually, $\frac{1}{2}\sigma^a$ corresponds to the generators). Infinitesimally, the two components of the spinor would rotate as

$$\delta_\epsilon \psi^i = i\left(\frac{1}{2}\,\sigma^a\epsilon^a\right)^{ij}\psi^j \qquad\qquad i,j = 1,2 \tag{5.4}$$

The $j = 1/2$ representation is $2j+1 = 2$ dimensional and is also called the fundamental representation of SU(2).

SU(2) is, of course, the simplest of the non-Abelian symmetries. In general, the algebra corresponding to a higher symmetry group SU(n) consists of $n^2 - 1$ Hermitian generators and satisfies an algebra of the form

$$[T^a, T^b] = if^{abc}T^c \qquad a, b, c = 1, 2, \ldots, n^2 - 1 \tag{5.5}$$

where the totally antisymmetric constants, f^{abc}, are known as the structure constants of the group. We can think of the generators T^a as generating rotations in a $(n^2 - 1)$ dimensional internal space. Therefore , we can readily generalize many of the results of SU(2) to the SU(n) case. For example, we note that if ψ is a function belonging to the fundamental representation of SU(n), then it will be a n-component object and under an infinitesimal SU(n) rotation, it would transform as

$$\delta_\epsilon \psi^i = i\,(T^a \epsilon^a)^{ij}\,\psi^j \qquad i, j = 1, 2, \ldots n \tag{5.6}$$

Here T^a corresponds to the SU(n) generators in the fundamental representation and ϵ^a is the parameter of rotation.

Let us next consider a free fermion theory where the fermion field belongs to the fundamental representation of an internal symmetry group SU(n). We are, of course, quite familiar with many such fermionic systems. We know that the three colored quarks belong to the fundamental representation of the color group SU(3). The up and down quarks belong to the fundamental representation of the isospin group SU(2) and so on. Such a system is, therefore, worth studying. The Lagrangian is given by

$$\mathcal{L} = i\bar{\psi}^i \gamma^\mu \partial_\mu \psi^i \qquad i = 1, 2, \ldots n \tag{5.7}$$

which is just a sum of n-free fermion Lagrangians. Note that the momenta conjugate to $\psi^i_\alpha(x)$ are

$$\Pi^i_\alpha(x) = \frac{\partial \mathcal{L}}{\partial \dot{\psi}^i_\alpha(x)} = i\psi^{i\dagger}_\alpha(x) \qquad \alpha = 1, 2, 3, 4 \tag{5.8}$$

so that the quantization rules become

$$\begin{aligned} \left\{\psi^i_\alpha(x), \psi^j_\beta(x')\right\}_{t=t'} &= 0 = \left\{\Pi^i_\alpha(x), \Pi^j_\beta(x')\right\}_{t=t'} \\ \left\{\psi^i_\alpha(x), \Pi^j_\beta(x')\right\}_{t=t'} &= i\delta^{ij}\delta_{\alpha\beta}\delta^{(3)}(x - x') \end{aligned} \tag{5.9}$$

The Lagrangian of Eq. (5.7) is, of course, invariant under the global U(1) transformation we have discussed earlier, namely,

$$\psi^i(x) \longrightarrow e^{i\alpha}\psi^i(x)$$
$$\bar{\psi}^i(x) \longrightarrow e^{-i\alpha}\bar{\psi}^i(x) \tag{5.10}$$

where α is a constant scalar parameter. But more importantly, the Lagrangian is also invariant under a global SU(n) rotation which has the infinitesimal form

$$\delta_\epsilon\psi^i(x) = i\left(T^a\epsilon^a\right)^{ij}\psi^j$$
$$\delta_\epsilon\bar{\psi}^i(x) = -i\bar{\psi}^j\left(T^a\epsilon^a\right)^{ji} \tag{5.11}$$

Here ϵ^a are constant, infinitesimal parameters of the SU(n) transformation. The invariance can be checked readily as

$$\begin{aligned}\delta_\epsilon\mathcal{L} &= i\delta_\epsilon\bar{\psi}^i\gamma^\mu\partial_\mu\psi^i + i\bar{\psi}^i\gamma^\mu\partial_\mu\delta_\epsilon\psi^i \\ &= \bar{\psi}^j\left(T^a\epsilon^a\right)^{ji}\gamma^\mu\partial_\mu\psi^i - \bar{\psi}^i\gamma^\mu\partial_\mu\left((T^a\epsilon^a)^{ij}\psi^j\right) \\ &= \bar{\psi}^i\left(T^a\epsilon^a\right)^{ij}\gamma^\mu\partial_\mu\psi^j - \bar{\psi}^i\left(T^a\epsilon^a\right)^{ij}\gamma^\mu\partial_\mu\psi^j\end{aligned}$$

$$\text{or,}\quad \delta_\epsilon\mathcal{L} = 0 \tag{5.12}$$

The conserved current can now be constructed from the Noether procedure and has the form

$$j^{\mu,a} = -\bar{\psi}^i\left(T^a\right)^{ij}\gamma^\mu\psi^j = -\bar{\psi}T^a\gamma^\mu\psi \tag{5.13}$$

The corresponding conserved charges

$$Q^a = \int d^3x\, j^{0,a}(x) \tag{5.14}$$

can be shown using Eqs. (5.5) and (5.9) to satisfy

$$\left[Q^a, Q^b\right] = if^{abc}Q^c \tag{5.15}$$

which, as we have seen, is the SU(n) algebra (see Eq. (5.5)).

The Lagrangian of Eq. (5.7) is, however, not invariant under the SU(n) transformations of Eq. (5.11) if the parameter ϵ^a are coordinate dependent. As we have seen earlier, invariance under a local transformation necessarily requires the introduction of a gauge field. In the present case, the Lagrangian

$$\mathcal{L} = i\bar{\psi}^i \gamma^\mu \left(\delta^{ij}\partial_\mu - igA^a_\mu (T^a)^{ij}\right)\psi^j \tag{5.16}$$

can be shown to be invariant under the local gauge transformations

$$\begin{aligned}
\delta_\epsilon \psi^i(x) &= i\left(T^a \epsilon^a(x)\right)^{ij}\psi^j(x) \\
\delta_\epsilon \bar{\psi}^i(x) &= -i\bar{\psi}^j(x)\left(T^a\epsilon^a(x)\right)^{ji} \\
\delta_\epsilon A^a_\mu(x) &= \frac{1}{g}\,\partial_\mu \epsilon^a(x) + f^{abc}A^b_\mu(x)\epsilon^c(x)
\end{aligned} \tag{5.17}$$

The invariance can, in fact, be readily checked as

$$\begin{aligned}
\delta_\epsilon \mathcal{L} = & \; i\delta_\epsilon \bar{\psi}^i \gamma^\mu \left(\delta^{ij}\partial_\mu - igA^a_\mu (T^a)^{ij}\right)\psi^j \\
& + i\bar{\psi}^i \gamma^\mu \left(\delta^{ij}\partial_\mu - igA^a_\mu (T^a)^{ij}\right)\delta_\epsilon \psi^j \\
& + g\bar{\psi}^i \gamma^\mu (T^a)^{ij}\delta_\epsilon A^a_\mu \psi^j(x) \\
= & \; \bar{\psi}^k \left(T^b\epsilon^b(x)\right)^{ki}\gamma^\mu \left(\delta^{ij}\partial_\mu - igA^a_\mu (T^a)^{ij}\right)\psi^j \\
& - \bar{\psi}^i \gamma^\mu \left(\delta^{ij}\partial_\mu - igA^a_\mu (T^a)^{ij}\right)\left(T^b\epsilon^b(x)\right)^{jk}\psi^k(x) \\
& + g\bar{\psi}^i \gamma^\mu (T^a)^{ij}\left(\frac{1}{g}\,\partial_\mu \epsilon^a(x) + f^{abc}A^b_\mu(x)\epsilon^c(x)\right)\psi^j(x) \\
= & \; ig\bar{\psi}^i\gamma^\mu \left[T^a, T^b\right]^{ij} A^a_\mu(x)\epsilon^b(x)\psi^j(x) \\
& + g\bar{\psi}^i\gamma^\mu f^{abc}(T^a)^{ij}A^b_\mu(x)\epsilon^c(x)\psi^j(x)
\end{aligned}$$

Using Eq. (5.5), we now obtain

$$\begin{aligned}
\delta_\epsilon \mathcal{L} = & - g\bar{\psi}^i\gamma^\mu f^{abc}(T^c)^{ij}A^a_\mu(x)\epsilon^b(x)\psi^j(x) \\
& + g\bar{\psi}^i\gamma^\mu f^{abc}(T^a)^{ij}A^b_\mu(x)\epsilon^c(x)\psi^j(x) = 0
\end{aligned} \tag{5.18}$$

Several comments are in order here. The parameter g can be thought of as the coupling constant for the SU(n) gauge group. Furthermore, in the present case we note that the gauge fields A_μ^a carry SU(n) quantum numbers and hence SU(n) charge. This behavior is quite distinct from the photon which does not carry electric charge. Furthermore, since the gauge fields couple to any source carrying the corresponding charge and since in the case of SU(n) the gauge fields themselves carry SU(n) charge, it is clear that the gauge fields of SU(n) must couple to themselves - that is, they must have self-interaction in contrast to the case of the photon. In fact, the dynamical Lagrangian for the SU(n) gauge fields invariant under the transformations of Eq. (5.17) can be shown to be

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4} F_{\mu\nu}^a(x) F^{\mu\nu,a}(x) \qquad a = 1, 2, \ldots n^2 - 1$$

where (5.19)

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$$

The self-coupling is now obvious. Thus for example, if we are considering Quantum Chromodynamics corresponding to the gauge group SU(3), there would be $3^2 - 1 = 8$ gauge fields or gluons which not only couple to the colored quarks but also to themselves. This, of course, has profound consequences leading to asymptotic freedom.

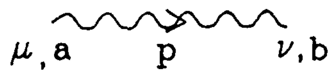
The complete Lagrangian including the fermions and the dynamics of the gauge fields which is invariant under the transformations of Eq. (5.17) is given by

$$\mathcal{L}_{\text{inv}} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu,a} + i\bar{\psi}^i \gamma^\mu \left(\delta^{ij} \partial_\mu - ig \left(T^a\right)^{ij} A_\mu^a\right) \psi^j(x) \qquad (5.20)$$

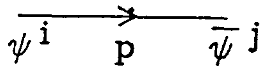
The gauge invariance, as we have seen, presents problems in quantizing the theory. Therefore, following the method due to Faddeev and Popov, we choose a gauge fixing and a ghost Lagrangian. A covariant choice of the gauge in the present case leads to the complete Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{inv}} - \frac{1}{2\alpha} \left(\partial_\mu A^{\mu,a}\right)^2 + \partial^\mu \bar{c}^a(x) \left(\partial_\mu \delta^{ac} + g f^{abc} A_\mu^b\right) c^c(x) \tag{5.21}$$

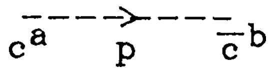
where $c^a(x)$ and $\bar{c}^a(x)$ are the respective ghost and anti-ghost fields. The Feynman rules for this Lagrangian can now be derived.



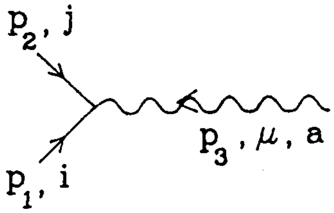
$$iG^{ab}_{\mu\nu}(p) = -\frac{i\delta^{ab}}{p^2} \left(\eta_{\mu\nu} - (1-\alpha)\frac{p_\mu p_\nu}{p^2}\right)$$



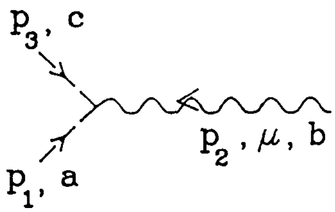
$$iS^{ij}(p) = \frac{i\delta^{ij}}{\not{p}}$$



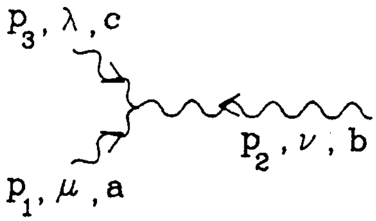
$$iG^{ab}(p) = \frac{i\delta^{ab}}{p^2}$$



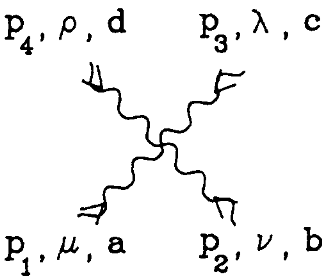
$$-i\Gamma^{a,ij}_\mu(p_1,p_2,p_3) = ig\gamma_\mu \left(T^a\right)^{ij} \delta^{(4)}(p_1+p_2+p_3) \tag{5.22}$$



$$-i\Gamma^{abc}_\mu(p_1,p_2,p_3) = g p_{1\mu} f^{abc} \delta^{(4)}(p_1+p_2+p_3)$$



$$-i\Gamma^{abc}_{\mu\nu\lambda}(p_1,p_2,p_3) = -g f^{abc}\Big[\left(p_1-p_2\right)_\lambda \eta_{\mu\nu} + \left(p_2-p_3\right)_\mu \eta_{\nu\lambda} + \left(p_3-p_1\right)_\nu \eta_{\lambda\mu}\Big]$$



$$-i\Gamma^{abcd}_{\mu\nu\lambda\rho}(p_1,p_2,p_3,p_4) = -ig^2\Big[f^{abp}f^{cdp}\left(\eta_{\mu\lambda}\eta_{\nu\rho} - \eta_{\mu\rho}\eta_{\nu\lambda}\right) + f^{acp}f^{dbp}\left(\eta_{\mu\rho}\eta_{\nu\lambda} - \eta_{\mu\nu}\eta_{\lambda\rho}\right) + f^{adp}f^{bcp}\left(\eta_{\mu\nu}\eta_{\lambda\rho} - \eta_{\mu\lambda}\eta_{\nu\rho}\right)\Big]$$

The Feynman rules clearly bring out the feature of pure gauge interactions. With these, one can now calculate any scattering amplitude.

The Lagrangian in Eq. (5.21) is no longer gauge invariant. But as in the case of QED, there is a residual (BRST) symmetry involving the anticommuting ghost fields. Thus the Lagrangian in Eq. (5.21) is invariant under the set of transformations

$$\begin{aligned} \delta_\beta A_\mu^a &= \beta\left(\frac{1}{g}\,\partial_\mu c^a(x) + f^{abc}A_\mu^b(x)c^c(x)\right) \\ \delta_\beta \psi^i &= i\beta c^a(x)\left(T^a\right)^{ij}\psi^j(x) \\ \delta_\beta \bar{\psi}^i &= -i\beta c^a(x)\bar{\psi}^j\left(T^a\right)^{ji} \\ \delta_\beta c^a(x) &= -\frac{\beta}{2}\,f^{abc}c^b(x)c^c(x) \\ \delta_\beta \bar{c}^a(x) &= -\frac{\beta}{\alpha}\,\left(\partial_\mu A^{\mu,a}(x)\right) \end{aligned} \tag{5.23}$$

Here β is an anticommuting constant parameter and the invariance of the Lagrangian can be checked in a straightforward manner. Once again the BRST symmetry leads to relations between different scattering amplitudes known as Ward identities or Slavnov-Taylor identities. These identities are much more complicated than the ones we encountered in the case of QED but are quite useful in renormalizing the theory. The conserved charge associated with the BRST symmetry in the present case can also be shown to be nilpotent. As in the case of QED, this helps us define a physical Hilbert space. In this space, the theory can again be shown to be independent of the choice of the gauge and the parameter α. Furthermore, perturbative unitarity can also be shown to hold in this space.

As we have seen earlier, renormalization introduces a mass scale μ and that all coupling constants become functions of μ. The μ-dependence of the SU(n) gauge coupling in the present case can be calculated since all the Feynman rules are known. In fact, if we assume that there are n_f fermion fields in the fundamental representation interacting with the SU(n) gauge field, then at one-loop level we find

$$\mu\,\frac{\partial g}{\partial \mu} = \beta(g) = -\frac{g^3}{16\pi^2}\cdot\frac{1}{3}\,(11n - 2n_f) \tag{5.24}$$

The first term on the right hand side comes from pure gauge interactions whereas the second term which depends on the number of fermion flavors comes from the fermionic interactions. Note that the two terms contribute with opposing signs. As in Eq. (2.29), we can solve Eq. (5.24) to obtain

$$g^2(\mu) = \frac{g^2(\bar{\mu})}{1 + \frac{g^2(\bar{\mu})}{48\pi^2}(11n - 2n_f)\ln\frac{\mu^2}{\bar{\mu}^2}} \tag{5.25}$$

This shows that if

$$11n - 2n_f > 0 \tag{5.26}$$

Then $g(\mu)$ decreases as μ increases with respect to $\bar{\mu}$. In other words, in such a case the coupling becomes weaker as the energy scale increases. In particular, for infinitely large energy values, the coupling vanishes leading us to conclude that such theories are asymptotically free.

Let us note, in particular, that when $n_f = 0$, namely, when no fermions are present, the scale dependence of the gauge coupling is given by

$$g^2(\mu) = \frac{g^2(\bar{\mu})}{1 + \frac{11ng^2(\bar{\mu})}{48\pi^2}\ln\frac{\mu^2}{\bar{\mu}^2}} \tag{5.27}$$

That is, in a pure non-Abelian gauge theory, the coupling is asymptotically free. It is the presence of fermions and other matter fields that spoils asymptotic freedom. Intuitively, one understands this as saying that fermions and other matter fields lead to a screening effect whereas a non Abelian gauge field leads to antiscreening which is responsible for asymptotic freedom. Note also that since in an asymptotically free theory, the coupling is weak at high energies, perturbative calculations can be trusted only at large energies. At low energies, however, the coupling constant grows and hence perturbation theory breaks down.

Let me conclude by pointing out that Quantum Chromodynamics which is the theory of strong interactions is a gauge theory based on the gauge group SU(3). The quarks which are the fermion fields in this theory come in threee colors and belong to the fundamental representation of SU(3). Thus specializing to $n = 3$ we would obtain all the necessary results for QCD. Since we do not see free quarks in nature, we can say that the color symmetry (SU(3)) is unbroken leading to the fact that observables must be color singlet states. As we have seen, since the coupling becomes stronger in non Abelian gauge theories, it supports this hypothesis that the quarks must be strongly bound. However, a conclusive proof of quark confinement is still lacking.

Lecture VI

Weinberg-Salam-Glashow Theory:

The strong force can be described by Quantum Chromodynamics which is a gauge theory based on the gauge group SU(3). As we have seen, we understand the basic features of this theory quite well. Thus let us ignore the strong interactions for a moment and try to understand the gauge structure of the other two fundamental interactions, namely, the weak interaction and the electromagnetic interaction. Let us recall that while leptons interact weakly as well as through electromagnetic interactions, they do not have any strong interaction. Consequently, we can, for simplicity, restrict ourselves to the gauge theory involving only leptons in order to understand the weak and electromagnetic forces.

To begin with, let us recall some facts about fundamental particles. We know that all elementary particles can be classified according to the representations of the weak isospin group, SU(2), which is very similar to the rotation group. (For clarity let me emphasize here that the weak isospin is different from the strong isospin which classifies observed hadrons.) Thus, let us list some of the more familiar particles all of which correspond to the $I = 1/2$ representation of the weak isospin group.

$$\begin{matrix} I_3 = \frac{1}{2} \\ I_3 = -\frac{1}{2} \end{matrix} \qquad \begin{pmatrix} \nu_e \\ e \end{pmatrix} \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix} \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} \tag{6.1}$$

The particles within a given multiplet are arranged so that the member with a higher $I_3(I_z)$ value has a larger electric charge. It is also known that we can assign to every elementary particle a U(1) quantum number known as the weak hypercharge and denoted by Y such that the electric charge of any given particle can be written as

$$Q = I_3 + \frac{Y}{2} \tag{6.2}$$

(Once again, a word of caution that the weak hypercharge is different from the strong hypercharge which can be identified with the sum of the baryon number and the strangeness number.) Eq. (6.2) can, in fact, be taken as defining the hypercharge of a given particle. Thus, the hypercharges of some of the particles in Eq. (6.1) are

$$Y_e = -1 = Y_{\nu_e}$$
$$Y_u = \frac{1}{3} = Y_d \tag{6.3}$$

Note here that the hypercharges of the particles within an isospin multiplet are the same.

Phenomenologically, we know that weak interactions are short ranged and, therefore, if they can be written as a gauge theory, the gauge bosons must be massive. Second, we know that they violate parity maximally and have a V−A structure. To understand this better, let us recall that the electromagnetic current, in the case of QED, has the form (see Eq. (3.41))

$$j^{\mu}_V = -\bar{\psi}\gamma^{\mu}\psi \tag{6.4}$$

This behaves like a vector under a Lorentz transformation as well as under a space reflection and is, therefore, called a vector current. An axial vector current, on the other hand, transforms like a vector under a Lorentz transformation but behaves like a pseudo-vector under a space reflection and has the form (see Eq. (3.42))

$$j^{\mu}_A = -\bar{\psi}\gamma^{\mu}\gamma_5\psi \tag{6.5}$$

A V−A current, as the name suggests, has the structure

$$\begin{aligned} j^{\mu}_{V-A} &= -\frac{1}{2}\,\bar{\psi}\gamma^{\mu}(1-\gamma_5)\psi \\ &= -\bar{\psi}_L\gamma^{\mu}\psi_L \end{aligned} \tag{6.6}$$

where we have defined

$$\begin{aligned} \psi_L &= \frac{1}{2}\,(1-\gamma_5)\psi \\ \bar{\psi}_L &= \frac{1}{2}\,\bar{\psi}(1+\gamma_5) \end{aligned} \tag{6.7}$$

The quantity $\frac{1}{2}(1-\gamma_5)$ is a projection operator which merely projects out the left handed component of a fermion field. Thus the V−A structure of weak interactions tantamounts to saying that only the left-handed components of particles participate in weak interactions. Consequently, we can think of weak isospin as a left-handed group.

With all this information, let us construct the simplest theory involving only one family of leptons, namely, the electron family. Let

$$\ell = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \tag{6.8}$$

This is an isospin doublet. However, whereas we know that the right handed component of the electron exists, right handed neutrinos are not seen in nature. Consequently, there will only be one right-handed lepton in this case which is the right-handed electron. By definition, this will be an isospin singlet.

$$r = e_R \tag{6.9}$$

All the fermions, of course, carry the hypercharge quantum number. We have already determined the hypercharge of the left-handed particles to be

$$Y_\ell = -1 \tag{6.10}$$

The hypercharge of the right-handed electron can, similarly, be determined to be

$$Y_r = -2 \tag{6.11}$$

Note that all the fermions carry both the isospin as well as the hypercharge quantum numbers. Thus the simplest gauge theory that we can think of constructing is one where both these symmetries are local. We can easily write down a fermionic Lagrangian which is invariant under the isospin ($\mathrm{SU_L}(2)$) and hypercharge ($\mathrm{U}_Y(1)$) gauge transformations.

$$\begin{aligned} \mathcal{L}_f = & \, i\bar{\ell}^i \gamma^\mu \left(\delta^{ij} \partial_\mu - \frac{ig'}{2} \delta^{ij} Y_\mu - \frac{ig}{2} \left(\sigma^a\right)^{ij} W_\mu^a \right) \ell^j \\ & + i\bar{r}\gamma^\mu (\partial_\mu - ig' Y_\mu) r \end{aligned} \tag{6.12}$$

Here $i, j = 1, 2$ and $a = 1, 2, 3$. We have introduced the gauge fields W_μ^a and Y_μ corresponding to the isospin and hypercharge transformations. g and g' denote respectively the strengths of the isospin and hypercharge interactions. Note that since the right-handed field does not carry any isospin quantum number, it does not couple to W_μ^a.

The dynamics of the gauge fields can now be introduced in a straightforward manner.

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}\, Y_{\mu\nu}Y^{\mu\nu} - \frac{1}{4}\, W_{\mu\nu}^a W^{\mu\nu,a} \tag{6.13}$$

where

$$\begin{aligned} Y_{\mu\nu} &= \partial_\mu Y_\nu - \partial_\nu Y_\mu \\ W_{\mu\nu}^a &= \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g\epsilon^{abc} W_\mu^b W_\nu^c \end{aligned} \tag{6.14}$$

Relation (6.14) emphasizes that Y_μ is an Abelian gauge field like the photon field since it corresponds to the group $\mathrm{U_Y}(1)$. W_μ^a, on the other hand, is a non-Abelian gauge field corresponding to the gauge group $\mathrm{SU_L}(2)$. Thus, together,

$$\mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_f \tag{6.15}$$

defines an interacting gauge theory of leptons based on the gauge group $\mathrm{SU_L}(2) \times \mathrm{U_Y}(1)$.

The weak interactions, on the other hand, are short ranged which amounts to the corresponding gauge bosons being massive. We can incorporate this into our theory by adding to our Lagrangian a part depending on scalar fields which, as we have seen, can give masses to the gauge bosons through the Higgs mechanism. Let

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \tag{6.16}$$

denote an isospin doublet of complex scalar fields with charges 1 and 0. Thus the hypercharge quantum number associated with this multiplet is 1. Let us also denote the Hermitian conjugate of ϕ as

$$\phi^\dagger = \phi^- \quad \bar{\phi}^0 \tag{6.17}$$

The scalar Lagrangian, invariant under $SU_L(2) \times U_Y(1)$ transformations, can now be written as

$$\begin{aligned}\mathcal{L}_{\text{Higgs}} = &\left(\left(\delta^{ij}\partial_\mu + \frac{ig'}{2}\,\delta^{ij}Y_\mu - \frac{ig}{2}\,(\sigma^a)^{ij}W^a_\mu\right)\phi^j\right)^\dagger \\ &\cdot\left(\delta^{ik}\partial^\mu + \frac{ig'}{2}\,\delta^{ik}Y^\mu - \frac{ig}{2}\,(\sigma^b)^{ik}W^{\mu,b}\right)\phi^k \\ &+ m^2\phi^\dagger\phi - \frac{\lambda}{4}\,(\phi^\dagger\phi)^2 - h(\bar{r}\phi^\dagger\ell + \bar{\ell}\phi r)\end{aligned} \tag{6.18}$$

This is the usual symmetry breaking Lagrangian for the scalar fields except for the last term representing the interaction between the scalar fields and the fermions in the theory. h is known as the strength of the Yukawa coupling. We can now write the total Lagrangian for the theory to be

$$\mathcal{L}_{\text{TOT}} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_f + \mathcal{L}_{\text{Higgs}} \tag{6.19}$$

Note that if we now define the combinations

$$W^\pm_\mu = \frac{1}{\sqrt{2}}\,(W^1_\mu \pm iW^2_\mu) \tag{6.20}$$

then $\mathcal{L}_{\text{Higgs}}$ can be written explicitly in terms of the components as

$$\begin{aligned}\mathcal{L}_{\text{Higgs}} = &\left((\partial_\mu - \frac{ig'}{2}\,Y_\mu)\phi^- + \frac{ig}{2}\,W^3_\mu\phi^- + \frac{ig}{\sqrt{2}}\,W^-_\mu\bar{\phi}^0\right)\left((\partial^\mu + \frac{ig'}{2}\,Y^\mu)\phi^+\right. \\ &\left.- \frac{ig}{2}\,W^{\mu 3}\phi^+ - \frac{ig}{\sqrt{2}}\,W^{\mu +}\phi^0\right) + \left((\partial_\mu - \frac{ig'}{2}\,Y_\mu)\bar{\phi}^0\right. \\ &\left.+ \frac{ig}{\sqrt{2}}\,W^+_\mu\phi^- - \frac{ig}{2}\,W^3_\mu\bar{\phi}^0\right)\left((\partial^\mu + \frac{ig'}{2}\,Y^\mu)\phi^0\right. \\ &\left.- \frac{ig}{\sqrt{2}}\,W^{\mu -}\phi^+ + \frac{ig}{2}\,W^{\mu 3}\phi^0\right) + m^2(\phi^-\phi^+ + \bar{\phi}^0\phi^0) \\ &- \frac{\lambda}{4}\,(\phi^-\phi^+ + \bar{\phi}^0\phi^0)^2 - h\bar{e}_R\nu_{eL}\phi^- - h\bar{e}_R e_L\bar{\phi}^0 \\ &- h\bar{\nu}_{eL}e_R\phi^+ - h\bar{e}_L e_R\phi^0\end{aligned} \tag{6.21}$$

As before, we can now calculate the minimum of the potential. To be consistent with our earlier notation, let me define

$$\phi^0 = \frac{1}{\sqrt{2}}(\sigma + i\zeta)$$
$$\bar{\phi}^0 = \frac{1}{\sqrt{2}}(\sigma - i\zeta) \tag{6.22}$$

In terms of these, the minimum of the potential can be shown to occur at (see Eq. (4.29))

$$\phi^+ = 0 = \phi^- = \zeta$$
$$\sigma = v = \frac{2m}{\sqrt{\lambda}} \tag{6.23}$$

We can now expand the theory around this classical minimum by letting

$$\sigma \longrightarrow \sigma + v \tag{6.24}$$

To bring out the essential features of the theory, let us first look at only the quadratic terms in $\mathcal{L}_{\mathrm{TOT}}$ after shifting. The quadratic Lagrangian takes the form

$$\begin{aligned}
\mathcal{L}_{\mathrm{Quad.}} = & -\frac{1}{2}\left(\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+\right)\left(\partial^\mu W^{\nu-} - \partial^\nu W^{\mu-}\right) \\
& -\frac{1}{4}\left(\partial_\mu W_\nu^3 - \partial_\nu W_\mu^3\right)\left(\partial^\mu W^{\nu 3} - \partial^\nu W^{\mu 3}\right) \\
& -\frac{1}{4}\left(\partial_\mu Y_\nu - \partial_\nu Y_\mu\right)\left(\partial^\mu Y^\nu - \partial^\nu Y^\mu\right) + i\bar{\nu}_{eL}\partial\!\!\!/ \nu_{eL} \\
& + i\bar{e}_L \partial\!\!\!/ e_L + i\bar{e}_R \partial\!\!\!/ e_R + \partial_\mu\phi^-\partial^\mu\phi^+ \\
& + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\zeta\partial^\mu\zeta - \frac{igv}{2} W_\mu^+\partial^\mu\phi^- \\
& + \frac{igv}{2} W_\mu^-\partial^\mu\phi^+ + \frac{v}{2}\left(g'Y_\mu + gW_\mu^3\right)\partial^\mu\zeta \\
& + \frac{g^2v^2}{4} W_\mu^+ W^{\mu-} + \frac{v^2}{8}\left(g'Y_\mu + gW_\mu^3\right)\left(g'Y^\mu + gW^{\mu 3}\right) \\
& - \frac{hv}{\sqrt{2}}\bar{e}_R e_L - \frac{hv}{\sqrt{2}}\bar{e}_L e_R - m^2\sigma^2
\end{aligned} \tag{6.25}$$

Although this looks complicated, it can be simplified by redefining variables in the following way. Let

$$\begin{aligned}
\sin\theta_W &= \frac{g'}{\sqrt{g^2+g'^2}} \qquad\qquad (\theta_W = \text{Weinberg angle}) \\
\cos\theta_W &= \frac{g}{\sqrt{g^2+g'^2}} \\
Z_\mu &= \frac{1}{\sqrt{g^2+g'^2}}\left(g'Y_\mu + gW^3_\mu\right) = \sin\theta_W Y_\mu + \cos\theta_W W^3_\mu \\
A_\mu &= \frac{1}{\sqrt{g^2+g'^2}}\left(gY_\mu - g'W^3_\mu\right) = \cos\theta_W Y_\mu - \sin\theta_W W^3_\mu
\end{aligned} \tag{6.26}$$

Then in terms of these variables, the quadratic Lagrangian takes the form

$$\begin{aligned}
\mathcal{L}_{\text{Quad.}} = &-\frac{1}{2}\left(\partial_\mu W^+_\nu - \partial_\nu W^+_\mu\right)\left(\partial^\mu W^{\nu-} - \partial^\nu W^{\mu-}\right) + \frac{g^2v^2}{4}\,W^+_\mu W^{\mu-} \\
&-\frac{1}{4}\left(\partial_\mu Z_\nu - \partial_\nu Z_\mu\right)\left(\partial^\mu Z^\nu - \partial^\nu Z^\mu\right) + \frac{(g^2+g'^2)v^2}{8}\,Z_\mu Z^\mu \\
&-\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) \\
&+ i\bar{\nu}_{eL}\partial\!\!\!/\,\nu_{eL} + i\bar{e}_L\partial\!\!\!/\,e_L + i\bar{e}_R\partial\!\!\!/\,e_R \\
&-\frac{hv}{\sqrt{2}}\,\bar{e}_R e_L - \frac{hv}{\sqrt{2}}\,\bar{e}_L e_R + \partial_\mu\phi^-\partial^\mu\phi^+ \\
&+\frac{1}{2}\,\partial_\mu\zeta\partial^\mu\zeta + \frac{1}{2}\,\partial_\mu\sigma\partial^\mu\sigma - m^2\sigma^2 \\
&-\frac{igv}{2}\,W^+_\mu\partial^\mu\phi^- + \frac{igv}{2}\,W^-_\mu\partial^\mu\phi^+ \\
&+\frac{v\sqrt{g^2+g'^2}}{2}\,Z_\mu\partial^\mu\zeta
\end{aligned} \tag{6.27}$$

We note here that three of the four gauge fields have become massive and only one gauge field remains massless. (This is particularly obvious if we choose the gauge $\phi^+ = \phi^- = \zeta = 0$.) Thus the original symmetry has spontaneously broken down to U(1). Thus we say

$$\mathrm{SU_L(2)} \times \mathrm{U_Y(1)} \longrightarrow \mathrm{U_{em}(1)} \tag{6.28}$$

The field A_μ can be identified with the photon so that we have the familiar result that in this theory even though isospin and hypercharge quantum numbers may be violated in some processes, the electric charge will be conserved. We also note that spontaneous symmetry breaking gives a mass to the electron through the Yukawa coupling whereas the neutrino remains massless. Note that the mass of the W and the Z-bosons are given by

$$\begin{aligned} M_W &= \frac{gv}{2} \\ M_Z &= \frac{(g^2+g'^2)^{1/2}v}{2} \end{aligned} \tag{6.29}$$

so that

$$\frac{M_W}{M_Z} = \frac{g}{\sqrt{g^2+g'^2}} = \cos\theta_W \tag{6.30}$$

Both these masses and the Weinberg angle are, of course, well measured experimentally.

Let us next look at the part of $\mathcal{L}_{\mathrm{TOT}}$ describing the interaction of the fermions with the gauge fields.

$$\begin{aligned} \mathcal{L}_{\text{int}} = \frac{g}{\sqrt{2}}\, W_\mu^+ \bar{\nu}_{eL}\gamma^\mu e_L + \frac{g}{\sqrt{2}}\, W_\mu^- \bar{e}_L\gamma^\mu \nu_{eL} \\ + \frac{g}{2}\, W_\mu^3 \Big(\bar{\nu}_{eL}\gamma^\mu \nu_{eL} - \bar{e}_L\gamma^\mu e_L \Big) \\ + \frac{g'}{2}\, Y_\mu \Big(\bar{\nu}_{eL}\gamma^\mu \nu_{eL} + \bar{e}_L\gamma^\mu e_L + 2\bar{e}_R\gamma^\mu e_R \Big) \end{aligned} \tag{6.31}$$

We can rewrite this in terms of the variables in Eq. (6.26) as

$$\begin{aligned} \mathcal{L}_{\text{int}} = \frac{g}{\sqrt{2}} \Big(W_\mu^+ \bar{\nu}_{eL}\gamma^\mu e_L + W_\mu^- \bar{e}_L\gamma^\mu \nu_{eL} \Big) \\ + \frac{Z_\mu}{2\sqrt{g^2+g'^2}} \Big(g^2 (\bar{\nu}_{eL}\gamma^\mu \nu_{eL} - \bar{e}_L\gamma^\mu e_L) \\ + g'^2 (\bar{\nu}_{eL}\gamma^\mu \nu_{eL} + \bar{e}_L\gamma^\mu e_L + 2\bar{e}_R\gamma^\mu e_R) \Big) \\ + \frac{gg'}{\sqrt{g^2+g'^2}}\, A_\mu (\bar{e}_L\gamma^\mu e_L + \bar{e}_R\gamma^\mu e_R) \end{aligned} \tag{6.32}$$

The first and the second terms in Eq. (6.32) exprerss the charged and neutral current structures of the weak interactions. The last term, on the other hand, has precisely the form of the coupling of electrons to photons if we identify the electromagnetic coupling to be

$$e = \frac{gg'}{\sqrt{g^2 + g'^2}} = g \sin\theta_W = g' \cos\theta_W \tag{6.33}$$

We have, of course, considered the simplest model with one family of leptons. One can add more families of leptons as well as quarks. We will then have a gauge theory of weak and electromagnetic interactions involving quarks and leptons. This is known as the standard model and seems to work well experimentally.