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MAXIMUM ALLOWABLE RESISTIVE ZONE IN A METASTABLE SUPERCONDUCTOR

L. Dresner

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Prepared by the
OAK RIDGE NATIONAL LABORATORY
Oak Ridge, Tennessee 37830
operated by
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ABSTRACT

If a metastable superconductor (i.e., not cryostable) has a small resistive zone (due, e.g., to a manufacturing defect), it may quench spontaneously at high transport current. Even if it does not quench, its stability is degraded by the defect, i.e., the existence of the defect decreases the energy needed to form the minimum propagating zone. This paper answers the following related questions: (1) how large is the maximum allowable resistive zone and (2) what is the effect on stability of a zone smaller than the maximum allowable one?

If the resistive fault is small enough, it creates a steady zone of elevated temperature with cold ends. The heat balance equation admits two steady solutions with cold ends — one stable against small disturbances and one unstable. The stable one is realized in nature; the unstable one is the minimum propagating zone. The difference in formation energy of the two zones, which can serve as a measure of the stability, decreases with increasing fault resistance. Eventually, it reaches zero for some maximum allowable fault resistance. When the fault resistance is beyond the maximum allowable value, there are no steady solutions of the heat balance equation with cold ends. This means the conductor must quench spontaneously when charged.

The steady solutions of the heat balance equation have been calculated in detail for the case of a constant heat transfer coefficient. The stability against small perturbations of the steady solutions has been studied by means of a Lyapunov function. Finally, the criterion for the maximum allowable fault resistance has been generalized to account for any curve of heat transfer coefficient versus temperature, including the usual three-part boiling curve.

In the simpler case of constant heat transfer coefficient, the maximum allowable fault resistance is given by

$$\beta = (\alpha i/[\alpha i-1)])^{1/2}(1-i) ,$$

where

$$\beta \equiv Q(Ph/Ak)^{1/2}/2Ph(T_c - T_b) ,$$

$$\alpha \equiv \rho I_c^2/PAh(T_c - T_b),$$

and

$$i \equiv I/I_c$$

and Q is the steady power produced in the plane of the fault, P is the cooled perimeter, A is the conductor cross section, k is the longitudinal thermal conductivity, h is the heat transfer coefficient, T_c is the critical temperature, T_b is the ambient helium temperature, ρ is the normal longitudinal resistivity, I_c is the critical current, and I is the transport current.

1. INTRODUCTION

If a metastable superconductor (i.e., not cryostable) has a small resistive zone (due, e.g., to a manufacturing defect), it may quench spontaneously at high transport current. This paper answers the related questions (1) how large is the maximum allowable resistive zone and (2) what is the effect on stability of a zone smaller than the maximum allowable one?

2. BASIC EQUATIONS

We make the simplifying assumption of constant physical properties (thermal conductivity, specific heat, heat transfer coefficient, matrix resistivity). We assume that at x = 0 there is a steady plane source of heat Q[W]. The heat balance equation is

$$AS\,\frac{\partial T}{\partial t} = A\,\frac{\partial}{\partial x}\left(k\,\frac{\partial T}{\partial x}\right) + \frac{\rho I^2}{A}\,g(T) - Ph(T - T_b) + Q\delta(x) \tag{1}$$

where g(T) is the current-sharing function

$$g(T) = \begin{cases} 0, & T_b < T < T_{cs} \\ (T - T_{cs})/(T_c - T_{cs}), & T_{cs} < T < T_c \\ 1, & T_c < T \end{cases} \tag{2}$$

and $\delta(x)$ is the Dirac delta function. If we introduce the dimensionless variables

$$\theta = (T - T_b)/(T_c - T_b) \tag{3a}$$

$$\tau = Pht/AS \tag{3b}$$

$$\xi = (Ph/Ak)^{1/2}x \tag{3c}$$

$$\alpha = \rho I_c^2/PAh(T_c - T_b) \tag{3d}$$

$$\beta = Q(Ph/Ak)^{1/2}/2Ph(T_c - T_b) \tag{3e}$$

Eq. (1) becomes

$$\frac{\partial\theta}{\partial\tau} = \frac{\partial^2\theta}{\partial\xi^2} + \alpha i^2 g(\theta) - \theta + 2\beta\delta(\xi) \tag{4a}$$

where now

$$g(\theta) = \begin{cases} 0, & \theta < 1-i \\ (\theta + i-1)/i, & 1-i < \theta < 1 \\ 1, & 1 < \theta \end{cases} \tag{4b}$$

[Remember $\delta(x)dx = \delta(\xi)d\xi$.] The delta function appearing in (4) can be replaced by the jump condition

$$\left(\frac{\partial\theta}{\partial\xi}\right)_{0+} - \left(\frac{\partial\theta}{\partial\xi}\right)_{0-} = -2\beta \tag{5a}$$

obtained by integrating (3) on an infinitesimal interval centered on $\xi = 0$. For solutions symmetric about $\xi = 0$, (5a) becomes

$$\left(\frac{\partial\theta}{\partial\xi}\right)_{0+} = -\left(\frac{\partial\theta}{\partial\xi}\right)_{0-} = -\beta \tag{5b}$$

3. STEADY-STATE SOLUTIONS

In steady state, (4a) can be written

$$\frac{d^2\theta}{d\xi^2} + h(\theta) = 0, \quad \xi > 0 \tag{6a}$$

$$\left(\frac{d\theta}{d\xi}\right)_{0+} = -\beta \tag{6b}$$

where

$$h(\theta) = \alpha i^2 g(\theta) - \theta \tag{6c}$$

To solve (6), multiply (6a) by $d\theta/d\xi$ and integrate from 0+ to ξ:

$$\dot{\theta}^2 \equiv \left(\frac{d\theta}{d\xi}\right)^2 = \beta^2 + 2\int_{\theta}^{\theta_m} h(\theta')d\theta' \tag{7}$$

where θ_m is the (maximum) value of θ at $\xi = 0$. If we look for solutions for which $\theta(\infty) = \dot{\theta}(\infty) = 0$, we see from Fig. 1 that there are at most three, for which β and θ_m are related by

$$0 = \beta^2 + 2\int_0^{\theta_m} h(\theta)d\theta \tag{8}$$

We can plot β^2 as a function of θ_m by evaluating the integral in (8):

$$\beta^2 = \begin{cases} \theta_m^2, \ \theta_m < 1-i & \text{(9a)} \\ \theta_m^2 - \alpha i(\theta_m + i-1)^2, \ 1-i < \theta_m < 1 & \text{(9b)} \\ \theta_m^2 - 2\alpha i^2\theta_m + \alpha i^2(2-i), \ 1 < \theta_m & \text{(9c)} \end{cases}$$

The form of the plot depends on whether $\alpha i^3 > 1$ or $\alpha i^3 < 1$. Figure 2 illustrates the form when $\alpha i^3 > 1$, the case of greatest importance. From Fig. 2, we see that when β is small θ_m has three values, but when β is large it has only one. Each value of θ_m corresponds to a steady solution of (6). Figure 2 also represents a plot of $\dot{\theta}^2$ versus θ, as we can see by subtracting (8) from (7) and comparing the result with (8). If we look for solutions of finite extent, i.e., solutions that have cold ends [$\theta(\infty) = 0$], we must choose the point (θ_m, β^2) on a branch of the curve that is continuously connected to the origin. The only such branch is OPQR. The branch ST cannot represent a steady state with cold ends.

From this analysis, we see that if β exceeds β_Q, where

$$\beta_Q = \left(\frac{\alpha i}{\alpha i - 1}\right)^{1/2}(1-i) \tag{10}$$

no steady normal zones having cold ends can exist. If $\beta < \beta_Q$, steady normal zones can exist <u>physically</u> only if they are stable against small perturbations. The next four sections of this paper are devoted to the analysis of the stability of the two steady-state solutions with cold ends represented by the branches OPQ and QR of the curve in Fig. 2. The upshot of this analysis is that the branch OPQ represents stable solutions and the branch QR unstable solutions. This means that steady normal zones exist for β right up to β_Q. Readers not interested in the details of the stability analysis can skip now to section 8.

4. STABILITY OF THE STEADY-STATE SOLUTIONS

The method used to study the stability of the steady-state solutions rests on the fact that the right-hand side of (4a) is the Euler-Lagrange equation of the Lagrangean

$$L(\theta,\dot{\theta}) = \frac{1}{2}\dot{\theta}^2 - \int_0^{\theta} h(\theta')d\theta' \tag{11}$$

In other words, for $\xi > 0$, (4a) can be written

$$\frac{\partial\theta}{\partial\tau} = \frac{\partial}{\partial\xi}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} \tag{12}$$

where $\dot{\theta}$ now denotes the partial derivative of θ with respect to ξ.

Let us now define the action A as the functional

$$A[\theta] = \int_{0+}^{\infty}\left[\frac{1}{2}\dot{\theta}^2 - \int_0^{\theta} h(\theta')d\theta'\right]d\xi - \beta\theta(\xi = 0) \tag{13}$$

and calculate the time rate of change of A:

$$\frac{dA}{d\tau} = \int_{0+}^{\infty} \left[\dot{\theta} \frac{\partial \dot{\theta}}{\partial \tau} - \frac{\partial \theta}{\partial \tau} h(\theta) \right] d\xi - \beta \left(\frac{\partial \theta}{\partial \tau}\right)_{\xi=0} \tag{14a}$$

$$= - \int_{0+}^{\infty} [\ddot{\theta} + h(\theta)] \frac{\partial \theta}{\partial \tau} d\xi \tag{14b}$$

$$= - \int_{0+}^{\infty} \left(\frac{\partial \theta}{\partial \tau}\right)^2 d\xi < 0 \tag{14c}$$

The passage from (14a) to (14b) is by integration by parts. The steady states make the action A a local extremum. If this extremum is a local minimum, it follows from (14c) that the steady state is stable against small perturbations. If the extremum is either a maximum or a saddle point, the steady state is unstable.

5. SECOND VARIATION OF THE ACTION

Suppose we set $\theta = \theta_0 + \eta$, where θ_0 is a steady-state solution and $\eta << \theta_0$ everywhere. Then, a short computation shows

$$A[\theta] = A[\theta_0] + \frac{1}{2} \int_{0+}^{\infty} [\dot{\eta}^2 - \dot{h}(\theta_0)\eta^2] d\xi + \ldots \tag{15}$$

where $\dot{h}(\theta_0)$ means $(dh/d\theta)_{\theta=\theta_0}$. The second term on the right-hand side of (15), $\delta^2 A$, is the second variation of A (i.e., the term of order η^2). The first variation δA (term of order η) can be shown to vanish by an integration by parts.

So far we have placed no restrictions on η, but we want θ to obey the same boundary conditions as θ_0, i.e., $\dot{\theta}(\xi = 0) = -\beta$ (source condition) and $\theta(\xi = \infty) = 0$ (cold-end condition). Then, $\dot{\eta}(\xi = 0) = 0$ and $\eta(\xi = \infty) = 0$. Now consider steady-state solutions corresponding to points on the segment OP in Fig. 2. Since $\theta_0 \leqslant \theta_{0m} \leqslant 1-i$ for these solutions, $\dot{h}(\theta_0) = -1$ for all ξ. If follows from (15) that the second variation $\delta^2 A$ is always positive no matter what the value of η. Clearly, then,

$A[\theta_0]$ is a local minimum. Any small perturbation must raise the value of A, and because A can only fall as time advances it must return eventually to the local minimum $A[\theta_0]$. This means the slightly perturbed state returns to θ_0, which is therefore stable.

6. STABILITY OF THE SEGMENT PQR

Let us now consider the stability of the steady-state solutions corresponding to segment PQR in Fig. 2. For these solutions

$$\delta^2 A = \frac{1}{2}\int_b^\infty (\dot{\eta}^2 + \eta^2)d\xi + \frac{1}{2}\int_0^b [\dot{\eta}^2 - (\alpha i - 1)\eta^2]d\xi \qquad (16)$$

where $\xi = b$ is the value of ξ at which $\theta_0 = 1-i$. The second term in the last integral offers the possibility of making $\delta^2 A$ negative. Can this actually be done?

The sign of $\delta^2 A$ depends on the shape of $\eta(\xi)$ and not on its magnitude since the right-hand side of (16) is a homogeneous function of η. Hence, we can fix $\eta(b) = 1$. Since $\eta(\xi)$ and its derivatives vanish at $\xi = \infty$, it is easy to prove by ordinary variational methods that the first integral will have its smallest value when $\eta = e^{b-\xi}$. {Proof: Set $a[\eta] = (1/2)\int_b^\infty (\dot{\eta}^2 + \eta^2)d\xi$. If $\eta = \eta_* + \varepsilon$, where η_* is the value of η that makes a a minimum, then $a[\eta] = a[\eta_*] + \int_b^\infty (\varepsilon\eta_* + \dot{\varepsilon}\dot{\eta}_*)d\xi + a[\varepsilon]$. The first-order a term becomes $\dot{\eta}\varepsilon|_b^\infty + \int_b^\infty \varepsilon(\eta_* - \ddot{\eta}_*)d\xi$ after partial integration. The integrated term vanishes because $\varepsilon(b) = 0$ [$\eta_*(b) = \eta(b) = 1$] and $\dot{\eta}_*(\infty) = 0$. The integral will vanish if η_* satisfies the Euler-Lagrange equation $\eta_* - \ddot{\eta}_* = 0$, i.e., if $\eta_* = e^{b-\xi}$. Since $a[\varepsilon] > 0$, $a[\eta_*]$ is a minimum. The minimum value is 1/2.}

To make the second term have its smallest (i.e., most negative) value, we want to make η large while keeping $\dot{\eta}$ as small as we can. Note that in choosing η in the interval $0 < \xi < b$ we must take $\eta(b) = 1$ but we need not specify $\dot{\eta}(0)$. Continuity of $\dot{\eta}$ at $\xi = 0$ can be assured by introducing a "fairing segment" in an arbitrarily small neighborhood of $\xi = 0$ whose effect on $\delta^2 A$ can be made as small as desired.

A somewhat tedious calculation (see Appendix) shows the second term has a minimum for

$$\eta = \frac{\cos(\xi\sqrt{\alpha i - 1})}{\cos(b\sqrt{\alpha i - 1})} \tag{17}$$

An integration then gives the following for the minimum value of $\delta^2 A$:

$$\delta^2 A = \frac{1}{2}\left[1 - \sqrt{\alpha i - 1}\,\tan(b\sqrt{\alpha i - 1})\right] \tag{18}$$

The value of b for which (18) first vanishes marks the separation between stable and unstable steady-state solutions. If b is larger than the first root of (18), $\delta^2 A$ will be negative for the trial function (17) and A can be $< A[\theta_0]$. Hence, perturbed states exist from which the system can never return to θ_0. If b is smaller than the first root of (18), then $\delta^2 A$ has a positive minimum for the trial function (17) and A is always $\geqslant A[\theta_0]$. The first root of (18) is

$$b_1 = \frac{\arctan(1/\sqrt{\alpha i - 1})}{\sqrt{\alpha i - 1}} = \frac{\arcsin(1/\sqrt{\alpha i})}{\sqrt{\alpha i - 1}} \tag{19}$$

As required in the Appendix, $b_1\sqrt{\alpha i - 1} < \pi/2$.

7. DIRECT CALCULATION OF b

We can find b for the various steady-state solutions from the relation

$$\xi = \int_{\theta=\theta_m}^{\theta=1-i} d\xi = \int_{1-i}^{\theta_m} \frac{d\theta}{(-\dot{\theta})} \tag{20}$$

Now, in the range $1-i < \theta, \theta_m < 1$, $\dot{\theta}$ is given by (7) as

$$\dot{\theta}^2 = \beta^2 + 2 \int_{\theta}^{\theta_m} [\alpha i(\theta' + i-1) - \theta']d\theta \tag{21a}$$

$$= \theta^2 - \alpha i(\theta + i-1)^2 \tag{21b}$$

where we have substituted from (9b) for β. Then,

$$b = \int_{1-i}^{\theta_m} \frac{d\theta}{[\theta^2 - \alpha i(\theta + i-1)^2]^{1/2}} \tag{22a}$$

$$= \frac{1}{\sqrt{\alpha i-1}}\left\{\arcsin\left(\frac{1}{\sqrt{\alpha i}}\right) - \arcsin\left[\frac{\alpha i(1-i)/\alpha i-1 - \theta_m}{\sqrt{\alpha i}(1-i)/\sqrt{\alpha i-1}}\right]\right\} \tag{22b}$$

Comparing (22b) with (19) we see that if $\theta_m > \alpha i(1-i)/(\alpha i-1)$, then $b > b_1$ and the steady-state solution is unstable, whereas if $\theta_m < \alpha i(1-i)/(\alpha i-1)$, then $b < b_1$ and the steady-state solution is stable. Thus, the entire branch OPQ corresponds to stable solutions and the branch QR to unstable solutions.

8. NUMERICAL EXAMPLE

As an example let us consider an early candidate EBT conductor studied by Elrod et al. (1).

Dimensions	2.26 mm × 2.70 mm
T_b	4.2 K
B	8 T
T_c	5.6 K
$J_{c(NbTi)}$	560 $A\cdot mm^{-2}$
RRR_{Cu}	160
Cu/NbTi	3
$J_{overall}$	90 $A\cdot mm^{-2}$
A	6.10 mm^2
P	2.7 mm

ρ	6.45×10^{-10} Ω m (including magnetoresistance)
k	1.60 $W \cdot cm^{-1} \cdot K^{-1}$ (Wiedemann-Franz law)
I_c	854 A
I	549 A
i	0.643
h	0.20 $W \cdot cm^{-2} \cdot K^{-1}$
α	10.2
β_Q	0.388
Q	78.8 mW
V	144 μV
R	0.262 $\mu\Omega$

This resistance corresponds to a 2.47-mm separation in the filaments, a rather substantial defect.

9. REDUCTION IN MPZ ENERGY DUE TO A RESISTIVE DEFECT

The presence of a local defect reduces the stability of the superconductor to pulsed energy input at the location of the defect compared with its stability elsewhere. In the absence of a defect, the usual way of estimating the stability against pulsed energy input (quench energy) is to use the MPZ energy as a figure of merit (2).* The MPZ energy is the energy required to form state R in Fig. 2. So, in the presence of a defect, we might be inclined to use the energy difference between the upper solution (branch QR) and the lower solution (branch OPQ) as a figure of merit. In this way, we get an idea of how stability declines as the size of the defect increases.

*That the quench energy is of the order of magnitude of the MPZ energy is reasonable, but no rigorous quantitative relation between them is presently known. It is not known, for example, if one can quench with less than the MPZ energy although it is easy to see that initial temperature distributions that recover can be chosen with more than the MPZ energy.

The energy required to form a steady state is

$$E = 2 \int_0^\infty S(T - T_b)A\,dx = 2SA(T_c - T_b)\left(\frac{Ak}{Ph}\right)^{1/2} \int_0^\infty \theta\,d\xi \qquad (23a)$$

$$= 2SA(T_c - T_b)\left(\frac{Ak}{Ph}\right)^{1/2} \int_0^{\theta_m} \frac{\theta\,d\theta}{(-\dot{\theta})} \qquad (23b)$$

$$= 2SA(T_c - T_b)\left(\frac{Ak}{Ph}\right)^{1/2} \cdot \varepsilon \qquad (23c)$$

The computation of $\Delta\varepsilon$, the difference of ε between the upper and lower states, as a function of β is straightforward but tedious, and we give only the results

$$\Delta\varepsilon = \frac{\alpha i}{\alpha i-1}(1-i)\left\{1 - \frac{\beta}{1-i} + (\alpha i-1)^{-1/2} \times \right.$$
$$\left. \times \left[\arcsin\frac{1}{\sqrt{\alpha i}} + \arcsin\sqrt{1 - \left(\frac{\beta}{\beta_Q}\right)^2}\right]\right\}, \quad 0 < \beta < 1-i \qquad (24a)$$

$$= \frac{\alpha i}{\alpha i-1}(1-i)\cdot 2(\alpha i-1)^{-1/2}\arcsin\sqrt{1 - \left(\frac{\beta}{\beta_Q}\right)^2}, \quad 1-i < \beta < \beta_Q \qquad (24b)$$

Figure 3 shows $\Delta\varepsilon$ versus β for the case $\alpha = 10$, $i = 0.65$. The dimensionless energy $\Delta\varepsilon$ falls almost linearly until β reaches $1-i$. At this point θ_m reaches $1-i$, too, and part of the conductor in the heated zone goes into current sharing. The Joule heat production then causes a faster than linear decline in $\Delta\varepsilon$ with increasing β. As a rough approximation one might use $\Delta\varepsilon = (1 - \beta/\beta_Q)\cdot\Delta\varepsilon(\beta = 0)$.

10. GENERALIZATION

The conclusions we have drawn so far apply to the special case governed by the forms (6c) and (4b) of $h(\theta)$ and were demonstrated by direct calculation, but some of these conclusions are more general and can be proved for functions $h(\theta)$ having only a few key properties of the functions prescribed by (6c) and (4b). These general conclusions are especially useful in dealing with the temperature-dependent heat transfer coefficients that characterize boiling heat transfer.

Suppose $h(\theta)$ has the form shown in Fig. 4. It has three lobes corresponding to three intersections of the heating and cooling curves (one is at the origin!). The area of the lobe AD'''D'B is larger than that of the lobe OFA — this means the conductor is not cold-end stable. When $\beta = 0$, θ_m, given by (8), can have three values: $\theta_m = 0$, the value at line DD' (for which area OFA = area ADD'), and the value at line EE' (for which area ABD'D''' = area OFA + area BEE'). When β is slightly greater than zero, these values of θ_m move to CC', D''D''', and E''E''', respectively. When $\beta^2/2$ becomes equal to the area under the first lobe OFA, the first two values of θ_m become equal at point A. For larger β, the only value of θ_m that occurs is that on the branch E''E'''. Thus, the plot of β^2 versus θ_m looks like Fig. 2 (see Fig. 5). The three branches of the curve of β^2 versus θ_m are labeled C, D, and E to conform with the notation of Fig. 4.

As before, this plot of β^2 versus θ_m is identical to the plot of $\dot{\theta}^2$ versus θ. As before, then, values of θ_m on branch E cannot occur in normal zones with cold ends because branch E is not connected continuously to the origin. Finally, then, we see that the maximum allowable value of β^2 is

$$\beta_A^2 = 2 \times \text{area of first lobe of OFA} \qquad (25)$$

If we apply this to the $h(\theta)$ shown in Fig. 1, we get (10), as we must. It is easy to see now that high heat transfer in the nucleate boiling regime will increase the maximum value of β^2 severalfold over the value that would be calculated with film boiling heat transfer coefficients.

What about the stability of the branches C and D? We can prove the instability of the branch D easily enough because we need to find only one trial function that makes $\delta^2 A$ in (15) negative. A discussion in the book of Altov et al. (3) suggests using $\eta = \dot{\theta}_0$, and this succeeds, as the short computation below shows:

$$2\delta^2 A = \int_{0+}^{\infty} (\dot{\eta}^2 - h'(\theta_0)\eta^2]d\xi \qquad (26a)$$

$$= \eta\dot{\eta}\Big|_{0+}^{\infty} - \int_{0+}^{\infty} \eta[\ddot{\eta} + h'(\theta_0)\eta]d\xi \qquad (26b)$$

$$= -\eta(0+)\ \dot{\eta}(0+) = -\dot{\theta}_0(0+)\ddot{\theta}_0(0+) \qquad (26c)$$

$$= -\beta h[\theta_0(0)] = -\beta h(\theta_m) \qquad (26d)$$

The integral in (26b) vanishes because $\dot{\theta}_0$ satisfies the equation $\ddot{\eta} + h'(\theta_0)\eta = 0$, as we can see by differentiating (6a). If $\theta_m > \theta_A$, then $h(\theta_m) > 0$ and $\delta^2 A < 0$, proving the instability of branch D. [The fact that the trial function $\eta = \dot{\theta}_0$ does not satisfy the boundary condition $\dot{\eta}(0) = 0$ is not of importance, for we can satisfy this condition by introducing a "fairing" segment in an arbitrarily small neighborhood of $\xi = 0$, whose effect on $\delta^2 A$ can be made as small as desired.]

Proving the stability of branch C is more difficult because we must show $\delta^2 A > 0$ for <u>all</u> trial functions η. For $\theta_m < \theta_F$, $h'(\theta_0) < 0$, so the branch C is stable for $\theta_m < \theta_F$. However, the region $\theta_F < \theta_m < \theta_A$ presents difficulties I have not overcome.

APPENDIX

In this appendix we consider the behavior of the functional $A[y] = (1/2) \int_0^b (\dot{y}^2 - k^2y^2)dx$ with y confined to the family of bounded, continuously differentiable functions for which $y(b) = 1$. (As we have seen, by adding "fairing" segments we can achieve any slope we want at the endpoints without changing the value of A. In fact, we can make the slope have any values we want at any finite number of points without changing the value of A.)

Suppose the intercept of y at $x = 0$ equals a. What curve $y(x)$ joining the points (0,a) and (b,1) makes $A[y]$ a minimum? This is a well-posed variational problem, the answer to which is

$$y = \frac{a \sin(k[b - x]) + \sin kx}{\sin kb} \tag{A1}$$

For this y, the corresponding value of A is

$$A = \frac{1}{2} \int_0^b (\dot{y}^2 - k^2 y^2) dx$$

$$= \frac{1}{2} y\dot{y} \Big|_0^b - \frac{1}{2} \int_0^b y(\ddot{y} + k^2 y) dx$$

$$= \frac{1}{2} [\dot{y}(b) - a\dot{y}(0)]$$

$$= \frac{k}{2 \sin kb} [a^2 \cos kb - 2a + \cos kb] \tag{A2}$$

This quantity will have a minimum (if $kb < \pi/2$) when $a = \sec kb$. The minimum value of A is then $-k(\tan kb)/2$, and the corresponding value of y is $\cos kx/\cos kb$. Interestingly, $\dot{y}(0) = 0$, so no "fairing" function is necessary.

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NOTATION

A	cross-sectional area of the conductor
$A[\theta]$	action functional defined in (13)
E	energy required to form a steady state
$g(T)$	current-sharing function defined in (2)
$g(\theta)$	current-sharing function defined in (4b)
h	heat transfer coefficient
$h(\theta)$	function defined in (6c)
I	transport current
I_c	critical current
i	I/I_c
k	average longitudinal thermal conductivity of the conductor
L	Lagrangean defined in (11)
P	cooled perimeter of the conductor
Q	steady plane source of heat
S	average volumetric heat capacity of the conductor
T	temperature of the conductor
T_b	helium temperature
T_c	critical temperature
T_{cs}	current-sharing threshold temperature
t	time
x	longitudinal distance along the conductor
α	Stekly parameter defined in (3d)
β	parameter defined in (3e)
ε	dimensionless energy defined in (23c); sometimes used to denote an arbitrary, small quantity
ξ	dimensionless distance defined in (3c)
θ	dimensionless temperature defined in (3a)
ρ	average longitudinal resistivity of the conductor in the normal state
τ	dimensionless time defined by (3b)

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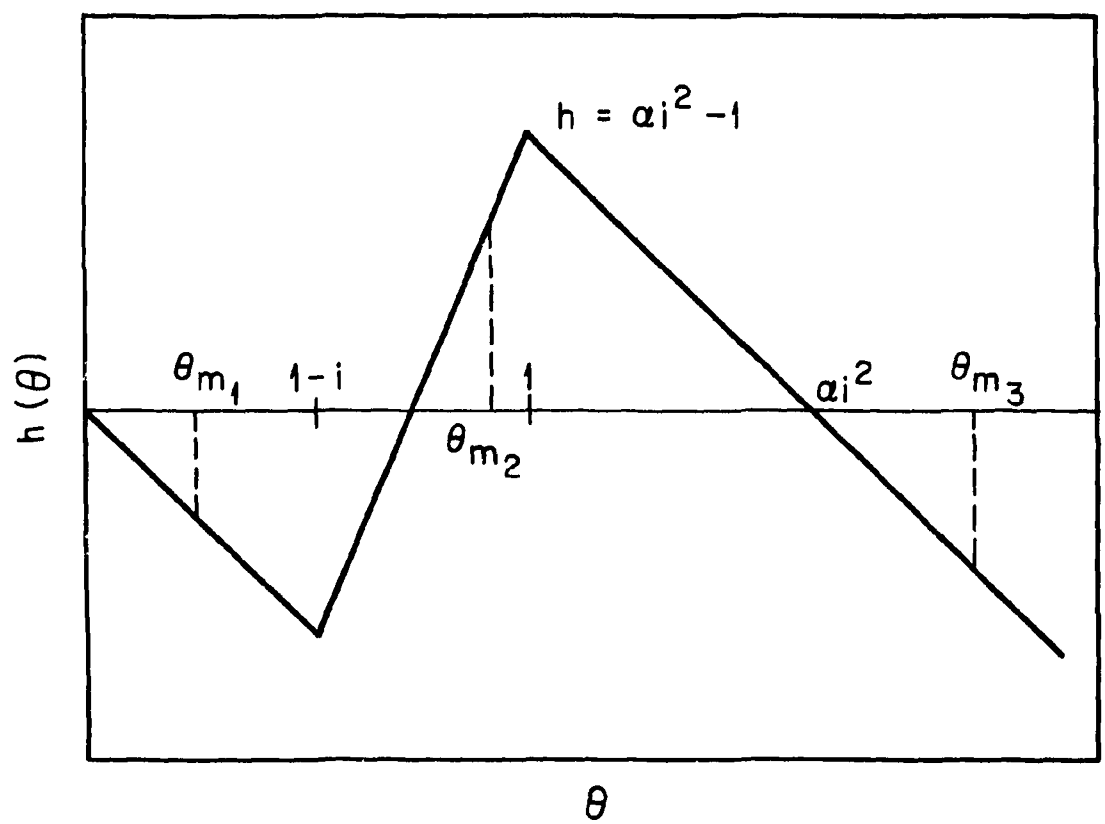

Fig. 1. Sketch of the function h(θ) showing that there are at most three steady-state solutions satisfying (8).

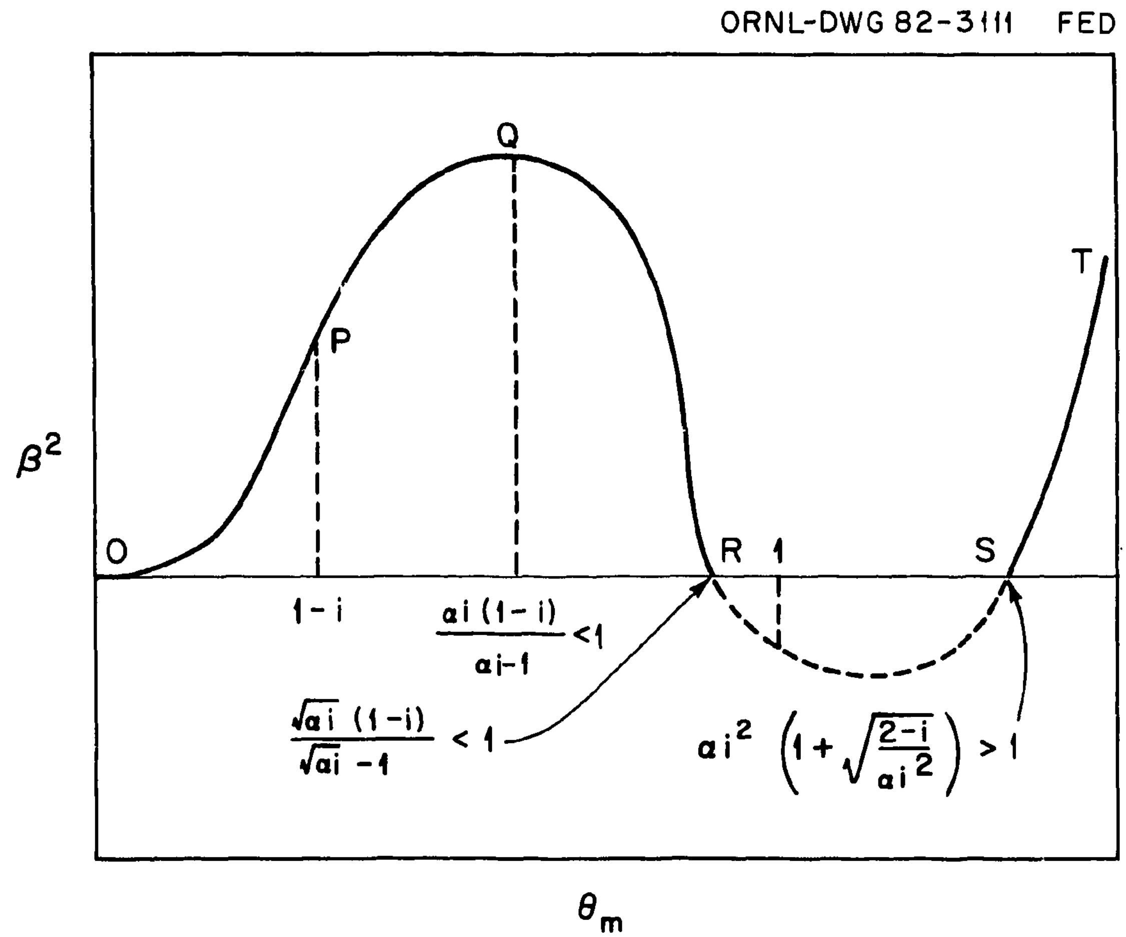

Fig. 2. Sketch of β^2 vs θ_m.

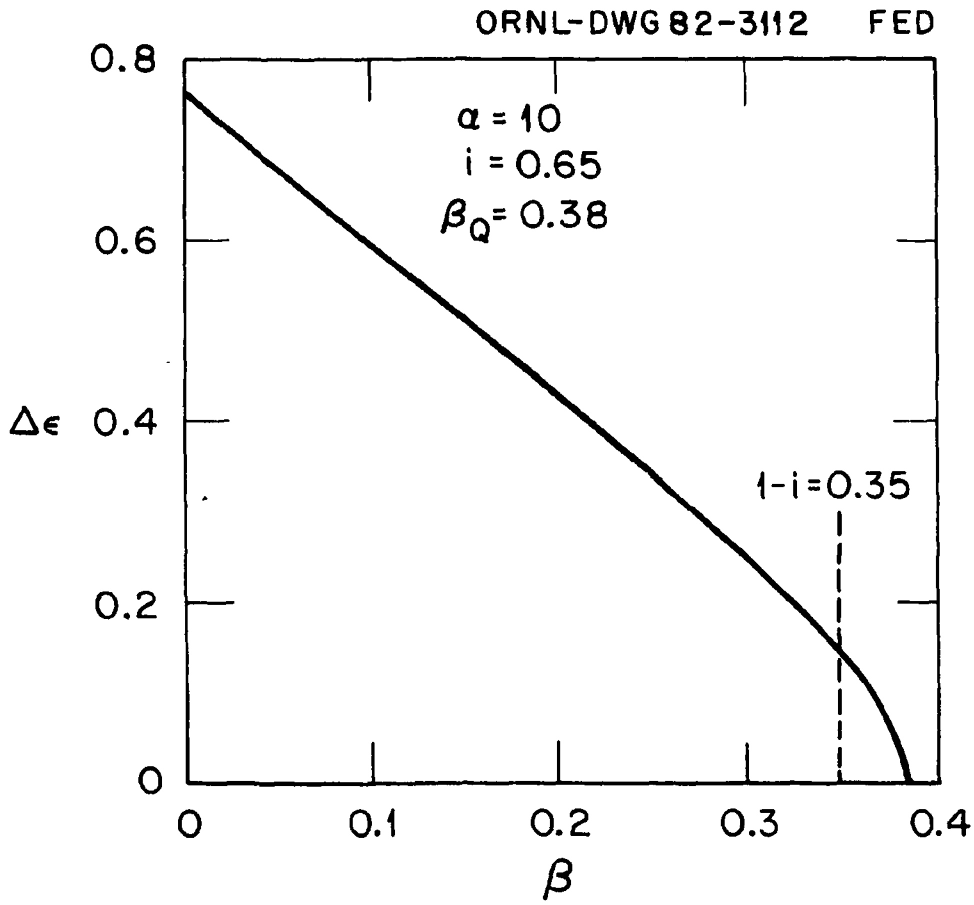

Fig. 3. The difference in the dimensionless MPZ energy $\Delta\varepsilon$ plotted vs β for the case $\alpha = 10$, $i = 0.65$.

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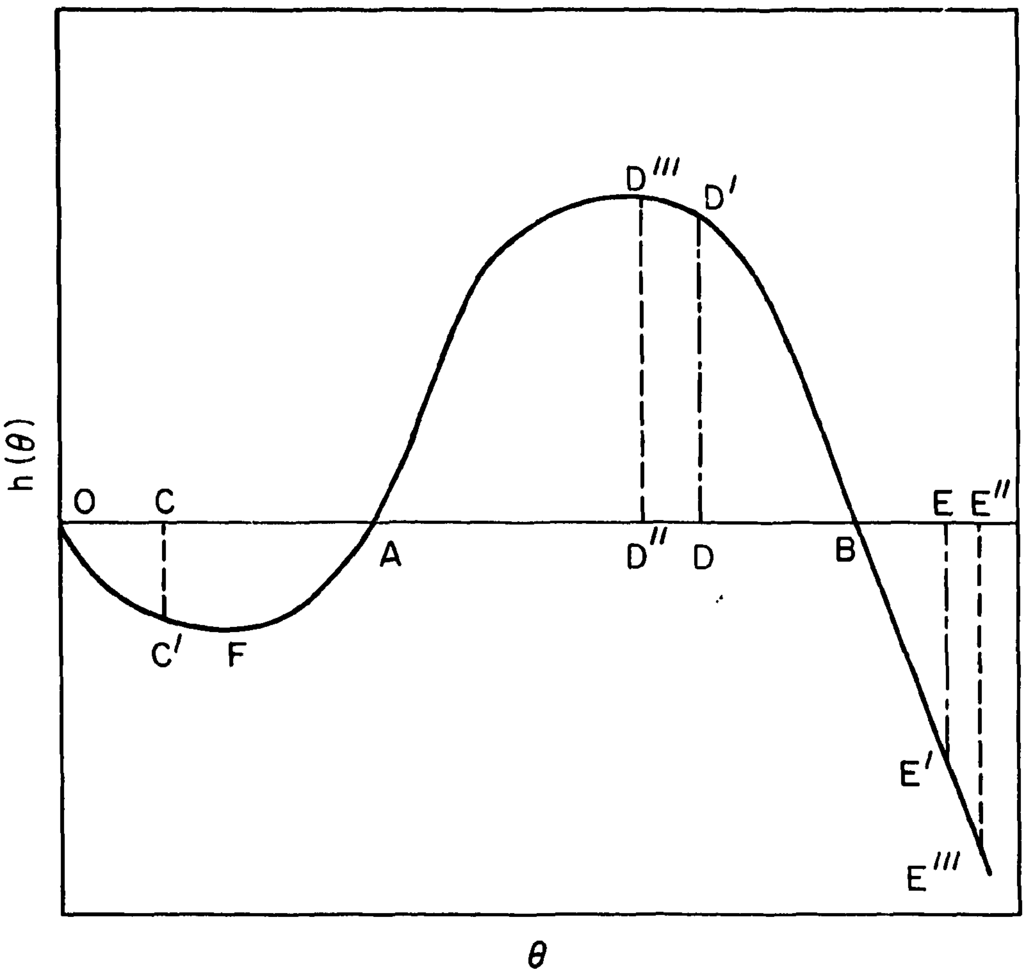

Fig. 4. Schematic drawing of h(θ) vs θ.

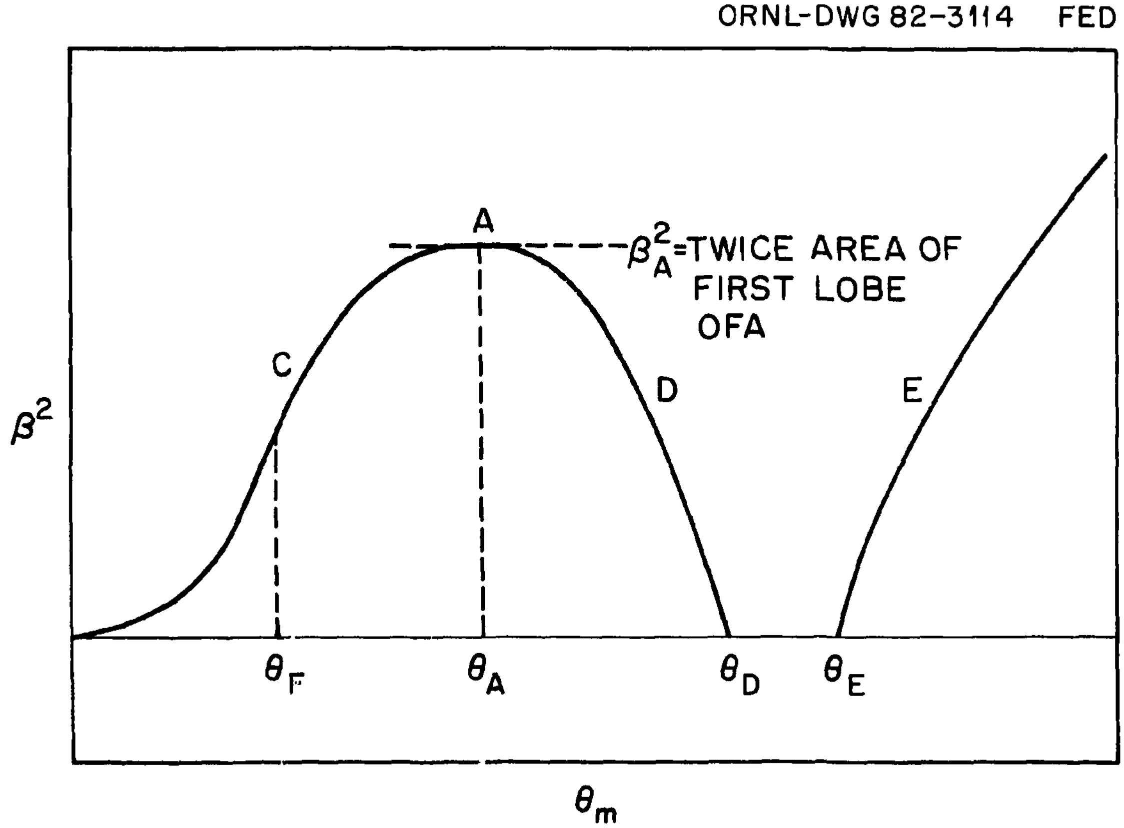

Fig. 5. Sketch of β^2 vs θ_m.